# Cocliques in the Kneser graph on line-plane flags in PG$(4, q)$

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**Abstract**

We determine the independence number of the Kneser graph on line-plane flags in the projective space PG$(4, q)$.

## 1 Introduction

Let $\Gamma$ be the graph with as vertices the line-plane flags (incident line-plane pairs) in PG$(4, q)$, where two flags $(L, A)$, $(L', A')$ are adjacent when they are in general position, i.e., when $L \cap A' = L' \cap A = 0$. In this note we determine the maximum-size cocliques in $\Gamma$. It will turn out that $\Gamma$ has independence number $(q^2 + q + 1)(q^3 + 2q^2 + q + 1)$. In PG$(4, q)$, a *solid* is the same as a hyperplane. The hypotheses are self-dual: true statements remain true if one interchanges the words 'line' and 'plane', and 'point' and 'solid'.

## 2 Lower bound

Cocliques of size $(q^2 + q + 1)(q^3 + 2q^2 + q + 1)$ can be constructed in the following four ways.

(i) Fix a solid $S_0$, and take all $(L, A)$ with $A \subseteq S_0$, together with (ia) all $(L, A)$ with $L = A \cap S_0$ and $P_0 \subseteq L$, for some fixed point $P_0 \subseteq S_0$, or (ib) all $(L, A)$ with $L = A \cap S_0$ and $L \subseteq A_0$, for some fixed plane $A_0 \subseteq S_0$.

(ii) Or, fix a point $P_0$, and take all $(L, A)$ with $P_0 \subseteq L$, together with (iia) all $(L, A)$ with $A = L + P_0$ and $A \subseteq S_0$ for some fixed solid $S_0 \supseteq P_0$, or (iib) all $(L, A)$ with $A = L + P_0$ and $L_0 \subseteq A$ for some fixed line $L_0 \supseteq P_0$.

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## 3 Structure

Let $\mathcal{C}$ be a maximal coclique in $\Gamma$. We show that $\mathcal{C}$ must have some linear structure.

**Lemma 3.1** *Let $L$ and $M$ be two lines in the plane $A$ meeting in the point $P$. If $(L, A) \in \mathcal{C}$ and $(M, A) \in \mathcal{C}$, then also $(K, A) \in \mathcal{C}$ for all lines $K$ in $A$ on the point $P$.*

**Proof.** If $(K, A) \notin \mathcal{C}$, then since $\mathcal{C}$ is maximal, there is a $(N, B) \in C$ with $N \cap A = K \cap B = 0$. Now $B$ meets both $L$ and $M$, and hence also $K$, a contradiction. $\square$

Call a line $L$ or a plane $A$ a $\mathcal{C}$-*line* or $\mathcal{C}$-*plane* when they occur in a flag $(L, A) \in \mathcal{C}$, and call such a flag a $\mathcal{C}$-*flag*. It immediately follows that

**Lemma 3.2** *Every $\mathcal{C}$-plane $A$ contains $1$, $q + 1$ or $q^2 + q + 1$ lines $L$ such that $(L, A) \in \mathcal{C}$. Dually, a $\mathcal{C}$-line $L$ is contained in $1$, $q + 1$ or $q^2 + q + 1$ planes $A$ such that $(L, A) \in \mathcal{C}$.* $\square$

Call a line or plane *red*, *yellow* or *white* when it occurs in $q^2+q+1$, $q+1$ or $1$ $\mathcal{C}$-flag(s). A yellow plane $A$ carries a pencil of $q+1$ lines on a fixed point $P_A$ in $A$. A yellow line $L$ carries a pencil of $q + 1$ planes in a fixed solid $S_L$ on $L$.

**Lemma 3.3** *Every $\mathcal{C}$-line meets each red plane. Dually, every $\mathcal{C}$-plane meets each red line. Two red planes meet in a line. Dually, two red lines meet in a point.*

**Proof.** If $(L, A) \in \mathcal{C}$ and $B$ is a red plane disjoint from $L$, then each line in $B$ meets $A$, but $A \cap B$ is a single point, contradiction. The other statements follow. $\square$

**Lemma 3.4** *In the pencil spanned by two intersecting red lines, all lines are red.*

**Proof.** Let the red lines $K, L$ meet in the point $P$, and let $M$ be a line on $P$ in the plane $K + L$. If $(M, A) \notin \mathcal{C}$ for some plane $A$ on $M$, then there is a flag $(N, B) \in \mathcal{C}$ with $N \cap A = M \cap B = 0$. If $B$ meets both $K$ and $L$, then also $M$, contradiction. So suppose $B \cap K = 0$. Since $K$ is red, $(K, C) \in \mathcal{C}$ for each plane $C$ on $K$. So $N$ meets each plane on $K$, and hence meets $K$. But that contradicts $B \cap K = 0$. $\square$

For a point-plane flag $(P, A)$, let $p(P, A)$ denote the collection of flags (pencil) $\{(K, A) \mid P \subseteq K \subseteq A\}$.

**Lemma 3.5** *If $A, B$ are yellow planes meeting in a single point, then $P_A \subseteq B$ or $P_B \subseteq A$. Dually, if $L, M$ are disjoint yellow lines, then $L \subseteq S_M$ or $M \subseteq S_L$.*

**Proof.** If $P := A \cap B$ is distinct from $P_A, P_B$, then $p(P_A, A) \cup p(P_B, B)$ is not independent. $\square$

# 4 Upper bound

Let $\mathcal{C}$ be a maximal coclique in $\Gamma$. We show that if $|\mathcal{C}| \geq (q^2 + q + 1)(q^3 + 2q^2 + q + 1)$, then equality holds and $\mathcal{C}$ is one of the examples.

Let there be $m_i$ lines, and $n_i$ planes, that occur in $i$ flags from $\mathcal{C}$. Then $\mathcal{C}$ has size $m_1 + (q+1)m_{q+1} + (q^2+q+1)m_{q^2+q+1} = n_1 + (q+1)n_{q+1} + (q^2+q+1)n_{q^2+q+1}$. In Examples (ia), (ib) we have $n_{q^2+q+1} = q^3+q^2+q+1$, $n_{q+1} = 0$, $n_1 = q^2(q^2+q+1)$, $m_{q^2+q+1} = q^2+q+1$, $m_{q+1} = q^2(q^2+q+1)$, $m_1 = 0$. And in Examples (iia), (iib) we find the same, with $m_i$ and $n_i$ interchanged.

Two red planes meet in a line, so either all red planes lie in a solid, or all contain a fixed line. Dually, all red lines pass through one point, or are all contained in a fixed plane. In Examples (ia), (ib), (iia) all red planes are in a solid. In Example (iib) all red planes contain a fixed line. In Examples (ia), (iia), (iib) all red lines pass through on a point. In Example (ib) all red lines lie in a fixed plane.

All $\mathcal{C}$-lines intersect all red planes, and dually all $\mathcal{C}$-planes intersect all red lines. For any two red lines, the pencil they span consists of red lines. If there is one more red line through the common point then there are at least $q^2 + q + 1$. If there is one more red line not through this point, then one has a plane full of red lines.

It follows that the collection of red lines (planes) is one of:

**A:** empty,

**B:** a single line (plane),

**C:** $q+1$ lines (planes) forming a pencil,

**D:** $q^2+q+1$ lines in a plane (planes on a line),

**E:** $q^2+q+1$ lines (planes) on a point in a solid,

**F:** all $q^3+q^2+q+1$ lines on a point (planes in a solid).

## 4.1 Many red lines or planes

In this section we investigate the cases with at least $q^2+q+1$ red lines, or red planes.

**F:** If there are $q^3+q^2+q+1$ red planes, all in the solid $S_0$, then all $\mathcal{C}$-lines are contained in $S_0$, since a $\mathcal{C}$-line and a red plane intersect. The $\mathcal{C}$-lines $L$ occurring in flags $(L,A)\in\mathcal{C}$ where $A \not\subseteq S_0$ must meet pairwise, and hence all such lines pass through a common point (and we have Example (ia)) or are contained in a common plane (and we have Example (ib)). The dual situation yields Examples (iia) and (iib).

We have settled Case F, and will from now on assume that there are at most $q^2+q+1$ red planes, and at most $q^2+q+1$ red lines. If all $\mathcal{C}$-lines lie in some fixed solid $S$, then all planes in $S$ are red and we are in Case F. If all $\mathcal{C}$-planes contain a fixed point $P$, we are in the dual situation. So, we may assume that neither happens.

**E:** If there are $q^2+q+1$ red planes, all on the point $P_0$ and contained in the solid $S_0$, then any $\mathcal{C}$-line either lies in $S_0$, or contains $P_0$. If $(L,A)\in\mathcal{C}$ and $P_0 \not\subseteq A \not\subseteq S_0$, then $L = A\cap S_0$. If we vary $(L,A)$, we find a collection $\mathcal{L}$ of pairwise intersecting lines $L$, hence all in a plane, or all on a point. Let $(M,B)\in C$ with $M\cap S_0 = P_0$ and let $K = B\cap S_0$. Now $K$ is a line in $S_0$, and at most $q^2$ lines in $\mathcal{L}$ are disjoint from $K$. On the other hand, there are $q^4$ lines in $S_0$ disjoint from $K$. Consider the at least $q^4-q^2$ lines $N$ disjoint from $K$ but not in a flag $(N,A)$ in $\mathcal{C}$ where $A \not\subseteq S_0$. If $(N,A)\in\mathcal{C}$, and $P_0 \not\subseteq A$, then $M\cap A = 0$, and $B\cap N = K\cap N = 0$, contradiction. So, such lines $N$ are in precisely one flag, namely $(N, P_0+N)$.

Now count flags on the various possible $\mathcal{C}$-lines. We have $q^3+q^2+q+1$ lines on $P_0$, and $q^4+q^3+q^2$ further lines in $S_0$. At most $q^2+q+1$ of these lines are red, and contribute $q^2+q+1$ flags each. At least $q^4-q^2$

of these lines contribute only a single flag. There remain $2q^3 + 2q^2$ lines (including the lines on $P_0$) that contribute at most $q + 1$ flags. Altogether $|\mathcal{C}| \leq (q^2 + q + 1)^2 + (q^4 - q^2) + (2q^3 + 2q^2)(q + 1)$. If we assume $|\mathcal{C}| \geq (q^2+q+1)(q^3+2q^2+q+1)$, then this is a contradiction, except when $q = 2$ where equality holds. Now $K$ is unique, so $M$ occurs only once, and equality does not hold. This settles Case E.

**D:** If there are $q^2 + q + 1$ red lines, all in a plane $A_0$, then all $\mathcal{C}$-planes must intersect $A_0$ in a line. If $(L, A) \in \mathcal{C}$, with $L \not\subseteq A_0$, and $P = L \cap A_0$, then $p(P, A) \subseteq \mathcal{C}$ (recall that $p(P, A) = \{(K, A) \,|\, P \subseteq K \subseteq A\}$).

There are at least $(q^2 + q + 1)(q^3 + q^2)$ flags $(L, A) \in \mathcal{C}$ with $L \not\subseteq A_0$. Consider the red planes. If there are $q^2 + q + 1$ of them, all on a given point and in a given solid, then we are in (the dual of) Case E, that we treated already. Otherwise, if there are $q^2 + q + 1$ of them, all on a fixed line $L_0$, then all $\mathcal{C}$-lines meet $L_0$, so necessarily $L_0 \subseteq A_0$. Apart from the flags $(L, A)$ with $L \subseteq A_0$ (a total of $(q^2 + q + 1)^2$) and those with $L_0 \subseteq A$ (an additional $(q^2 + q)^2$) we need $q^5 + q^4$ more flags $(K, A)$ where $A \cap A_0$ is a line $M$ and $K \cap A_0 = K \cap L_0$ is a point $P$. If $M_1, M_2$ are two lines in $A_0$ meeting $L_0$ in distinct points $P_1, P_2$, and $(K_i, A_i) \in \mathcal{C}$ with $A_i \cap A_0 = M_i$, $i = 1, 2$, then $K_1$ must lie in the solid $A_0 + A_2$, or $K_2$ in the solid $A_0 + A_1$, that is, we must have $A_0 + A_1 = A_0 + A_2$. But such a solid has $q^3 + q^2 + q + 1$ planes $A$, $q^3 + q^2$ not on $L_0$, and each plane $A$ contains $q$ lines on $A \cap L_0$ distinct from $A \cap A_0$, so there would be at most $q^4 + q^3$ flags, not enough. So, all lines $M_i$ have to be concurrent, and again there are at most $q^4 + q^3$ flags, not enough.

If there are $q+1$ red planes (one of them $A_0$), forming a pencil on the line $L_0$ in $A_0$, then these contribute $q(q^2 + q)$ flags, and we still need $q(q^2 + q)^2$, precisely what we would get with one $p(P, A)$ in each further plane $A$ meeting $A_0$ in a line, impossible. Finally, if $A_0$ is the only red plane, then we still need $(q^2 + q + 1)(q^3 + q^2)$ flags, and obtain the same contradiction.

## 4.2 Few red lines and planes

We have settled the cases D, E and F with at least $q^2 + q + 1$ red lines, or at least $q^2 + q + 1$ red planes. From now on we assume that there are at most $q + 1$ red lines, and at most $q + 1$ red planes.

**Lemma 4.1** *Let $P$ be a point and $A$ a plane containing $P$ such that $p(P, A) \subseteq \mathcal{C}$. If $P$ is in $r$ red planes that meet $A$ in a single point, then there are at most $q^3 + r(q^2 - q)$ flags $(M, B)$ with $M \cap A = 0$. Each such flag has $B = M + P$.*

**Proof.** Suppose the $\mathcal{C}$-plane $A$ occurs in $q+1$ $\mathcal{C}$-flags $(L, A)$ (with the $q+1$ lines $L$ forming a pencil on a fixed point $P$), and that the $\mathcal{C}$-line $M$ has zero intersection with $A$. If $(M, B) \in \mathcal{C}$, then $B$ must meet all lines $L$, and hence contains $P$, so that $B = M + P$. In the local $PG(3, q)$ at $P$, such planes $B$ form pairwise intersecting lines, so that they all lie in one plane or all pass through the same point. Hence all planes $B$ lie in a solid $S$ on $P$, or all contain a line $K$ on $P$. In the former case the planes $B$ become locally at $P$ lines in the plane $S/P$ disjoint from the point $(A \cap S)/P$, so there are at most $q^2$. In the latter case the planes $B$ become locally at $K$ points disjoint from the line $(K + A)/K$, again at most $q^2$. Each plane $B$ contains at most $q^2$ lines $M$ not on $P$, and if $B$ is not red, then at most $q$ such lines. □

The next step settles the case of three pairwise disjoint yellow lines not in a single solid, or, dually, three yellow planes that pairwise meet in a single point, where these points are distinct.

**Lemma 4.2** *Suppose* $|\mathcal{C}| \geq (q^2 + q + 1)(q^3 + 2q^2 + q + 1)$. *Then there is no triple of yellow or red planes that pairwise meet in a single point, where these three points are distinct.*

**Proof.** Suppose there are three yellow or red planes $A, B, C$ that pairwise meet in a single point, where these three points of intersection are distinct. Then none of these planes is red. Each point of intersection is the center of a pencil, so w.l.o.g. $A \cap B = P_A$, $B \cap C = P_B$, and $C \cap A = P_C$. Count $\mathcal{C}$-flags. The number of $\mathcal{C}$-flags with a line disjoint from $A$, $B$, or $C$, is at most $3q^3 + (q+1)(q^2 - q)$ where the second term bounds the number of additional flags $(M, D)$ for a red plane $D$. All remaining $\mathcal{C}$-flags have a line that meets each of $A$, $B$, and $C$. For $Q$ in $A$ distinct from $P_A, P_C$, there are $q+1$ lines on $Q$ meeting both $B$ and $C$. For $Q = P_A$ or $Q = P_C$, there are $q^2+q+1$ lines on $Q$ meeting $C$ or $B$ (and the line $P_A + P_B$ is counted twice). Altogether, there are at most $(q^2 + q + 1)(q + 1) + 2q^2 - 1$ lines meeting each of $A$, $B$, and $C$. At most $q+1$ of these are red. It follows that $(q^2 + q + 1)(q^3 + 2q^2 + q + 1) \leq |\mathcal{C}| \leq 3q^3 + (q+1)(q^2 - q) + ((q^2 + q + 1)(q + 1) + 2q^2 - 1)(q + 1) + (q + 1)q^2$, a contradiction. □

After this preparation, it also follows that there are no two disjoint yellow lines, or, dually, no two yellow planes meeting in a single point.

**Lemma 4.3** *Suppose* $|\mathcal{C}| \geq (q^2 + q + 1)(q^3 + 2q^2 + q + 1)$. *Then there are no two yellow or red planes that meet in a single point.*

**Proof.** Suppose $A, B$ are red or yellow planes and $A \cap B = P$, a single point. Count $\mathcal{C}$-flags. The number of $\mathcal{C}$-flags with a line disjoint from $A$ or $B$ is at most $2q^3 + (q+1)(q^2-q)$ where the second term bounds the number of additional flags $(M, D)$ for a red plane $D = M + P$. All remaining $\mathcal{C}$-flags have a line that meets both $A$ and $B$.

Make a directed graph on the yellow lines not on $P$ meeting both $A$ and $B$, with an edge $L \to M$ if $L \cap M \neq 0$ or $L \subseteq S_M$. If $P \not\subseteq S_M$, then lines meeting $A, B$ and contained in $S_M$ are lines meeting both lines $S_M \cap A$ and $S_M \cap B$. There are $(q+1)^2$ of those, $q^2$ disjoint from $M$. If $P \subseteq S_M$ and both $S_M \cap A$, $S_M \cap B$ are lines, then these two lines span a plane containing $M$, and there are no lines meeting both, not on $P$, and disjoint from $M$. Finally, if $A \subseteq S_M$ and $B \cap S_M$ is a line (or vice versa) then there are $(q-1)q^2$ lines meeting $A$ and $B \cap S_M$ but not $P$ or $M$.

In each case, there are at most $(q-1)q^2$ lines in $S_M$, disjoint from $M$ and not on $P$ that meet both $A$ and $B$. The number of lines distinct from $M$ meeting each of $A, B$ and $M$ but not on $P$ is $2(q^2+q-1)+(q-1)^2 = 3q^2-1$. If we count edges for intersecting pairs of lines for $\frac{1}{2}$ at both ends, we find that the vertex $M$ has indegree at most $d := (q-1)q^2 + (3q^2-1)/2$. The underlying undirected graph is complete, so if there are $s$ vertices, we have $s(s-1)/2 \leq sd$ and hence $s \leq 2d+1 = (2q+1)q^2$.

In other words, of the $(q^2+q)^2$ lines meeting both planes outside $P$, at most $(2q+1)q^2$ are yellow. There are at most $q+1$ red lines. The red lines contribute $q^2+q+1$ flags, the yellow lines $q+1$ flags, and the white lines 1 flag, so we find $|\mathcal{C}| \leq 2q^3 + (q+1)(q^2-q) + (q+1)q^2 + (q^3+q^2+q+1)(q+1) + (2q+1)q^3 + (q^2+q)^2$. This is a contradiction for $q > 2$, so assume $q = 2$.

For $q = 2$ we need an auxiliary step, and first show that a red plane and a yellow plane cannot meet in a single point. So, let us suppose that $B$ is red. Since a red plane meets all $\mathcal{C}$-lines, there are at most $q^3 + (q+1)(q^2-q) = 14$ $\mathcal{C}$-flags with a line disjoint from $A$ or $B$. If there are two disjoint yellow or red lines $L, M$ not on $P$ meeting both $A$ and $B$, then since there cannot be three pairwise disjoint yellow or red lines not in a single solid, the point $P$ is on at most 7 yellow or red lines (and on none at all if $P \not\subseteq L+M$). Our count becomes $133 \leq |\mathcal{C}| \leq 14+12+(15+14)+40+36 = 131$, contradiction. This shows that every two yellow or red lines not on $P$ and meeting both $A$ and $B$ must meet. Our count becomes $133 \leq |\mathcal{C}| \leq 14+12+45+12+36 = 119$, contradiction again. So, a red plane and a yellow plane cannot meet in a single point. And dually, a red line and a yellow line cannot be disjoint.

Now let $A, B$ again be yellow planes meeting in the point $P = P_A$. If there is a red line not on $P$, then $P$ is on at most 3 red or yellow lines. Our count becomes $133 \le |\mathcal{C}| \le 22+12+(15+6)+40+36 = 131$, contradiction. Hence all red lines contain $P$.

If there are at last two red lines, then there are at most four yellow lines not on $P$, and our count becomes $133 \le |\mathcal{C}| \le 22+12+45+8+36 = 123$, contradiction. If there is precisely one red line, our count becomes $133 \le |\mathcal{C}| \le 22+4+45+24+36 = 131$, contradiction. So there are no red lines and no red planes.

If $P_A = P_B$ for every choice of two yellow planes $A, B$ that meet in a single point, then for every $\mathcal{C}$-line $M$ disjoint from a yellow plane $A$ the plane $M+P_A$ is not yellow and hence occurs in a unique $\mathcal{C}$-flag. Our count becomes $133 \le |\mathcal{C}| \le 8+45+40+36 = 129$, contradiction. So, we may suppose that $P = P_A \neq P_B$. Now if $M$ is a $\mathcal{C}$-line disjoint from $B$, the plane $M + P_B$ meets $A$ in a point other than $P_A$, so this plane is not yellow. Our count becomes $133 \le |\mathcal{C}| \le (8+4)+45+40+36 = 133$ and equality holds. In the directed graph, each of the 20 vertices has 4 inarrows and 4 outarrows and 11 neighbours that are intersecting lines. So if two lines $L, M$ meet, and both meet $A$ and $B$ but not $P$, and $L$ is yellow, then also $M$ is yellow. But that means that the number of such yellow lines is either 0 or 36, contradiction. $\square$

At this point we know that no two yellow or red planes meet in a single point, and dually no two yellow or red lines are disjoint. Now all yellow or red lines pass through one point, or all are in one plane. Dually, all yellow or red planes are contained in one solid, or all have a common line.

After removing at most $2q^2(q+1)$ flags, no red lines or planes are left. After removing at most $2(q^3+q^2+q+1)q$ further flags, no yellow lines or planes are left. How big can a coclique $\mathcal{C}'$ be when all lines and planes are white?

Define a directed graph on the $\mathcal{C}'$-flags: $F = (L, A)$, and for $F = (L, A)$, $F' = (L', A')$, let $F \to F'$ if $L$ meets $A'$.

For a point $P$, the $\mathcal{C}$-planes on $P$ with line not on $P$ meet pairwise in a line, so that there are at most $q^2+q+1$ of them. So given a line $L$, there are at most $(q+1)(q^2+q+1)$ $\mathcal{C}'$-flags of which the plane meets $L$, but the line does not.

For any two flags $F, F' \in \mathcal{C}'$, there is at least one arrow between them. Moreover, for each vertex, the number of out-neighbours that are not also in-neighbours, is at most $(q+1)(q^2+q+1)$. It follows that the number of directed edges is at least $c(c-1-(q+1)(q^2+q+1))$, where $c = |\mathcal{C}'|$.

Given $L$, there are at most $(q+1)(q^2+q+1)$ flags $F'$ where $L$ meets $A'$ but not $L'$. And at most $(q+1)(q^3+q^2+q)$ flags $F'$ where $L$ meets $L'$. Counting pairs $(F, F')$ with $F \to F'$, we find $c - 1 - (q+1)(q^2+q+1) \leq (q+1)(q^2+q+1)+(q+1)(q^3+q^2+q)$, so that $c \leq (q+1)(q+2)(q^2+q+1)+1$. Now $|\mathcal{C}| \leq 2(q+1)q^2+2(q^3+q^2+q+1)q+c$ yields a contradiction for $q > 2$.

If $C$ is a red plane, then each $\mathcal{C}$-line meets $C$. Dually, if $N$ is a red line, then each $\mathcal{C}$-plane meets $N$. Suppose $C$ is a red plane and $N$ is a red line and $N \not\subseteq C$. We can bound $c$ by the at most $3 \cdot 7 = 21$ flags $F = (L, A)$ where $N$ meets $A$ but not $L$, together with the at most $6+6+14$ such flags where $N$ meets $L$, and at most one flag with $N = L$. Now $133 \leq |\mathcal{C}| \leq 24+60+48 = 132$, a contradiction. It follows that each red plane contains each red line, and $|\mathcal{C}| \leq 16+60+c$.

Suppose there are two intersecting red lines $M, N$. We can bound $c$ by the at most $5 \cdot 7 = 35$ flags $F = (L, A)$ where $L$ is disjoint from $M$ or from $N$, together with the at most $15+4 = 19$ such flags where $L$ meets both $M$ and $N$. Now $133 \leq |\mathcal{C}| \leq 16+60+54 = 130$, a contradiction. It follows that there is at most one red line, and at most one red plane, and $|\mathcal{C}| \leq 8+60+c$.

Suppose that there is a red line $N$. We can bound $c$ by the at most $3 \cdot 7 = 21$ flags $F = (L, A)$ where $L$ is disjoint from $N$, together with the at most $3 \cdot 14 + 1 = 43$ such flags where $L$ meets $N$. Now $133 \leq |\mathcal{C}| \leq 8+60+64 = 132$, a contradiction. It follows that there are no red lines or red planes, and $|\mathcal{C}| \leq 60+c$.

Suppose there is a solid $S$ that contains at least 8 yellow planes. Then for each $\mathcal{C}$-flag $(L, A)$ with $A \not\subseteq S$ we must have $L = A \cap S$. Now we can bound $c$ by the at most 35 $\mathcal{C}$-lines in $S$, together with the at most 15 $\mathcal{C}$-planes in $S$, so that $133 \leq |\mathcal{C}| \leq 60+50 = 110$, a contradiction. It follows that there are at most 7 yellow planes, and at most 7 yellow lines, and $133 \leq |\mathcal{C}| \leq 28+c \leq 28+85 = 113$, a contradiction.

This rules out $q = 2$ and shows that there are no other cocliques of size at least $(q^2+q+1)(q^3+2q^2+q+1)$ than the ones described above.

# 5 Generalizations

The problem studied in this note is a special case of a variation on the Erdős-Ko-Rado theme. The Erdős-Ko-Rado theorem [3] determines the largest cocliques in the Kneser graph $K(n,m)$ of which the vertices are the $m$-subsets of an $n$-set, adjacent when disjoint (assuming $n > 2m$). The Frankl-Wilson theorem [4] determines the largest cocliques in the Kneser graph $K_q(n,m)$ of which the vertices are the $m$-subspaces of an $n$-space, adjacent when they have zero intersection (assuming $n \geq 2m$). Much more generally, one can take as vertices the flags of a certain type in a spherical building, where two flags are adjacent when they are in mutual general position.

Earlier, there was an obvious candidate for the largest coclique, and the work was to prove that this candidate is indeed the largest, or even the only largest. Here, it is already nontrivial to come up with plausible candidates for the largest coclique. Below we give a construction that yields the largest cocliques in the case of point-hyperplane flags [1], and in our present case of line-plane flags in $PG(4,q)$. For a much more thorough investigation of the case of point-plane flags in $PG(4,q)$ see [2].

## 5.1 A construction

Let $V$ be a vector space of dimension $n$ over the field $\mathbb{F}_q$. Let $0 < i < n/2$, and let $\Gamma$ be the Kneser graph on the flags $(A,B)$, where $A, B$ are subspaces of $V$ of respective dimensions $i$ and $n-i$, and $(A,B)$ is adjacent to $(A',B')$ when $A \cap B' = A' \cap B = 0$. We construct large cocliques in $\Gamma$.

Let $X = (X_0, \ldots, X_n)$ be a series of subspaces of $V$, with $\dim X_i = i$ and $X_i \subseteq X_j$ when $i \leq j$. Fix an integer $h$ with $i \leq h \leq n-i+1$.

Let $\mathcal{C}_h(X)$ consist of all flags $(A,B)$ where $A$ is an $i$-space and $B$ an $(n-i)$-space, and $A \subseteq T \subseteq B$ for some $j$-space $T$, where either $j < h$ and $X_{j-i+1} \subseteq T$, or $j \geq h$ and $X_{h-i} \subseteq T \subseteq X_{j+i-1}$.

Let $\mathcal{C}'_h(X)$ consist of all flags $(A,B)$ where $A$ is an $i$-space and $B$ an $(n-i)$-space, and $A \subseteq T \subseteq B$ for some $j$-space $T$, where either $j < h$ and $X_{j-i+1} \subseteq T \subseteq X_{h+i-1}$, or $j \geq h$ and $T \subseteq X_{j+i-1}$.

**Theorem 5.1** *$\mathcal{C}_h(X)$ and $\mathcal{C}'_h(X)$ are cocliques in $\Gamma$.*

**Proof.** Since $\mathcal{C}'_h(X)$ is the dual of $\mathcal{C}_h(X)$, it suffices to look at $\mathcal{C}_h(X)$. If $A \subseteq T \subseteq B$ and $A' \subseteq T' \subseteq B'$ and $\dim T = j$, $\dim T' = j'$, and $j \leq j'$, then either $j < h$ and $X_{j-i+1} \subseteq T \cap T'$, or $j \geq h$ and $T + T' \subseteq X_{j'+i-1}$, so that in both cases $\dim(T \cap T') \geq j-i+1$, or, equivalently, $\dim(T+T') \leq j'+i-1$. It follows that $A$ and $T \cap T'$ cannot be disjoint inside $T$, so that $0 \neq A \cap T' \subseteq A \cap B'$, and the flags are not in general position. $\square$

In the cases of point-hyperplane flags in $PG(n-1,q)$, and of line-plane flags in $PG(4,q)$, these cocliques $\mathcal{C}_h(X)$ and $\mathcal{C}'_h(X)$ are precisely all the largest cocliques.

# References


[1] A. Blokhuis, A.E. Brouwer & Ç. Güven, *Cocliques in the Kneser graph on the point-hyperplane flags*, accepted for publication in Combinatorica.

[2] A. Blokhuis, A.E. Brouwer & T. Szőnyi, *Maximal cocliques in the Kneser graph on point-plane flags in $PG(4,q)$*, accepted for publication in the Europ. J. Comb.

[3] P. Erdős, C. Ko & R. Rado, *Intersection theorems for systems of finite sets*, Quart. J. Math. Oxford Ser. (2) **12** (1961) 313–320.

[4] P. Frankl & R. M. Wilson, *The Erdős-Ko-Rado theorem for vector spaces*, J. Combin. Theory (A) **43** (1986) 228–236.



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