# IMPROVED BOUNDS FOR ROTA'S BASIS CONJECTURE 

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#### Abstract

We prove that, if $B_{1}, \ldots, B_{n}$ are disjoint bases of a rank- $n$ matroid, then there are at least $\frac{n}{7 \log n}$ disjoint transversals of $\left(B_{1}, \ldots, B_{n}\right)$ that are also bases.


## 1. Introduction

A transversal basis of a collection $\left(B_{1}, \ldots, B_{n}\right)$ of sets of elements in a rank- $n$ matroid is a basis containing exactly one element from each of $B_{1}, \ldots, B_{n}$. Rota's Basis Conjecture, which first appeared in [1], is receiving renewed interest as the topic of Polymath 12 [2].
Conjecture 1.1 (Rota's Basis Conjecture). Given disjoint bases $B_{1}, \ldots, B_{n}$ of a rank-n matroid, there exist $n$ disjoint transversal bases.

Geelen and Webb [3 showed that it is possible to get $\lceil\sqrt{n-1}\rceil$ disjoint transversal bases; our main result improves on their bound.
Theorem 1.2. Given disjoint bases $B_{1}, \ldots, B_{n}$ of a rank-n matroid, where $n \geq 2$, there are at least $\left\lfloor\frac{n}{6[\log n\rceil}\right\rfloor$ disjoint transversal bases.

Throughout the paper we use the natural logarithm. Using the same methods, but taking more care with the calculations, it should be possible to improve on our bound of $\left\lfloor\frac{n}{6\lceil\log n\rceil}\right\rfloor$; however, new ideas will be needed to beat $\frac{n}{\log n}$. The bound of $\frac{n}{7 \log n}$, claimed in the abstract, is obtained by combining the bound $\lceil\sqrt{n-1}\rceil$, when $n \leq 3000$, with the bound $\left\lfloor\frac{n}{6\lceil\log n\rceil}\right\rfloor$, when $n>3000$.

We present the central ideas of the proof here in the introduction, leaving the technical details for the next section. We deduce Theorem 1.2 from the following key result.

[^0]Theorem 1.3. Let $B_{1}, \ldots, B_{n}$ be disjoint bases of a rank-n matroid, where $n \geq 2$, and let $\alpha=3\lceil\log n\rceil$. If we choose $\alpha$-element subsets $S_{1}, \ldots, S_{n}$ independently and uniformly at random from $B_{1}, \ldots, B_{n}$, respectively, then $\left(S_{1}, \ldots, S_{n}\right)$ contains a transversal basis with probability at least $1 / 2$.

We start by showing that Theorem 1.3 implies Theorem 1.2.
Proof of Theorem [1.2. Let $m=\left\lfloor\frac{n}{6\lceil\log n\rceil}\right\rfloor$. For each $i \in\{1, \ldots, n\}$, let $S_{i, 1}, \ldots, S_{i, 2 m}$ be disjoint $\alpha$-element subsets of $B_{i}$ chosen at random. For each $j \in\{1, \ldots, 2 m\}$, the sets $S_{1, j}, \ldots, S_{n, j}$ are subsets of $B_{1}, \ldots, B_{n}$ that are chosen independently and uniformly at random, so, by Theorem 1.3, $\left(S_{1, j}, \ldots, S_{n, j}\right)$ contains a transversal basis with probability at least $1 / 2$. By the linearity of expectation, the expected number of disjoint transversal bases of $\left(B_{1}, \ldots, B_{n}\right)$ is at least $\frac{1}{2} \cdot 2 m$. So there exist at least $m$ disjoint transversal bases.

To prove Theorem [1.3, we use the following result of Rado [4] which characterizes the existence of a transversal basis.

Theorem 1.4 (Rado's Theorem). Let $\left(S_{1}, \ldots, S_{n}\right)$ be sets of elements in a rank-n matroid. Then there is a transversal basis of $\left(S_{1}, \ldots, S_{n}\right)$ if and only if $r\left(\cup_{i \in X} S_{i}\right) \geq|X|$ for all $X \subseteq\{1, \ldots, n\}$.

In order to prove Theorem 1.3, we will focus on the probability of failure of each of the conditions in Rado's Theorem. Let $B_{1}, \ldots, B_{k}$ be bases (not necessarily disjoint) of a rank-n matroid and let $\alpha=$ $3\lceil\log n\rceil$. We let $Q\left(B_{1}, \ldots, B_{k}\right)$ denote the probability that, when $\alpha$ element subsets $S_{1}, \ldots, S_{k}$ are chosen independently and uniformly at random from $B_{1}, \ldots, B_{k}$, respectively, we have $r\left(S_{1} \cup \cdots \cup S_{k}\right)<k$.

Note that we do not require the sets $B_{1}, \ldots, B_{k}$ to be disjoint. In fact, the case that $B_{1}=\cdots=B_{k}$ is interesting and plays an important role in the proof. In this case we have $r\left(S_{1} \cup \cdots \cup S_{k}\right)=\left|S_{1} \cup \cdots \cup S_{k}\right|$, and hence the failure probability $Q\left(B_{1}, \ldots, B_{k}\right)$ depends only on $k$ and $n$; we let $Q_{k, n}=Q\left(B_{1}, \ldots, B_{k}\right)$. Thus $Q_{k, n}$ denotes the probability that, when $\alpha$-element sets $S_{1}, \ldots, S_{k}$ are chosen independently and uniformly at random from the set $\{1, \ldots, n\}$ we have $\left|S_{1} \cup \cdots \cup S_{k}\right|<k$.

The following key lemma shows that the failure probability $Q\left(B_{1}, \ldots, B_{k}\right)$ is worst when $B_{1}=\cdots=B_{k}$; we postpone the proof of this result until Section 2.

Lemma 1.5. Let $n$ and $k$ be positive integers with $k \leq n$ and let $B_{1}, \ldots, B_{k}$ be bases of a rank-n matroid. Then $Q\left(B_{1}, \ldots, B_{k}\right) \leq Q_{k, n}$.

Computing $Q_{k, n}$ is closely related to the Coupon Collector's Problem, as well as a bipartite matching problem considered by Erdős and Renyi [5].

Lemma 1.6. Let $n$ and $k$ be positive integers with $k \leq n$ and let $\alpha=3\lceil\log n\rceil$. Then

$$
Q_{k, n} \leq\binom{ n}{k-1}\left(\frac{\binom{k-1}{\alpha}}{\binom{n}{\alpha}}\right)^{k}
$$

Proof. There are $\binom{n}{\alpha}^{k}$ ways to choose $\alpha$-element sets $S_{1}, \ldots, S_{k}$ from $\{1, \ldots, n\}$. To bound the number of such $\left(S_{1}, \ldots, S_{k}\right)$ with $\mid S_{1} \cup \cdots \cup$ $S_{k} \mid<k$, we sum, over all $(k-1)$-element subsets $X$ of $\{1, \ldots, n\}$, the number of ways to choose $\left(S_{1}, \ldots, S_{k}\right)$ from $X$.

Combining the above results gives us an upper bound on the failure probability in Theorem 1.3,

Lemma 1.7. Let $B_{1}, \ldots, B_{n}$ be disjoint bases of a rank-n matroid, where $n \geq 2$, and let $\alpha=3\lceil\log n\rceil$. If we choose $\alpha$-element sets $S_{1} \subseteq B_{1}, \ldots, S_{n} \subseteq B_{n}$ independently and uniformly at random, then the probability that $\left(S_{1}, \ldots, S_{n}\right)$ does not contain a transversal basis is at most

$$
\sum_{k=1}^{n}\binom{n}{k}\binom{n}{k-1}\left(\frac{k-1}{n}\right)^{k \alpha}
$$

Proof. By the union bound, the failure probability is at most the sum of the failure probabilities of each of the conditions in Rado's Theorem, so, by Lemmas 1.5 and 1.6, the probability that $\left(S_{1}, \ldots, S_{n}\right)$ does not contain a transversal basis is at most

$$
\sum_{k=1}^{n}\binom{n}{k}\binom{n}{k-1}\left(\frac{\binom{k-1}{\alpha}}{\binom{n}{\alpha}}\right)^{k}
$$

Moreover

$$
\begin{aligned}
\frac{\binom{k-1}{\alpha}}{\binom{n}{\alpha}} & =\left(\frac{k-1}{n}\right)\left(\frac{k-2}{n-1}\right) \cdots\left(\frac{k-\alpha}{n-\alpha+1}\right) \\
& \leq\left(\frac{k-1}{n}\right)^{\alpha}
\end{aligned}
$$

since $k-1 \leq n$.
Theorem 1.3 follows via a routine technical calculation which we complete in Section 2.

## 2. Technical details

We start with the proof of Lemma 1.5.
Proof of Lemma 1.5. Let $B_{1}, \ldots, B_{k}$ be bases of a rank- $n$ matroid and let $\alpha=3\lceil\log n\rceil$. Recall $Q_{k}\left(B_{1}, \ldots, B_{k}\right)$ is the probability that $r\left(S_{1} \cup\right.$ $\left.\cdots \cup S_{k}\right)<k$ in an experiment $\mathcal{E}$ where we choose $\alpha$-element subsets $S_{1} \subset B_{1}, \ldots, S_{k} \subset B_{k}$ independently and uniformly at random. We obtain a lower bound on $r\left(S_{1} \cup \cdots \cup S_{k}\right)$ by constructing an independent set in a naive way. Given an outcome $\left(S_{1}, \ldots, S_{k}\right)$ of $\mathcal{E}$, we construct bases $B_{1}^{\prime}, \ldots, B_{k}^{\prime}$ and independent sets $I_{1}, \ldots, I_{k}$ iteratively, such that:

- $B_{1}^{\prime}=B_{1}, I_{1}=S_{1}$, and
- for each $i \in\{2, \ldots, k\}$, the set $B_{i}^{\prime}$ is an arbitrary basis with $I_{i-1} \subseteq$ $B_{i}^{\prime} \subseteq B_{i} \cup I_{i-1}$, and $I_{i}=I_{i-1} \cup\left(S_{i} \cap B_{i}^{\prime}\right)$.
Observe that the independent set $I_{i-1}$ can be extended to a basis $B_{i}^{\prime}$ with $I_{i-1} \subseteq B_{i}^{\prime} \subseteq B_{i} \cup I_{i-1}$ and that $I_{i}=I_{i-1} \cup\left(S_{i} \cap B_{i}^{\prime}\right) \subseteq B_{i}^{\prime}$, so $I_{i}$ is independent. Thus, given $\left(S_{1}, \ldots, S_{k}\right)$, the required bases $B_{1}^{\prime}, \ldots, B_{k}^{\prime}$ and independent sets $I_{1}, \ldots, I_{k}$ exist. Note that $r\left(S_{1} \cup \cdots \cup S_{k}\right) \geq\left|I_{k}\right|$. It suffices to prove that $\left|I_{k}\right|<k$ with probability equal to $Q_{k, n}$. To see this we will describe an equivalent random process for generating $B_{1}^{\prime}, \ldots, B_{k}^{\prime}$ and $I_{1}, \ldots, I_{k}$ based on a collection $\left(S_{1}^{\prime}, \ldots, S_{k}^{\prime}\right)$ of $\alpha$-element sets chosen independently and uniformly at random from $\{1, \ldots, n\}$ such that $\left|I_{k}\right|=\left|S_{1}^{\prime} \cup \cdots \cup S_{k}^{\prime}\right|$.

We start with an observation regarding the construction of the sets $B_{1}^{\prime}, \ldots, B_{k}^{\prime}$ and $I_{1}, \ldots, I_{k}$. Suppose, for some $i \geq 2$, we have already created $B_{1}^{\prime}, \ldots, B_{i-1}^{\prime}$ and $I_{1}, \ldots, I_{i-1}$. We construct $B_{i}^{\prime}$ by extending $I_{i-1}$ to a basis within $I_{i-1} \cup B_{i}$. Up to this point we have not used the set $S_{i}$, so we may suppose that it is randomly generated at this time. Moreover we claim that, for the purpose of constructing $I_{i}$, we may choose $S_{i}$ randomly from $B_{i}^{\prime}$ instead of $B_{i}$. To see this, consider a bijection from $B_{i}$ to $B_{i}^{\prime}$ that fixes the elements in $B_{i} \cap B_{i}^{\prime}$, and let $S_{i}^{\prime \prime}$ denote the image of $S_{i}$ under this bijection. Since $B_{i}^{\prime}-B_{i} \subseteq I_{i-1}$, we have $I_{i-1} \cup\left(S_{i} \cap B_{i}^{\prime}\right)=I_{i-1} \cup\left(S_{i}^{\prime \prime} \cap B_{i}^{\prime}\right)$, so the set $I_{i}$, considered as a random variable, has the same distribution when we choose $S_{i}$ from $B_{i}$ as it does when we choose $S_{i}$ from $B_{i}^{\prime}$.

In the following process, we will assume that sets $\left(S_{1}^{\prime}, \ldots, S_{k}^{\prime}\right)$ are only generated upon request. Initially we set $B_{1}^{\prime}=B_{1}$ and choose an arbitrary bijection $\psi_{1}:\{1, \ldots, n\} \rightarrow B_{1}^{\prime}$. Now request $S_{1}^{\prime}$. Note that $\psi_{1}\left(S_{1}^{\prime}\right)$ is chosen uniformly at random from the $\alpha$-element subsets of $B_{1}^{\prime}$. Set $S_{1}=\psi_{1}\left(S_{1}^{\prime}\right)$ and $I_{1}=S_{1}$. For some $i \geq 2$, suppose that we have already created $B_{1}^{\prime}, \ldots, B_{i-1}^{\prime}, \psi_{1}, \ldots, \psi_{i-1}, S_{1}, \ldots, S_{i-1}$ and $I_{1}, \ldots, I_{i-1}$. As before, we construct $B_{i}^{\prime}$ by extending $I_{i-1}$ to a basis
within $I_{i-1} \cup B_{i}$. Construct a bijection $\psi_{i}:\{1, \ldots, n\} \rightarrow B_{i}^{\prime}$ such that $\psi_{i}^{-1}(e)=\psi_{i-1}^{-1}(e)$ for all $e \in I_{i-1}$. Now request $S_{i}^{\prime}$. Note that $\psi_{i}\left(S_{i}^{\prime}\right)$ is chosen uniformly at random from the $\alpha$-element subsets of $B_{i}^{\prime}$. Set $S_{i}=\psi_{i}\left(S_{i}^{\prime}\right)$ and $I_{i}=I_{i-1} \cup S_{i}$.

A simple inductive argument shows that $\left|I_{i}\right|=\left|S_{1}^{\prime} \cup \cdots \cup S_{i}^{\prime}\right|$ for each $i \in\{1, \ldots, n\}$. In particular, $\left|I_{k}\right|=\left|S_{1}^{\prime} \cup \cdots \cup S_{k}^{\prime}\right|$, as required.

It remains to prove Theorem 1.3,
Proof of Theorem 1.3. Let

$$
q_{n}=\sum_{k=1}^{n}\binom{n}{k}\binom{n}{k-1}\left(\frac{k-1}{n}\right)^{k \alpha} .
$$

By Lemma 1.7, it suffices to prove that $q_{n} \leq 1 / 2$. We have verified this numerically for all $n \in\{2, \ldots, 59\}$ using Maple, so we may assume that $n \geq 60$.

Now we split the sum in two parts, change the index of summation in the second part, and apply the inequality $1+x \leq e^{x}$, after which the terms in the two parts become identical.

$$
\begin{aligned}
& q_{n}=\sum_{k=1}^{\lfloor n / 2\rfloor}\binom{n}{k}\binom{n}{k-1}\left(\frac{k-1}{n}\right)^{k \alpha}+ \\
& \sum_{k=\lfloor n / 2\rfloor+1}^{n}\binom{n}{k}\binom{n}{k-1}\left(\frac{k-1}{n}\right)^{k \alpha} \\
&= \sum_{k=1}^{\lfloor n / 2\rfloor}\binom{n}{k}\binom{n}{k-1}\left(\frac{k-1}{n}\right)^{k \alpha}+ \\
& \sum_{k=1}^{\lceil n / 2\rceil}\binom{n}{n-k+1}\binom{n}{n-k}\left(\frac{n-k}{n}\right)^{(n-k+1) \alpha} \\
&= \sum_{k=1}^{\lfloor n / 2\rfloor}\binom{n}{k}\binom{n}{k-1}\left(1-\frac{n-k+1}{n}\right)^{k \alpha}+ \\
& \leq \sum_{k=1}^{\lceil n / 2\rceil}\binom{n}{k-1}\binom{n}{k}\left(1-\frac{k}{n}\right)^{(n-k+1) \alpha} \\
& \sum_{k=1}^{\lceil n / 2\rceil}\binom{n}{k}^{2} e^{-\frac{n-k+1}{n} \cdot k \alpha}+\sum_{k=1}^{\lceil n / 2\rceil}\binom{n}{k}^{2} e^{-\frac{k}{n} \cdot(n-k+1) \alpha}
\end{aligned}
$$

$$
\leq 2 \sum_{k=1}^{\lceil n / 2\rceil}\left(\frac{e n}{k}\right)^{2 k} e^{-\frac{n-k+1}{n} \cdot k \alpha}
$$

as $\binom{n}{k} \leq\left(\frac{e n}{k}\right)^{k}$. This bound is decreasing as a function of $\alpha$, so we can replace $\alpha$ with $3 \log n$; after simplifying we get

$$
q_{n} \leq 2 \sum_{k=1}^{\lceil n / 2\rceil}\left(\frac{e}{k}\right)^{2 k} n^{-k+\frac{3 k(k-1)}{n}}
$$

Let $t_{k}=\left(\frac{e}{k}\right)^{2 k} n^{-k+\frac{3 k(k-1)}{n}}$.
Claim. For each $k \in\left\{1, \ldots,\left\lceil\frac{n}{2}\right\rceil\right\}$ we have $t_{k} \leq\left(\frac{1}{2}\right)^{k+2}$.
Proof of claim. We have numerically verified, for $k \in\{1,2,3\}$ and $n=$ 60 , that $t_{k} \leq\left(\frac{1}{2}\right)^{k+2}$ (the $k=1$ case is where we require $n \geq 60$ ). Since $t_{k}$ is non-increasing as a function of $n$, the claim holds for $k \in\{1,2,3\}$.

Now consider the claim for $4 \leq k \leq \frac{n}{3}$. Note that when $k \geq 4$, the term $\left(\frac{e}{k}\right)^{2}$ can be bounded above by $\frac{1}{2}$. Furthermore, when $k \leq \frac{n}{3}$, we have $-k+\frac{3 k(k-1)}{n} \leq-1$. Hence,

$$
t_{k} \leq\left(\frac{e}{k}\right)^{2 k} \frac{1}{n} \leq\left(\frac{1}{2}\right)^{k} \frac{1}{60}<\left(\frac{1}{2}\right)^{k+2}
$$

It remains to prove the claim for $\frac{n}{3}<k \leq \frac{n}{2}+1$. Observe that $\frac{9 e^{2}}{n^{3 / 2-3 / n}}$ is decreasing in $n$ when $n \geq 2$, so it is routine to verify that $\frac{9 e^{2}}{n^{3 / 2-3 / n}}<\frac{1}{2}$ for all $n \geq 60$. Now,

$$
\begin{aligned}
t_{k} & =\left(\left(\frac{e}{k}\right)^{2} n^{-1+\frac{3 k}{n}}\right)^{k} n^{-\frac{3 k}{n}} \\
& \leq\left(\left(\frac{e}{n / 3}\right)^{2} n^{-1+\frac{3(n / 2+1)}{n}}\right)^{k} n^{-1} \\
& =\left(\frac{9 e^{2}}{n^{3 / 2-3 / n}}\right)^{k} n^{-1} \\
& \leq\left(\frac{1}{2}\right)^{k} \frac{1}{60} \\
& <\left(\frac{1}{2}\right)^{k+2}
\end{aligned}
$$

as required.

By the above claim,

$$
q_{n} \leq 2 \sum_{k=1}^{\lceil n / 2\rceil} t_{k} \leq 2 \sum_{k \geq 1}\left(\frac{1}{2}\right)^{k+2}=\frac{1}{2}
$$

which completes the proof of Theorem 1.3,

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