IMPROVED BOUNDS FOR ROTA'S BASIS CONJECTURE

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ABSTRACT. We prove that, if B_1, \ldots, B_n are disjoint bases of a rank-*n* matroid, then there are at least $\frac{n}{7 \log n}$ disjoint transversals of (B_1, \ldots, B_n) that are also bases.

1. INTRODUCTION

A transversal basis of a collection (B_1, \ldots, B_n) of sets of elements in a rank-*n* matroid is a basis containing exactly one element from each of B_1, \ldots, B_n . Rota's Basis Conjecture, which first appeared in [1], is receiving renewed interest as the topic of Polymath 12 [2].

Conjecture 1.1 (Rota's Basis Conjecture). Given disjoint bases B_1, \ldots, B_n of a rank-n matroid, there exist n disjoint transversal bases.

Geelen and Webb [3] showed that it is possible to get $\lceil \sqrt{n-1} \rceil$ disjoint transversal bases; our main result improves on their bound.

Theorem 1.2. Given disjoint bases B_1, \ldots, B_n of a rank-n matroid, where $n \ge 2$, there are at least $\left|\frac{n}{6\lceil \log n\rceil}\right|$ disjoint transversal bases.

Throughout the paper we use the natural logarithm. Using the same methods, but taking more care with the calculations, it should be possible to improve on our bound of $\left\lfloor \frac{n}{6\lceil \log n \rceil} \right\rfloor$; however, new ideas will be needed to beat $\frac{n}{\log n}$. The bound of $\frac{n}{7\log n}$, claimed in the abstract, is obtained by combining the bound $\lceil \sqrt{n-1} \rceil$, when $n \leq 3000$, with the bound $\left\lfloor \frac{n}{6\lceil \log n \rceil} \right\rfloor$, when n > 3000.

We present the central ideas of the proof here in the introduction, leaving the technical details for the next section. We deduce Theorem 1.2 from the following key result.

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Theorem 1.3. Let B_1, \ldots, B_n be disjoint bases of a rank-n matroid, where $n \geq 2$, and let $\alpha = 3\lceil \log n \rceil$. If we choose α -element subsets S_1, \ldots, S_n independently and uniformly at random from B_1, \ldots, B_n , respectively, then (S_1, \ldots, S_n) contains a transversal basis with probability at least 1/2.

We start by showing that Theorem 1.3 implies Theorem 1.2.

Proof of Theorem 1.2. Let $m = \left\lfloor \frac{n}{6\lceil \log n \rceil} \right\rfloor$. For each $i \in \{1, \ldots, n\}$, let $S_{i,1}, \ldots, S_{i,2m}$ be disjoint α -element subsets of B_i chosen at random. For each $j \in \{1, \ldots, 2m\}$, the sets $S_{1,j}, \ldots, S_{n,j}$ are subsets of B_1, \ldots, B_n that are chosen independently and uniformly at random, so, by Theorem 1.3, $(S_{1,j}, \ldots, S_{n,j})$ contains a transversal basis with probability at least 1/2. By the linearity of expectation, the expected number of disjoint transversal bases of (B_1, \ldots, B_n) is at least $\frac{1}{2} \cdot 2m$. So there exist at least m disjoint transversal bases. \Box

To prove Theorem 1.3, we use the following result of Rado [4] which characterizes the existence of a transversal basis.

Theorem 1.4 (Rado's Theorem). Let (S_1, \ldots, S_n) be sets of elements in a rank-n matroid. Then there is a transversal basis of (S_1, \ldots, S_n) if and only if $r(\bigcup_{i \in X} S_i) \ge |X|$ for all $X \subseteq \{1, \ldots, n\}$.

In order to prove Theorem 1.3, we will focus on the probability of failure of each of the conditions in Rado's Theorem. Let B_1, \ldots, B_k be bases (not necessarily disjoint) of a rank-*n* matroid and let $\alpha = 3\lceil \log n \rceil$. We let $Q(B_1, \ldots, B_k)$ denote the probability that, when α -element subsets S_1, \ldots, S_k are chosen independently and uniformly at random from B_1, \ldots, B_k , respectively, we have $r(S_1 \cup \cdots \cup S_k) < k$.

Note that we do not require the sets B_1, \ldots, B_k to be disjoint. In fact, the case that $B_1 = \cdots = B_k$ is interesting and plays an important role in the proof. In this case we have $r(S_1 \cup \cdots \cup S_k) = |S_1 \cup \cdots \cup S_k|$, and hence the failure probability $Q(B_1, \ldots, B_k)$ depends only on k and n; we let $Q_{k,n} = Q(B_1, \ldots, B_k)$. Thus $Q_{k,n}$ denotes the probability that, when α -element sets S_1, \ldots, S_k are chosen independently and uniformly at random from the set $\{1, \ldots, n\}$ we have $|S_1 \cup \cdots \cup S_k| < k$.

The following key lemma shows that the failure probability $Q(B_1, \ldots, B_k)$ is worst when $B_1 = \cdots = B_k$; we postpone the proof of this result until Section 2.

Lemma 1.5. Let n and k be positive integers with $k \leq n$ and let B_1, \ldots, B_k be bases of a rank-n matroid. Then $Q(B_1, \ldots, B_k) \leq Q_{k,n}$.

Computing $Q_{k,n}$ is closely related to the Coupon Collector's Problem, as well as a bipartite matching problem considered by Erdős and Renyi [5].

Lemma 1.6. Let n and k be positive integers with $k \leq n$ and let $\alpha = 3 \lceil \log n \rceil$. Then

$$Q_{k,n} \leq \binom{n}{k-1} \left(\frac{\binom{k-1}{\alpha}}{\binom{n}{\alpha}}\right)^k.$$

Proof. There are $\binom{n}{\alpha}^k$ ways to choose α -element sets S_1, \ldots, S_k from $\{1, \ldots, n\}$. To bound the number of such (S_1, \ldots, S_k) with $|S_1 \cup \cdots \cup S_k| < k$, we sum, over all (k-1)-element subsets X of $\{1, \ldots, n\}$, the number of ways to choose (S_1, \ldots, S_k) from X.

Combining the above results gives us an upper bound on the failure probability in Theorem 1.3.

Lemma 1.7. Let B_1, \ldots, B_n be disjoint bases of a rank-n matroid, where $n \geq 2$, and let $\alpha = 3\lceil \log n \rceil$. If we choose α -element sets $S_1 \subseteq B_1, \ldots, S_n \subseteq B_n$ independently and uniformly at random, then the probability that (S_1, \ldots, S_n) does not contain a transversal basis is at most

$$\sum_{k=1}^{n} \binom{n}{k} \binom{n}{k-1} \left(\frac{k-1}{n}\right)^{k\alpha}$$

Proof. By the union bound, the failure probability is at most the sum of the failure probabilities of each of the conditions in Rado's Theorem, so, by Lemmas 1.5 and 1.6, the probability that (S_1, \ldots, S_n) does not contain a transversal basis is at most

$$\sum_{k=1}^{n} \binom{n}{k} \binom{n}{k-1} \left(\frac{\binom{k-1}{\alpha}}{\binom{n}{\alpha}}\right)^{k}$$

Moreover

$$\frac{\binom{k-1}{\alpha}}{\binom{n}{\alpha}} = \left(\frac{k-1}{n}\right) \left(\frac{k-2}{n-1}\right) \cdots \left(\frac{k-\alpha}{n-\alpha+1}\right) \\ \leq \left(\frac{k-1}{n}\right)^{\alpha},$$

since $k - 1 \leq n$.

Theorem 1.3 follows via a routine technical calculation which we complete in Section 2.

2. Technical details

We start with the proof of Lemma 1.5.

Proof of Lemma 1.5. Let B_1, \ldots, B_k be bases of a rank-*n* matroid and let $\alpha = 3\lceil \log n \rceil$. Recall $Q_k(B_1, \ldots, B_k)$ is the probability that $r(S_1 \cup \cdots \cup S_k) < k$ in an experiment \mathcal{E} where we choose α -element subsets $S_1 \subset B_1, \ldots, S_k \subset B_k$ independently and uniformly at random. We obtain a lower bound on $r(S_1 \cup \cdots \cup S_k)$ by constructing an independent set in a naive way. Given an outcome (S_1, \ldots, S_k) of \mathcal{E} , we construct bases B'_1, \ldots, B'_k and independent sets I_1, \ldots, I_k iteratively, such that:

- $B'_1 = B_1, I_1 = S_1$, and
- for each $i \in \{2, \ldots, k\}$, the set B'_i is an arbitrary basis with $I_{i-1} \subseteq B'_i \subseteq B_i \cup I_{i-1}$, and $I_i = I_{i-1} \cup (S_i \cap B'_i)$.

Observe that the independent set I_{i-1} can be extended to a basis B'_i with $I_{i-1} \subseteq B'_i \subseteq B_i \cup I_{i-1}$ and that $I_i = I_{i-1} \cup (S_i \cap B'_i) \subseteq B'_i$, so I_i is independent. Thus, given (S_1, \ldots, S_k) , the required bases B'_1, \ldots, B'_k and independent sets I_1, \ldots, I_k exist. Note that $r(S_1 \cup \cdots \cup S_k) \ge |I_k|$. It suffices to prove that $|I_k| < k$ with probability equal to $Q_{k,n}$. To see this we will describe an equivalent random process for generating B'_1, \ldots, B'_k and I_1, \ldots, I_k based on a collection (S'_1, \ldots, S'_k) of α -element sets chosen independently and uniformly at random from $\{1, \ldots, n\}$ such that $|I_k| = |S'_1 \cup \cdots \cup S'_k|$.

We start with an observation regarding the construction of the sets B'_1, \ldots, B'_k and I_1, \ldots, I_k . Suppose, for some $i \ge 2$, we have already created B'_1, \ldots, B'_{i-1} and I_1, \ldots, I_{i-1} . We construct B'_i by extending I_{i-1} to a basis within $I_{i-1} \cup B_i$. Up to this point we have not used the set S_i , so we may suppose that it is randomly generated at this time. Moreover we claim that, for the purpose of constructing I_i , we may choose S_i randomly from B'_i instead of B_i . To see this, consider a bijection from B_i to B'_i that fixes the elements in $B_i \cap B'_i$, and let S''_i denote the image of S_i under this bijection. Since $B'_i - B_i \subseteq I_{i-1}$, we have $I_{i-1} \cup (S_i \cap B'_i) = I_{i-1} \cup (S''_i \cap B'_i)$, so the set I_i , considered as a random variable, has the same distribution when we choose S_i from B'_i .

In the following process, we will assume that sets (S'_1, \ldots, S'_k) are only generated upon request. Initially we set $B'_1 = B_1$ and choose an arbitrary bijection $\psi_1 : \{1, \ldots, n\} \to B'_1$. Now request S'_1 . Note that $\psi_1(S'_1)$ is chosen uniformly at random from the α -element subsets of B'_1 . Set $S_1 = \psi_1(S'_1)$ and $I_1 = S_1$. For some $i \ge 2$, suppose that we have already created $B'_1, \ldots, B'_{i-1}, \psi_1, \ldots, \psi_{i-1}, S_1, \ldots, S_{i-1}$ and I_1, \ldots, I_{i-1} . As before, we construct B'_i by extending I_{i-1} to a basis within $I_{i-1} \cup B_i$. Construct a bijection $\psi_i : \{1, \ldots, n\} \to B'_i$ such that $\psi_i^{-1}(e) = \psi_{i-1}^{-1}(e)$ for all $e \in I_{i-1}$. Now request S'_i . Note that $\psi_i(S'_i)$ is chosen uniformly at random from the α -element subsets of B'_i . Set $S_i = \psi_i(S'_i)$ and $I_i = I_{i-1} \cup S_i$.

A simple inductive argument shows that $|I_i| = |S'_1 \cup \cdots \cup S'_i|$ for each $i \in \{1, \ldots, n\}$. In particular, $|I_k| = |S'_1 \cup \cdots \cup S'_k|$, as required. \Box

It remains to prove Theorem 1.3.

Proof of Theorem 1.3. Let

$$q_n = \sum_{k=1}^n \binom{n}{k} \binom{n}{k-1} \left(\frac{k-1}{n}\right)^{k\alpha}$$

By Lemma 1.7, it suffices to prove that $q_n \leq 1/2$. We have verified this numerically for all $n \in \{2, \ldots, 59\}$ using Maple, so we may assume that $n \geq 60$.

Now we split the sum in two parts, change the index of summation in the second part, and apply the inequality $1 + x \leq e^x$, after which the terms in the two parts become identical.

$$q_{n} = \sum_{k=1}^{\lfloor n/2 \rfloor} {\binom{n}{k}} {\binom{n}{k-1}} {\binom{k-1}{n}}^{k\alpha} + \\ \sum_{k=\lfloor n/2 \rfloor+1}^{n} {\binom{n}{k}} {\binom{n}{k-1}} {\binom{k-1}{n}}^{k\alpha} \\ = \sum_{k=1}^{\lfloor n/2 \rfloor} {\binom{n}{k}} {\binom{n}{k-1}} {\binom{k-1}{n-k}}^{k\alpha} + \\ \sum_{k=1}^{\lfloor n/2 \rfloor} {\binom{n}{n-k+1}} {\binom{n}{n-k+1}} {\binom{n}{n-k}} {\binom{n-k+1}{n}}^{(n-k+1)\alpha} \\ = \sum_{k=1}^{\lfloor n/2 \rfloor} {\binom{n}{k}} {\binom{n}{k-1}} {\binom{1-\frac{n-k+1}{n}}{n-k}}^{k\alpha} + \\ \sum_{k=1}^{\lfloor n/2 \rfloor} {\binom{n}{k}}^{2} e^{-\frac{n-k+1}{n} \cdot k\alpha} + \\ \sum_{k=1}^{\lfloor n/2 \rfloor} {\binom{n}{k}}^{2} e^{-\frac{k+1}{n} \cdot k\alpha} + \\ \sum_{k=1}^{\lfloor n/2 \rfloor} {\binom{n}{k}}^{2} e^{-\frac{k}{n} \cdot (n-k+1)\alpha} \end{cases}$$

$$\leq 2\sum_{k=1}^{\lceil n/2\rceil} \left(\frac{en}{k}\right)^{2k} e^{-\frac{n-k+1}{n} \cdot k\alpha}$$

as $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$. This bound is decreasing as a function of α , so we can replace α with $3 \log n$; after simplifying we get

$$q_n \le 2\sum_{k=1}^{\lceil n/2 \rceil} \left(\frac{e}{k}\right)^{2k} n^{-k + \frac{3k(k-1)}{n}}.$$

Let $t_k = \left(\frac{e}{k}\right)^{2k} n^{-k + \frac{3k(k-1)}{n}}$.

Claim. For each $k \in \{1, \ldots, \lceil \frac{n}{2} \rceil\}$ we have $t_k \leq (\frac{1}{2})^{k+2}$.

Proof of claim. We have numerically verified, for $k \in \{1, 2, 3\}$ and n = 60, that $t_k \leq \left(\frac{1}{2}\right)^{k+2}$ (the k = 1 case is where we require $n \geq 60$). Since t_k is non-increasing as a function of n, the claim holds for $k \in \{1, 2, 3\}$.

Now consider the claim for $4 \le k \le \frac{n}{3}$. Note that when $k \ge 4$, the term $\left(\frac{e}{k}\right)^2$ can be bounded above by $\frac{1}{2}$. Furthermore, when $k \le \frac{n}{3}$, we have $-k + \frac{3k(k-1)}{n} \le -1$. Hence,

$$t_k \le \left(\frac{e}{k}\right)^{2k} \frac{1}{n} \le \left(\frac{1}{2}\right)^k \frac{1}{60} < \left(\frac{1}{2}\right)^{k+2}.$$

It remains to prove the claim for $\frac{n}{3} < k \leq \frac{n}{2} + 1$. Observe that $\frac{9e^2}{n^{3/2-3/n}}$ is decreasing in n when $n \geq 2$, so it is routine to verify that $\frac{9e^2}{n^{3/2-3/n}} < \frac{1}{2}$ for all $n \geq 60$. Now,

$$t_{k} = \left(\left(\frac{e}{k}\right)^{2} n^{-1 + \frac{3k}{n}} \right)^{k} n^{-\frac{3k}{n}}$$

$$\leq \left(\left(\frac{e}{n/3}\right)^{2} n^{-1 + \frac{3(n/2+1)}{n}} \right)^{k} n^{-1}$$

$$= \left(\frac{9e^{2}}{n^{3/2 - 3/n}}\right)^{k} n^{-1}$$

$$\leq \left(\frac{1}{2}\right)^{k} \frac{1}{60}$$

$$< \left(\frac{1}{2}\right)^{k+2},$$

as required.

By the above claim,

$$q_n \le 2\sum_{k=1}^{\lceil n/2 \rceil} t_k \le 2\sum_{k\ge 1} \left(\frac{1}{2}\right)^{k+2} = \frac{1}{2},$$

which completes the proof of Theorem 1.3.

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