CONNECTED BARANYAI'S THEOREM

AMIN BAHMANIAN

ABSTRACT. Let $K_n^h = (V, {V \choose h})$ be the complete *h*-uniform hypergraph on vertex set V with |V| = n. Baranyai showed that K_n^h can be expressed as the union of edge-disjoint *r*-regular factors if and only if *h* divides rn and *r* divides ${n-1 \choose h-1}$. Using a new proof technique, in this paper we prove that λK_n^h can be expressed as the union $\mathcal{G}_1 \cup \ldots \cup \mathcal{G}_k$ of *k* edge-disjoint factors, where for $1 \leq i \leq k$, \mathcal{G}_i is r_i -regular, if and only if (i) *h* divides r_in for $1 \leq i \leq k$, and (ii) $\sum_{i=1}^k r_i = \lambda {n-1 \choose h-1}$. Moreover, for any i $(1 \leq i \leq k)$ for which $r_i \geq 2$, this new technique allows us to guarantee that \mathcal{G}_i is connected, generalizing Baranyai's theorem, and answering a question by Katona.

1. INTRODUCTION

A hypergraph \mathcal{G} is a pair (V, E) where V is a finite set called the vertex set, E is the edge multiset, where every edge is itself a multi-subset of V. This means that not only can an edge occur multiple times in E, but also each vertex can have multiple occurrences within an edge. The total number of occurrences of a vertex v among all edges of E is called the degree, $d_{\mathcal{G}}(v)$ of v in \mathcal{G} . For a positive integer h, \mathcal{G} is said to be h-uniform if |e| = h for each $e \in E$. For positive integers r, r_1, \ldots, r_k , an r-factor in a hypergraph \mathcal{G} is a spanning r-regular sub-hypergraph, and an (r_1, \ldots, r_k) -factorization is a partition of the edge set of \mathcal{G} into F_1, \ldots, F_k where F_i is an r_i -factor for $1 \leq i \leq k$, abbreviate (r, \ldots, r) -factorization to r-factorization. The hypergraph $K_n^h := (V, {V \choose h})$ with |V| = n (by ${V \choose h}$ we mean the collection of all h-subsets of V) is called a complete h-uniform hypergraph. Avoiding trivial cases, we assume that n > h. Baranyai proved that:

Theorem 1.1. (Baranyai [6]) If a_1, \ldots, a_s are positive integers such that $\sum_{i=1}^s a_i = \binom{n}{h}$, then the edges of $K_n^h = (V, E)$ can be partitioned into almost regular hypergraphs (V, E_i) so that $|E_i| = a_i$ for $1 \le i \le s$.

In particular, if $h \mid r_i n$ and $\sum_{i=1}^k r_i = \lambda \binom{n-1}{h-1}$, then K_n^h is (r_1, \ldots, r_k) -factorizable. It is natural to ask if we can obtain a connected factorization; that is, a factorization in which each factor is a connected hypergraph. Let m be the least common multiple of h and n, and let a = m/h. Define the set of edges

$$\mathscr{K} = \{\{1, \dots, h\}, \{h+1, \dots, 2h\}, \dots, \{(a-1)h+1, (a-1)h+2, \dots, ah\}\},\$$

where the elements of the edges are considered mod n. The families obtained from \mathscr{K} by permuting the elements of the underlying set $\{n\}$ are called *wreaths*. If h divides n, then a wreath is just a partition. Baranyai and Katona conjectured that the edge set of K_n^h can be decomposed into disjoint wreaths [10]. In connection with this conjecture, Katona (private

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communication) suggested the problem of finding a connected factorization for K_n^h . In this paper, we solve this problem.

If we replace every edge e of K_n^h by λ copies of e, then we denote the new hypergraph by λK_n^h . In this paper, the main result is the following theorem:

Theorem 1.2. λK_n^h is (r_1, \ldots, r_k) -factorizable if and only if $h \mid r_i n$ for $1 \leq i \leq k$, and $\sum_{i=1}^k r_i = \lambda \binom{n-1}{h-1}$. Moreover, for $1 \leq i \leq k$, if $r_i \geq 2$, then we can guarantee that the r_i -factor is connected.

In particular if $\lambda = 1$, and $h = r_1 = \cdots = r_k = 2$, Theorem 1.2 implies the classical result of Walecki [11] that the edge set of K_n can be partitioned into Hamiltonian cycles if and only if n is odd. Here we list some other interesting special consequences of Theorem 1.2:

Corollary 1.3. K_n^h is connected 2-factorizable if and only if $\binom{n-1}{h-1}$ is even and $h \mid 2n$.

Corollary 1.4. K_n^h has a connected $\frac{h}{\gcd(n,h)}$ -factorization.

We note that the idea behind the proof of Theorem 1.2 is based on the amalgamation technique; for some graph amalgamation results, see [1, 4, 7, 8, 9, 12] and for hypergraph amalgamations, see [2, 3, 5]. Preliminaries are given in Section 2, followed by the proof of Theorem 1.2 in Section 3.

We end this section with some notation we need to be able to describe hypergraphs that arise in this setting.

Let $\mathcal{G} = (V, E)$ be a hypergraph with $\alpha \in V$, and let $U = \{u_1, \ldots, u_z\} \subset V \setminus \{\alpha\}$. Recall that each edge is a multi-subset of V. We abbreviate an edge of the form $\{\underbrace{\alpha, \ldots, \alpha}_{u_1, \ldots, u_z}, u_1, \ldots, u_z\}$

to $\{\alpha^p, u_1, \ldots, u_z\}$. An *h*-loop incident with α is an edge of the form $\{\alpha^h\}$, and $m(\alpha^p, U)$ denotes the multiplicity of an edge of the form $\{\alpha^p\} \cup U$. A *k*-edge-coloring of \mathcal{G} is a mapping $f: E \to C$, where C is a set of k colors (often we use $C = \{1, \ldots, k\}$), and the edges of one color form a color class. The sub-hypergraph of \mathcal{G} induced by the color class *i* is denoted by \mathcal{G}_i , abbreviate $d_{\mathcal{G}_i}(\alpha)$ to $d_i(\alpha)$ and $m_{\mathcal{G}_i}(\alpha^p, U)$ to $m_i(\alpha^p, U)$.

2. Preliminaries

A hypergraph is said to be *non-trivial* if it has at least one edge. A vertex α in a connected hypergraph \mathcal{G} is a *cut vertex* if there exist two non-trivial sub-hypergraphs I, J of \mathcal{G} such that $I \cup J = \mathcal{G}, V(I \cap J) = \alpha$ and $E(I \cap J) = \emptyset$. A non-trivial connected sub-hypergraph W of a connected hypergraph \mathcal{G} is said to be an α -wing of \mathcal{G} , if α is not a cut vertex of Wand no edge in $E(\mathcal{G}) \setminus E(W)$ is incident with a vertex in $V(W) \setminus \{\alpha\}$. The set of all α -wings of \mathcal{G} is denoted by $\mathscr{W}_{\alpha}(\mathcal{G})$. We remark that $\mathscr{W}_{\alpha}(\mathcal{G}) = \{\mathcal{G}\}$ if \mathcal{G} is non-trivial and connected and α is not a cut vertex of \mathcal{G} . Figure 1 illustrates an example of a hypergraph and the set of all its α -wings. If the multiplicity of a vertex α in an edge e is p, we say that α is *incident* with p distinct objects, say $h_1(\alpha, e), \ldots, h_p(\alpha, e)$. We call these objects *hinges*, and we say that eis *incident* with $h_1(\alpha, e), \ldots, h_p(\alpha, e)$. The set of all hinges in \mathcal{G} incident with α is denoted by $H_{\mathcal{G}}(\alpha)$; so $|H_{\mathcal{G}}(\alpha)|$ is in fact the degree of α .

Intuitively speaking, an α -detachment of \mathcal{G} is a hypergraph obtained by splitting a vertex α into one or more vertices and sharing the incident hinges and edges among the subvertices. That is, in an α -detachment \mathcal{G}' of \mathcal{G} in which we split α into α and β , an edge of the form $\{\alpha^{p-i}, \beta^i, u_1, \ldots, u_z\}$ in \mathcal{G} will be of the form $\{\alpha^{p-i}, \beta^i, u_1, \ldots, u_z\}$ in \mathcal{G}' for some $i, 0 \leq i \leq p$. Note

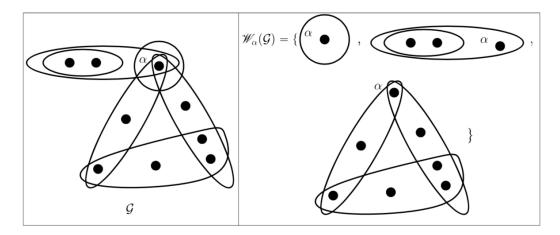


FIGURE 1. A hypergraph \mathcal{G} and the set of all its α -wings

that a hypergraph and its detachments have the same hinges. Whenever it is not ambiguous, we use d', m', etc. for degree, multiplicity and other hypergraph parameters in \mathcal{G}' . Also, for an α -wing W in \mathcal{G} and an α -detachment \mathcal{G}' , let W' denote the sub-hypergraph of \mathcal{G}' whose hinges are the same as those in W. Figure 2 illustrates a detachment \mathcal{G}' of the hypergraph \mathcal{G} in Figure 1 and the set of all its α -wings. We shall present three lemmas, all of which follow

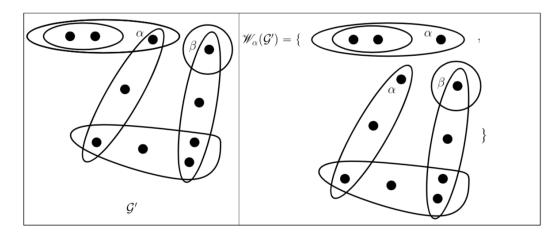


FIGURE 2. A detachment \mathcal{G}' of \mathcal{G} in Figure 1 and the set of all its α -wings

immediately from definitions.

Lemma 2.1. Let \mathcal{G} be a connected hypergraph. Let \mathcal{G}' be an α -detachment of \mathcal{G} obtained by splitting a vertex α into two vertices α and β . Then \mathcal{G}' is connected if and only if for some α -wing $W \in \mathscr{W}_{\alpha}(\mathcal{G})$ with $d_W(\alpha) \ge 2$,

$$1 \leq |H_W(\alpha) \cap H_{\mathcal{G}'}(\beta)| < d_W(\alpha).$$

Informally speaking, Lemma 2.1 says that for some α -wing W with $d_W(\alpha) \ge 2$, at least one but not all the hinges incident with α in W must be incident with β in \mathcal{G}' .

A family \mathscr{A} of sets is *laminar* if, for every pair A, B of sets belonging to $\mathscr{A}: A \subset B$, or $B \subset A$, or $A \cap B = \varnothing$.

Let us fix a vertex α of a k-edge-colored hypergraph $\mathcal{G} = (V, E)$. For $1 \leq i \leq k$, let H_i be the set of hinges each of which is incident with both α and an edge of color i (so $d_i(\alpha) = |H_i|$. For any edge $e \in E$, let H_e be the collection of hinges incident with both α and e. Clearly, if e is of color i, then $H_e \subset H_i$. For an α -wing W, let $H_W = H_W(\alpha)$. For $1 \leq i \leq k$, let

$$H^i = \bigcup_{W \in \mathscr{W}_{\alpha}(\mathcal{G}_i), d_W(\alpha) \ge 2} H_W.$$

Lemma 2.2. Let

$$\mathscr{A} = \{H_1, \dots, H_k\} \cup \{H^1, \dots, H^k\}$$
$$\cup \{H_W : W \in \mathscr{W}_{\alpha}(\mathcal{G}_i), 1 \leq i \leq k\} \cup \{H_e : e \in E\}.$$

Then \mathscr{A} is a laminar family of subsets of $H(\alpha)$.

For each $p \ge 1$, and each $U \subset V \setminus \{\alpha\}$, let H_p^U be the set of hinges each of which is incident with both α and an edge of the form $\{\alpha^p\} \cup U$ in \mathcal{G} (so $|H_p^U| = pm(\alpha^p, U)$).

Lemma 2.3. Let

$$\mathscr{B} = \{H_p^U : p \ge 1, U \subset V \setminus \{\alpha\}\}$$

Then \mathscr{B} is a laminar family of disjoint subsets of $H(\alpha)$.

If x, y are real numbers, $x \approx y$ means $|y| \leq x \leq [y]$. We need the following powerful lemma:

Lemma 2.4. (Nash-Williams [12, Lemma 2]) If \mathscr{A}, \mathscr{B} are two laminar families of subsets of a finite set S, and n is a positive integer, then there exist a subset A of S such that

 $|A \cap P| \approx |P|/n$ for every $P \in \mathscr{A} \cup \mathscr{B}$.

3. PROOF OF THE MAIN THEOREM

To prove Theorem 1.2, first we look at the obvious necessary conditions:

Lemma 3.1. If λK_n^h is connected (r_1, \ldots, r_k) -factorizable, then

- (i) $r_i \ge 2$ for $1 \le i \le k$,
- (ii) $h \mid r_i n \text{ for } 1 \leq i \leq k, \text{ and}$ (iii) $\sum_{i=1}^k r_i = \lambda {n-1 \choose h-1}.$

Proof. Suppose that λK_n^h is connected (r_1, \ldots, r_k) -factorizable. The necessity of (i) is sufficiently obvious. Since each edge contributes h to the sum of the degrees of the vertices in an r_i -factor for $1 \leq i \leq k$, we must have (ii). Since each r_i -factor is an r_i -regular spanning sub-hypergraph for $1 \leq i \leq k$, and λK_n^h is $\lambda \binom{n-1}{h-1}$ -regular, we must have (iii).

In order to get an inductive proof of Theorem 1.2 to work, we actually prove the following seemingly stronger result:

Theorem 3.2. Let $n, h, \lambda, k, r_1, \ldots, r_k$ be positive integers with n > h satisfying (i)–(iii). For any integer $1 \leq \ell \leq n$, there exists an ℓ -vertex k-edge-colored h-uniform hypergraph \mathcal{G} with vertex set V ($\alpha \in V$) such that

(1)
$$d_i(u) = \begin{cases} r_i(n-\ell+1) & \text{if } u = \alpha \\ r_i & \text{if } u \neq \alpha \end{cases} \text{ for } u \in V, 1 \leq i \leq k,$$

(2)
$$m(\alpha^p, U) = \lambda \binom{n-\ell+1}{p} \text{ for } p \ge 0, U \subset V \setminus \{\alpha\} \text{ with } |U| = h-p, \text{ and}$$

(3)
$$\mathcal{G}_i \text{ is connected if } r_i \ge 2, \text{ for } 1 \le i \le k.$$

Remark 3.3. Theorem 1.2 follows from Theorem 3.2 in the case where $\ell = n$ as the following argument shows. If $\ell = n$, then conditions (1)–(3) imply that we have an *n*-vertex *k*-edge-colored hypergraph \mathcal{G} in which the *i*th color class is r_i -regular by (1), and connected by (3). Moreover, (2) implies that for $U \subset V \setminus \{\alpha\}$, (i) $m(U) = \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda$ if |U| = h (when p = 0), (ii) $m(\alpha, U) = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda$ if |U| = h - 1 (when p = 1), and (iii) $m(\alpha^p, U) = \lambda \begin{pmatrix} 1 \\ p \end{pmatrix} = 0$ for $p \ge 2$, and |U| = h - p. Therefore $\mathcal{G} \cong \lambda K_n^h$.

Proof. The proof is by induction on ℓ . At each step we will assume not only that \mathcal{G} is an ℓ -vertex k-edge-colored hypergraph with vertex set V ($\alpha \in V$) satisfying conditions (1)–(3), but that \mathcal{G} also satisfies the two additional properties

(4)
$$|H_e| \leq n - \ell + 1$$
 for each edge e of \mathcal{G} , and

(5) for
$$1 \le i \le k$$
, if $r_i \ge 2$ and if $\ell \le n-1$, then $\delta_i = r_i(n-\ell+1)$

where $\delta_i = |H^i|$ for $1 \leq i \leq k$.

First consider the base case when $\ell = 1$. Let \mathcal{F} be a hypergraph with a single vertex α incident with $\lambda {n \choose h} h$ -loops; i.e. $m(\alpha^h) = \lambda {n \choose h}$. Color the edges of \mathcal{F} such that $m_i(\alpha^h) = r_i n/h$ for $1 \leq i \leq k$. This is possible since by (ii) $h \mid r_i n$, and by (iii) $\sum_{i=1}^k m_i(\alpha^h) = \sum_{i=1}^k r_i n/h = n/h \sum_{i=1}^k r_i = \lambda n {n-1 \choose h-1}/h = \lambda {n \choose h} = m(\alpha^h)$. Also, note that for $\ell = 1$, the hypergraph \mathcal{F} trivially satisfies (4), and since each h-loop is an α -wing, \mathcal{F} also satisfies (5). Therefore, \mathcal{F} shows that conditions (1)–(5) holds for $\ell = 1$.

Now suppose that $1 \leq \ell < n$, and that \mathcal{G} satisfies (1)–(5). The proof is completed by showing that \mathcal{G} has an $(\ell + 1)$ -vertex α -detachment \mathcal{G}' with vertex set $V' = V \cup \{\beta\}$ satisfying

(6)
$$|H'_e| \leq n - \ell$$
 for each edge e of \mathcal{G}' ,

(7)
$$d'_{i}(u) = \begin{cases} r_{i}(n-\ell) & \text{if } u = \alpha \\ r_{i} & \text{if } u \neq \alpha \end{cases} \text{ for } u \in V', 1 \leq i \leq k,$$

(8)
$$m'(\alpha^p, U) = \lambda \binom{n-\ell}{p} \text{ for } p \ge 0, U \subset V' \setminus \{\alpha\} \text{ with } |U| = h - p,$$

(9)
$$\mathcal{G}'(i)$$
 is connected if $r_i \ge 2$, for $1 \le i \le k$, and

for $1 \leq i \leq k$, if $r_i \geq 2$ and if $\ell < n - 1$, then (10) $\delta'_i = r_i(n - \ell).$

Let \mathscr{A} and \mathscr{B} be the laminar families in Lemmas 2.2, and 2.3. By Lemma 2.4, there exists a subset A of $H(\alpha)$ such that

(11)
$$|A \cap P| \approx |P|/(n-\ell+1)$$
 for every $P \in \mathscr{A} \cup \mathscr{B}$.

Let \mathcal{G}' be the hypergraph obtained from \mathcal{G} by splitting α into two vertices α and β in such a way that hinges which were incident with α in \mathcal{G} become incident in \mathcal{G}' with α or β according as they do not or do belong to A, respectively. More precisely,

(12)
$$H'(\beta) = A, \quad H'(\alpha) = H(\alpha) \backslash A.$$

Since $H_i \in \mathscr{A}$ for $1 \leq i \leq k$, we have

$$\begin{aligned} d'_{i}(\beta) &= |A \cap H_{i}| \\ &\approx |H_{i}|/(n-\ell+1) = d_{i}(\alpha)/(n-\ell+1) \\ &= r_{i}(n-\ell+1)/(n-\ell+1) = r_{i}, \\ d'_{i}(\alpha) &= d_{i}(\alpha) - d'_{i}(\beta) \\ &= r_{i}(n-\ell+1) - r_{i} = r_{i}(n-\ell), \end{aligned}$$

and for $u \notin \{\alpha, \beta\}, d'_i(u) = d_i(u) = r_i$. Therefore \mathcal{G}' satisfies (7).

Let e be an edge in \mathcal{G} incident with α . Then $H_e \in \mathscr{A}$, and so

$$|A \cap H_e| \approx |H_e|/(n-\ell+1) \leq 1,$$

observing that the last inequality implies from (4). This means that either $A \cap H_e = \emptyset$ or $|A \cap H_e| = 1$. Therefore $m'(\beta^q, U) = 0$ for $q \ge 2$ and $U \subset V'$. Also, note that if $|H_e| = n - \ell + 1$, then $|A \cap H_e| = 1$ and thus $|H'_e| = n - \ell$, and if $|H_e| < n - \ell + 1$, then $|H'_e| \le |H_e| \le n - \ell$, both cases together proving (6).

Since for $p \ge 1$, and $U \subset V \setminus \{\alpha\}$, $H_p^U \in \mathscr{B}$, we have

$$m'(\alpha^{p-1},\beta,U) = |A \cap H_p^U|$$

$$\approx |H_p^U|/(n-\ell+1) = pm(\alpha^p,U)/(n-\ell+1)$$

$$= \lambda p \binom{n-\ell+1}{p}/(n-\ell+1) = \lambda \binom{n-\ell}{p-1},$$

$$m'(\alpha^p,U) = m(\alpha^p,U) - m'(\alpha^{p-1},\beta,U)$$

$$= \lambda \binom{n-\ell+1}{p} - \lambda \binom{n-\ell}{p-1} = \lambda \binom{n-\ell}{p}.$$

Therefore \mathcal{G}' satisfies (8).

Let us fix an $i, 1 \leq i \leq k$ such that $r_i \geq 2$. Let W be an α -wing of \mathcal{G}_i with $d_W(\alpha) \geq 2$. Then $H_W \in \mathscr{A}$, and so

(13)
$$|A \cap H_W| \approx |H_W|/(n-\ell+1) = d_W(\alpha)/(n-\ell+1),$$

which implies that (noting that $n - \ell + 1 \ge 2$)

$$(14) |A \cap H_W| < |H_W|.$$

Moreover,

(15)
$$|A \cap H^i| \approx |H^i|/(n-\ell+1) = \delta_i/(n-\ell+1) = r_i \ge 2,$$

and therefore there exists an α -wing W in \mathcal{G}_i with $d_W(\alpha) \ge 2$, such that $A \cap H_W \neq \emptyset$. Therefore by Lemma 2.1, \mathcal{G}'_i is connected.

Now, suppose that $\ell \leq n-2$, or equivalently that $n-\ell+1 \geq 3$. Since $\delta_i = d_i$ by (1) and (5), we have that for every $W \in \mathscr{W}_{\alpha}(\mathcal{G}_i)$, $d_W(\alpha) \geq 2$. So there is no α -wing W in \mathcal{G}_i with $d_W(\alpha) = 1$. Let us fix an α -wing W in \mathcal{G}_i . There are two cases to consider:

- Case 1: If $|H_W| \ge 3$, then since $|A \cap H_W| \approx |H_W|/(n-\ell+1) \le |H_W|/3$, we have that $d'_{W'}(\alpha) \ge 2$, and thus $\delta'_i = d'_i(\alpha) = r_i(n-\ell)$. Note that W' is a sub-hypergraph of some α -wing S in \mathcal{G}' with $d'_S(\alpha) \ge 2$.
- Case 2: If $|H_W| = 2$, then $|A \cap H_W| \approx |H_W|/(n-\ell+1) = 2/(n-\ell+1) \leq 2/3$. So $|A \cap H_W| \in \{0, 1\}$. If $A \cap H_W = \emptyset$, we are done. So let us assume that $|A \cap H_W| = 1$. Recall from (15) that $|A \cap H^i| \geq 2$. Therefore, there is another α -wing T in \mathcal{G}_i with $|H_T| \geq 2$ such that $1 \leq |A \cap H_T| < |H_T|$. Therefore, there exists an α -wing S in \mathcal{G}' with $W' \cup T' \subset S$, and $d'_S(\alpha) \geq 2$. Thus, in this case also we have $\delta'_i = \delta_i - r_i = r_i(n-\ell)$.

Therefore \mathcal{G}' satisfies (10) and the proof is complete.

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DEPARTMENT OF MATHEMATICS AND STATISTICS AUBURN UNIVERSITY, AUBURN, AL USA 36849-5310

Current address: Department of Mathematics and Statistics, University of Ottawa, 585 King Edward, Ottawa, ON Canada K1N 6N5

E-mail address: mbahmani@uottawa.ca