Treewidth computation and extremal combinatorics^{*}

Fedor V. Fomin and Yngve Villanger

Department of Informatics, University of Bergen, N-5020 Bergen, Norway {fedor.fomin,yngve.villanger}@ii.uib.no

Abstract. For a given graph G and integers $b, f \geq 0$, let S be a subset of vertices of G of size b+1 such that the subgraph of G induced by S is connected and S can be separated from other vertices of G by removing f vertices. We prove that every graph on n vertices contains at most $n\binom{b+f}{b}$ such vertex subsets. This result from extremal combinatorics appears to be very useful in the design of several enumeration and exact algorithms. In particular, we use it to provide algorithms that for a given n-vertex graph G

- compute the treewidth of G in time $\mathcal{O}(1.7549^n)$ by making use of exponential space and in time $\mathcal{O}(2.6151^n)$ and polynomial space;
- decide in time $\mathcal{O}((\frac{2n+k+1}{3})^{k+1} \cdot n^6)$ if the treewidth of G is at most k;
- list all minimal separators of G in time $\mathcal{O}(1.6181^n)$ and all potential maximal cliques of G in time $\mathcal{O}(1.7549^n)$.

This significantly improves previous algorithms for these problems.

1 Introduction

The aim of exact algorithms is to solve exactly hard problems exponentially faster than bruteforce search. The first papers in the area date back to the sixties and seventies [18,27]. For the
last two decades the amount of literature devoted to this topic has been tremendous and it is
impossible to give here a list of representative references without missing significant results. Recent
surveys [14, 20, 25, 29] provide a comprehensive information on exact algorithms. It is very natural
to assume the existence of strong links between the area of exact algorithms and some areas of
extremal combinatorics, especially the part of extremal combinatorics which studies the maximum
(minimum) cardinalities of a system of subsets of some set satisfying certain properties. Strangely
enough, there are not so many examples of such links in the literature, and the majority of exact
algorithms are based on the so-called branching (backtracking) technique which traces back to the
works of Davis, Putnam, Logemann, and Loveland [11, 12].

In this paper, we prove a combinatorial lemma which appears to be very useful in the analysis of certain enumeration and exact algorithms. For a vertex v of a graph G and integers $b, f \geq 0$, let t(b,f) be the maximum number of connected induced subgraphs of G of size b+1 such that the intersection of all these subgraphs is nonempty and each such a subgraph has exactly f neighbors (a neighbor of a subgraph H is a vertex of $G \setminus H$ which is adjacent to a vertex of H). Then Combinatorial Lemma states that $t(b,f) \leq {b+f \choose b}$ (and it is easy to check that this bound is tight). This can be seen as a variation of Bollobáss Theorem [7], which is one of the corner-stones in extremal set theory. (See Section 9.2.2 of [21] for detailed discussions on Bollobáss Theorem and its variants.)

We use Combinatorial Lemma to obtain faster algorithm for a number of problems related to the treewidth of a graph. The treewidth is a fundamental graph parameter from Graph Minors Theory by Robertson and Seymour [24] and it has numerous algorithmic applications, see the surveys [4,6]. Despite of the importance of treewidth almost nothing is known on how to cope

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with its intractability. The problems to compute the treewidth is known to be NP-hard [1] and the best known approximation algorithm for treewidth has a factor $\sqrt{\log OPT}$ [13]. It is an old open question whether the treewidth can be approximated within a constant factor. Treewidth is known to be fixed parameter tractable. Moreover, for any fixed k, there is a linear time algorithm due to Bodlaender [3] computing the treewidth of graphs of treewidth at most k. Unfortunately, huge hidden constants in the running time of Bodlaender's algorithm are a serious obstacle to its implementation. For small values of k, the classical algorithm of Arnborg, Corneil and Proskurowski [1] from 1987 which runs in time $\mathcal{O}(n^{k+2})$ can be used to decide if the treewidth of a graph is at most k. For example, a modified version of this algorithms has been implemented by Shoikhet and Geiger [26] to resolve the treewidth of randomly generated graphs of treewidth at most 10 with up to 100 vertices. The first exact algorithm computing the treewidth of an n-vertex graph is due to Fomin et al. [15] and has running time $\mathcal{O}(1.9601^n)$. Later these results were improved in [16, 28] to $\mathcal{O}(1.8899^n)$. Both algorithms use exponential space. The fastest polynomial space algorithm for treewidth prior to this work is due to Bodlaender et al. [5] and runs in time $\mathcal{O}(2.9512^n)$.

Our results. We introduce a new (exponential space) algorithm computing the treewidth of a graph G on n vertices in time $\mathcal{O}(1.7549^n)$ and a polynomial space algorithm computing the treewidth in time $\mathcal{O}(2.6151^n)$. We also show that if the treewidth of G is at most k, then it can be computed in time $\mathcal{O}((\frac{2n+k+1}{3})^{k+1} \cdot n^6)$. For $k \geq 5$, this is an improvement over the running time $\mathcal{O}(n^{k+2})$ from the classical algorithm of Arnborg et al. Running times of all these algorithms strongly depend on possibilities of fast enumeration of specific structures in a graph, namely, potential maximal cliques, and minimal separators [5, 8, 9, 15, 28]. Combinatorial Lemma becomes crucial in obtaining new combinatorial bounds and enumeration algorithms for minimal separators and potential maximal cliques, which, in turn, provides faster algorithms for treewidth.

Similar improvements in running times from $\mathcal{O}(1.8899^n)$ to $\mathcal{O}(1.7549^n)$ can be obtained for a number of results in the literature on problems related to treewidth (we skip definitions here). For example, by combining the ideas from [15] it is possible to compute the fill-in of a graph in time $\mathcal{O}(1.7549^n)$. Another example are the treelength and the Chordal Sandwich problem [23] which also can be solved in time $\mathcal{O}(1.7549^n)$ by making use of our technique.

The remaining part of the paper is organized as follows. In the next section we provide definitions and preliminary results. In Section 3, we prove our main combinatorial tool, Combinatorial Lemma. By making use of this tool, in Section 4, we prove combinatorial bounds on the number of minimal separators and potential maximal cliques and obtain algorithm enumerating these structures. These results form the basis for all our algorithms computing the treewidth of a graph presented in Sections 5, 6, and 7.

2 Preliminaries

We denote by G = (V, E) a finite, undirected and simple graph with |V| = n vertices and |E| = m edges. For any non-empty subset $W \subseteq V$, the subgraph of G induced by W is denoted by G[W]. We say that a vertex set $S \subseteq V$ is connected if G[S] is connected.

The neighborhood of a vertex v is $N(v) = \{u \in V : \{u, v\} \in E\}$ and for a vertex set $S \subseteq V$ we set $N(S) = \bigcup_{v \in S} N(v) \setminus S$. A clique C of a graph G is a subset of V such that all the vertices of C are pairwise adjacent.

Minimal separators. Let u and v be two non adjacent vertices of a graph G = (V, E). A set of vertices $S \subseteq V$ is an u, v-separator if u and v are in different connected components of the graph $G[V \setminus S]$. A connected component C of $G[V \setminus S]$ is a full component (associated to S) if N(C) = S. S is a minimal u, v-separator of G if no proper subset of S is an u, v-separator. We say that S is a

minimal separator of G if there are two vertices u and v such that S is a minimal u, v-separator. Notice that a minimal separator can be strictly included in another one. We denote by Δ_G the set of all minimal separators of G.

We need the following result due to Berry et al. [2] (see also Kloks et al. [22])

Proposition 1 ([2]). There is an algorithm listing all minimal separators of an input graph G in $\mathcal{O}(n^3|\Delta_G|)$ time.

The following proposition is an exercise in [17].

Proposition 2 (Folklore). A set S of vertices of G is a minimal a, b-separator if and only if a and b are in different full components of S. In particular, S is a minimal separator if and only if there are at least two distinct full components of S.

Potential maximal cliques. A graph H is chordal (or triangulated) if every cycle of length at least four has a chord, i.e. an edge between two non-consecutive vertices of the cycle. A triangulation of a graph G = (V, E) is a chordal graph H = (V, E') such that $E \subseteq E'$. H is a minimal triangulation if for any set E'' with $E \subseteq E'' \subset E'$, the graph F = (V, E'') is not chordal.

A set of vertices $\Omega \subseteq V$ of a graph G is called a *potential maximal clique* if there is a minimal triangulation H of G such that Ω is a maximal clique of H. We denote by Π_G the set of all potential maximal cliques of G.

The following result on the structure of potential maximal cliques is due to Bouchitté and Todinca.

Proposition 3 ([8]). Let $K \subseteq V$ be a set of vertices of the graph G = (V, E). Let $C(K) = \{C_1(K), \ldots, C_p(K)\}$ be the set of the connected components of $G[V \setminus K]$ and let $S(K) = \{S_1(K), S_2(K), \ldots, S_p(K)\}$ where $S_i(K)$, $i \in \{1, 2, \ldots, p\}$, is the set of those vertices of K which are adjacent to at least one vertex of the component $C_i(K)$. Then K is a potential maximal clique of G if and only if:

- 1. $G[V \setminus K]$ has no full component associated to K, and
- 2. the graph on the vertex set K obtained from G[K] by completing each $S_i \in \mathcal{S}(K)$ into a clique, is a complete graph.

The following result is also due to Bouchitté and Todinca.

Proposition 4 ([8]). There is an algorithm that, given a graph G = (V, E) and a set of vertices $K \subseteq V$, verifies if K is a potential maximal clique of G. The time complexity of the algorithm is $\mathcal{O}(nm)$.

Treewidth. A tree decomposition of a graph G = (V, E) is a pair (χ, T) in which $T = (V_T, E_T)$ is a tree and $\chi = \{\chi_i | i \in V_T\}$ is a family of subsets of V such that: (1) $\bigcup_{i \in V_T} \chi_i = V$; (2) for each edge $e = \{u, v\} \in E$ there exists an $i \in V_T$ such that both u and v belong to χ_i ; and (3) for all $v \in V$, the set of nodes $\{i \in V_T | v \in \chi_i\}$ forms a connected subtree of T. To distinguish between vertices of the original graph G and vertices of T, we call vertices of T nodes and their corresponding χ_i 's bags. The maximum size of a bag minus one is called the width of the tree decomposition. The treewidth of a graph G, tw(G), is the minimum width over all possible tree decompositions of G.

An alternative definition of treewidth is via minimal triangulations. The *treewidth* of a graph G is the minimum of $\omega(H)-1$ taken over all triangulations H of G. (By $\omega(H)$ we denote the maximum clique-size of a graph H.)

Our algorithm for treewidth is based on the following result.

Proposition 5 ([15]). There is an algorithm that, given a graph G together with the list of its minimal separators Δ_G and the list of its potential maximal cliques Π_G , computes the treewidth of G in $\mathcal{O}(n^3(|\Pi_G|+|\Delta_G|))$ time. Moreover, the algorithm constructs an optimal triangulation for the treewidth.

3 Combinatorial Lemma

The following lemma is our main combinatorial tool.

Lemma 1 (Combinatorial Lemma). Let G = (V, E) be a graph. For every $v \in V$, and $b, f \ge 0$, the number of connected vertex subsets $B \subseteq V$ such that

(i) $v \in B$,

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- (ii) |B| = b + 1, and
- (iii) |N(B)| = f

is at most $\binom{b+f}{b}$.

Proof. Let v be a vertex of a graph G=(V,E). For b+f=0 Lemma trivially holds. We proceed by induction assuming that for some k>0 and every b and f such that $b+f\leq k-1$, Lemma holds. For b and f such that b+f=k we define $\mathcal B$ as the set of sets B satisfying (i),(ii),(iii). We claim that

$$|\mathcal{B}| \le \binom{b+f}{b}.$$

Since the claim always holds for b = 0, let us assume that b > 0.

Let $N(v) = \{v_1, v_2, \dots, v_p\}$. For $1 \le i \le p$, we define \mathcal{B}_i as the set of all connected subsets B such that

- Vertices $v, v_i \in B$,
- For every $j < i, v_j \notin B$,
- -|B| = b + 1,
- -|N(B)| = f.

Let us note, that every set B satisfying the conditions of the lemma is in some set \mathcal{B}_i for some i, and that for $i \neq j$, $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$. Therefore,

$$|\mathcal{B}| = \sum_{i=1}^{p} |\mathcal{B}_i| \tag{1}$$

For every i > f+1, $|\mathcal{B}_i| = 0$ (this is because for every $B \in B_i$, the set N(B) contains vertices v_1, \ldots, v_{i-1} and thus is of size at least f+1.) Thus (1) can be rewritten as follows

$$|\mathcal{B}| = \sum_{i=1}^{f+1} |\mathcal{B}_i| \tag{2}$$

Let G_i be the graph obtained from G by contracting edge $\{v, v_i\}$ (removing the loop, reduce double edges to single edges, and calling the new vertex by v) and removing vertices v_1, \ldots, v_{i-1} . Then the cardinality of \mathcal{B}_i is equal to the number of the connected vertex subsets B of G_i such that

- $-v \in B$,
- -|B|=b,
- -|N(B)| = f i + 1.

By the induction assumption, this number is at most $\binom{f+b-i}{b-1}$ and (2) yields that

$$|\mathcal{B}| = \sum_{i=1}^{f+1} |\mathcal{B}_i| \le \sum_{i=1}^{f+1} {f+b-i \choose b-1} = {b+f \choose b}.$$

The inductive proof of Combinatorial Lemma can be easily turned into recursive enumeration algorithm (we skip the proof here).

Lemma 2. All vertex sets of size b+1 with f neighbors in a graph G can be enumerated in time $\mathcal{O}(n\binom{b+f}{b})$ by making use of polynomial space.

4 Combinatorial bounds

In this section we provide combinatorial bounds on the number of minimal separators and potential maximal cliques in a graph. Both bounds are the applications of Combinatorial Lemma and will be used in the algorithmic results in the next sections.

4.1 Minimal separators

Theorem 1. Let Δ_G be the set of all minimal separators in a graph G on n vertices. Then $|\Delta_G| = \mathcal{O}(1.6181^n)$.

Proof. For $1 \le i \le n$, let f(i) be the number of all minimal separators in G of size i. Then

$$|\Delta_G| = \sum_{1}^{n} f(i) \tag{3}$$

Let S be a minimal separator of size αn , where $0 < \alpha < 1$. By Proposition 2, S has two full components C_1 and C_2 . Let us assume that $|C_1| \le |C_2|$. Then $|C_1| \le (1-\alpha)n/2$. By the definition of a full component, we have $N(C_1) = S$. Thus $f(\alpha n)$ is at most the number of connected sets C of size at most $(1-\alpha)n/2$ with neighborhoods of size $|N(C)| = \alpha n$. Hence, to bound $f(\alpha n)$ we can use Combinatorial Lemma for every vertex of G.

By Lemma 1, for every vertex v, the number of full components of size $b+1=(1-\alpha)n/2$ containing v and with neighborhoods of size αn is at most

$$\binom{b+1+\alpha n}{b} \leq \binom{(1+\alpha)n/2}{b}.$$

Therefore

$$f(\alpha n) \le n \cdot \sum_{i=1}^{(1-\alpha)n/2} {i+\alpha n \choose i} < n \cdot \sum_{i=1}^{(1-\alpha)n/2} {(1+\alpha)n/2 \choose i}$$

$$\tag{4}$$

For $\alpha \leq 1/3$, we have

$$\sum_{i=1}^{(1-\alpha)n/2} \binom{(1+\alpha)n/2}{i} < 2^{(1+\alpha)n/2} < 2^{2n/3} < 1.59^n,$$

and thus

$$\sum_{i=1}^{n/3} f(i) = \mathcal{O}(1.59^n) \tag{5}$$

For $\alpha \geq 1/3$,

$$\sum_{i=1}^{(1-\alpha)n/2} \binom{(1+\alpha)n/2}{i} < (1-\alpha)n/2 \cdot \binom{(1+\alpha)n/2}{(1-\alpha)n/2} < n \cdot \binom{(1+\alpha)n/2}{(1-\alpha)n/2},$$

and thus

$$f(\alpha n) < n^2 \cdot \binom{(1+\alpha)n/2}{(1-\alpha)n/2}.$$

By making use of the fact that the number of subsets of size $\beta \cdot n$ of a set of size n is of size $O^*((\beta^{-\beta} \cdot (1-\beta)^{\beta-1})^n)^1$ and computer, one can show that

$$n^3 \binom{(1+\alpha)n/2}{(1-\alpha)n/2} = \mathcal{O}(1.6181^n).$$

Therefore,

$$\sum_{i=n/3}^{n} f(i) = \mathcal{O}(1.6181^{n}) \tag{6}$$

Finally, the theorem follows from (3),(5) and (6).

4.2 Potential maximal cliques

Definition 1 ([8]). Let Ω be a potential maximal clique of a graph G and let $S \subset \Omega$ be a minimal separator of G. We say that S is an active separator for Ω , if Ω is not a clique in the graph obtained from G by completing all the minimal separators contained in Ω , except S. A potential maximal clique Ω containing an active separator (for Ω) is called a nice potential maximal clique.

We need the following result by Bouchitté and Todinca.

Proposition 6 ([9]). Let Ω be a potential maximal clique of G = (V, E), let u be a vertex of G, and let $G' = G[V \setminus \{u\}]$. Then one of the following holds:

- 1. Either Ω , or $\Omega \setminus \{u\}$ is a potential maximal clique of G';
- 2. $\Omega = S \cup \{u\}$, where S is a minimal separator of G;
- 3. Ω is a nice potential maximal clique.

Let Π_n be the maximum number of nice potential maximal cliques that can be contained in a graph on n vertices. Proposition 6 is useful to bound the number of potential maximal cliques in a graph by the number of minimal separators Δ_G and Π_n .

Lemma 3. For any graph G = (V, E), $|\Pi_G| \le n(n|\Delta_G| + \Pi_n)$.

Proof. Let $v_1, v_2, ..., v_n$ be an ordering of V and let $V_i = \bigcup_{j=1}^i v_j$. The proof of the lemma follows from the following claim $\Pi_{G[V_{i+1}]} \leq \Pi_{G[V_i]} + n|\Delta_G| + \Pi_n$ which can be proved by making inductive use of Proposition 6.

Definition 2. Let $\Omega \in \Pi_G$, $v \in \Omega$, and C_v be the connected component of $G[V \setminus (\Omega \setminus \{v\})]$ containing v. We call the pair (C_v, v) by vertex representation of Ω .

Lemma 4. Let (C_v, v) be a vertex representation of Ω . Then $\Omega = N(C_v) \cup \{v\}$.

¹ To suppress polynomially bounded factors we use a modified big-Oh notation. For functions f and g we write $f(n) = O^*(g(n))$ if f(n) = O(g(n)poly(n)), where poly(n) is a polynomial.

Proof. By Proposition 3, every vertex $u \in \Omega \setminus \{v\}$, is either adjacent to v, or there exists a connected component C of $G[V \setminus \Omega]$ such that $u, v \in N(C)$. Since $C \subset C_v$, we have that $\Omega \setminus \{v\} \subseteq N(C_v)$. Every connected component C of $G[V \setminus \Omega]$ that contains $v \in N(C)$ is contained in C_v and $N(C) \subset \Omega$ for every C, therefore $\Omega \setminus \{v\} = N(C_v)$.

We need also the following result from [28].

Proposition 7 ([28]). Let Ω be a nice potential maximal clique of size αn in a graph G. There exists a vertex representation (C_v, v) of Ω such that $|C_v| \leq \lceil \frac{2(1-\alpha)n}{3} \rceil$.

Now everything is settled to apply Combinatorial Lemma.

Lemma 5. The number of nice potential maximal cliques in a graph G = (V, E) is $\mathcal{O}(1.7549^n)$.

Proof. By Proposition 7, for every nice potential maximal clique Ω of cardinality αn , there exists a vertex representation (C_v, v) of Ω such that $|C_v| \leq \lceil 2n(1-\alpha)/3 \rceil$. Let b+1 be the number of vertices in C_v . By Lemma 1, for every vertex v, the number of such pairs (C_v, v) is at most

$$\sum_{i=1}^{2(1-\alpha)n/3} \binom{(2+\alpha)n/3}{i}$$

As in the proof of Theorem 1, one can compute that the value of the above sum, and thus the number of nice potential maximal cliques, is $\mathcal{O}(1.7549^n)$.

By combining Lemma 3, 5 and Theorem 1 we arrive at the main result of this subsection.

Theorem 2. For any graph G, $|\Pi_G| = \mathcal{O}(1.7549^n)$.

5 Exponential space exact algorithm for treewidth

To compute the treewidth of a graph we want to use Proposition 5. To be able to do this we should list minimal separators and potential maximal cliques. While by Proposition 1 and Theorem 1, all minimal separators can be listed in time $\mathcal{O}(1.6181^n)$, we have to explain here how to list potential maximal cliques.

Lemma 6. For any graph G, all potential maximal cliques of G can be listed in time $\mathcal{O}(1.7549^n)$.

Proof. The algorithm listing potential maximal cliques has two phases. First it generates all nice potential maximal cliques (and here again we use Combinatorial Lemma). In the second phase by making use of Proposition 6, we generate potential maximal cliques from minimal separators and the nice ones.

Generating nice potential maximal cliques. For every vertex v and $0 < \alpha < 1$ we generate all nice potential maximal cliques of size αn containing vertex v. By Lemma 4, to generate all such potential maximal cliques it is sufficient to generate all their vertex representations. By Proposition 7, we can restrict our search to vertex representations of size at most $\lceil 2n(1-\alpha)/3 \rceil$. For every vertex representation C, we have that $|N(C) \cup \{v\}| = \alpha n$ (the set $N(C) \cup \{v\}$ should form a potential maximal clique). For each such subset we use Proposition 4 to check if $N(C) \cup \{v\}$ is a potential maximal clique.

So finally, the problem of generating all nice potential maximal cliques boils down to the following problem: List all connected sets B such that $|B| \leq \lceil 2n(1-\alpha)/3 \rceil$ and $|N(B) \cup \{v\}| \leq \alpha n$. By plugging

into algorithmic version of Combinatorial Lemma (Lemma 2) $b \leq \lceil 2n(1-\alpha)/3 \rceil - 1$ and $f = \alpha n$, we obtain that for each $0 < \alpha < 1$ all nice potential maximal cliques of size αn can be listed in time

$$\mathcal{O}\left(n\sum_{i=1}^{2(1-\alpha)n/3} \binom{(2+\alpha)n/3}{i}\right) \tag{7}$$

By using Combinatorial Lemma for each value of α and by bounding the sum as in the proof of Lemma 5, we end up with an algorithm that lists all nice potential maximal cliques in time $\mathcal{O}(1.7549^n)$.

Generating potential maximal cliques. First we generate all minimal separators and all potential maximal cliques that can be obtained from a minimal separator by adding one vertex. Then we list all nice potential maximal cliques. As we already have shown, all these sets can be listed in time $\mathcal{O}(1.7549^n)$.

Let $v_1, v_2, ..., v_n$ be an ordering of V and let $V_i = \bigcup_{j=1}^i v_j$. By Proposition 6, every potential maximal clique Ω in $G[V_{i+1}]$ that is not yet listed (i.e. is not obtained by adding a vertex to a minimal separator or which is not nice) is either also potential maximal clique in $G[V_i]$ or obtained from a potential maximal clique in $G[V_i]$ by adding v_{i+1} . Thus by performing dynamic programming one can list all remaining potential maximal cliques in time $\mathcal{O}(1.7549^n)$.

As an immediate corollary of Proposition 1 and Lemma 6, we have the following result

Theorem 3. The treewidth of a graph on n vertices can be computed in time $\mathcal{O}(1.7549^n)$.

6 Computing treewidth at most k

In this section we show how Combinatorial Lemma can be used to refine the classical result of Arnborg et al. [1].

By Proposition 5, the treewidth of a graph can be computed in $\mathcal{O}(n^3 (|\Pi_G| + |\Delta_G|))$ time if the list of all minimal separators Δ_G and the list of all potential maximal cliques Π_G for the graph is given. Actually, the results of Proposition 5 can be strengthened (with almost the same proof as in [16]) as follows. Let $\Delta_G[k]$ be the set of minimal separators and let $\Pi_G[k]$ be the set of potential maximal cliques of size at most k.

Lemma 7. Given a graph G with sets $\Delta_G[k]$ and $\Pi_G[k+1]$, it can be decided in time $\mathcal{O}(n^3 (|\Pi_G[k+1]| + |\Delta_G[k]|))$ if the treewidth of G is at most k. Moreover, if the treewidth of G is at most k, an optimal tree decomposition can be constructed within the same time.

By Lemma 2 and Equation (4),

$$|\Delta_G[k]| \le n \cdot \sum_{i=1}^{(n-k)/2} {\binom{(n+k)/2}{i}} \le n^2 \cdot {\binom{(n+k)/2}{k}}$$
 (8)

and to list all vertex subsets containing all separators from $\Delta_G[k]$ in time $\mathcal{O}(n^2 \cdot \binom{(n+k)/2}{k})$). For each such a subset one can check in time $\mathcal{O}(n^2)$ if it is a minimal separator or not, and thus all minimal separators of size at most k can be listed in time $\mathcal{O}(n^4 \cdot \binom{(n+k)/2}{k})$.

Let $\Pi_n[k]$ be the maximum number of nice potential maximal cliques of size at most k that can be in a graph on n vertices. By (7),

$$|\Pi_n[k]| \le n \cdot \sum_{i=1}^{(2(n-k))/3} {(2n+k)/3 \choose i} \le n^2 \cdot {(2n+k)/3 \choose k},$$

and by making use of Proposition 4, all nice potential maximal cliques of size at most k can be listed in time $\mathcal{O}(n^5 \cdot \binom{(2n+k)/3}{k})$.

Finally, we use nice potential maximal cliques and minimal separators of size k to generate all potential maximal cliques of size at most k.

Lemma 8. For every graph G on n vertices, $|\Pi_G[k]| \le n(|\Delta_G[k]| + |\Pi_n[k]|)$ and all potential maximal cliques of G of size at most k can be listed in time $\mathcal{O}(n^6 \cdot \binom{(2n+k)/3}{k})$.

Proof. Let $v_1, v_2, ..., v_n$ be an ordering of V and let $V_i = \bigcup_{j=1}^i v_j$. By Proposition 6 and Lemma 3, every potential maximal clique of $G[V_i]$ either is a nice potential maximal clique of $G[V_i]$, or is obtained by adding v_i to a minimal separator or a potential maximal clique of $G[V_{i-1}]$. This yields that $|\Pi_G[k]| \leq n(|\Delta_G[k]| + |\Pi_n[k]|)$. To list all potential maximal cliques, for each $i, 1 \leq i \leq n$, we list all minimal separators and nice potential maximal cliques in $G[V_i]$. This can be done in time $O(n^6 \cdot \binom{(2n+k)/3}{k})$. The total number of all such structures is at most $n^3 \cdot \binom{(2n+k)/3}{k}$. By making use of dynamic programing, one can check if adding v_i to a minimal separator or potential maximal clique of $G[V_{i-1}]$ creates a potential maximal clique in $G[V_i]$, which by Proposition 4 can be done in time $O(n^3)$. Thus dynamic programming can be done in $O(n^6 \cdot \binom{(2n+k)/3}{k})$ steps.

Now putting Lemma 7, Lemma 8 and Equation (8) together, we obtain the main result of this section.

Theorem 4. There exists an algorithm that for a given graph G and integer $k \geq 0$, either computes a tree decomposition of G of the minimum width, or correctly concludes that the treewidth of G is at least k+1. The running time of this algorithm is $\mathcal{O}(n^6 \cdot \binom{(2n+k+1)/3}{k+1}) = \mathcal{O}(n^6 \cdot (\frac{2n+k+1}{3})^{k+1})$.

7 Polynomial space exact algorithm for treewidth

The algorithm used in Proposition 1 requires exponential space because it is based on dynamic programming which keeps a table with all potential maximal cliques. As a consequence of that, our time $\mathcal{O}(1.7549^n)$ algorithm computing treewidth also uses space $\mathcal{O}(1.7549^n)$.

When restricting to polynomial space, we cannot store all minimal separators and potential maximal cliques. Thus the idea to compute the treewidth of a graph in polynomial space is to search for a "central" potential maximal clique or a minimal separator in the graph which can safely be completed into a clique. Similar idea is used in [5], however the improvement in the running time of our algorithm, is due to the following lemma and the technique used for listing minimal separators. Both results are, again, based on Combinatorial Lemma.

Lemma 9. For a given graph G = (V, E) and $0 < \alpha < 1$, one can list in time $\mathcal{O}(mn^2 \cdot 2^{n(1-\alpha)})$ and polynomial space all potential maximal cliques of G such that for every potential maximal clique Ω from this list, there is a connected component of $G[V \setminus \Omega]$ of size at least αn .

Proof. Let Ω be a potential maximal clique satisfying the conditions of the lemma, and let C be the connected component of size at least αn . By Proposition 3, N(C) is a minimal separator containing in Ω and $\Omega \setminus N(C) \neq \emptyset$. Let (C_u, u) be a vertex representation of Ω , where $u \in \Omega \setminus N(C)$. Since u is not adjacent to any vertex in C, we have that $C_u \cap C = \emptyset$. Thus to find Ω , we try to find its vertex representation by a connected vertex set such that the closed neighborhood of this set is of size at most $n(1-\alpha)$. By the Combinatorial Lemma, the number of such sets is at most

$$n \cdot \sum_{i=1}^{n(1-\alpha)} \binom{n(1-\alpha)}{i} = n \cdot 2^{n(1-\alpha)}$$

and by Lemma 2, all these sets can be listed in $\mathcal{O}(n \cdot 2^{n(1-\alpha)})$ steps and within polynomial space. Finally, for each set we use Lemma 4 and Proposition 4 to check in time $\mathcal{O}(mn)$ if the set is a potential maximal clique.

We also use the following result which is a slight modification of the result from [5], where it is stated in terms of elimination orderings.

Proposition 8 ([5]). For a given graph G = (V, E) and a clique $K \subset V$, there exists a polynomial space algorithm, that computes the optimum tree decomposition (χ, T) of G, subject to the condition that the vertices of K form a bag which is a leaf of T. This algorithm runs in time $\mathcal{O}^*(4^{n-|K|})$.

Theorem 5. The treewidth of a graph G = (V, E) can be computed in $\mathcal{O}(2.6151^n)$ time and polynomial space.

Proof. It is well known (and follows from the properties of clique trees of chordal graphs), that there is an optimal tree decomposition (χ, T) , $\{\chi_i : i \in V_T\}$, $T = (V_T, E_T)$, of G, where every bag is a potential maximal clique [8, 10, 19]. Among all the bags of χ , let χ_i be a bag such that the largest connected component of $G[V \setminus \chi_i]$ is of minimum size, i.e. χ_i is a bag with the minimum value of

$$\max\{|C|: C \text{ is a connected component of } G[V \setminus \chi_i]\},$$

where minimum is taken over all bags of χ . Let C_i be the connected component of $G - \chi_i$ of maximum size.

Our further strategy depends on the size of $|C_i|$. Let us assume first that $|C_i| < 0.38685n$. In this case, by Lemma 9, the set of potential maximal cliques S such that for every $\Omega \in S$ the maximum size of a component of $G[V \setminus \Omega]$ is $|C_i|$, can be listed in time $\mathcal{O}(mn^2 \cdot 2^{n-|C_i|})$ and polynomial space. Since $\chi_i \in S$, we have that there is a potential maximal clique $\Omega \in S$ such that $tw(G_\Omega) = tw(G)$, where G_Ω is obtained from G by turning Ω into a clique. The treewidth of G_Ω is equal to the maximum of minimum width of decompositions of $G_\Omega[C \cup \Omega]$ with Ω forming a leaf bag, where C is a connected component of $G_\Omega[V \setminus \Omega]$. Let us remind that the size of each such component is at most $|C_i|$.

By Proposition 8, the optimum width of $G_{\Omega}[C \cup \Omega]$ for every connected component C of $G_{\Omega}[C \cup \Omega]$ (and with Ω forming a leaf bag) can be computed in $\mathcal{O}^*(4^{|C|}) = \mathcal{O}^*(4^{|C_i|})$, time and thus the treewidth of G can be found in time

$$\mathcal{O}^*(2^{n-|C_i|} \cdot 4^{|C_i|}) = \mathcal{O}^*(2^{(1-0.38685)n} \cdot 4^{0.38685n}) = \mathcal{O}(2.6151^n).$$

Thus if $|C_i| < 0.38685n$, we compute the treewidth of G, and the running time of this polynomial space procedure is $\mathcal{O}(2.6151^n)$.

Let us consider the case $|C_i| \geq 0.38685n$. For each connected component C of $G[V \setminus \chi_i]$, there exists a bag $\chi_{i'} \subset N(C) \cup C$ and a minimal separator $S = \chi_i \cap \chi_{i'}$ in χ_i that separates C from the rest of the graph. Let $S = \chi_i \cap \chi_j$ be the separator in χ_i that separates C_i from the rest of the graph. Let G_S be the graph obtained from G by turning S into a clique. Then $tw(G_S) = tw(G)$. To compute the treewidth of G_S we compute the minimum width of decompositions of $G_S[C \cup S]$ with S forming a leaf bag, where C is a connected component of $G_S[V \setminus S]$, and then take the maximum of these values.

By the definition of χ_i , there exists a connected component C_j of $G[V \setminus \chi_j]$, such that $|C_j| \geq |C_i|$. By Proposition 3, $\chi_j \not\subseteq \chi_i$. Thus $\chi_j \setminus \chi_i \neq \emptyset$, and the size of every connected component in $G[C_i \setminus \chi_j]$ is at most $|C_i| - 1$. Furthermore, since $S = \chi_i \cap \chi_j$, we have that every connected component of $G[C_i \setminus \chi_j]$ is also a connected component of $G[V \setminus \chi_j]$. This yields that $C_j \cap C_i = \emptyset$ and that both C_i and C_j are full connected components of $G[V \setminus S]$. Thus $|C_j| + |C_i| \leq n - |S|$. Every connected

component of $G[V \setminus S]$, except C_i , is a connected component of $G[V \setminus \chi_j]$. Because $|C_i| \leq |C_j|$, this implies that C_j is the largest component of $G[V \setminus S]$. Both C_i and C_j contain at least 0.38685n vertices, thus the size of S is at most $n(1-2\cdot 0.38685)=0.2263n$. By the algorithmic version of Combinatorial Lemma, all sets of such size (and which form the neighborhood of a set of size $|C_i|$) can be listed in polynomial space and time

$$\mathcal{O}(nm \cdot \sum_{p=1}^{0.2263n} \binom{|C_i|+p}{p}).$$

By Proposition 8, we can compute the minimum width of decompositions of $G_S[C \cup S]$ with S forming a leaf bag, where C is a connected component of $G_S[V \setminus S]$, in time

$$\mathcal{O}^*(4^{|C|}) = \mathcal{O}^*(4^{|C_j|})$$

and polynomial space. Because $|C_j| \le n - |S| - |C_i|$, we have that for |S| = p,

$$\mathcal{O}^*(4^{|C_j|}) = \mathcal{O}^*(4^{n-|C_i|-p}).$$

Thus to compute the treewidth of G_S (and the treewidth of G), we list all sets S and for each such a set we use Proposition 8 for all graphs $G_S[C \cup S]$. The running time of this procedure is

$$\mathcal{O}^* \left(\sum_{p=1}^{0.2263n} \binom{|C_i|+p}{p} \cdot 4^{n-|C_i|-p} \right).$$

By Vandermonde's identity, we have that

$$\binom{|C_i|+p}{p} = \sum_{k=0}^{p} \binom{0.38685n+p}{k} \binom{|C_i|-0.38685n}{k} < \sum_{k=0}^{p} \binom{0.38685n+p}{k} 2^{|C_i|-0.38685n}.$$

Thus

$$\sum_{p=1}^{0.2263n} {\binom{|C_i|+p}{p} \cdot 4^{n-|C_i|-p}} < \sum_{p=1}^{0.2263n} \sum_{k=0}^{p} {\binom{0.38685n+p}{k}} 2^{|C_i|-0.38685n} \cdot 4^{n-|C_i|-p}$$

$$\leq p \cdot \sum_{p=1}^{0.2263n} {\binom{0.38685n+p}{p}} \cdot 2^{2((1-0.38685)n-p)} = \mathcal{O}(2.6151^n)$$

To conclude, if $|C_i| \ge 0.38685n$, we compute the treewidth of G in polynomial space within $\mathcal{O}(2.6151^n)$ steps.

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