# PATHS OF HOMOMORPHISMS FROM STABLE KNESER GRAPHS 

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#### Abstract

We denote by $S G_{n, k}$ the stable Kneser graph (Schrijver graph) of stable $n$-subsets of a set of cardinality $2 n+k$. For $k \equiv 3(\bmod 4)$ and $n \geq 2$ we show that there is a component of the $\chi$-colouring graph of $S G_{n, k}$ which is invariant under the action of the automorphism group of $S G_{n, k}$. We derive that there is a graph $G$ with $\chi(G)=\chi\left(S G_{n, k}\right)$ such that the complex $\operatorname{Hom}\left(S G_{n, k}, G\right)$ is non-empty and connected. In particular, for $k \equiv 3(\bmod 4)$ and $n \geq 2$ the graph $S G_{n, k}$ is not a test graph.


## 1. Introduction

For graphs $G$ and $H$, the complex $\operatorname{Hom}(G, H)$ is a cell complex whose vertices are the graph homomorphisms from $G$ to $H$ and whose topology captures global properties of the set of these homomorphisms. Research on these complexes in recent years has been driven by the concept of a test graph. In this work we present a result in this area which can be formulated naturally in the category of graphs without mentioning complexes.
Definition (BK06). A graph $T$ is a test graph if for all graphs $G$ and $r \geq 0$ such that the cell complex $\operatorname{Hom}(T, G)$ is $(r-1)$-connected we have $\chi(G) \geq r+\chi(T)$.

A cell complex is said to be 0 -connected if it is non-empty. If there is a graph homomorphism from $T$ to $G$, then the chromatic number of $G$ is at least as large as that of $T$. Therefore the condition for $r=0$ is always satisfied.

A cell complex is said to be 1-connected if it is non-empty and path-connected. If $f, g: T \rightarrow G$ are graph homomorphism, then by definition there is an edge from $f$ to $g$ in $\operatorname{Hom}(T, G)$ if and only if $f$ and $g$ differ at exactly one vertex of $T$. This is all that is needed to understand the condition for $r=1$, and since that is all that we will be interested in in this work, we omit the description of the higher dimensional cells of $\operatorname{Hom}(T, G)$.

The seminal result regarding test graphs is that $K_{2}$ is a test graph. This is a translation of a result by Lovász Lov78] which predates the definiton of the complex $\operatorname{Hom}(T, G)$. While some effort has gone into proving that certain graphs are test graphs BK06, BK07, Sch09, DS10, Sch10, it was not known from the beginning if possibly every graph is a test graph. An example of a graph which is not a test graph was given by Hoory and Linial.
Theorem ([甘05). There is a graph $T$ such that $\operatorname{Hom}\left(T, K_{\chi(T)}\right)$ is connected.
The graph $T$ in this example fails to be a test graph already for $r=1$. Since $\operatorname{Hom}\left(T, K_{\chi(T)}\right)$ is non-empty by definition, we would otherwise have $\chi\left(K_{\chi(T)}\right) \geq$ $\chi(T)+1$, which is absurd.

[^0]So this graph $T$ fails to be a test graph in the most fundamental way. Furthermore, the fact that $\operatorname{Hom}\left(T, K_{\chi(T)}\right)$ is connected also implies that $\operatorname{Hom}\left(K_{2}, T\right)$ cannot be $(\chi(T)-3)$-connected, i.e. the chromatic number of $T$ is not detected by the test graph $K_{2}$. The proof of this uses some easy topology and functorial properties of Hom. One might therefore ask if there are graphs $T$ whose chromatic numbers are detected by $K_{2}$ and which still fail to be test graphs for $r=1$. We will find such examples among the stable Kneser graphs.

The stable Kneser graphs, first introduced by Schrijver, form a two parameter family of graphs $S G_{n, k}$, see Definition 4.1. We list some facts that are known of them.
$\triangleright$ The chromatic number of $S G_{n, k}$ equals $k+2$ (Schrijver Sch78, extending work of Lovasz Lov78 and Bárány Bár78).
$\triangleright$ Indeed, $\operatorname{Hom}\left(K_{2}, S G_{n, k}\right)$ is homotopy equivalent to a $k$-sphere as shown by Björner and de Longueville BDL03] (a simplified proof can be found in Sch10).
$\triangleright$ The graph $S G_{n, k}$ is vertex critical, i.e. every induced subgraph on a proper subset of its set of vertices is $(k+1)$-colourable Sch78.
$\triangleright$ The stable Kneser graph $S G_{1, k}$ is a complete graph and hence a test graph by work of Babson and Kozlov [BK06]. The stable Kneser graph $S G_{n, 1}$ is a cycle on $2 n+1$ vertices and hence also a test graph by their work BK07.
$\triangleright$ For $n \geq 2$ and $k \geq 1$, Braun Bra09 has shown the automorphism group of $S G_{n, k}$ to be the symmetry group of a $(2 n+k)$-gon.
Since $\operatorname{Hom}\left(K_{2}, S G_{n, k}\right)$ is topologically as nice as one might hope, one might have thought that all stable Kneser graphs are test graphs. It turns out, however, that very few of them are.

Theorem ([Sch10, 10.5-10.10]). If $k \notin\{0,1,2,4,8\}$ then there is an $N(k)$ such that for all $n \geq N(k)$ the graph $S G_{n, k}$ is not a test graph. For $k \equiv 3(\bmod 4)$ the graph $S G_{n, k}$ fails to be a test graph for $r=1$.

The proof of this theorem studies the action of the automorphism group of $S G_{n, k}$ on the space $\operatorname{Hom}\left(K_{2}, S G_{n, k}\right)$ using methods from algebraic topology. The current work gives an elementary proof for the case $k \equiv 3(\bmod 4)$, which yields the stronger result that in these cases we can actually set $N(k)=2$. Its main result is thus the following.

Theorem. Let $k \equiv 3(\bmod 4)$ and $n \geq 2$. There is a graph $G$ with $\chi(G)=$ $\chi\left(S G_{n, k}\right)=k+2$ and such that $\operatorname{Hom}\left(S G_{n, k}, G\right)$ is non-empty and connected.

In the proof, which is the combination of the following two theorems, the automorphism group of $S G_{n, k}$ again plays an important role. However, we only have to study its action on the set of path components of $\operatorname{Hom}\left(S G_{n, k}, K_{k+2}\right)$. While we know from the above discussion that the complex $\operatorname{Hom}\left(S G_{n, k}, K_{k+2}\right)$ cannot be connected, we will show in Section 4 the following weaker result.
Theorem (4.3). Let $k \equiv 3(\bmod 4)$ and $n \geq 2$. Then there is a component of $\operatorname{Hom}\left(S G_{n, k}, K_{k+2}\right)$ which is invariant under the action of the automorphism group of $S G_{n, k}$.

This of course relies on Braun's result on the structure of Aut $\left(S G_{n, k}\right)$. In Section 3 we give a self-contained proof the following general criterion from Sch10.

Theorem (3.1). Let $T$ be a finite, vertex critical graph. If there is a component of $\operatorname{Hom}\left(T, K_{\chi(T)}\right)$ which is $\operatorname{Aut}(T)$-invariant, then there exists a graph $G$ such that $\operatorname{Hom}(T, G)$ is non-empty and connected and $\chi(G)=\chi(T)$.

That every endomorphism of $S G_{n, k}$ is an automorphism follows immediately from vertex criticality. In this sense, our proof also relies on Schrijver's result that $S G_{n, k}$ is vertex crtitical.

## 2. Constructions in the category of graphs

We recall some definitions related to the category of graphs. Details can be found in Doc09a.

A graph $G$ consists of a vertex set $V(G)$ and a symmetric binary relation $E(G) \subset$ $V(G) \times V(G)$. The relation is called adjacency, adjacent vertices are also called neighbours and elements of $E(G)$ edges. We also write $u \sim v$ for $(u, v) \in E(G)$. We point out that we allow loops, i.e. edges of the form $(v, v)$. A graph with a loop at every vertex is called reflexive.

A graph homomorphism $f: G \rightarrow H$ is a function $f: V(G) \rightarrow V(H)$ between the vertex sets which preserves the adjacency relation, $(f(u), f(v)) \in E(H)$ for all $(u, v) \in E(G)$. When discussing the structure of the set of graph homomorphisms from $G$ to $H$, it will be useful to not only consider the cell complex $\operatorname{Hom}(G, H)$, but also the closely related graph $[G, H]$. This graph, sometimes also written $H^{G}$, is defined by

$$
\begin{aligned}
& V([G, H])=V(H)^{V(G)} \\
& E([G, H])=\{(f, g):(f(u), g(v)) \in E(H) \text { for all }(u, v) \in E(G)\}
\end{aligned}
$$

In particular, we have $f \sim f$ if and only if $f$ is a graph homomorphism. Furthermore, we have the following.
2.1. Lemma. Let $f, g: G \rightarrow H$ be graph homomorphisms and $G$ loopless. Then $f \sim g$ in $[G, H]$ if and only if each $h: V(G) \rightarrow V(H)$ with $h(u) \in\{f(u), g(u)\}$ for all $u \in V(G)$ is a graph homomorphism.

It follows that $f \sim g$ if an edge joins the two graph homomorphism $f$ and $g$ in $\operatorname{Hom}(G, H)$. If on the other hand $f \sim g$, then we can get from $f$ to $g$ in $\operatorname{Hom}(G, H)$ by changing the values at the vertices of $G$ in any order. Therefore, for questions of connectivity it does not matter whether we work in $\operatorname{Hom}(G, H)$ or in the induced subgraph of looped vertices of $[G, H]$. That two graph homomorphisms are in the same component can now be reformulated as follows.
2.2. Definition. For $n \geq 0$ we define a reflexive graph $I_{n}$ by $V\left(I_{n}\right)=\{0, \ldots, n\}$, $i \sim j \Longleftrightarrow|i-j| \leq 1$. For graph homomorphisms $f, g: G \rightarrow H$ we write $f \simeq g$ if and only if there is an $n \geq 0$ and a graph homomorphism $p: I_{n} \rightarrow[G, H]$ with $p(0)=f, p(n)=g$.

The construction $[\bullet, \bullet]$ is an inner hom, intimately related to products. The product of two graphs in the category $\mathcal{G}$ of graphs and graph homomorphisms is given by

$$
\begin{aligned}
& V(G \times H)=V(G) \times V(H) \\
& E(G \times H)=\left\{\left(\left(u, u^{\prime}\right),\left(v, v^{\prime}\right)\right):(u, v) \in E(G),\left(u^{\prime}, v^{\prime}\right) \in E(H)\right\}
\end{aligned}
$$

For every graph $G$, the functor $[G, \bullet]$ is a right adjoint to $\bullet \times G$, i.e. there is a natural equivalence

$$
\begin{equation*}
\mathcal{G}(Z \times G, H) \cong \mathcal{G}(Z,[G, H]) \tag{1}
\end{equation*}
$$

For example, that looped vertices of $[G, H]$ correspond to graph homomorphisms can be derived as a formal consequence of this adjunction. If $\mathbf{1}$ denotes the terminal object of $\mathcal{G}$, we obtain

$$
\mathcal{G}(G, H) \cong \mathcal{G}(\mathbf{1} \times G, H) \cong \mathcal{G}(\mathbf{1},[G, H])
$$

and since $\mathbf{1}$ is a graph consisting of one vertex and one loop, the graph homomorphisms from 1 to some other graph correspond to the looped vertices of that graph.

## 3. A criterion for not being a test graph

If $T$ is a vertex critical finite graph, then every graph homomorphism $T \rightarrow T$ will have to be surjective and hence bijective. In other words, every endomorphism of $T$ is an automorphism. For graphs with this property we obtain the following criterion to decide whether they satisfy the test graph property for $r=1$.
3.1. Theorem. Let $T$ be a finite graph such that $\operatorname{End}(T)=\operatorname{Aut}(T)$. If there is a component of $\operatorname{Hom}\left(T, K_{\chi(T)}\right)$ which is $\operatorname{Aut}(T)$-invariant, then there exists a graph $G$ with $\chi(G)=\chi(T)$ such that $\operatorname{Hom}(T, G)$ is non-empty and connected.
3.2. Remark. This is the case $r=1, s=\chi(T)+1$ of the implication (iv) $\Rightarrow$ (ii) of [Sch10, Thm 10.1]. The proof there uses results from [Doc09b] and [DS10]. Here we are only interested in dimension 1, and for this easier case we can give a self-contained proof which follows the same lines.
3.3. Remark. The converse holds in general without conditions on $T$. If $f: T \rightarrow G$ and $c: G \rightarrow K_{\chi(T)}$, then $c \circ f$ is a vertex of $\operatorname{Hom}\left(T, K_{\chi(T)}\right)$, and if $\operatorname{Hom}(T, G)$ is connected, then the component of that vertex will be invariant under $\operatorname{Aut}(T)$ : If $\gamma \in \operatorname{Aut}(T)$ then $f \circ \gamma \simeq f$ implies $c \circ f \circ \gamma \simeq c \circ f$.
Proof of Theorem 3.1. Let $c: T \rightarrow K_{\chi(T)}$ be a colouring which lies in the invariant component. Then for every $\gamma \in \operatorname{Aut}(T)$ there is an $n \geq 0$ and a graph homomorphism $I_{n} \rightarrow\left[T, K_{\chi(T)}\right]$ with $0 \mapsto c$ and $n \mapsto c \gamma$. We can assemble these into a graph homomorphism $X \rightarrow\left[T, K_{\chi(T)}\right]$, where $X$ is a reflexive graph consisting of a vertex $u$ and for every $\gamma \in \operatorname{Aut}(T)$ a path from $u$ to a vertex $v_{\gamma}$ and the graph homomorphism such that $u \mapsto c$ and $v_{\gamma} \mapsto c \gamma$. We will later require that the paths have a certain minimal length, which we can arrange.

The graph homomorphism which we have just constructed gives rise via (1) to a graph homomorphism $f: X \times T \rightarrow K_{\chi(T)}$ with $f\left(v_{\gamma}, t\right)=c(\gamma(t))=f(u, \gamma(t))$ for all $\gamma \in \operatorname{Aut}(T)$ and $t \in V(T)$. We define an equivalence relation on $V(X \times T)$ such that $\left(v_{\gamma}, t\right) \sim(u, \gamma(t))$ and obtain a commutative diagram

with $q$ the quotient map. Now let $j: T \rightarrow G$ be the inclusion at $u$, more formally $j=q \circ\left(\operatorname{const}_{u}, \mathrm{id}_{T}\right)$. Now we already know that $\operatorname{Hom}(T, G) \neq \varnothing$ and $\chi(G)=\chi(T)$. We will proceed to show that $\operatorname{Hom}(T, G)$ is connected.

We first note that $\operatorname{const}_{u} \simeq \operatorname{const}_{v_{\gamma}}: T \rightarrow X$ for all $\gamma \in \operatorname{Aut}(T)$, and hence

$$
j \simeq q \circ\left(\operatorname{const}_{v_{\gamma}}, \mathrm{id}_{T}\right)=q \circ\left(\operatorname{const}_{u}, \gamma\right)=j \gamma
$$

Now let $g: T \rightarrow G$ be an arbitrary graph homomorphism. Let $\sim^{\prime}$ be the equivalence relation on $V(X)$ given by $v_{\gamma} \sim^{\prime} u$. The reflexive graph $X / \sim^{\prime}$ is a bouquet of circles, and there is a natural surjection $G \rightarrow X / \sim^{\prime}$. Since we may assume to haven chosen the graph $X$ large enough (this depending on $T$, not on $g$ ), the image of the composition $T \xrightarrow{g} G \rightarrow X / \sim^{\prime}$ will miss at least one vertex of each of the circles. Let $X^{\prime}$ be the graph obtained from $X / \sim^{\prime}$ by removing one these vertices from each circle. We denote the vertex corresponding to $u$ by $u^{\prime}$. The preimage of $X^{\prime}$ in $G$ is isomorphic to $X^{\prime} \times T$. Let $h: X^{\prime} \times T \rightarrow G$ be the corresponding embedding such that $j=h \circ\left(\operatorname{const}_{u^{\prime}}, \mathrm{id}_{T}\right)$. This defines a commutative diagram


Now $\tilde{g}=\left(\tilde{g}_{1}, \tilde{g}_{2}\right)$. Since $X^{\prime}$ is a reflexive tree with loops, we have $\tilde{g}_{1} \simeq$ const $_{u^{\prime}}$. Since $\operatorname{End}(T)=\operatorname{Aut}(T)$, there is a $\gamma \in \operatorname{Aut}(T)$ such that $\tilde{g}_{2}=\gamma$. Therefore

$$
g \simeq h \circ\left(\operatorname{const}_{u^{\prime}}, \gamma\right)=h \circ\left(\operatorname{const}_{u^{\prime}}, \operatorname{id}_{T}\right) \circ \gamma=j \gamma \simeq j .
$$

Since $g$ was arbitrary, this shows that $\operatorname{Hom}(T, G)$ is connected.

## 4. Paths of colourings of stable Kneser graphs

4.1. Definition. Let $n \geq 1, k \geq 0$ and $m=2 n+k$. The Kneser graph $K G_{n, k}$ is the graph whose vertices are the $n$-element subsets of $\{0, \ldots, m-1\}$ and in which two of them are adjacent if and only if they are disjoint. We call a subset $S$ of $\{0, \ldots, m-1\}$ semi-stable, if $\{i, i+1\} \not \subset S$ for all $0 \leq i \leq m-2$, and stable, if additionally $\{0, m-1\} \not \subset S$. The stable Kneser graph $S G_{n, k}$ is the induced subgraph of $K G_{n, k}$ on the set of stable sets. The semi-stable Kneser graph $\overline{S G}_{n, k}$ is the induced subgraph of $K G_{n, k}$ on the set of semi-stable sets.
4.2. Definition. We call the graph homomorphism

$$
\begin{aligned}
c_{n, k}: \overline{S G}_{n, k} & \rightarrow K_{k+2}, \\
S & \mapsto \min S
\end{aligned}
$$

the canonical colouring of $\overline{S G}_{n, k}$ and its restriction the canonical colouring of $S G_{n, k}$.

We will prove the following.
4.3. Theorem. Let $k \equiv 3(\bmod 4)$ and $n \geq 2$. Then for any $\gamma \in \operatorname{Aut}\left(S G_{n, k}\right)$ and $c: S G_{n, k} \rightarrow K_{k+2}$ the canonical colouring there is a path in $\operatorname{Hom}\left(S G_{n, k}\right)$ from $c$ to $c \gamma$.

We consider automorphisms of $K_{k+2}$ before turning to automorphisms of $S G_{n, k}$.
4.4. Proposition. Let $n \geq 2, k \geq 1$. Let $\pi \in A_{k+2}$ be an even permutation of the vertices of $K_{k+2}$. Then $c_{n, k} \simeq \pi \circ c_{n, k}: \overline{S G}_{n, k} \rightarrow K_{k+2}$.

Proof. Let $m=2 n+k$. We will assume that $\pi$ is a cycle of the form $(i i+1 i+2)$ with $0 \leq i<k$. This is possible, since these permutations generate $A_{k+2}$. The proof will be by induction on $k$. We distinguish three cases.
$\underline{i>0}$. We note that $c(S) \geq i$ if and only if $S \subset\{i, \ldots, m-1\}$. The induced subgraph on the set of these vertices is isomorphic to $\overline{S G}_{n, k-i}$ via $S \mapsto S-i$. Therefore a path from $c_{n, k-i}$ to $\left(\begin{array}{ll}0 & 1\end{array}\right) \circ c_{m, k-i}$, which exists by induction, can be extended to a path from $c_{n, k}$ to $\pi \circ c_{n, k}$ by fixing all colours less than $i$.
$i=0, k>1$. We define

$$
\begin{aligned}
c^{\prime}: \overline{S G}_{n, k} & \rightarrow K_{k+2}, \\
S & \mapsto \begin{cases}k+1, & m-1 \in S, \\
\min S, & \text { otherwise } .\end{cases}
\end{aligned}
$$

Obviously $c^{\prime}$ is a graph homomorphism. If $S \sim S^{\prime}$ then $m-1 \notin \emptyset=S \cap S^{\prime}$ and hence $c_{n, k}(S)=c^{\prime}(S)$ or $c_{n, k}\left(S^{\prime}\right)=c^{\prime}\left(S^{\prime}\right)$. This shows $c^{\prime} \sim c_{n, k}$. The induced subgraph on those $S$ for which $c^{\prime}(S)<k+1$ equals $\overline{S G}_{n, k-1}$. The restriction of $c^{\prime}$ to that subgraph equals $c_{n, k-1}$. Therefore a path from $c_{n, k-1}$ to $\pi \circ c_{n, k-1}$ extends to one from $c^{\prime}$ to $\pi \circ c^{\prime}$. Hence $c_{n, k} \sim c^{\prime} \simeq \pi \circ c^{\prime} \sim \pi \circ c_{n, k}$.
$i=0, k=1$. There is a unique function $h: V\left(\overline{S G}_{n, 1}\right) \rightarrow V\left(S G_{2,1}\right)$ with $h(S) \cap$ $\{0,1,2,3\}=S \cap\{0,1,2,3\}$ for all $S$. It satisfies $h(S) \subset S$ and is therefore a graph homomorphism. Also, $c_{n, 1}=c_{2,1} \circ h$. It therefore suffices to show that $c_{2,1} \simeq(012) \circ c_{2,1}$, where $c_{2,1}$ is defined on $S G_{2,1}$. We note that $S G_{2,1}$ is a cycle of length 5 . Homomorphisms between cycles have been studied in more detail by Čukić and Koylov ČK06. For our needs the following explicit construction suffices.

| $\{0,3\}$ | $\{1,4\}$ | $\{0,2\}$ | $\{1,3\}$ | $\{2,4\}$ |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 1 | 0 | 1 | 2 |
| 0 | 2 | 0 | 1 | 2 |
| 1 | 2 | 0 | 1 | 2 |
| 1 | 2 | 0 | 1 | 0 |
| 1 | 2 | 0 | 2 | 0 |
| 1 | 2 | 1 | 2 | 0 |

Each row of the table defines a graph homomorphism $S G_{2,1} \rightarrow K_{3}$. Adjacent rows define homomorphisms which are adjacent in $\left[S G_{2,1}, K_{3}\right]$.
4.5. Lemma. Let $n \geq 2, k \geq 1, m=2 n+k$. Let $\tau, \rho \in \operatorname{Aut}\left(S G_{n, k}\right)$ be defined by

$$
\tau S=S+1, \quad \rho S=k-S
$$

with arithmetic modulo $m$, and $\bar{\tau}, \bar{\rho} \in \operatorname{Aut}\left(K_{k+2}\right)$ by

$$
\bar{\tau} x=x+1, \quad \bar{\rho} x=k-x,
$$

with arithmetic modulo $k+2$. Then $c \tau \sim \bar{\tau} c$ and $c \rho \sim \bar{\rho} c$, where $c$ is the canonical colouring.
Proof. Let $S, S^{\prime} \in V\left(S G_{n, k}\right)$.
If $S \sim S^{\prime}$, then $S$ and $S^{\prime}$ cannot both contain $m-1$. Assume $m-1 \notin S$. Then $c \tau\left(S^{\prime}\right) \sim c \tau(S)=\bar{\tau} c(S)$ and $c \tau(S)=\bar{\tau} c(S) \sim \bar{\tau} c\left(S^{\prime}\right)$. This shows $c \tau \sim \bar{\tau} c$.

Assume $c \rho(S) \nsim \bar{\rho} c\left(S^{\prime}\right)$, i.e. $c \rho(S)=i=\bar{\rho} c\left(S^{\prime}\right)$ for some $0 \leq i<k+2$. If $i=k+1$ then $S=S^{\prime}=\{k+1, k+3, \ldots, k+2 n-1\}$. If $0 \leq i<k$, then $k-i \in S \cap S^{\prime}$. In both cases $S \nsim S^{\prime}$. This shows $c \rho \sim \bar{\rho} c$.
Proof of Theorem 4.3. By a theorem of Braun Bra09, every automorphism of $S G_{n, k}$ is induced by a permutation of the base set which preserves its cyclic adjacency relation. Therefore the elements $\tau$ and $\rho$ of Lemma 4.5 generate Aut $\left(S G_{n, k}\right)$, and it suffices to show that $c \simeq c \tau$ and $c \simeq c \rho$. By Lemma 4.5 this reduces to $c \simeq \bar{\tau} c$ and $c \simeq \bar{\rho} c$. By Proposition 4.4 these will be true if $\bar{\tau}$ and $\bar{\rho}$ are even permutations. Now sign $\bar{\tau}=(-1)^{k+1}=1$ and $\operatorname{sign} \bar{\rho}=(-1)^{\frac{k(k+1)}{2}}=1$.

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