# Cycles in triangle-free graphs of large chromatic number* 

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#### Abstract

More than twenty years ago Erdős conjectured [4] that a triangle-free graph $G$ of chromatic number $k \geq k_{0}(\varepsilon)$ contains cycles of at least $k^{2-\varepsilon}$ different lengths as $k \rightarrow \infty$. In this paper, we prove the stronger fact that every triangle-free graph $G$ of chromatic number $k \geq k_{0}(\varepsilon)$ contains cycles of $\left(\frac{1}{64}-\varepsilon\right) k^{2} \log k$ consecutive lengths, and a cycle of length at least $\left(\frac{1}{4}-\varepsilon\right) k^{2} \log k$. As there exist triangle-free graphs of chromatic number $k$ with at most roughly $4 k^{2} \log k$ vertices for large $k$, theses results are tight up to a constant factor. We also give new lower bounds on the circumference and the number of different cycle lengths for $k$-chromatic graphs in other monotone classes, in particular, for $K_{r}$-free graphs and graphs without odd cycles $C_{2 s+1}$.


## 1 Introduction

It is well-known that every $k$-chromatic graph has a cycle of length at least $k$ for $k \geq 3$. In 1991, Gyarfás [6] proved a stronger statement, namely, the conjecture by Bollobás and Erdős that every graph of chromatic number $k \geq 3$ contains cycles of at least $\left\lfloor\frac{1}{2}(k-1)\right\rfloor$ odd lengths. This is best possible in view of any graph whose blocks are complete graphs of order $k$. Mihok and Schiermeyer [9] proved a similar result for even cycles: every graph $G$ of chromatic number $k \geq 3$ contains cycles of at least $\left\lfloor\frac{k}{2}\right\rfloor-1$ even lengths. A consequence of the main result in [14] is that a graph of chromatic number $k \geq 3$ contains cycles of $\left\lfloor\frac{1}{2}(k-1)\right\rfloor$ consecutive lengths. Erdős [4] made the following conjecture:

Conjecture 1. For every $\varepsilon>0$, there exists $k_{0}(\varepsilon)$ such that for $k \geq k_{0}(\varepsilon)$, every triangle-free $k$-chromatic graph contains more than $k^{2-\epsilon}$ odd cycles of different lengths.

The second and third authors proved [12] that if $G$ is a graph of average degree $k$ and girth at least five, then $G$ contains cycles of $\Omega\left(k^{2}\right)$ consecutive even lengths, and in [13] it was shown that if an $n$-vertex graph of independence number at most $\frac{n}{k}$ is triangle-free, then it contains cycles of $\Omega\left(k^{2} \log k\right)$ consecutive lengths.

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### 1.1 Main Result

In this paper, we prove Conjecture 1 in the following stronger form:
Theorem 1. For all $\varepsilon>0$, there exists $k_{0}(\varepsilon)$ such that for $k \geq k_{0}(\varepsilon)$, every triangle-free $k$-chromatic graph $G$ contains a cycle of length at least $\left(\frac{1}{4}-\varepsilon\right) k^{2} \log k$ as well as cycles of at least $\left(\frac{1}{64}-\varepsilon\right) k^{2} \log k$ consecutive lengths.

Kim [8] was the first to construct a triangle-free graph with chromatic number $k$ and $\Theta\left(k^{2} \log k\right)$ vertices. Bohman and Keevash [2] and Fiz Pontiveros, Griffiths and Morris [5] independently constructed a $k$-chromatic triangle-free graph with at most $(4+o(1)) k^{2} \log k$ vertices as $k \rightarrow \infty$, refining the earlier construction of Kim [8]. These constructions show that the bound in Theorem is tight up to a constant factor.

### 1.2 Monotone Properties

Theorem $\mathbb{1}$ is a special case of a more general theorem on monotone properties. A graph property is called monotone if it holds for all subgraphs of a graph which has this property, i.e., is preserved under deletion of edges and vertices. Throughout this section, let $n_{\mathcal{P}}(k)$ denote the smallest possible order of a $k$-chromatic graph in a monotone property $\mathcal{P}$.
Definition 1. Let $\alpha \geq 1$ and let $f:[3, \infty) \rightarrow \mathbb{R}^{+}$. Then $f$ is $\alpha$-bounded if $f$ is non-decreasing and whenever $y \geq x \geq 3, y^{\alpha} f(x) \geq x^{\alpha} f(y)$.

For instance, any polynomial with positive coefficients is $\alpha$-bounded for some $\alpha \geq 1$. We stress that an $\alpha$-bounded function is required to be a non-decreasing positive real-valued function with domain $[3, \infty)$.
Theorem 2. For all $\varepsilon>0$ and $\alpha, m \geq 1$, there exists $k_{1}=k_{1}(\varepsilon, \alpha, m)$ such that the following holds. If $\mathcal{P}$ is a monotone property of graphs with $n_{\mathcal{P}}(k) \geq f(k)$ for $k \geq m$ and some $\alpha$ bounded function $f$, then for $k \geq k_{1}$, every $k$-chromatic graph $G \in \mathcal{P}$ contains ( $i$ ) a cycle of length at least $(1-\varepsilon) f(k)$ and $(i i)$ cycles of at least $(1-\varepsilon) f\left(\frac{k}{4}\right)$ consecutive lengths.

If $n_{\mathcal{P}}(k)$ itself is $\alpha$-bounded for some $\alpha$, then we obtain from Theorem 2 a tight result that a $k$-chromatic graph in $\mathcal{P}$ contains a cycle of length asymptotic to $n_{\mathcal{P}}(k)$ as $k \rightarrow \infty$. But proving that $n_{\mathcal{P}}(k)$ is $\alpha$-bounded for some $\alpha$ is probably difficult for many properties, and in the case $\mathcal{P}$ is the property of $F$-free graphs, perhaps is as difficult as obtaining asymptotic formulas for certain Ramsey numbers. Even in the case of the property of triangle-free graphs, we have seen $n_{\mathcal{P}}(k)$ is known only up to a constant factor. We remark that in Theorems $\mathbb{1}$ and 2, we have not attempted to optimize the quantities $k_{0}(\varepsilon)$ and $k_{1}(\varepsilon, \alpha, m)$.

### 1.3 An application: $K_{r}$-free graphs

As an example of an application of Theorem 2, we consider the property $\mathcal{P}$ of $K_{r}$-free graphs. A lower bound for the quantity $n_{\mathcal{P}}(k)$ can be obtained by combining upper bounds for Ramsey
numbers together with a lemma on colorings obtained by removing maximum independent sets - see Section [5. In particular, we shall obtain the following from Theorem 2.

Theorem 3. If $G$ is a $k$-chromatic $K_{r+1}$-free graph, where $r, k \geq 3$, then $G$ contains a cycle of length $\Omega\left(k^{\frac{r}{r-1}}\right)$, and cycles of $\Omega\left(k^{\frac{r}{r-1}}\right)$ consecutive lengths as $k \rightarrow \infty$.

Theorem 3 is derived from upper bounds on the Ramsey numbers $r\left(K_{r}, K_{t}\right)$ combined with Theorem [2. In general, if for a graph $F$ one has $r\left(F, K_{t}\right)=O\left(t^{a}(\log t)^{-b}\right)$ for some $a>1$ and $b>0$, then any $k$-chromatic $F$-free graph has cycles of

$$
\Omega\left(k^{\frac{a}{a-1}}(\log k)^{\frac{b}{a-1}}\right)
$$

consecutive lengths. We omit the technical details, since the ideas of the proof are identical to those used for Theorem 3. These technical details are presented in the proof of Theorem 1 in Section 4, where $F$ is a triangle (in which case $a=2$ and $b=1$ ), and the same ideas can be used to slightly improve Theorem 3 by logarithmic factors using better bounds on $r\left(K_{s}, K_{t}\right)$ from results of Ajtai, Komlós and Szemerédi [1]. Similarly, if $C_{\ell}$ denotes the cycle of length $\ell$, then it is known that $r\left(C_{2 s+1}, K_{t}\right)=O\left(t^{1+1 / s}(\log t)^{-1 / s}\right)$ - see [11]. This in turn provides cycles of $\Omega\left(k^{s+1} \log k\right)$ consecutive lengths in any $C_{2 s+1}$-free $k$-chromatic graph, extending Theorem 1

Notation and terminology. For a graph $G$, let $c(G)$ denote the length of a longest cycle in $G$ and $\chi(G)$ the chromatic number of $G$. If $F \subset G$ and $S \subset V(G)$, let $G[F]$ and $G[S]$ respectively denote the subgraphs of $G$ induced by $V(F)$ and $S$. A chord of a cycle $C$ in a graph is an edge of the graph joining two non-adjacent vertices on the cycle. All logarithms in this paper are with the natural base.

Organization. In the next section, we present the lemmas which will be used to prove Theorem 2. Then in Section 4, we apply Theorem 2 to obtain the proof of Theorem 1 , Theorem 3 is proved in Section 5.

## 2 Lemmas

### 2.1 Vertex cuts in $k$-critical graphs

When a small vertex cut is removed from a $k$-critical graph, all the resulting components still have relatively high chromatic number:

Lemma 1. Let $G$ be a $k$-critical graph and let $S$ be a vertex cut of $G$. Then for any component $H$ of $G-S, \chi(H) \geq k-|S|$.

Proof. If $|S|+\chi(H) \leq k-1$, then a $(k-1)$-coloring of $G-H$ (existing by the criticality of $G$ ) can be extended to a ( $k-1$ )-coloring of $G$.

### 2.2 Nearly 3-connected subgraphs

Our second lemma finds an almost 3-connected subgraph with high chromatic number in a graph with high chromatic number.

Lemma 2. Let $k \geq 4$. For every $k$-chromatic graph $G$, there is a graph $G^{*}$ and an edge $e^{*} \in E\left(G^{*}\right)$ such that
(a) $G^{*}-e^{*} \subset G$ and $\chi\left(G^{*}-e^{*}\right) \geq k-1$.
(b) $G^{*}$ is 3-connected.
(c) $\quad c\left(G^{*}\right) \leq c(G)$.

Proof. Let $G^{\prime}$ be a $k$-critical subgraph of $G$. Then $G^{\prime}$ is 2 -connected. If $G^{\prime}$ is 3-connected, then the lemma holds for $G^{*}=G^{\prime}$ with any $e \in E\left(G^{\prime}\right)$ as $e^{*}$. So suppose $G^{\prime}$ is not 3-connected. Among all separating sets $S$ in $G^{\prime}$ of size 2 and components $F$ of $G^{\prime}-S$, choose a pair ( $S, F$ ) with the minimum $|V(F)|$. If $S=\{u, v\}$, then we let $G^{*}$ be induced by $V(F) \cup S$ plus the edge $e^{*}=u v$. We claim $\chi\left(G^{*}-e^{*}\right) \geq k-1$. Since $G^{\prime}$ is $k$-critical, there is a $(k-1)$-coloring $\varphi: V\left(G^{\prime}\right) \backslash V(F) \rightarrow\{1,2, \ldots, k-1\}$ of $G^{\prime}-V(F)$ and we may assume $\varphi(u)=k-1$ and by renaming colors $\varphi(v) \in\{1, k-1\}$. Suppose for a contradiction that there is a coloring $\varphi^{*}: V\left(G^{*}-e^{*}\right) \rightarrow\{1,2, \ldots, k-2\}$ of $G^{*}-e^{*}$. If $\varphi(v)=k-1$, then we let $\varphi^{\prime}(x)=\varphi(x)$ if $x \in V\left(G^{\prime}\right)-V(F)$ and $\varphi^{\prime}(x)=\varphi^{*}(x)$ if $x \in V(F)$, and this $\varphi^{\prime}$ is a proper $(k-1)$-coloring of $G^{\prime}$, a contradiction. Otherwise $\varphi(v)=1$. Then we change the names of colors in $\varphi^{*}$ so that $\varphi^{*}(v)=1$ and again let $\varphi^{\prime}(x)=\varphi(x)$ if $x \in V\left(G^{\prime}\right)-V(F)$ and $\varphi^{\prime}(x)=\varphi^{*}(x)$ if $x \in V(F)$. Again we have a proper $(k-1)$-coloring of $G^{\prime}$. This contradiction proves (a).

To prove (b), if $G^{*}$ has a separating set $S^{\prime}$ with $\left|S^{\prime}\right|=2$, then, since $u v \in E\left(G^{*}\right)$, it is also a separating set in $G$ and at least one component of $G^{\prime}-S^{\prime}$ is strictly contained in $F$. This contradicts the choice of $F$ and $S$.

For (c), let $C$ be a cycle in $G^{*}$ with $|C|=c\left(G^{*}\right)$. If $e^{*} \notin E(C)$, then $C$ is also a cycle in $G$, and thus $c(G) \geq|C|=c\left(G^{*}\right)$. If $e^{*} \in E(C)$ and $G^{*} \neq G^{\prime}$, then we obtain a longer cycle $C$ in $G^{\prime}$ by replacing $e^{*}$ with a uv-path in $G^{\prime}-V(F)$ - note such a path exists since $G^{\prime}$ is 2 -connected. This proves (c).

### 2.3 Finding cycles of consecutive lengths

In this subsection we show how to go from longest cycles in graphs to cycles of many consecutive lengths. We will need the following result from [14], which is also implicit in the paper of Bondy and Simonovits [3]:

Lemma 3 (Lemma 2 in [14]). Let $H$ be a graph comprising a cycle with a chord. Let $(A, B)$ ) be a nontrivial partition of $V(H)$. Then $H$ contains $A, B$-paths of every positive length less than $|H|$, unless $H$ is bipartite with bipartition $(A, B)$.

Lemma 4. Let $k \geq 4$ and $\mathcal{Q}$ be a monotone class of graphs. Let $h(k, \mathcal{Q})$ denote the smallest possible length of a longest cycle in any $k$-chromatic graph in $\mathcal{Q}$. Then every $4 k$-chromatic graph in $\mathcal{Q}$ contains cycles of at least $h(k, \mathcal{Q})$ consecutive lengths.

Proof. Let $F$ be a connected subgraph of $G \in \mathcal{Q}$ with chromatic number at least $4 k$ and let $T$ be a breadth-first search tree in $F$. Let $L_{i}$ be the set of vertices at distance exactly $i$ from the root of $T$ in $F$. Then for some $i, H=F\left[L_{i}\right]$ has chromatic number at least $2 k$. Let $U$ be a breadth-first search tree in a component of $H$ with chromatic number at least $2 k$ and let $M_{i}$ be the set of vertices at distance exactly $i$ from the root of $U$ in $H$. Then for some $i, J=H\left[M_{i}\right]$ has chromatic number at least $k$. Let $J^{\prime}$ be a $k$-critical subgraph of $J$. Let $P$ be a longest path in $J^{\prime}$, so that $|P| \geq h(k, \mathcal{Q})$. Since $J^{\prime}$ has minimum degree at least $k-1 \geq 3$, each of the ends of $P$ has at least two neighbors on $P$. In particular, there is a path $P^{\prime} \subset P$ of odd length with at least one chord, obtained by deleting at most one end of $P$, and $\left|P^{\prime}\right| \geq h(k, \mathcal{Q})-1$. Then the ends of $P^{\prime}$ are joined by an even length path $Q \subset U$ that is internally disjoint from $P^{\prime}$, and $C=Q \cup P^{\prime}$ is a cycle of odd length plus a chord, with $|C| \geq h(k, \mathcal{Q})+1$. Let $\ell:=|C|$ and $H^{\prime}=G[C]$. Now $V\left(H^{\prime}\right) \subset L_{i}$ by construction. Let $T^{\prime}$ be a minimal subtree of $T$ whose set of leaves is $V\left(H^{\prime}\right)$. Then $T^{\prime}$ branches at its root. Let $A$ be the set of leaves in some branch of $T^{\prime}$, and let $B=V\left(H^{\prime}\right) \backslash A$. Then $(A, B)$ is not a bipartition of $H^{\prime}$, since $C$ has odd length, and therefore by Lemma 1 in [14], there exist paths $P_{1}, P_{2}, \ldots, P_{\ell-1} \subset H^{\prime}$ such that $P_{i}$ has length $i$ and one end of $P_{i}$ is in $A$ and one end of $P_{i}$ is in $B$, for $i=1,2, \ldots, \ell-1$. Now for each path $P_{i}$, the ends of $P_{i}$ are joined by a path $Q_{i}$ of length $2 r$, where $r$ is the height of $T^{\prime}$ and $Q_{i}$ and $P_{i}$ are internally disjoint. Therefore $P_{i} \cup Q_{i}$ is a cycle of length $2 r+i$ for $i=1,2, \ldots, \ell-1$, as required.

### 2.4 A lemma on $\alpha$-bounded functions

The following technical lemma is required for the proof of Theorem 2.
Lemma 5. Let $\alpha, x_{0} \geq 1$, and let $f$ be $\alpha$-bounded. Then the function

$$
g(x)=\frac{x f(x)}{x+f\left(x_{0}\right)} .
$$

is $(\alpha+1)$-bounded, $g(x) \leq x$ for $x \in\left[3, x_{0}\right]$, and $g(x) \leq f(x)$ for all $x \in[3, \infty)$.
Proof. By definition, $g(x) \leq f(x)$ for $x \in[3, \infty)$ and $g(x) \leq x$ for $x \in\left[3, x_{0}\right]$. Also, since $f$ is non-decreasing and positive on $[3, \infty), g$ is non-decreasing on $[3, \infty)$. It remains to check that $g$ is $(\alpha+1)$-bounded. For $y \geq x \geq 3$, using that $y^{\alpha} f(x) \geq x^{\alpha} f(y)$, we find

$$
y^{\alpha+1} g(x)=\frac{y^{\alpha+1} x f(x)}{x+f\left(x_{0}\right)} \geq \frac{x^{\alpha+1} y f(y)}{x+f\left(x_{0}\right)} \geq x^{\alpha+1} g(y)
$$

Therefore $g$ is $(\alpha+1)$-bounded.

## 3 Proof of Theorem 2

It is enough to prove Theorem [2 for all $\varepsilon<1 / 2$. Let $\beta=\alpha+1, \eta=\frac{\varepsilon}{2}$ and $x_{0}=$ $\max \left\{2 m,\left(\frac{12 \beta}{\eta}\right)^{\beta+1}\right\}$. Define $k_{1}=k_{1}(\varepsilon, \alpha, m)=\frac{8}{\varepsilon} f\left(x_{0}\right)$. Let $g$ be a $\beta$-bounded function in Lemma 5 We prove the following claim:

Claim. For $k \geq 3$, every $k$-chromatic graph $G \in \mathcal{P}$ has a cycle of length at least $(1-\eta) g(k)$.
Once this claim is proved, Theorem 2(i) follows since for $k \geq k_{1}$,

$$
(1-\eta) g(k)=\left(1-\frac{\varepsilon}{2}\right) \frac{k f(k)}{k+f\left(x_{0}\right)} \geq\left(1-\frac{\varepsilon}{2}\right) \frac{k f(k)}{k+\varepsilon k / 8} \geq(1-\varepsilon) f(k),
$$

as required. Also, if $k \geq k_{1}$, then by Lemma 4 every $k$-chromatic graph in $\mathcal{P}$ contains cycles of at least $(1-\eta) g\left(\frac{k}{4}\right) \geq(1-\varepsilon) f\left(\frac{k}{4}\right)$ consecutive lengths, which gives Theorem 2(ii). We prove the claim by induction on $k \geq 3$. For $k \leq x_{0}, g(k) \leq k$ from Lemma [5, so in that case $G$ contains a $k$-critical subgraph which has minimum degree at least $k-1$ and therefore also a cycle of length at least $k$. This proves the claim for $k \leq x_{0}$. Now suppose $k>x_{0}$. Let $G^{*}$ be the graph obtained from $G$ in Lemma 2. By Lemma 2(c), it is sufficient to show that $G^{*}$ has a cycle of length at least $(1-\eta) g(k)$. Let $C$ be a longest cycle in $G^{*}-e^{*}$. By induction, $|C| \geq(1-\eta) g(k-1)$. Let $G_{1}=G^{*}[C]$ and $\chi_{1}=\chi\left(G_{1}\right)$, and let $G_{2}=G^{*}-G_{1}-e^{*}$ and $\chi_{2}:=\chi\left(G_{2}\right)$. Take $C^{\prime}$ to be a longest cycle in $G_{2}$. Let $S$ be a minimum vertex set covering all paths from $C$ to $C^{\prime}$. Either $S$ separates $C^{\prime}-S$ from $C-S$ or $S=V\left(C^{\prime}\right)$. Let $|S|=\ell$. By Menger's Theorem, $G^{*}$ has $\ell$ vertex-disjoint paths $P_{1}, P_{2}, \ldots, P_{\ell}$ between $C$ and $C^{\prime}$ - note $\ell \geq 3$, as $G^{*}$ is 3 -connected. Let $H=\bigcup_{i=1}^{\ell} P_{i} \cup C \cup C^{\prime}$. We find a cycle $C^{*} \subset H$ with

$$
\begin{equation*}
\left|C^{*}\right| \geq \frac{\ell-1}{\ell}|C|+\frac{1}{2}\left|C^{\prime}\right| . \tag{1}
\end{equation*}
$$

To see this, first note that two of the paths, say $P_{i}$ and $P_{j}$, contain ends at distance at most $\frac{1}{\ell}|C|$ on $C$, and now $P_{i} \cup P_{j} \cup C \cup C^{\prime}$ contains a cycle $C^{*}$ of length at least

$$
\frac{\ell-1}{\ell}|C|+\frac{1}{2}\left|C^{\prime}\right|+\left|P_{i}\right|+\left|P_{j}\right| \geq \frac{\ell-1}{\ell}|C|+\frac{1}{2}\left|C^{\prime}\right| .
$$

At the same time, $H$ contains a cycle $C^{* *}$ with

$$
\begin{equation*}
\left|C^{* *}\right| \geq \frac{2}{3}\left(|C|+\left|C^{\prime}\right|\right) \tag{2}
\end{equation*}
$$

since there exist three cycles that together cover every edge of $P_{1} \cup P_{2} \cup P_{3} \cup C \cup C^{\prime}$ exactly twice, and one of them has the required length. Now we complete the proof in three cases.

Case 1. $\chi_{1} \geq\left(1-\frac{\eta}{\beta}\right) k$. Then $\chi_{1} \geq(1-\eta) k \geq \frac{k}{2} \geq m \geq 3$, which implies $n_{\mathcal{P}}\left(\chi_{1}\right) \geq f\left(\chi_{1}\right)$. Since $\eta \leq \frac{1}{2}<\beta$, we have $\left(1-\frac{\eta}{\beta}\right)^{\alpha} \geq 1-\eta$. Since $f$ is $\alpha$-bounded,

$$
|C| \geq n_{\mathcal{P}}\left(\chi_{1}\right) \geq f\left(\chi_{1}\right) \geq\left(1-\frac{\eta}{\beta}\right)^{\alpha} f(k) \geq(1-\eta) f(k) \geq(1-\eta) g(k) .
$$

Case 2. $\chi_{1}<\left(1-\frac{\eta}{\beta}\right) k$ and $\chi_{2} \geq\left(1-\frac{1}{4 \beta}\right) k$. Since $\chi_{2} \geq 3$ and $g$ is $\beta$-bounded,

$$
g\left(\chi_{2}\right) \geq\left(1-\frac{1}{4 \beta}\right)^{\beta} g(k) \geq \frac{3}{4} g(k) \quad \text { and } \quad g(k-1) \geq\left(\frac{k-1}{k}\right)^{\beta} g(k) \geq\left(1-\frac{\beta}{k}\right) g(k) .
$$

As $k>x_{0}>6 \beta$ and $\left|C^{\prime}\right| \geq(1-\eta) g\left(\chi_{2}\right)$, we obtain from (2):

$$
\left|C^{* *}\right| \geq \frac{2}{3}(1-\eta) g(k-1)+\frac{2}{3}(1-\eta) g\left(\chi_{2}\right) \geq(1-\eta) g(k) \cdot\left(\frac{7}{6}-\frac{\beta}{k}\right)>(1-\eta) g(k) .
$$

Case 3. $\chi_{1}<\left(1-\frac{\eta}{\beta}\right) k$ and $\chi_{2}<\left(1-\frac{1}{4 \beta}\right) k$. Then $\chi_{2} \geq k-1-\chi_{1}>\frac{\eta}{5 \beta} k \geq 3$. Since every $\chi_{2}$-chromatic graph contains a cycle of length at least $\chi_{2}$, we have that $\left|C^{\prime}\right| \geq \chi_{2}$. If $S=V\left(C^{\prime}\right)$, then $\ell=\left|C^{\prime}\right| \geq \chi_{2}>\frac{\eta}{5 \beta} k$. Otherwise, by Lemma回, $\chi_{2} \geq k-\min \left\{\chi_{1}, \ell\right\}-1$, and so $\ell>k-\chi_{2}-1>\frac{\eta}{5 \beta} k$. By (II), and since $g$ is $\beta$-bounded,

$$
\begin{aligned}
\left|C^{*}\right| & \geq \frac{\ell-1}{\ell}(1-\eta) g(k-1)+\frac{1}{2} g\left(\chi_{2}\right) \\
& \geq(1-\eta) g(k) \cdot\left(\frac{\ell-1}{\ell}\left(\frac{k-1}{k}\right)^{\beta}+\frac{1}{2}\left(\frac{\eta}{4 \beta}\right)^{\beta}\right) \\
& \geq(1-\eta) g(k) \cdot\left(1-\frac{\beta}{k}-\frac{1}{\ell}+\frac{1}{2}\left(\frac{\eta}{6 \beta}\right)^{\beta}\right) .
\end{aligned}
$$

Since $k \geq\left(\frac{12 \beta}{\eta}\right)^{\beta+1}$ and $\ell>\frac{\eta}{5 \beta} k, \frac{\beta}{k}+\frac{1}{\ell} \leq \frac{6 \beta}{\eta k}<\frac{1}{2}\left(\frac{\eta}{6 \beta}\right)^{\beta}$, and $\left|C^{*}\right|>(1-\eta) g(k)$, as required.

## 4 Proof of Theorem 1

For the proof of Theorem 1 , we use Theorem 2 with some specific choice of the function $f(k)$. Let $\alpha(G)$ denote the independence number of a graph $G$. Shearer [10] showed the following:

Lemma 6. For every n-vertex triangle-free graph $G$ with average degree $d$,

$$
\begin{equation*}
\alpha(G)>n \frac{\log (d / e)}{d} . \tag{3}
\end{equation*}
$$

This implies the following simple fact. In what follows, let $\varphi(x)=\left(\frac{1}{2} x \log x\right)^{1 / 2}$.
Lemma 7. If $G$ is an $n$-vertex triangle-free graph and $n \geq e^{2 e^{3}}$, then $\alpha(G) \geq \varphi(n)$.
Proof. Let $d=\varphi(n)$. Then if $n \geq e^{2 e^{3}}, d>e(e n)^{1 / 2}$. Since $G$ is triangle-free, the neighborhood of any vertex is an independent set, so we may assume $G$ has maximum degree less than $d$. Then by (3) and since $d>e(e n)^{1 / 2}$,

$$
d>n \frac{\log (d / e)}{d}>n \frac{\log (e n)^{1 / 2}}{\varphi(n)}=\varphi(n)=d,
$$

a contradiction.

To find a lower bound on the number of vertices of a triangle-free $k$-chromatic graph, we require a lemma of Jensen and Toft 7 (they took $s=2$, but their proof works for each positive integer $s$ ):
Lemma 8 ([7], Problem 7.3). Let $s \geq 1$ and let $\psi:[s, \infty) \rightarrow(0, \infty)$ be a positive continuous nondecreasing function. Let $\mathcal{P}$ be a monotone class of graphs such that $\alpha(G) \geq \psi(|V(G)|)$ for every $G \in \mathcal{P}$ with $|V(G)| \geq s$. Then for every such $G$ with $|V(G)|=n$,

$$
\chi(G) \leq s+\int_{s}^{n} \frac{1}{\psi(x)} d x
$$

Lemma 9. Let $f(x)=c x^{2} \log x$ where $x \geq 3$ and $c>0$. Then $f$ is 3-bounded.
Proof. The function $f$ is positive and non-decreasing, so one only has to check $y^{3} f(x) \geq$ $x^{3} f(y)$ whenever $y \geq x \geq 3$. This follows from $y \log x \geq x \log y$ for $y \geq x \geq 3$, since the function $\frac{y}{\log y}$ is increasing for $y \geq 3$.

Lemma 10. For every $\delta>0$, there exists $k_{2}(\delta)$ such that if $k \geq k_{2}(\delta)$, then every $k$-chromatic triangle-free graph has at least $\left(\frac{1}{4}-\delta\right) k^{2} \log k$ vertices.
Proof. If $\delta \geq \frac{1}{4}$ the lemma is trivial, so suppose $\delta<\frac{1}{4}$. Let $\gamma(x)=\frac{1}{2}-\frac{1}{2 \log e x}$. Let $G$ be a $k$-chromatic triangle-free $n$-vertex graph. We apply the preceding lemma with $\psi(x)=\varphi(x)$ supplied by Lemma 7. For $s \geq e^{2 e^{3}}$, and using $\gamma(s) \leq \gamma(x)$ for $x \geq s$ :

$$
\begin{aligned}
\chi(G) & \leq s+\sqrt{2} \int_{s}^{n}(x \log e x)^{-1 / 2} d x=s+\frac{\sqrt{2}}{\gamma(s)} \int_{s}^{n}(x \log e x)^{-1 / 2} \gamma(s) d x \\
& \leq s+\frac{\sqrt{2}}{\gamma(s)} \int_{s}^{n}(x \log x)^{-1 / 2} \gamma(x) d x
\end{aligned}
$$

An antiderivative for the integrand is exactly $x^{1 / 2}(\log e x)^{-1 / 2}$, and therefore

$$
\chi(G) \leq s+\frac{\sqrt{2}}{\gamma(s)} n^{1 / 2}(\log e n)^{-1 / 2} .
$$

On the other hand, $\chi(G) \geq k$ so if $j=k-s$,

$$
n \geq \gamma(s)^{2} j^{2} \log (\gamma(s) j)
$$

If $s=\left\lceil\max \left\{e^{2 e^{3}}, e^{1 / \delta}\right\}\right\rceil$, then $\gamma(s) \geq \frac{1}{2}(1-\delta)$, so since $\delta<\frac{1}{4}$,

$$
n \geq \frac{1}{4}(1-\delta)^{2} j^{2} \log \frac{1}{4} j \geq \frac{1}{4}(1-\delta)^{2} j^{2} \log j-k^{2} .
$$

If $j \geq 3$, then by Lemma 9

$$
n \geq \frac{1}{4}(1-\delta)^{2}\left(\frac{j}{k}\right)^{3} k^{2} \log k-k^{2} \geq \frac{1}{4}\left(1-2 \delta-\frac{3 s}{k}\right) k^{2} \log k-k^{2} .
$$

Let $k_{2}(\delta)=s^{4} \geq \max \left\{e^{8 e^{3}}, e^{4 / \delta}\right\}$. Since $k \geq e^{4 / \delta}, k^{2} \leq \frac{1}{4} \delta k^{2} \log k$ and $3 s \leq \delta k$. Therefore

$$
n \geq \frac{1}{4}\left(1-2 \delta-\frac{3 s}{k}\right) k^{2} \log k-k^{2} \geq \frac{1}{4}(1-4 \delta) k^{2} \log k .
$$

This completes the proof.

Proof of Theorem 1. The theorem is trivial if $\varepsilon \geq \frac{1}{4}$, so we assume $\varepsilon<\frac{1}{4}$. We will derive Theorem 1 from Theorem 2, Let $\eta=2 \varepsilon, \delta=\frac{\varepsilon}{2}$. By Lemma 10, for $k \geq m:=k_{2}(\delta)$, every triangle-free $k$-chromatic graph $G$ has at least $\left(\frac{1}{4}-\delta\right) k^{2} \log k$ vertices. Then $f(x)=$ $\left(\frac{1}{4}-\delta\right) x^{2} \log x$ is 3 -bounded, by Lemma 9 . By Theorem 2, with $\alpha=3, G$ contains a cycle of length at least $(1-\eta) f(k)$ as well as cycles of at least $(1-\eta) f\left(\frac{k}{4}\right)$ consecutive lengths in $G$, provided $k \geq k_{1}(\eta, \alpha, m)$. Letting $k_{0}(\varepsilon)=k_{1}(\eta, \alpha, m)=k_{1}\left(2 \varepsilon, 3, k_{2}\left(\frac{\varepsilon}{2}\right)\right)$, and noting $(1-\eta) f(k) \geq\left(\frac{1}{4}-\varepsilon\right) k^{2} \log k$ by the choice of $\eta$ and $\delta$, we have a cycle of length at least $\left(\frac{1}{4}-\varepsilon\right) k^{2} \log k$ in $G$ whenever $k \geq k_{0}(\varepsilon)$. Similarly, if $k$ is large enough relative to $\varepsilon$, then $G$ contains cycles of at least $(1-\eta) f\left(\frac{k}{4}\right) \geq\left(\frac{1}{64}-\varepsilon\right) k^{2} \log k$ consecutive lengths. This completes the proof.

## 5 Proof of Theorem 3

Lemma 11. Let $r \geq 3$ and $G$ be a $K_{r}$-free n-vertex graph. Then $\alpha(G) \geq n^{1 /(r-1)}-1$.
Proof. If $r \geq 3$ and $n \leq 2^{r-1}$, then $n^{1 /(r-1)}-1 \leq 1$, so the claim holds. Let $n>2^{r-1}$. If $r=3$, either $\Delta(G) \geq n^{1 / 2}$ or the graph is greedily $\left\lfloor n^{1 / 2}\right\rfloor+1$-colorable. Since vertex neighborhoods are independent sets in a triangle-free graph, either case gives an independent set of size at least $n^{1 / 2}-1$. For $r>3$, either the graph has a vertex $v$ of degree at least $d \geq n^{(r-2) /(r-1)}$, or the graph is $n^{1-1 /(r-1)}+1$-colorable. In the latter case, the largest color class is an independent set of size at least $n^{1 /(r-1)}-1$. In the former case, since the neighborhood of $v$ induces a $K_{r-1}$-free graph, by induction it contains an independent set of size at least $d^{1 /(r-2)}-1 \geq n^{1 /(r-1)}-1$, as required.

Lemma 12. Let $r \geq 3$ and $G$ be a $K_{r}$-free n-vertex graph. Then

$$
\chi(G)<4 n^{1-1 /(r-1)} .
$$

Proof. The function $f(x)=\max \left\{1, x^{1 /(r-1)}-1\right\}$ is positive continuous and nondecreasing. Since each nontrivial graph has an independent set of size 1, by Lemmas 8 and 11 ,

$$
\chi(G) \leq 1+\int_{1}^{n} \frac{1}{f(x)} \mathrm{d} x \leq 1+\int_{1}^{n} \frac{2}{x^{1 /(r-1)}} \mathrm{d} x<4 n^{1-1 /(r-1)} .
$$

Proof of Theorem 3. We prove the first claim of the theorem for all $k, r \geq 3$, and then apply Lemma 4 to prove the second claim. Let $G$ be an $n$-vertex $K_{r+1}$-free graph with $\chi(G)=k \geq 3$. By Lemma 12, $k<4 n^{1-\frac{1}{r}}$, so $|V(G)| \geq\left(\frac{k}{4}\right)^{\frac{r}{r-1}}:=f(k)$. Since $f$ is $\frac{r}{r-1}$-bounded, the proof is complete by Theorem 2 with $\mathcal{P}$ the property of $K_{r+1}$-free graphs.

## 6 Concluding remarks

- In this paper, we have shown that the length of a longest cycle and the length of a longest interval of lengths of cycles in $k$-chromatic graphs $G$ are large when $G$ lacks certain subgraphs. In particular, when $G$ has no triangles, this yields a proof of Conjecture 1 (in a stronger form). We believe that the following holds.

Conjecture 2. Let $G$ be a $k$-chromatic triangle-free graph and let $n_{k}$ be the minimum number of vertices in a $k$-chromatic triangle-free graph. Then $G$ contains a cycle of length at least $n_{k}-o\left(n_{k}\right)$.

- If Shearer's bound [10] is tight, i.e., $n_{k} \sim \frac{1}{4} k^{2} \log k$, then the lower bound on the length of the longest cycle in any $k$-chromatic triangle-free graph in Theorem 1 would be tight.


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