INTERNALLY 4-CONNECTED BINARY MATROIDS WITH EVERY ELEMENT IN THREE TRIANGLES

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ABSTRACT. Let M be an internally 4-connected binary matroid with every element in three triangles. Then M has at least four elements esuch that si(M/e) is internally 4-connected.

1. INTRODUCTION

Terminology in this note will follow [1]. A matroid is *internally* 4connected if it is 3-connected and, for every 3-separation (X, Y) of M, either X or Y is a triangle or a triad of M.

The purpose of this note is to prove the following technical result.

Theorem 1.1. Let M be a binary internally 4-connected matroid in which every element is in exactly three triangles. Then M has at least four elements e such that si(M/e) is internally 4-connected. Morever, if M has fewer than six such elements, then these elements are in a 4-element cocircuit.

2. Preliminaries

This section introduces some basic material relating to matroid connectivity. For a matroid M, let E be the ground set of M and r be its rank function. The connectivity function λ_M of M is defined on all subsets X of E by $\lambda_M(X) = r(X) + r(E - X) - r(M)$. Equivalently, $\lambda_M(X) = r(X) + r^*(X) - |X|$. We will sometimes abbreviate λ_M as λ . For a positive integer k, a subset X or a partition (X, E - X) of E is k-separating if $\lambda_M(X) \leq k-1$. A k-separating partition (X, E - X) of E is a k-separation if $|X|, |E - X| \geq k$. If n is an integer exceeding one, a matroid is n-connected if it has no k-separations for all k < n. Let (X, Y) be a 3-separation in a matroid M. If $|X|, |Y| \geq 4$, then we call X, Y, or (X, Y) a (4, 3)-violator since it certifies that M is not internally 4-connected. For example, if X is a 4-fan, that is, a 4-element set containing a triangle and a triad, then X is a (4, 3)-violator provided $|Y| \geq 4$.

In a matroid M, a set U is *fully closed* if it is closed in both M and M^* . The *full closure* fcl(Z) of a set Z in M is the intersection of all fully closed sets containing Z. The full closure of Z may be obtained by alternating

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between taking the closure and the coclosure until both operations leave the set unchanged. Let (X, Y) be a partition of E(M). If (X, Y) is k-separating in M for some positive integer k, and y is an element of Y that is also in cl(X) or $cl^*(X)$, then it is well known and easily checked that $(X \cup y, Y - y)$ is k-separating, and we say that we have *moved* y into X. More generally, (fcl(X), Y - fcl(X)) is k-separating in M.

The following elementary result will be used repeatedly.

Lemma 2.1. If M is an internally 4-connected binary matroid and $e \in E(M)$, then si(M/e) is 3-connected.

Proof. The result is easily checked if |E(M)| < 4, so we may assume that $|E(M)| \ge 4$. Since M is 3-connected and binary, $|E(M)| \ge 6$ and both M/e and $\operatorname{si}(M/e)$ are 2-connected. If $|E(M)| \in \{6,7\}$, then M is isomorphic to $M(K_4), F_7$, or F_7^* and again the result is easily checked. Thus we may assume that $|E(M)| \ge 8$.

Now let $M' = \operatorname{si}(M/e)$ and suppose that M' has a 2-separation (X, Y). We may assume that $|X| \ge |Y|$. Suppose |Y| = 2. Then Y is a 2-cocircuit $\{y_1, y_2\}$ of M'. As $\{y_1, y_2\}$ is not a 2-cocircuit of M/e and M is binary, we see that, in M/e, either one or both of y_1 and y_2 is in a 2-element parallel class. Thus we may assume that M/e has $\{y_1, y_1'\}$ as a circuit and $\{y_1, y_1', y_2\}$ as a cocircuit, or M/e has $\{y_1, y_1'\}$ and $\{y_2, y_2'\}$ as circuits and has $\{y_1, y_1', y_2, y_2'\}$ as a cocircuit. Hence M has $\{e, y_1, y_1', y_2\}$ as a 4-fan or has $\{y_1, y_1', y_2, y_2'\}$ as both a circuit and a cocircuit. Since $|E(M)| \ge 8$, each possibility contradicts the fact that M is internally 4-connected. We conclude that $|Y| \ge 3$.

Let (X', Y') be obtained from (X, Y) by adjoining each element of E(M/e) - E(M') to the side of (X, Y) that contains an element parallel to it. Then $r_{M/e}(X') = r_{M'}(X)$ and $r_{M/e}(Y') = r_{M'}(Y)$, so (X', Y') is a 2-separation of M/e. Hence $(X', Y' \cup e)$ and $(X' \cup e, Y')$ are 3-separations of M. As $|Y' \cup e| \ge 4$ and $|E(M) \ge 8$, this gives a contradiction. \Box

Let *n* be an integer exceeding one. If *M* is *n*-connected, an *n*-separation (U, V) of *M* is sequential if fcl(U) or fcl(V) is E(M). In particular, when fcl(U) = E(M), there is an ordering (v_1, v_2, \ldots, v_m) of the elements of *V* such that $U \cup \{v_m, v_{m-1}, \ldots, v_i\}$ is *n*-separating for all *i* in $\{1, 2, \ldots, m\}$. When this occurs, the set *V* is called sequential.

3. Small matroids

We begin this section by noting two useful results.

Lemma 3.1. Let M be a matroid in which every element is in exactly three triangles. Then M has exactly |E(M)| triangles.

Proof. Consider the set of ordered pairs (e, T) where $e \in E(M)$ and T is a triangle of M containing e. The number of such pairs is 3|E(M)| since each element is in exactly three triangles. As each triangle contains exactly three elements, this number is also three times the number of triangles of M. \Box

Lemma 3.2. Let M be an internally 4-connected binary matroid in which every element is in exactly three triangles. Then M has no cocircuits of odd size.

Proof. For a cocircuit C^* of M, we construct an auxiliary graph G as follows. Let C^* be the vertex set of G, and let c_1c_2 be an edge exactly when c_1 and c_2 are members of C^* that are contained in a triangle of M. Since every element in is three triangles of M, every vertex in G has degree three by orthogonality and the fact that M is binary. Hence $|C^*|$, which equals the number of vertices of G with odd degree, is even.

To prove the next lemma, we shall use the following theorem of Qin and Zhou [2].

Theorem 3.3. Let M be an internally 4-connected binary matroid with no minor isomorphic to any of $M(K_{3,3})$, $M^*(K_{3,3})$, $M(K_5)$, or $M^*(K_5)$. Then either M is isomorphic to the cycle matroid of a planar graph, or Mis isomorphic to F_7 or F_7^* .

Lemma 3.4. Let M be an internally 4-connected binary matroid in which every element is in exactly three triangles and $|E(M)| \leq 13$. Then M is isomorphic to F_7 or $M(K_5)$. Hence $\operatorname{si}(M/e)$ is internally 4-connected for all elements e of M.

Proof. Assume that M is not isomorphic to F_7 or $M(K_5)$. Suppose first that M has none of $M(K_{3,3})$, $M^*(K_{3,3})$, $M(K_5)$, or $M^*(K_5)$ as a minor. As F_7^* has no triangles, it follows that M is isomorphic to the cycle matroid of a planar graph G. As every edge of G is in exactly three triangles, but M(G) is internally 4-connected, every vertex has degree at least four. Hence $|E(G)| \geq 2|V(G)|$. Moreover, by Lemma 3.2, every vertex of G has even degree. Clearly $|V(G)| \neq 4$. Moreover, $|V(G)| \neq 5$, otherwise $M \cong M(K_5)$; a contradiction. As $|E(G)| \leq 13$, it follows that |V(G)| = 6 and |E(G)| = 12. Then G is obtained from K_6 by deleting the edges of a perfect matching. But no edge of this graph is in exactly three triangles.

We may now assume that M has an N-minor for some N in $\{M(K_{3,3}), M^*(K_{3,3}), M(K_5), M^*(K_5)\}$. By the Splitter Theorem for 3connected matroids, there is a sequence M_0, M_1, \ldots, M_k of 3-connected matroids such that $M_0 \cong N$ and $M_k \cong M$, while $|E(M_{i+1}) - E(M_i)| = 1$ for all i in $\{0, 1, \ldots, k-1\}$. Since $|E(M) \ge 9$ and $|E(M)| \le 13$, it follows that $k \in \{0, 1, 2, 3, 4\}$.

Suppose that some M_i is obtained from its successor by contracting an element e. Then M/e has an N-minor. But $\operatorname{si}(M/e)$ has at most nine elements. Thus |E(M)| = 13 and N is $M(K_{3,3})$ or $M^*(K_{3,3})$. Since $\operatorname{si}(M/e)$ must contain triangles, N is $M^*(K_{3,3})$. Now, by Lemma 3.2, every cocircuit of M/e is even. Moreover, M/e has exactly three 2-circuits. The union of these three 2-circuits cannot have rank two in M/e otherwise M has F_7 as a restriction but the remaining six elements of M cannot all be in exactly three triangles of M. Let a, b and c be the three elements of $M^*(K_{3,3})$ that are

in 2-circuits in M/e. Then one easily checks that there are two intersecting triangles of $M^*(K_{3,3})$ whose union contains exactly two elements of $\{a, b, c\}$. The cocircuit of M/e whose complement is the union of the closure of these two triangles is odd; a contradiction.

We now know that M is an extension of N by at most four elements. Let $N = M \setminus D$. Then $|D| \ge 1$ so $|E(M)| \ge 10$. Moreover, N has at least |E(M)| - 3|D| triangles. It is straightforward to check that the last number is positive, so N cannot be $M(K_{3,3})$ or $M^*(K_5)$. Thus N is $M^*(K_{3,3})$ or $M(K_5)$. Each element of $M(K_5)$ is in three triangles, so $N \neq M(K_5)$ since each element of E(M) - E(N) must be in a triangle with some element of $M(K_5)$; a contradiction. We deduce that $N = M^*(K_{3,3})$. Now $M^*(K_{3,3})$ has exactly six triangles with each element being in precisely two triangles. Thus, in M, there are six triangles each containing a single element of $M^*(K_{3,3})$ and two elements of E(M) - E(N). As $|E(M)| - E(N)| \le 4$, there are at most six triangles containing exactly two elements of E(M) - E(N). We deduce that |E(M)| = 13 so M can be obtained from PG(3,2) by deleting exactly two elements. As PG(3,2) has exactly seven triangles containing each element, deleting two elements leaves each element in at least five triangles; a contradiction.

4. Small cocircuits

In this section, we move towards proving the main result by dealing with 4-cocircuits and certain special 6-cocircuits in M. Throughout the section, we will assume that M is an internally 4-connected binary matroid in which every element is in exactly three triangles, and $|E(M)| \ge 14$.

Lemma 4.1. If C^* is a 4-element cocircuit of M, then, for all e in C^* , the matroid $\operatorname{si}(M/e)$ is internally 4-connected having no triads.

Proof. Suppose that $C^* = \{e, f_1, f_2, f_3\}$ and $\operatorname{si}(M/e)$ is not internally 4-connected. As M is internally 4-connected, $r(C^*) = 4$. As e is in three triangles of M, there are elements $\{g_1, g_2, g_3\}$ such that $\{e, f_i, g_i\}$ is a triangle for all i. As f_i is in three triangles for all i, by orthogonality and the fact that M is binary, there are elements $\{h_1, h_2, h_3\}$ such that $\{f_1, f_2, h_1\}, \{f_1, f_3, h_3\}$, and $\{f_2, f_3, h_2\}$ are triangles. This forces $\{g_1, g_2, h_1\}, \{g_1, g_3, h_3\}$, and $\{g_2, g_3, h_2\}$ to be triangles, so g_i is in no other triangle of M for all i.

Let $M' = \operatorname{si}(M/e) = M/e \setminus f_1, f_2, f_3$. Lemma 2.1 implies that M' is 3-connected. The set $\{g_1, g_2, g_3, h_1, h_2, h_3\}$ forms an $M(K_4)$ -restriction in M'. Suppose M' has a non-sequential 3-separation. Then we may assume that $\{g_1, g_2, g_3, h_1, h_2, h_3\}$ is contained in one side of the 3-separation. Since $\{f_i, g_i\}$ is a circuit in M/e, we may add f_1, f_2 , and f_3 to the side containing the $M(K_4)$ -restriction, and then add e to get a (4, 3)-violator of M; a contradiction. We deduce that a (4, 3)-violator of $\operatorname{si}(M/e)$ is a sequential 3-separation.

We show next that

4.1.1. $M/e \setminus f_1, f_2, f_3$ has no triads.

Suppose $M/e \setminus f_1, f_2, f_3$ has a triad $\{\beta, \gamma, \delta\}$. Then $M \setminus f_1, f_2, f_3$ has $\{\beta, \gamma, \delta\}$ as a cocircuit. By Lemma 3.2, we may assume that $\{\beta, \gamma, \delta, f_1, f_2, f_3\}$ or $\{\beta, \gamma, \delta, f_1\}$ is a cocircuit of M. By orthogonality, in the first case, $\{\beta, \gamma, \delta\} = \{g_1, g_2, g_3\}$ while, in the second case, $g_1 \in \{\beta, \gamma, \delta\}$. In the first case, let $Z = \{e, f_1, f_2, f_3, g_1, g_2, g_3\}$. Then $r(Z) \leq 4$ while $|Z| - r^*(Z) \geq 2$, so $\lambda(Z) \leq 2$, a contradiction as $|E(M)| \geq 14$.

In the second case, M has a 4-cocircuit D^* such that $C^* \cap D^* = \{f_1\}$ and $g_1 \in D^*$. Apart from $\{f_1, e, g_1\}$, the other triangles containing f_1 must meet $C^* - \{f_1, e\}$ in distinct elements and must meet $D^* - \{f_1, g_1\}$ in distinct elements. Thus $r(C^* \cup D^*) \leq 4$ and $|C^* \cup D^*| - r^*(C^* \cup D^*) \geq 2$, so $\lambda(C^* \cup D^*) \leq 2$; a contradiction since $|E(M)| \geq 14$. Thus 4.1.1 holds.

By 4.1.1, $M/e \setminus f_1, f_2, f_3$ has no 4-fans and so has no sequential 3-separation that is a (4,3)-violator. This contradiction completes the proof.

Lemma 4.2. Take $e \in E(M)$ and the three triangles T_1, T_2 , and T_3 containing e. If $(T_1 \cup T_2 \cup T_3) - e$ is a cocircuit C^* , then $\operatorname{si}(M/x)$ is internally 4-connected for every element x of C^* .

Proof. Let $T_i = \{e, f_i, g_i\}$ for each $i \in \{1, 2, 3\}$. Note that T_1, T_2 , and T_3 are not coplanar, otherwise their union forms an F_7 -restriction, and C^* contains a triangle; a contradiction to the fact that M is binary. Suppose the lemma fails. Then we may assume that $\operatorname{si}(M/f_3)$ is not internally 4-connected.

As f_1 is in two triangles other than T_1 , orthogonality and the fact that M is binary imply that each of these triangles contains an element of $\{f_2, g_2, f_3, g_3\}$. If $\{f_1, f_2\}$ and $\{f_1, g_2\}$ are each contained in a triangle, then the plane containing T_1 and T_2 is an F_7 -restriction, so e is in a fourth triangle; a contradiction. Hence f_1 is in a single triangle with an element of $\{f_3, g_3\}$. Without loss of generality, $\{f_1, g_2, x_1\}$ and $\{f_1, g_3, x_2\}$ are triangles. By taking the symmetric difference of these triangles with the circuits $\{f_1, g_1, f_2, g_2\}$ and $\{f_1, g_1, f_3, g_3\}$, respectively, we see that $\{g_1, f_2, x_1\}$ and $\{g_1, f_3, x_2\}$ are also triangles. We have now identified all three of the triangles containing each element in $\{f_1, g_1\}$. But, for each element in $\{f_2, g_2, f_3, g_3\}$, one of the triangles containing the element remains undetermined.

Either $\{f_2, g_3, x_3\}$ and $\{g_2, f_3, x_3\}$ are triangles, or $\{f_2, f_3, y_3\}$ and $\{g_2, g_3, y_3\}$ are triangles. In each of these cases, we will obtain the contradiction that $\operatorname{si}(M/f_3)$ is internally 4-connected. By Lemma 2.1, $M' = \operatorname{si}(M/f_3)$ is 3-connected. Take (U, V) to be a (4, 3)-violator in M'.

Let $X = \{e, f_1, f_2, g_1, g_2, x_1\}$. Clearly the restriction of M/f_3 to X is isomorphic to $M(K_4)$. We may assume that $M' = M/f_3 \setminus Y$ where Y is $\{g_3, x_2, x_3\}$ or $\{g_3, x_2, y_3\}$ depending on whether $\{f_3, g_2, x_3\}$ or $\{f_3, f_2, y_3\}$ is a triangle of M. Without loss of generality, we may also assume that U spans X in M'. Then $(U \cup X, V - X)$ is 3-separating in M' and it follows that $(U \cup X \cup Y \cup f_3, V - X)$ is 3-separating in M. Since M has no (4, 3)-violator, we deduce that V is a sequential 3-separating set in M'. Thus M' has a triad $\{\beta, \gamma, \delta\}$. By Lemma 3.2, M has a cocircuit D^* where D^* is $\{\beta, \gamma, \delta\} \cup Y$ or $\{\beta, \gamma, \delta\} \cup y$ for some y in Y. In the first case, by orthogonality, $\{\beta, \gamma, \delta\} \subseteq X$. The last inclusion also follows by orthogonality in the second case since $\{\beta, \gamma, \delta\}$ must meet X and $M|X \cong M(K_4)$. Hence $X \cup Y \cup f_3$ contains at least two cocircuits. Since $r(X \cup Y \cup f_3) = 4$, it follows that $\lambda(X \cup Y \cup f_3) \leq 2$; a contradiction as $|E(M)| \geq 14$.

Lemma 4.3. Let (X, Y) be an exact 4-separation in M with $X \subseteq fcl(Y)$. If M has no 4-cocircuits, then X is coindependent, r(X) = 3, and $X \subseteq cl(Y)$.

Proof. If $X \subseteq cl(Y)$, then Y contains a basis of M, and X is coindependent. As $r(X) + r^*(X) - |X| \leq 3$, the rank of X is at most three, and the result holds. If $X \subseteq cl^*(Y)$, then X is independent, so $r^*(X) = 3$. As $|X| \geq 4$, it follows that X is a 4-cocircuit; a contradiction.

Beginning with Y, look at $cl(Y), cl^*(cl(Y)), cl(cl^*(cl(Y))), \ldots$ until the first time we get E(M). Consider the set Y' that occurs before E(M) in this sequence, let X' = E(M) - Y', and let e be the last element that was added in taking the closure or coclosure that equals Y'. Then either Y' is a hyperplane and X' is a cocircuit, or Y' is a cohyperplane and X' is a circuit.

Suppose X' is a circuit. As $r(X') + r^*(X') - |X'| \leq 3$, we see that $r^*(X') \leq 4$. Thus, as X' does not contain a 4-cocircuit, it is coindependent, so it has size at most four. We may assume that $X' \subsetneq X$, otherwise the lemma holds. Suppose |X'| = 4. Then both $(X' \cup e, Y' - e)$ and (X', Y') are exact 4-separations. Thus $e \in cl^*(X') \cap cl^*(Y' - e)$ or $e \in cl(X') \cap cl(Y' - e)$. The latter holds otherwise M has a 4-cocircuit; a contradiction. But Y' is coclosed, so e was added by coclosure; that is, $e \in cl^*(Y - e)$ and we have a contradiction to orthogonality since $e \in cl(X)$. It remains to consider the case when |X'| = 3. Then $|X' \cup e| = 4$. The lemma holds if $X' \cup e = X$, so there is an element f of Y' - e that was added immediately before e in the construction of Y'. Now if f is added via closure, then we can also add e and X' via closure, so we violate our choice of Y'. Thus f is added via coclosure so $f \in cl^*(Y' - e - f) \cap cl^*(X' \cup e)$. Hence M has a 4-cocircuit; a contradiction.

We may now assume that X' is a cocircuit. Then X' has at least six elements. As X' is 4-separating, $3 = r(X') + r^*(X') - |X'| = r(X') - 1$. Hence r(X') = 4, so M|X' is a restriction of PG(3,2). As M is binary, X' contains no triangle and no 5-circuits, so M|X' is a restriction of AG(3,2). As X' has six or eight elements, it follows that X' is a union of 4-circuits so fcl(Y') cannot contain X'; a contradiction.

Lemma 4.4. Assume M has no 4-cocircuits. If every exact 4-separation in M is sequential, then, for every element $e \in E(M)$, the matroid $\operatorname{si}(M/e)$ is internally 4-connected with no triads.



FIGURE 1. A skew plane and line in a binary matroid. Squares indicate positions that may be occupied by elements of M.

Proof. Let $\{e, f_i, g_i\}$ be a triangle for all $i \in \{1, 2, 3\}$. The matroid $M' = \operatorname{si}(M/e) = M/e \setminus f_1, f_2, f_3$ is 3-connected by Lemma 2.1. Let (U, V) be a (4, 3)-violator in M'. Then $|U|, |V| \ge 4$. Add f_i to the side of the 3-separation containing g_i for all $i \in \{1, 2, 3\}$ to obtain (U', V'), a 3-separation in M/e. Neither $(U' \cup e, V')$ nor $(U', V' \cup e)$ is a 3-separation in M. Hence both are 4-separations in M. Thus, by hypothesis, each is a sequential 4-separation in M. Lemma 4.3 implies that, without loss of generality, either $U' \cup e$ is coindependent and has rank at most three in M; or both U' and V' have rank at most three and are contained in $\operatorname{cl}(V' \cup e)$ and $\operatorname{cl}(U' \cup e)$, respectively. In the first case, as $U' \cup e$ is contained in a plane, U is contained in a triangle in $\operatorname{si}(M/e)$; a contradiction. In the second case, r(M) = 4, so U' and V' span planes in PG(3, 2). These planes meet in a line, so $|U' \cup V'| \leq 7 + 7 - 3 = 11$. Hence $E(M) \leq 12$; a contradiction.

Suppose $M/e \setminus f_1, f_2, f_3$ has a triad $\{a, b, c\}$. Then, by Lemma 4.1, M has $\{a, b, c, f_1, f_2, f_3\}$ as a cocircuit, so we may assume that $(a, b, c) = (g_1, g_2, g_3)$. Now M has a triangle containing f_1 and exactly one of f_2, g_2, f_3 , or g_3 . It follows that $\operatorname{si}(M/e)$ has a triangle meeting $\{g_1, g_2, g_3\}$, so $\operatorname{si}(M/e)$ has a 4-fan; a contradiction.

The next three lemmas deal with a plane and a line in M.

Lemma 4.5. Suppose M contains a plane P and a line L that are skew and are labelled as in Figure 1 where not every element in the figure must be in M. If a, b, c, d, e, f, x, y, and z are in M, and $\{x, y, a, b, d, e\}$ and $\{y, z, b, c, e, f\}$ are cocircuits in M, then si(M/w) is internally 4-connected for all w in $\{a, b, c, d, e, f\}$.

Proof. By symmetric difference, $\{x, z, a, c, d, f\}$ is a cocircuit. As z is in three triangles of M, orthogonality implies that z is in a triangle with c, say $\{z, c, c'\}$, and a triangle with f, say $\{z, f, f'\}$. Likewise, x is in triangles $\{x, a, a'\}$ and $\{x, d, d'\}$, while y is in triangles $\{y, b, b'\}$ and $\{y, e, e'\}$, for some elements a', d', b', e'. As P and L are skew, all of a', b', c', d', e', f' are distinct and none is in P or L.

By symmetry, it suffices to show that si(M/a) is internally 4-connected. Let $M' = si(M/a) = M/a \backslash a', b, f$. Let $Z = \{c, d, e, x, y, z, d', b', f'\}$. The restriction of M' to Z is isomorphic to $M^*(K_{3,3})$. Suppose (U, V) is a (4, 3)violator of M'. Without loss of generality, U spans Z in M'. Thus U spans $\{c', e'\}$. Hence $(U \cup Z \cup \{c', e'\} \cup \{a', b, f\}, V - Z - \{c', e'\})$ is 3-separating in M/a, so $(U \cup Z \cup \{c', e'\} \cup \{a', b, f\} \cup a, V - Z - \{c', e'\})$ is 3-separating in M. Thus V is a sequential 3-separating set in M', so V contains a triad $\{\beta, \gamma, \delta\}$. Thus either $\{x, c, e, a', b, f\}$ or $\{\beta, \gamma, \delta\} \cup t$ is a cocircuit of M for some t in $\{a', b, f\}$. The first possibility gives a contradiction to orthogonality with $\{y, b, b'\}$. Thus $\{\beta, \gamma, \delta, b\}$, $\{\beta, \gamma, \delta, f\}$, or $\{\beta, \gamma, \delta, a'\}$ is a cocircuit. Suppose $\{\beta, \gamma, \delta, b\}$ or $\{\beta, \gamma, \delta, f\}$ is a cocircuit. Then orthogonality implies that $\{\beta, \gamma, \delta\}$ contains $\{b, c, d\}$ or $\{f, e, d\}$ and so we get a contradiction to orthogonality with at least one of $\{x, d, d'\}, \{z, c, c'\}, \{z, f, f'\}, \{y, b, b'\}$ and $\{y, e, e'\}$. Thus $\{\beta, \gamma, \delta, a'\}$ is a cocircuit. This cocircuit also contains x so either contains y and elements from each of $\{b, b'\}$ and $\{e, e'\}$, or contains z and elements from each of $\{f, f'\}$ and $\{c, c'\}$. Each case gives a contradiction to orthogonality. We conclude that si(M/a) is internally 4-connected, so the lemma holds.

Lemma 4.6. Assume M has no 4-cocircuits. Let (U, V) be a non-sequential 4-separation of M where U is closed and V is contained in the union of a plane P and a line L of M. Then either V is 6-cocircuit, or |V| = 9 and |P| = 6. Moreover, $\operatorname{si}(M/v)$ is internally 4-connected for at least six elements v of V.

Proof. By Lemma 3.2, each cocircuit contained in V has exactly six elements otherwise it contains a triangle. Suppose r(V) = 3. As $r(V) + r^*(V) - |V| = 3$, we know that V is coindependent. Hence it is contained in cl(U); a contradiction. Evidently $r(V) \ge 4$. We use Figure 1 as a guide for the points that may exist in V. We consider which positions are filled, keeping in mind that V is the union of circuits and the union of cocircuits.

Suppose V has rank four and view V as a restriction of Q = PG(3, 2). Then $\operatorname{cl}_Q(P) \cap \operatorname{cl}_Q(L)$ is a point of Q, so we may suppose e = z. Furthermore, as $r(V) + r^*(V) - |V| = 3$, we know that V contains, and therefore is, a cocircuit. Thus |V| = 6. As V contains no triangles, $|(P \cup L) \cap \operatorname{cl}_Q(P)| \le 4$, and $|(P \cup L) \cap \operatorname{cl}_Q(L)| \le 2$. Thus $e \notin P \cup L$. Without loss of generality, the points in V are a, b, f, g, x, and y, and the result follows by Lemma 4.2 provided $e \in E(M)$.

We assume therefore that $e \notin E(M)$. We know that $V = \{x, y, a, b, f, g\}$. By orthogonality, without loss of generality, the three triangles of M containing x are $\{x, a, a'\}, \{x, f, f'\}$, and $\{x, b, b'\}$. Thus M has as triangles each of $\{y, a', f\}, \{y, a, f'\}, \text{ and } \{y, b', g\}$. Hence M has no other triangles containing x or y. Thus the remaining triangles containing g must be in P, and so contain c and d. But then $\{a, b, c\}$ and $\{a, g, d\}$ are triangles of M, so a is in four triangles; a contradiction.

Suppose that r(V) = 5. Then P and L are skew, and V is the union of two 6-cocircuits, C^* and D^* . By orthogonality, each of C^* and D^* contains at most four elements of P. Thus, by orthogonality, $|P| \le 6$ so $|C^* \cup D^*| \le 9$.

Hence $|C^* \triangle D^*| = 6$ and |V| = 9. Then, without loss of generality, each of C^* and D^* meets P in four elements and L in two elements. The result now follows by Lemma 4.5.

Lemma 4.7. If M has a 6-element cocircuit $C^* = \{a, b, c, d, e, f\}$ where $\{a, b, c, d\}$ and $\{a, b, e, f\}$ are circuits, then si(M/x) is internally 4-connected for all x in C^* .

Proof. By symmetric difference, $\{c, d, e, f\}$ is also a circuit. Thus C^* is the union of three disjoint pairs, $\{a, b\}$, $\{c, d\}$, and $\{e, f\}$ such that the union of any two of these pairs is a circuit. If one of these pairs is in a triangle with some element x, then each of the pairs is in a triangle with x and the lemma follows by Lemma 4.2. Thus we may assume that each of $\{a, c\}$ and $\{a, d\}$ is in a triangle. Hence so are $\{b, c\}$ and $\{b, d\}$. Thus each of a, b, c and d is in exactly one triangle with an element of $\{e, f\}$. Hence e and f cannot both be in exactly three triangles; a contradiction.

Lemma 4.8. Let (J, K) be an exact 4-separation of M such that J is closed. If $|K| \leq 6$, then K is a 6-cocircuit and $\operatorname{si}(M/k)$ is internally 4-connected for all k in K.

Proof. We have $r(K) + r^*(K) - |K| = 3$ and $|K| \ge 4$. If |K| = 4, then K is a cocircuit; a contradiction. Thus $|K| \ge 5$. Since K is a union of cocircuits each of which has even cardinality, it follows that $|K| \ge 6$. Hence K is a 6-cocircuit. Thus r(K) = 4 so K contains two circuits such that they and their symmetric difference have even cardinality. Hence K is the union of two 4-circuits that meet in exactly two elements and the result follows by Lemma 4.6.

5. The proof of the main result

The next lemma essentially completes the proof of Theorem 1.1.

Lemma 5.1. Let M be an internally 4-connected binary matroid in which every element is in exactly three triangles. Suppose M has no 4-cocircuits. Then M has at least six elements e such that si(M/e) is internally 4connected.

Proof. By Lemma 3.4, we know that $|E(M)| \ge 14$. Assume that the lemma fails. By Lemma 4.4, M has a non-sequential 4-separation (X, Y) where X is minimal. Then Y is fully closed. By Lemma 4.8, $|X| \ge 7$ and X contains an element α such that $\operatorname{si}(M/\alpha)$ is not internally 4-connected. Let $\{\alpha, f_i, g_i\}$ be a triangle for all $i \in \{1, 2, 3\}$. Now $M' = \operatorname{si}(M/\alpha) = M/\alpha \setminus f_1, f_2, f_3$ is not internally 4-connected. By Lemma 2.1, it is 3-connected. Take a (4,3)-violator (U', V') in M'. Then $|U'|, |V'| \ge 4$. Hence $r_{M/\alpha}(U')$ and $r_{M/\alpha}(V')$ exceed two. Add f_i to the side containing g_i for all $i \in \{1, 2, 3\}$ to obtain (U'', V''). Then both $(U'' \cup \alpha, V'')$ and $(U'', V'' \cup \alpha)$ are exact 4-separations of M. Since $\alpha \in \operatorname{cl}(U'')$ and $\alpha \in \operatorname{cl}(V'')$, we deduce that $r_M(U'') \ge 4$ and $r_M(V'') \ge 4$. Moreover, by Lemma 4.3, both $(U'' \cup \alpha, V'')$ and $(U'', V'' \cup \alpha)$ are non-sequential. Without loss of generality, we may assume that $r(U'' \cap X) \ge r(V'' \cap X)$ and, when equality holds, $|U'' \cap X| \ge |V'' \cap X|$. Let $(U, V) = (\operatorname{cl}(U''), V'' - \operatorname{cl}(U''))$. Then

5.1.1. $r_M(U \cap X) \ge r_M(V \cap X)$, and, when equality holds, $|U \cap X| > |V \cap X|$.

We show next that

5.1.2. $X \cap U, X \cap V, Y \cap U$, and $Y \cap V$ are all non-empty.

As $\alpha \in X \cap U$, the first set is not empty. If the second is empty, then, as α is in the closure of $V = V \cap Y$, we can move α to Y to get $(X - \alpha, Y \cup \alpha)$ as a non-sequential 4-separation of M; a contradiction to our choice of (X, Y). If the third is empty, then $U = X \cap U$, and $(X \cap U, Y \cup V)$ contradicts our choice of (X, Y). Likewise, if the fourth set is empty, then $V = X \cap V$, and $(X \cap V, Y \cup U)$ violates our choice of (X, Y). This completes our proof of 5.1.2.

By submodularity of the connectivity function, $\lambda_M(X \cup U) + \lambda_M(X \cap U) \leq \lambda_M(X) + \lambda_M(U) = 3 + 3$. We now break the rest of the argument into the following two cases, which we shall then consecutively eliminate.

(A) $\lambda(X \cap U) \ge 4$ and $\lambda(X \cup U) = \lambda(Y \cap V) \le 2$; or (B) $\lambda(X \cap U) \le 3$.

5.1.3. (A) does not hold.

Suppose that (A) holds. As M is internally 4-connected, $Y \cap V$ is a triangle, or a triad, or contains at most two elements. Clearly, this set is not a triad. Suppose $\lambda(X \cap V) \ge 4$. Then, by submodularity again, $\lambda(Y \cap U) \le 2$, so $|Y \cap U| \le 3$. Then $|Y| \le 6$, so Y contains and so is a cocircuit. As this cocircuit cannot contain a triangle, it follows that $|Y \cap V| \le 2$, so $|Y| \le 5$; a contradiction. Thus $\lambda(X \cap V) \le 3$. If $\lambda(X \cap V) \le 2$, then $X \cap V$ is contained in a triangle, so V is contained in the union of two lines; a contradiction since V contains a cocircuit that must have six elements and so contain a triangle. We deduce that $\lambda(X \cap V) = 3$. Hence $X \cap V \subseteq \text{fcl}(Y \cup U)$. Lemma 4.3 implies that $X \cap V$ has rank at most three. Thus V is contained in the union of a line L and a plane P. It now follows by Lemma 4.8 that 5.1.3 holds.

Next we show that

5.1.4. (B) does not hold.

Assume that (B) holds. Since $\lambda(X \cap U) \leq 3$ and $X \cap U$ is properly contained in X, either $X \cap U \subseteq \operatorname{fcl}(Y \cup V)$), or $\lambda(X \cap U) \leq 2$. It follows using Lemma 4.3 that $r(X \cap U) \leq 3$. Thus, by 5.1.1, $r(X \cap V) \leq 3$. If $r(X \cap V) \leq 2$, then X is contained in the union of a plane and a line. Then, arguing as in (A), it follows that |X| = 6 or |X| = 9 and $\operatorname{si}(M/x)$ is internally 4-connected for all x in X. Each alternative gives a contradiction. Thus, by 5.1.1, $r(X \cap V) = 3 = r(X \cap U)$ and $|X \cap V| < |X \cap U| \leq 7$. Hence $4 \leq r(X) \leq 6$.

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Now view M as a restriction of Q = PG(r-1,2), where r = r(M). As (X,Y) is an exact 4-separation, $cl_Q(X) \cap cl_Q(Y)$ is a plane P of Q. Because Y is fully closed, no element of X is in P. It follows by orthogonality, since X is a union of cocircuits of M, that each triangle that meets an element of X is either contained in X or contains exactly two elements of X with the third element being in P.

We show that

5.1.5. $r(X) \in \{5, 6\}.$

Suppose not. Then r(X) = 4 and $X \subseteq cl_Q(X) - P$. So X is contained in an AG(3, 2)-restriction of M. As X is a cocircuit, |X| = 6 or |X| = 8. Since $|X \cap U| \neq |X \cap V|$ and each is at least three, it follows that |X| = 8. To have a triangle meeting X, there must be an element y of Y in P. But y is the tip of a binary spike in $X \cup y$ so it is in at least four triangles. This contradiction proves 5.1.5.

We show next that

5.1.6. r(X) = 5.

Suppose not. Then r(X) = 6. As $r(X \cap U) = r(X \cap V) = 3$, we deduce that $\operatorname{cl}_Q(X \cap U) \cap \operatorname{cl}_Q(X \cap V) = \emptyset$, where we recall that Q = PG(r-1,2)and $P = \operatorname{cl}_Q(X) \cap \operatorname{cl}_Q(Y)$.

Suppose $cl_Q(X \cap V)$ meets P. As $3 = \lambda(X) = r(X) + r(Y) - r(M)$, we know that r(Y) = r(M) - 3. Then $cl_M(Y \cup (X \cap V))$ is a flat with rank at most r(M) - 1. Hence its complement, which is contained in $X \cap U$, contains a cocircuit. But this cocircuit contains at least six elements by Lemma 4.1, so it contains a triangle in $X \cap U$. We deduce that $cl_Q(X \cap V)$ avoids P. By symmetry, so does $cl_Q(X \cap U)$. It follows that each triangle that meets X is either contained in $X \cap U$ or $X \cap V$, or contains an element of each of $X \cap U, X \cap V$, and P. If $|X \cap U| = 7$, then $M|(X \cap U) \cong F_7$ -restriction, so each element in $X \cap U$ is in three triangles contained in $X \cap U$. Then each element in $X \cap V$ is contained in three triangles in $X \cap V$, so $M|(X \cap V) \cong F_7$, and $|X \cap U| = |X \cap V|$; a contradiction to 5.1.1. Thus $|X \cap U| \leq 6$ and 5.1.1 implies that $|X \cap V| \leq 5$. Thus $X \cap V$ contains an element v that is in at most one triangle in $X \cap V$. Hence v is in triangles $\{v, u_1, p_1\}$ and $\{v, u_2, p_2\}$ for some u_1 and u_2 in $X \cap U$, and p_1 and p_2 in P. Take u_3 in $X \cap U$ such that $\{u_1, u_2, u_3\}$ is a basis for $X \cap U$. Then $cl(Y \cup \{v, u_3\})$ is a flat of rank at most r(M) - 1 whose complement, which is contained in $X \cap V$, contains a cocircuit. This cocircuit has at most five elements; a contradiction to Lemma 4.1. Hence 5.1.6 holds.

We now know that r(X) = 5. It follows, since $r(X \cap U) = r(X \cap V) = 3$, that $cl_Q(X \cap U) \cap cl_Q(X \cap V)$ is a point p of Q. Moreover, r(Y) = r(M) - 2, so $r(cl_Q(Y) \cap cl_Q(X \cap U)) = 1$ since $r(cl_Q(Y \cup (X \cap U))) = r(M)$, otherwise $X \cap V$ contains a cocircuit of M that either has fewer than six elements or contains a triangle. Similarly, $r(cl_Q(Y) \cap cl_Q(X \cap V)) = 1$.

The following is an immediate consequence of the fact that U is closed.

5.1.7. If $p \in X$, then $p \in X \cap U$.

Let $\operatorname{cl}_Q(Y) \cap \operatorname{cl}_Q(X \cap U) = \{s\}$ and $\operatorname{cl}_Q(Y) \cap \operatorname{cl}_Q(X \cap V) = \{t\}$. Neither s nor t is in X, so

$$|X \cap U| \le 6.$$

Hence $|X \cap V| \le |X \cap U| - 1 \le 5$. Recall that $|X| \ge 9$. As $|X \cap U| \ge |X \cap V|$, it follows that $|X \cap U| \ge 5$. Hence

5.1.8. $|X \cap U| \in \{5, 6\}.$

Call a triangle of M special if it contains an element of $X \cap U$, an element of $X \cap V$, and an element of P. Construct a bipartite graph H with vertex classes $X \cap U$ and $X \cap V$ with uv being an edge, where $u \in X \cap U$ and $v \in X \cap V$, precisely when $\{u, v\}$ is contained in a special triangle. Clearly

(1)
$$\sum_{u \in X \cap U} d_H(u) = \sum_{v \in X \cap V} d_H(v).$$

Next we show the following.

5.1.9. Every vertex x of $V(H) - \{p\}$ has its degree in $\{1, 2\}$.

Let $\{X', X''\} = \{X \cap U, X \cap V\}$ and take $x \in X'$ such that $x \neq p$. Let x'' be the element of $\operatorname{cl}_Q(X'') \cap P$. Thus $x'' \in \{s, t\}$. Clearly $d_H(x) \leq 3$. Assume $d_H(x) = 3$. Then $\operatorname{cl}_Q(Y \cup x)$ contains x', at least three distinct elements of X'', and x''. Thus $\operatorname{cl}_Q(Y \cup x)$ contains X''. Hence $E(M) - \operatorname{cl}_M(Y \cup x)$ contains at most five elements of M; a contradiction to the fact that every cocircuit of M has at least six elements. Thus $d_H(x) < 3$.

Next suppose that $d_H(x) = 0$. Then all three triangles containing x are contained in $\operatorname{cl}_M(X')$. Thus $M|\operatorname{cl}_M(X') \cong F_7$. Hence, for $z \in X'' - \operatorname{cl}_M(X')$, the three triangles containing z are contained in $\operatorname{cl}_M(X'')$. Thus $M|\operatorname{cl}_M(X'') \cong F_7$. Hence $\operatorname{cl}_M(X') \cap \operatorname{cl}_M(X'')$ contains a point of M that is in six triangles; a contradiction. Thus 5.1.9 holds.

Now either

- (i) s = t = p; or
- (ii) s, t, and p are distinct.

Suppose that (i) holds. Assume first that $p \notin Y$. By 5.1.9, for $W \in \{U, V\}$, every element of $M|(X \cap W)$ is in a triangle contained in $X \cap W$. Thus either $M|(X \cap W) \cong M(K_4)$ and $\sum_{w \in X \cap W} d_H(w) = 6$; or $M|(X \cap W) \cong M(K_4 \setminus e)$ and $\sum_{w \in X \cap W} d_H(w) = 9$. Since $|X \cap U| > |X \cap V|$, we obtain a contradiction using (1). Thus $p \in Y$.

As $|X \cap U| \in \{5, 6\}$ by 5.1.8, we see that $|X \cap U| = 5$, otherwise $M|((X \cap U) \cup p) \cong F_7$, and $d_H(x) = 0$ for every $x \in X \cap V$; a contradiction to 5.1.9. We deduce that $M|((X \cap U) \cup p) \cong M(K_4)$, and $5 = \sum_{u \in X \cap U} d_H(u)$. Now p is in two triangles in $(X \cap U) \cup p$. Thus, of the three triangles in $cl_Q(X \cap V)$ containing p, at most one contains two elements of $X \cap V$. Hence, using 5.1.9, we see that $M|cl_M(X \cap V)$ comprises two triangles with a single element,

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not p, in common. Thus $\sum_{v \in X \cap V} d_H(v) = 7$; a contradiction to Equation 1. We conclude that (i) does not hold.

We now know that s, t, and p are distinct. We show next that

5.1.10. $p \in X$.

Suppose $p \notin X$. Then $|X \cap U| = 5$ so $|X \cap V| = 4$. Thus $\sum_{u \in X \cap U} d_H(u)$ is five when $s \in Y$ and nine otherwise. As $d_H(v) < 3$ for each $v \in X \cap V$ by 5.1.9, it follows that $t \in Y$. Then $\sum_{v \in X \cap V} d_H(v)$ is eight or seven depending on whether $M|(X \cap V)$ is $U_{3,4}$ or $U_{2,3} \oplus U_{1,1}$. Thus, by (1), we have a contradiction. Hence 5.1.10 holds.

Suppose $|X \cap U| = 6$. Then $s \notin Y$, otherwise there is an element of $(X \cap U) - p$ with degree zero in H; a contradiction to 5.1.9. Then $\sum_{u \in X \cap U} d_H(u) = 6$. Suppose $t \in Y$. If the line through $\{p, t\}$ contains a third point of M, say q, then each of the other two lines through p in $\operatorname{cl}_Q(X \cap V)$ contains at most one point of M. Thus $|X \cap V| = 3$ and, as $r(X \cap V) = 3$, we see that $\{p, q, t\}$ is the unique triangle in $M|\operatorname{cl}_M(X \cap V)$ containing q. As this triangle is special, it follows that $d_H(q) = 3$; a contradiction to 5.1.9. Evidently the line through $\{p, t\}$ does not contain a third point of M. We deduce that $M|\operatorname{cl}_M(X \cap V)$ comprises two triangles that have one element, not p or t, in common. Then $\sum_{v \in X \cap V} d_H(v) = 5$; a contradiction. We deduce that $t \notin Y$. Then exactly one of the lines in $\operatorname{cl}_M(X \cap V)$ through p contains exactly three points. Since no point of $X \cap V$ has degree three in H, it follows that $M|\operatorname{cl}_M(X \cap V)$ comprises two triangles with a point, not p, in common. As $p \notin X \cap V$, it follows that $\sum_{v \in X \cap V} d_H(v) = 7$; a contradiction. We conclude that $|X \cap U| \neq 6$.

It remains to consider the case that $|X \cap U| = 5$ and $|X \cap V| = 4$. Then $\sum_{u \in X \cap U} d_H(u)$ is five or nine depending on whether or not s is in Y. From 5.1.10, $p \in X$. Thus $M|[(X \cap V) \cup p]$ consists of two three-point lines meeting in a point z. If z = p, then $\sum_{v \in X \cap V} d_H(v)$ is four or eight, depending on whether or not t is in Y; a contradiction. Hence $z \neq p$. Thus the third element on the line containing $\{p, t\}$ is in X. Again $\sum_{v \in X \cap V} d_H(v)$ is seven, if $t \notin Y$, or four, if $t \in Y$; a contradiction to (1). We conclude that 5.1.4 holds and the lemma follows.

It is now straightforward to complete the proof of our main result.

Proof of Theorem 1.1. If M has a 4-cocircuit, then the result follows by Lemma 4.1. If M has no 4-cocircuits, then the theorem follows by Lemma 5.1.

6. A (NON)-EXTENSION

It is natural to ask whether, for an internally 4-connected binary matroid M with every element in exactly three triangles, si(M/e) is internally 4-connected for every element e. We now describe an example where this is not the case.

Begin with $K_{3,3}$ having vertex classes $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$. Form the graph G by adjoining three new vertices u, v, and w, each adjacent to all of a_1, a_2, a_3, b_1, b_2 , and b_3 but not to each other. The vertex-edge incidence matrix of G is the matrix A shown below.

a_1	1	1	1	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0 \
a_2	0	0	0	1	1	1	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0
a_3	0	0	0	0	0	0	1	1	1	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0
b_1	1	0	0	1	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0
b_2	0	1	0	0	1	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0
b_3	0	0	1	0	0	1	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1
u	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
v	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0
w	10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1/

Then M(G) is an internally 4-connected matroid in which every element is in exactly three triangles. Now adjoin the matrix B to A where B is shown below.

The matroid N that is represented by [A|B] has each element in M(G) in exactly three triangles, and each element of $\{a, b, c, d, e, f\}$ is in exactly two triangles. To see this, observe that $N|\{a, b, c, d, e, f\} \cong M(K_4)$. Moreover, no element of M(G) lies on a line with two elements of $\{a, b, c, d, e, f\}$ and it is straightforward to check that no element of $\{a, b, c, d, e, f\}$ is in a triangle with two elements of M(G).

Now take the generalized parallel connection of $M(K_5)$ and N across $\{a, b, c, d, e, f\}$ to get an internally 4-connected binary matroid M in which every element is in exactly three triangles. Evidently $\operatorname{si}(M/z)$ is not internally 4-connected for all z in $\{a, b, c, d, e, f\}$.

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