# INTERNALLY 4-CONNECTED BINARY MATROIDS WITH EVERY ELEMENT IN THREE TRIANGLES 

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#### Abstract

Let $M$ be an internally 4-connected binary matroid with every element in three triangles. Then $M$ has at least four elements $e$ such that $\operatorname{si}(M / e)$ is internally 4 -connected.


## 1. Introduction

Terminology in this note will follow [1]. A matroid is internally 4connected if it is 3 -connected and, for every 3 -separation $(X, Y)$ of $M$, either $X$ or $Y$ is a triangle or a triad of $M$.

The purpose of this note is to prove the following technical result.
Theorem 1.1. Let $M$ be a binary internally 4-connected matroid in which every element is in exactly three triangles. Then $M$ has at least four elements $e$ such that si(M/e) is internally 4-connected. Morever, if M has fewer than six such elements, then these elements are in a 4-element cocircuit.

## 2. Preliminaries

This section introduces some basic material relating to matroid connectivity. For a matroid $M$, let $E$ be the ground set of $M$ and $r$ be its rank function. The connectivity function $\lambda_{M}$ of $M$ is defined on all subsets $X$ of $E$ by $\lambda_{M}(X)=r(X)+r(E-X)-r(M)$. Equivalently, $\lambda_{M}(X)=r(X)+r^{*}(X)-|X|$. We will sometimes abbreviate $\lambda_{M}$ as $\lambda$. For a positive integer $k$, a subset $X$ or a partition $(X, E-X)$ of $E$ is $k$-separating if $\lambda_{M}(X) \leq k-1$. A $k$-separating partition $(X, E-X)$ of $E$ is a $k$-separation if $|X|,|E-X| \geq k$. If $n$ is an integer exceeding one, a matroid is $n$-connected if it has no $k$-separations for all $k<n$. Let $(X, Y)$ be a 3 -separation in a matroid $M$. If $|X|,|Y| \geq 4$, then we call $X, Y$, or $(X, Y)$ a (4, 3)-violator since it certifies that $M$ is not internally 4 -connected. For example, if $X$ is a 4 -fan, that is, a 4 -element set containing a triangle and a triad, then $X$ is a (4,3)-violator provided $|Y| \geq 4$.

In a matroid $M$, a set $U$ is fully closed if it is closed in both $M$ and $M^{*}$. The full closure $\mathrm{fcl}(Z)$ of a set $Z$ in $M$ is the intersection of all fully closed sets containing $Z$. The full closure of $Z$ may be obtained by alternating

[^0]between taking the closure and the coclosure until both operations leave the set unchanged. Let $(X, Y)$ be a partition of $E(M)$. If $(X, Y)$ is $k$-separating in $M$ for some positive integer $k$, and $y$ is an element of $Y$ that is also in $\operatorname{cl}(X)$ or $\mathrm{cl}^{*}(X)$, then it is well known and easily checked that ( $\left.X \cup y, Y-y\right)$ is $k$-separating, and we say that we have moved $y$ into $X$. More generally, $(\mathrm{fcl}(X), Y-\mathrm{fcl}(X))$ is $k$-separating in $M$.

The following elementary result will be used repeatedly.
Lemma 2.1. If $M$ is an internally 4-connected binary matroid and $e \in$ $E(M)$, then $\operatorname{si}(M / e)$ is 3-connected.
Proof. The result is easily checked if $|E(M)|<4$, so we may assume that $|E(M)| \geq 4$. Since $M$ is 3 -connected and binary, $|E(M)| \geq 6$ and both $M / e$ and $\operatorname{si}(M / e)$ are 2-connected. If $|E(M)| \in\{6,7\}$, then $M$ is isomorphic to $M\left(K_{4}\right), F_{7}$, or $F_{7}^{*}$ and again the result is easily checked. Thus we may assume that $|E(M)| \geq 8$.

Now let $M^{\prime}=\operatorname{si}(M / e)$ and suppose that $M^{\prime}$ has a 2-separation $(X, Y)$. We may assume that $|X| \geq|Y|$. Suppose $|Y|=2$. Then $Y$ is a 2-cocircuit $\left\{y_{1}, y_{2}\right\}$ of $M^{\prime}$. As $\left\{y_{1}, y_{2}\right\}$ is not a 2 -cocircuit of $M / e$ and $M$ is binary, we see that, in $M / e$, either one or both of $y_{1}$ and $y_{2}$ is in a 2 -element parallel class. Thus we may assume that $M / e$ has $\left\{y_{1}, y_{1}^{\prime}\right\}$ as a circuit and $\left\{y_{1}, y_{1}^{\prime}, y_{2}\right\}$ as a cocircuit, or $M / e$ has $\left\{y_{1}, y_{1}^{\prime}\right\}$ and $\left\{y_{2}, y_{2}^{\prime}\right\}$ as circuits and has $\left\{y_{1}, y_{1}^{\prime}, y_{2}, y_{2}^{\prime}\right\}$ as a cocircuit. Hence $M$ has $\left\{e, y_{1}, y_{1}^{\prime}, y_{2}\right\}$ as a 4 -fan or has $\left\{y_{1}, y_{1}^{\prime}, y_{2}, y_{2}^{\prime}\right\}$ as both a circuit and a cocircuit. Since $|E(M)| \geq 8$, each possibility contradicts the fact that $M$ is internally 4 -connected. We conclude that $|Y| \geq 3$.

Let $\left(X^{\prime}, Y^{\prime}\right)$ be obtained from $(X, Y)$ by adjoining each element of $E(M / e)-E\left(M^{\prime}\right)$ to the side of $(X, Y)$ that contains an element parallel to it. Then $r_{M / e}\left(X^{\prime}\right)=r_{M^{\prime}}(X)$ and $r_{M / e}\left(Y^{\prime}\right)=r_{M^{\prime}}(Y)$, so $\left(X^{\prime}, Y^{\prime}\right)$ is a 2-separation of $M / e$. Hence $\left(X^{\prime}, Y^{\prime} \cup e\right)$ and $\left(X^{\prime} \cup e, Y^{\prime}\right)$ are 3 -separations of $M$. As $\left|Y^{\prime} \cup e\right| \geq 4$ and $\mid E(M) \geq 8$, this gives a contradiction.

Let $n$ be an integer exceeding one. If $M$ is $n$-connected, an $n$-separation $(U, V)$ of $M$ is sequential if $\operatorname{fcl}(U)$ or $\operatorname{fcl}(V)$ is $E(M)$. In particular, when $\mathrm{fcl}(U)=E(M)$, there is an ordering $\left(v_{1}, v_{2}, \ldots, v_{m}\right)$ of the elements of $V$ such that $U \cup\left\{v_{m}, v_{m-1}, \ldots, v_{i}\right\}$ is $n$-separating for all $i$ in $\{1,2, \ldots, m\}$. When this occurs, the set $V$ is called sequential.

## 3. Small matroids

We begin this section by noting two useful results.
Lemma 3.1. Let $M$ be a matroid in which every element is in exactly three triangles. Then $M$ has exactly $|E(M)|$ triangles.
Proof. Consider the set of ordered pairs $(e, T)$ where $e \in E(M)$ and $T$ is a triangle of $M$ containing $e$. The number of such pairs is $3|E(M)|$ since each element is in exactly three triangles. As each triangle contains exactly three elements, this number is also three times the number of triangles of $M$.

Lemma 3.2. Let $M$ be an internally 4-connected binary matroid in which every element is in exactly three triangles. Then $M$ has no cocircuits of odd size.

Proof. For a cocircuit $C^{*}$ of $M$, we construct an auxiliary graph $G$ as follows. Let $C^{*}$ be the vertex set of $G$, and let $c_{1} c_{2}$ be an edge exactly when $c_{1}$ and $c_{2}$ are members of $C^{*}$ that are contained in a triangle of $M$. Since every element in is three triangles of $M$, every vertex in $G$ has degree three by orthogonality and the fact that $M$ is binary. Hence $\left|C^{*}\right|$, which equals the number of vertices of $G$ with odd degree, is even.

To prove the next lemma, we shall use the following theorem of Qin and Zhou [2].
Theorem 3.3. Let $M$ be an internally 4-connected binary matroid with no minor isomorphic to any of $M\left(K_{3,3}\right), M^{*}\left(K_{3,3}\right), M\left(K_{5}\right)$, or $M^{*}\left(K_{5}\right)$. Then either $M$ is isomorphic to the cycle matroid of a planar graph, or $M$ is isomorphic to $F_{7}$ or $F_{7}^{*}$.
Lemma 3.4. Let $M$ be an internally 4-connected binary matroid in which every element is in exactly three triangles and $|E(M)| \leq 13$. Then $M$ is isomorphic to $F_{7}$ or $M\left(K_{5}\right)$. Hence $\operatorname{si}(M / e)$ is internally 4-connected for all elements e of $M$.
Proof. Assume that $M$ is not isomorphic to $F_{7}$ or $M\left(K_{5}\right)$. Suppose first that $M$ has none of $M\left(K_{3,3}\right), M^{*}\left(K_{3,3}\right), M\left(K_{5}\right)$, or $M^{*}\left(K_{5}\right)$ as a minor. As $F_{7}^{*}$ has no triangles, it follows that $M$ is isomorphic to the cycle matroid of a planar graph $G$. As every edge of $G$ is in exactly three triangles, but $M(G)$ is internally 4 -connected, every vertex has degree at least four. Hence $|E(G)| \geq 2|V(G)|$. Moreover, by Lemma 3.2 , every vertex of $G$ has even degree. Clearly $|V(G)| \neq 4$. Moreover, $|V(G)| \neq 5$, otherwise $M \cong M\left(K_{5}\right)$; a contradiction. As $|E(G)| \leq 13$, it follows that $|V(G)|=6$ and $|E(G)|=12$. Then $G$ is obtained from $K_{6}$ by deleting the edges of a perfect matching. But no edge of this graph is in exactly three triangles.

We may now assume that $M$ has an $N$-minor for some $N$ in $\left\{M\left(K_{3,3}\right), M^{*}\left(K_{3,3}\right), M\left(K_{5}\right), M^{*}\left(K_{5}\right)\right\} . \quad$ By the Splitter Theorem for 3connected matroids, there is a sequence $M_{0}, M_{1}, \ldots, M_{k}$ of 3-connected matroids such that $M_{0} \cong N$ and $M_{k} \cong M$, while $\left|E\left(M_{i+1}\right)-E\left(M_{i}\right)\right|=1$ for all $i$ in $\{0,1, \ldots, k-1\}$. Since $\mid E(M) \geq 9$ and $|E(M)| \leq 13$, it follows that $k \in\{0,1,2,3,4\}$.

Suppose that some $M_{i}$ is obtained from its successor by contracting an element $e$. Then $M / e$ has an $N$-minor. But $\operatorname{si}(M / e)$ has at most nine elements. Thus $|E(M)|=13$ and $N$ is $M\left(K_{3,3}\right)$ or $M^{*}\left(K_{3,3}\right)$. Since si $(M / e)$ must contain triangles, $N$ is $M^{*}\left(K_{3,3}\right)$. Now, by Lemma 3.2 , every cocircuit of $M / e$ is even. Moreover, $M / e$ has exactly three 2 -circuits. The union of these three 2-circuits cannot have rank two in $M / e$ otherwise $M$ has $F_{7}$ as a restriction but the remaining six elements of $M$ cannot all be in exactly three triangles of $M$. Let $a, b$ and $c$ be the three elements of $M^{*}\left(K_{3,3}\right)$ that are
in 2-circuits in $M / e$. Then one easily checks that there are two intersecting triangles of $M^{*}\left(K_{3,3}\right)$ whose union contains exactly two elements of $\{a, b, c\}$. The cocircuit of $M / e$ whose complement is the union of the closure of these two triangles is odd; a contradiction.

We now know that $M$ is an extension of $N$ by at most four elements. Let $N=M \backslash D$. Then $|D| \geq 1$ so $|E(M)| \geq 10$. Moreover, $N$ has at least $|E(M)|-3|D|$ triangles. It is straightforward to check that the last number is positive, so $N$ cannot be $M\left(K_{3,3}\right)$ or $M^{*}\left(K_{5}\right)$. Thus $N$ is $M^{*}\left(K_{3,3}\right)$ or $M\left(K_{5}\right)$. Each element of $M\left(K_{5}\right)$ is in three triangles, so $N \neq M\left(K_{5}\right)$ since each element of $E(M)-E(N)$ must be in a triangle with some element of $M\left(K_{5}\right)$; a contradiction. We deduce that $N=M^{*}\left(K_{3,3}\right)$. Now $M^{*}\left(K_{3,3}\right)$ has exactly six triangles with each element being in precisely two triangles. Thus, in $M$, there are six triangles each containing a single element of $M^{*}\left(K_{3,3}\right)$ and two elements of $E(M)-E(N)$. As $|E(M)|-E(N) \mid \leq 4$, there are at most six triangles containing exactly two elements of $E(M)-E(N)$. We deduce that $|E(M)|=13$ so $M$ can be obtained from $P G(3,2)$ by deleting exactly two elements. As $P G(3,2)$ has exactly seven triangles containing each element, deleting two elements leaves each element in at least five triangles; a contradiction.

## 4. Small cocircuits

In this section, we move towards proving the main result by dealing with 4 -cocircuits and certain special 6 -cocircuits in $M$. Throughout the section, we will assume that $M$ is an internally 4 -connected binary matroid in which every element is in exactly three triangles, and $|E(M)| \geq 14$.

Lemma 4.1. If $C^{*}$ is a 4-element cocircuit of $M$, then, for all $e$ in $C^{*}$, the matroid $\operatorname{si}(M / e)$ is internally 4-connected having no triads.
Proof. Suppose that $C^{*}=\left\{e, f_{1}, f_{2}, f_{3}\right\}$ and $\operatorname{si}(M / e)$ is not internally 4connected. As $M$ is internally 4 -connected, $r\left(C^{*}\right)=4$. As $e$ is in three triangles of $M$, there are elements $\left\{g_{1}, g_{2}, g_{3}\right\}$ such that $\left\{e, f_{i}, g_{i}\right\}$ is a triangle for all $i$. As $f_{i}$ is in three triangles for all $i$, by orthogonality and the fact that $M$ is binary, there are elements $\left\{h_{1}, h_{2}, h_{3}\right\}$ such that $\left\{f_{1}, f_{2}, h_{1}\right\},\left\{f_{1}, f_{3}, h_{3}\right\}$, and $\left\{f_{2}, f_{3}, h_{2}\right\}$ are triangles. This forces $\left\{g_{1}, g_{2}, h_{1}\right\},\left\{g_{1}, g_{3}, h_{3}\right\}$, and $\left\{g_{2}, g_{3}, h_{2}\right\}$ to be triangles, so $g_{i}$ is in no other triangle of $M$ for all $i$.

Let $M^{\prime}=\operatorname{si}(M / e)=M / e \backslash f_{1}, f_{2}, f_{3}$. Lemma 2.1 implies that $M^{\prime}$ is 3 -connected. The set $\left\{g_{1}, g_{2}, g_{3}, h_{1}, h_{2}, h_{3}\right\}$ forms an $M\left(K_{4}\right)$-restriction in $M^{\prime}$. Suppose $M^{\prime}$ has a non-sequential 3 -separation. Then we may assume that $\left\{g_{1}, g_{2}, g_{3}, h_{1}, h_{2}, h_{3}\right\}$ is contained in one side of the 3 -separation. Since $\left\{f_{i}, g_{i}\right\}$ is a circuit in $M / e$, we may add $f_{1}, f_{2}$, and $f_{3}$ to the side containing the $M\left(K_{4}\right)$-restriction, and then add $e$ to get a $(4,3)$-violator of $M$; a contradiction. We deduce that a $(4,3)$-violator of $\operatorname{si}(M / e)$ is a sequential 3 -separation.

We show next that
4.1.1. $M / e \backslash f_{1}, f_{2}, f_{3}$ has no triads.

Suppose $M / e \backslash f_{1}, f_{2}, f_{3}$ has a triad $\{\beta, \gamma, \delta\}$. Then $M \backslash f_{1}, f_{2}, f_{3}$ has $\{\beta, \gamma, \delta\}$ as a cocircuit. By Lemma 3.2, we may assume that $\left\{\beta, \gamma, \delta, f_{1}, f_{2}, f_{3}\right\}$ or $\left\{\beta, \gamma, \delta, f_{1}\right\}$ is a cocircuit of $M$. By orthogonality, in the first case, $\{\beta, \gamma, \delta\}=\left\{g_{1}, g_{2}, g_{3}\right\}$ while, in the second case, $g_{1} \in\{\beta, \gamma, \delta\}$. In the first case, let $Z=\left\{e, f_{1}, f_{2}, f_{3}, g_{1}, g_{2}, g_{3}\right\}$. Then $r(Z) \leq 4$ while $|Z|-r^{*}(Z) \geq 2$, so $\lambda(Z) \leq 2$, a contradiction as $|E(M)| \geq 14$.

In the second case, $M$ has a 4 -cocircuit $D^{*}$ such that $C^{*} \cap D^{*}=\left\{f_{1}\right\}$ and $g_{1} \in D^{*}$. Apart from $\left\{f_{1}, e, g_{1}\right\}$, the other triangles containing $f_{1}$ must meet $C^{*}-\left\{f_{1}, e\right\}$ in distinct elements and must meet $D^{*}-\left\{f_{1}, g_{1}\right\}$ in distinct elements. Thus $r\left(C^{*} \cup D^{*}\right) \leq 4$ and $\left|C^{*} \cup D^{*}\right|-r^{*}\left(C^{*} \cup D^{*}\right) \geq 2$, so $\lambda\left(C^{*} \cup D^{*}\right) \leq 2$; a contradiction since $|E(M)| \geq 14$. Thus 4.1.1 holds.

By 4.1.1, $M / e \backslash f_{1}, f_{2}, f_{3}$ has no 4 -fans and so has no sequential 3separation that is a $(4,3)$-violator. This contradiction completes the proof.

Lemma 4.2. Take $e \in E(M)$ and the three triangles $T_{1}, T_{2}$, and $T_{3}$ containing e. If $\left(T_{1} \cup T_{2} \cup T_{3}\right)-e$ is a cocircuit $C^{*}$, then $\operatorname{si}(M / x)$ is internally 4 -connected for every element $x$ of $C^{*}$.

Proof. Let $T_{i}=\left\{e, f_{i}, g_{i}\right\}$ for each $i \in\{1,2,3\}$. Note that $T_{1}, T_{2}$, and $T_{3}$ are not coplanar, otherwise their union forms an $F_{7}$-restriction, and $C^{*}$ contains a triangle; a contradiction to the fact that $M$ is binary. Suppose the lemma fails. Then we may assume that $\operatorname{si}\left(M / f_{3}\right)$ is not internally 4 -connected.

As $f_{1}$ is in two triangles other than $T_{1}$, orthogonality and the fact that $M$ is binary imply that each of these triangles contains an element of $\left\{f_{2}, g_{2}, f_{3}, g_{3}\right\}$. If $\left\{f_{1}, f_{2}\right\}$ and $\left\{f_{1}, g_{2}\right\}$ are each contained in a triangle, then the plane containing $T_{1}$ and $T_{2}$ is an $F_{7}$-restriction, so $e$ is in a fourth triangle; a contradiction. Hence $f_{1}$ is in a single triangle with an element of $\left\{f_{2}, g_{2}\right\}$ and a single triangle with an element of $\left\{f_{3}, g_{3}\right\}$. Without loss of generality, $\left\{f_{1}, g_{2}, x_{1}\right\}$ and $\left\{f_{1}, g_{3}, x_{2}\right\}$ are triangles. By taking the symmetric difference of these triangles with the circuits $\left\{f_{1}, g_{1}, f_{2}, g_{2}\right\}$ and $\left\{f_{1}, g_{1}, f_{3}, g_{3}\right\}$, respectively, we see that $\left\{g_{1}, f_{2}, x_{1}\right\}$ and $\left\{g_{1}, f_{3}, x_{2}\right\}$ are also triangles. We have now identified all three of the triangles containing each element in $\left\{f_{1}, g_{1}\right\}$. But, for each element in $\left\{f_{2}, g_{2}, f_{3}, g_{3}\right\}$, one of the triangles containing the element remains undetermined.

Either $\left\{f_{2}, g_{3}, x_{3}\right\}$ and $\left\{g_{2}, f_{3}, x_{3}\right\}$ are triangles, or $\left\{f_{2}, f_{3}, y_{3}\right\}$ and $\left\{g_{2}, g_{3}, y_{3}\right\}$ are triangles. In each of these cases, we will obtain the contradiction that $\operatorname{si}\left(M / f_{3}\right)$ is internally 4 -connected. By Lemma 2.1, $M^{\prime}=\operatorname{si}\left(M / f_{3}\right)$ is 3 -connected. Take $(U, V)$ to be a $(4,3)$-violator in $M^{\prime}$.

Let $X=\left\{e, f_{1}, f_{2}, g_{1}, g_{2}, x_{1}\right\}$. Clearly the restriction of $M / f_{3}$ to $X$ is isomorphic to $M\left(K_{4}\right)$. We may assume that $M^{\prime}=M / f_{3} \backslash Y$ where $Y$ is $\left\{g_{3}, x_{2}, x_{3}\right\}$ or $\left\{g_{3}, x_{2}, y_{3}\right\}$ depending on whether $\left\{f_{3}, g_{2}, x_{3}\right\}$ or $\left\{f_{3}, f_{2}, y_{3}\right\}$ is a triangle of $M$. Without loss of generality, we may also assume that $U$ spans $X$ in $M^{\prime}$. Then $(U \cup X, V-X)$ is 3 -separating in $M^{\prime}$ and it follows
that $\left(U \cup X \cup Y \cup f_{3}, V-X\right)$ is 3 -separating in $M$. Since $M$ has no (4,3)violator, we deduce that $V$ is a sequential 3 -separating set in $M^{\prime}$. Thus $M^{\prime}$ has a triad $\{\beta, \gamma, \delta\}$. By Lemma $3.2, M$ has a cocircuit $D^{*}$ where $D^{*}$ is $\{\beta, \gamma, \delta\} \cup Y$ or $\{\beta, \gamma, \delta\} \cup y$ for some $y$ in $Y$. In the first case, by orthogonality, $\{\beta, \gamma, \delta\} \subseteq X$. The last inclusion also follows by orthogonality in the second case since $\{\beta, \gamma, \delta\}$ must meet $X$ and $M \mid X \cong M\left(K_{4}\right)$. Hence $X \cup Y \cup f_{3}$ contains at least two cocircuits. Since $r\left(X \cup Y \cup f_{3}\right)=4$, it follows that $\lambda\left(X \cup Y \cup f_{3}\right) \leq 2$; a contradiction as $|E(M)| \geq 14$.

Lemma 4.3. Let $(X, Y)$ be an exact 4-separation in $M$ with $X \subseteq \mathrm{fcl}(Y)$. If $M$ has no 4-cocircuits, then $X$ is coindependent, $r(X)=3$, and $X \subseteq \operatorname{cl}(Y)$.

Proof. If $X \subseteq \operatorname{cl}(Y)$, then $Y$ contains a basis of $M$, and $X$ is coindependent. As $r(X)+r^{*}(X)-|X| \leq 3$, the rank of $X$ is at most three, and the result holds. If $X \subseteq \mathrm{cl}^{*}(Y)$, then $X$ is independent, so $r^{*}(X)=3$. As $|X| \geq 4$, it follows that $X$ is a 4-cocircuit; a contradiction.

Beginning with $Y$, look at $\left.\operatorname{cl}(Y), \operatorname{cl}^{*}(\operatorname{cl}(Y)), \operatorname{cl}^{\left(\mathrm{cl}^{*}\right.}(\operatorname{cl}(Y))\right), \ldots$ until the first time we get $E(M)$. Consider the set $Y^{\prime}$ that occurs before $E(M)$ in this sequence, let $X^{\prime}=E(M)-Y^{\prime}$, and let $e$ be the last element that was added in taking the closure or coclosure that equals $Y^{\prime}$. Then either $Y^{\prime}$ is a hyperplane and $X^{\prime}$ is a cocircuit, or $Y^{\prime}$ is a cohyperplane and $X^{\prime}$ is a circuit.

Suppose $X^{\prime}$ is a circuit. As $r\left(X^{\prime}\right)+r^{*}\left(X^{\prime}\right)-\left|X^{\prime}\right| \leq 3$, we see that $r^{*}\left(X^{\prime}\right) \leq 4$. Thus, as $X^{\prime}$ does not contain a 4-cocircuit, it is coindependent, so it has size at most four. We may assume that $X^{\prime} \varsubsetneqq X$, otherwise the lemma holds. Suppose $\left|X^{\prime}\right|=4$. Then both $\left(X^{\prime} \cup e, Y^{\prime}-e\right)$ and $\left(X^{\prime}, Y^{\prime}\right)$ are exact 4-separations. Thus $e \in \operatorname{cl}^{*}\left(X^{\prime}\right) \cap \operatorname{cl}^{*}\left(Y^{\prime}-e\right)$ or $e \in \operatorname{cl}\left(X^{\prime}\right) \cap \operatorname{cl}\left(Y^{\prime}-e\right)$. The latter holds otherwise $M$ has a 4 -cocircuit; a contradiction. But $Y^{\prime}$ is coclosed, so $e$ was added by coclosure; that is, $e \in \operatorname{cl}^{*}(Y-e)$ and we have a contradiction to orthogonality since $e \in \operatorname{cl}(X)$. It remains to consider the case when $\left|X^{\prime}\right|=3$. Then $\left|X^{\prime} \cup e\right|=4$. The lemma holds if $X^{\prime} \cup e=X$, so there is an element $f$ of $Y^{\prime}-e$ that was added immediately before $e$ in the construction of $Y^{\prime}$. Now if $f$ is added via closure, then we can also add $e$ and $X^{\prime}$ via closure, so we violate our choice of $Y^{\prime}$. Thus $f$ is added via coclosure so $f \in \operatorname{cl}^{*}\left(Y^{\prime}-e-f\right) \cap \operatorname{cl}^{*}\left(X^{\prime} \cup e\right)$. Hence $M$ has a 4-cocircuit; a contradiction.

We may now assume that $X^{\prime}$ is a cocircuit. Then $X^{\prime}$ has at least six elements. As $X^{\prime}$ is 4-separating, $3=r\left(X^{\prime}\right)+r^{*}\left(X^{\prime}\right)-\left|X^{\prime}\right|=r\left(X^{\prime}\right)-1$. Hence $r\left(X^{\prime}\right)=4$, so $M \mid X^{\prime}$ is a restriction of $P G(3,2)$. As $M$ is binary, $X^{\prime}$ contains no triangle and no 5 -circuits, so $M \mid X^{\prime}$ is a restriction of $A G(3,2)$. As $X^{\prime}$ has six or eight elements, it follows that $X^{\prime}$ is a union of 4-circuits so $\mathrm{fcl}\left(Y^{\prime}\right)$ cannot contain $X^{\prime}$; a contradiction.

Lemma 4.4. Assume $M$ has no 4-cocircuits. If every exact 4-separation in $M$ is sequential, then, for every element $e \in E(M)$, the matroid $\operatorname{si}(M / e)$ is internally 4-connected with no triads.


Figure 1. A skew plane and line in a binary matroid. Squares indicate positions that may be occupied by elements of $M$.

Proof. Let $\left\{e, f_{i}, g_{i}\right\}$ be a triangle for all $i \in\{1,2,3\}$. The matroid $M^{\prime}=$ $\operatorname{si}(M / e)=M / e \backslash f_{1}, f_{2}, f_{3}$ is 3 -connected by Lemma 2.1. Let $(U, V)$ be a $(4,3)$-violator in $M^{\prime}$. Then $|U|,|V| \geq 4$. Add $f_{i}$ to the side of the $3-$ separation containing $g_{i}$ for all $i \in\{1,2,3\}$ to obtain $\left(U^{\prime}, V^{\prime}\right)$, a 3 -separation in $M / e$. Neither $\left(U^{\prime} \cup e, V^{\prime}\right)$ nor $\left(U^{\prime}, V^{\prime} \cup e\right)$ is a 3-separation in $M$. Hence both are 4 -separations in $M$. Thus, by hypothesis, each is a sequential 4 -separation in $M$. Lemma 4.3 implies that, without loss of generality, either $U^{\prime} \cup e$ is coindependent and has rank at most three in $M$; or both $U^{\prime}$ and $V^{\prime}$ have rank at most three and are contained in $\operatorname{cl}\left(V^{\prime} \cup e\right)$ and $\operatorname{cl}\left(U^{\prime} \cup e\right)$, respectively. In the first case, as $U^{\prime} \cup e$ is contained in a plane, $U$ is contained in a triangle in $\operatorname{si}(M / e)$; a contradiction. In the second case, $r(M)=4$, so $U^{\prime}$ and $V^{\prime}$ span planes in $P G(3,2)$. These planes meet in a line, so $\left|U^{\prime} \cup V^{\prime}\right| \leq 7+7-3=11$. Hence $E(M) \leq 12$; a contradiction.

Suppose $M / e \backslash f_{1}, f_{2}, f_{3}$ has a triad $\{a, b, c\}$. Then, by Lemma 4.1, $M$ has $\left\{a, b, c, f_{1}, f_{2}, f_{3}\right\}$ as a cocircuit, so we may assume that $(a, b, c)=\left(g_{1}, g_{2}, g_{3}\right)$. Now $M$ has a triangle containing $f_{1}$ and exactly one of $f_{2}, g_{2}, f_{3}$, or $g_{3}$. It follows that $\operatorname{si}(M / e)$ has a triangle meeting $\left\{g_{1}, g_{2}, g_{3}\right\}$, so $\mathrm{si}(M / e)$ has a 4 -fan; a contradiction.

The next three lemmas deal with a plane and a line in $M$.
Lemma 4.5. Suppose $M$ contains a plane $P$ and a line $L$ that are skew and are labelled as in Figure 1 where not every element in the figure must be in $M$. If $a, b, c, d, e, f, x, y$, and $z$ are in $M$, and $\{x, y, a, b, d, e\}$ and $\{y, z, b, c, e, f\}$ are cocircuits in $M$, then $\operatorname{si}(M / w)$ is internally 4-connected for all $w$ in $\{a, b, c, d, e, f\}$.

Proof. By symmetric difference, $\{x, z, a, c, d, f\}$ is a cocircuit. As $z$ is in three triangles of $M$, orthogonality implies that $z$ is in a triangle with $c$, say $\left\{z, c, c^{\prime}\right\}$, and a triangle with $f$, say $\left\{z, f, f^{\prime}\right\}$. Likewise, $x$ is in triangles $\left\{x, a, a^{\prime}\right\}$ and $\left\{x, d, d^{\prime}\right\}$, while $y$ is in triangles $\left\{y, b, b^{\prime}\right\}$ and $\left\{y, e, e^{\prime}\right\}$, for some elements $a^{\prime}, d^{\prime}, b^{\prime}, e^{\prime}$. As $P$ and $L$ are skew, all of $a^{\prime}, b^{\prime}, c^{\prime}, d^{\prime}, e^{\prime}, f^{\prime}$ are distinct and none is in $P$ or $L$.

By symmetry, it suffices to show that $\operatorname{si}(M / a)$ is internally 4 -connected. Let $M^{\prime}=\operatorname{si}(M / a)=M / a \backslash a^{\prime}, b, f$. Let $Z=\left\{c, d, e, x, y, z, d^{\prime}, b^{\prime}, f^{\prime}\right\}$. The
restriction of $M^{\prime}$ to $Z$ is isomorphic to $M^{*}\left(K_{3,3}\right)$. Suppose $(U, V)$ is a $(4,3)$ violator of $M^{\prime}$. Without loss of generality, $U$ spans $Z$ in $M^{\prime}$. Thus $U$ spans $\left\{c^{\prime}, e^{\prime}\right\}$. Hence $\left(U \cup Z \cup\left\{c^{\prime}, e^{\prime}\right\} \cup\left\{a^{\prime}, b, f\right\}, V-Z-\left\{c^{\prime}, e^{\prime}\right\}\right)$ is 3-separating in $M / a$, so $\left(U \cup Z \cup\left\{c^{\prime}, e^{\prime}\right\} \cup\left\{a^{\prime}, b, f\right\} \cup a, V-Z-\left\{c^{\prime}, e^{\prime}\right\}\right)$ is 3-separating in $M$. Thus $V$ is a sequential 3-separating set in $M^{\prime}$, so $V$ contains a triad $\{\beta, \gamma, \delta\}$. Thus either $\left\{x, c, e, a^{\prime}, b, f\right\}$ or $\{\beta, \gamma, \delta\} \cup t$ is a cocircuit of $M$ for some $t$ in $\left\{a^{\prime}, b, f\right\}$. The first possibility gives a contradiction to orthogonality with $\left\{y, b, b^{\prime}\right\}$. Thus $\{\beta, \gamma, \delta, b\},\{\beta, \gamma, \delta, f\}$, or $\left\{\beta, \gamma, \delta, a^{\prime}\right\}$ is a cocircuit. Suppose $\{\beta, \gamma, \delta, b\}$ or $\{\beta, \gamma, \delta, f\}$ is a cocircuit. Then orthogonality implies that $\{\beta, \gamma, \delta\}$ contains $\{b, c, d\}$ or $\{f, e, d\}$ and so we get a contradiction to orthogonality with at least one of $\left\{x, d, d^{\prime}\right\},\left\{z, c, c^{\prime}\right\},\left\{z, f, f^{\prime}\right\},\left\{y, b, b^{\prime}\right\}$ and $\left\{y, e, e^{\prime}\right\}$. Thus $\left\{\beta, \gamma, \delta, a^{\prime}\right\}$ is a cocircuit. This cocircuit also contains $x$ so either contains $y$ and elements from each of $\left\{b, b^{\prime}\right\}$ and $\left\{e, e^{\prime}\right\}$, or contains $z$ and elements from each of $\left\{f, f^{\prime}\right\}$ and $\left\{c, c^{\prime}\right\}$. Each case gives a contradiction to orthogonality. We conclude that $\operatorname{si}(M / a)$ is internally 4-connected, so the lemma holds.

Lemma 4.6. Assume $M$ has no 4-cocircuits. Let $(U, V)$ be a non-sequential 4-separation of $M$ where $U$ is closed and $V$ is contained in the union of a plane $P$ and a line $L$ of $M$. Then either $V$ is 6-cocircuit, or $|V|=9$ and $|P|=6$. Moreover, $\operatorname{si}(M / v)$ is internally 4-connected for at least six elements $v$ of $V$.

Proof. By Lemma 3.2 , each cocircuit contained in $V$ has exactly six elements otherwise it contains a triangle. Suppose $r(V)=3$. As $r(V)+r^{*}(V)-|V|=$ 3, we know that $V$ is coindependent. Hence it is contained in $\operatorname{cl}(U)$; a contradiction. Evidently $r(V) \geq 4$. We use Figure 1 as a guide for the points that may exist in $V$. We consider which positions are filled, keeping in mind that $V$ is the union of circuits and the union of cocircuits.

Suppose $V$ has rank four and view $V$ as a restriction of $Q=P G(3,2)$. Then $\operatorname{cl}_{Q}(P) \cap \operatorname{cl}_{Q}(L)$ is a point of $Q$, so we may suppose $e=z$. Furthermore, as $r(V)+r^{*}(V)-|V|=3$, we know that $V$ contains, and therefore is, a cocircuit. Thus $|V|=6$. As $V$ contains no triangles, $\left|(P \cup L) \cap \operatorname{cl}_{Q}(P)\right| \leq 4$, and $\left|(P \cup L) \cap \operatorname{cl}_{Q}(L)\right| \leq 2$. Thus $e \notin P \cup L$. Without loss of generality, the points in $V$ are $a, b, f, g, x$, and $y$, and the result follows by Lemma 4.2 provided $e \in E(M)$.

We assume therefore that $e \notin E(M)$. We know that $V=\{x, y, a, b, f, g\}$. By orthogonality, without loss of generality, the three triangles of $M$ containing $x$ are $\left\{x, a, a^{\prime}\right\},\left\{x, f, f^{\prime}\right\}$, and $\left\{x, b, b^{\prime}\right\}$. Thus $M$ has as triangles each of $\left\{y, a^{\prime}, f\right\},\left\{y, a, f^{\prime}\right\}$, and $\left\{y, b^{\prime}, g\right\}$. Hence $M$ has no other triangles containing $x$ or $y$. Thus the remaining triangles containing $g$ must be in $P$, and so contain $c$ and $d$. But then $\{a, b, c\}$ and $\{a, g, d\}$ are triangles of $M$, so $a$ is in four triangles; a contradiction.

Suppose that $r(V)=5$. Then $P$ and $L$ are skew, and $V$ is the union of two 6-cocircuits, $C^{*}$ and $D^{*}$. By orthogonality, each of $C^{*}$ and $D^{*}$ contains at most four elements of $P$. Thus, by orthogonality, $|P| \leq 6$ so $\left|C^{*} \cup D^{*}\right| \leq 9$.

Hence $\left|C^{*} \triangle D^{*}\right|=6$ and $|V|=9$. Then, without loss of generality, each of $C^{*}$ and $D^{*}$ meets $P$ in four elements and $L$ in two elements. The result now follows by Lemma 4.5 .

Lemma 4.7. If $M$ has a 6 -element cocircuit $C^{*}=\{a, b, c, d, e, f\}$ where $\{a, b, c, d\}$ and $\{a, b, e, f\}$ are circuits, then $\operatorname{si}(M / x)$ is internally 4-connected for all $x$ in $C^{*}$.

Proof. By symmetric difference, $\{c, d, e, f\}$ is also a circuit. Thus $C^{*}$ is the union of three disjoint pairs, $\{a, b\},\{c, d\}$, and $\{e, f\}$ such that the union of any two of these pairs is a circuit. If one of these pairs is in a triangle with some element $x$, then each of the pairs is in a triangle with $x$ and the lemma follows by Lemma 4.2. Thus we may assume that each of $\{a, c\}$ and $\{a, d\}$ is in a triangle. Hence so are $\{b, c\}$ and $\{b, d\}$. Thus each of $a, b, c$ and $d$ is in exactly one triangle with an element of $\{e, f\}$. Hence $e$ and $f$ cannot both be in exactly three triangles; a contradiction.

Lemma 4.8. Let $(J, K)$ be an exact 4-separation of $M$ such that $J$ is closed. If $|K| \leq 6$, then $K$ is a 6 -cocircuit and $\operatorname{si}(M / k)$ is internally 4-connected for all $k$ in $K$.

Proof. We have $r(K)+r^{*}(K)-|K|=3$ and $|K| \geq 4$. If $|K|=4$, then $K$ is a cocircuit; a contradiction. Thus $|K| \geq 5$. Since $K$ is a union of cocircuits each of which has even cardinality, it follows that $|K| \geq 6$. Hence $K$ is a 6 -cocircuit. Thus $r(K)=4$ so $K$ contains two circuits such that they and their symmetric difference have even cardinality. Hence $K$ is the union of two 4-circuits that meet in exactly two elements and the result follows by Lemma 4.6

## 5. The proof of the main result

The next lemma essentially completes the proof of Theorem 1.1.
Lemma 5.1. Let $M$ be an internally 4-connected binary matroid in which every element is in exactly three triangles. Suppose $M$ has no 4-cocircuits. Then $M$ has at least six elements $e$ such that $\operatorname{si}(M / e)$ is internally 4connected.

Proof. By Lemma 3.4 , we know that $|E(M)| \geq 14$. Assume that the lemma fails. By Lemma 4.4, $M$ has a non-sequential 4-separation $(X, Y)$ where $X$ is minimal. Then $Y$ is fully closed. By Lemma $4.8,|X| \geq 7$ and $X$ contains an element $\alpha$ such that $\operatorname{si}(M / \alpha)$ is not internally 4-connected. Let $\left\{\alpha, f_{i}, g_{i}\right\}$ be a triangle for all $i \in\{1,2,3\}$. Now $M^{\prime}=\operatorname{si}(M / \alpha)=M / \alpha \backslash f_{1}, f_{2}, f_{3}$ is not internally 4 -connected. By Lemma 2.1, it is 3 -connected. Take a (4,3)-violator $\left(U^{\prime}, V^{\prime}\right)$ in $M^{\prime}$. Then $\left|U^{\prime}\right|,\left|V^{\prime}\right| \geq 4$. Hence $r_{M / \alpha}\left(U^{\prime}\right)$ and $r_{M / \alpha}\left(V^{\prime}\right)$ exceed two. Add $f_{i}$ to the side containing $g_{i}$ for all $i \in\{1,2,3\}$ to obtain $\left(U^{\prime \prime}, V^{\prime \prime}\right)$. Then both $\left(U^{\prime \prime} \cup \alpha, V^{\prime \prime}\right)$ and $\left(U^{\prime \prime}, V^{\prime \prime} \cup \alpha\right)$ are exact 4-separations of $M$. Since $\alpha \in \operatorname{cl}\left(U^{\prime \prime}\right)$ and $\alpha \in \operatorname{cl}\left(V^{\prime \prime}\right)$, we deduce that $r_{M}\left(U^{\prime \prime}\right) \geq 4$ and $r_{M}\left(V^{\prime \prime}\right) \geq 4$. Moreover, by Lemma 4.3, both $\left(U^{\prime \prime} \cup \alpha, V^{\prime \prime}\right)$
and $\left(U^{\prime \prime}, V^{\prime \prime} \cup \alpha\right)$ are non-sequential. Without loss of generality, we may assume that $r\left(U^{\prime \prime} \cap X\right) \geq r\left(V^{\prime \prime} \cap X\right)$ and, when equality holds, $\left|U^{\prime \prime} \cap X\right| \geq$ $\left|V^{\prime \prime} \cap X\right|$. Let $(U, V)=\left(\operatorname{cl}\left(U^{\prime \prime}\right), V^{\prime \prime}-\operatorname{cl}\left(U^{\prime \prime}\right)\right)$. Then
5.1.1. $r_{M}(U \cap X) \geq r_{M}(V \cap X)$, and, when equality holds, $|U \cap X|>|V \cap X|$.

We show next that

### 5.1.2. $X \cap U, X \cap V, Y \cap U$, and $Y \cap V$ are all non-empty.

As $\alpha \in X \cap U$, the first set is not empty. If the second is empty, then, as $\alpha$ is in the closure of $V=V \cap Y$, we can move $\alpha$ to $Y$ to get $(X-\alpha, Y \cup \alpha)$ as a non-sequential 4 -separation of $M$; a contradiction to our choice of $(X, Y)$. If the third is empty, then $U=X \cap U$, and $(X \cap U, Y \cup V)$ contradicts our choice of $(X, Y)$. Likewise, if the fourth set is empty, then $V=X \cap V$, and $(X \cap V, Y \cup U)$ violates our choice of $(X, Y)$. This completes our proof of 5.1.2.

By submodularity of the connectivity function, $\lambda_{M}(X \cup U)+\lambda_{M}(X \cap U) \leq$ $\lambda_{M}(X)+\lambda_{M}(U)=3+3$. We now break the rest of the argument into the following two cases, which we shall then consecutively eliminate.
(A) $\lambda(X \cap U) \geq 4$ and $\lambda(X \cup U)=\lambda(Y \cap V) \leq 2$; or
(B) $\lambda(X \cap U) \leq 3$.

### 5.1.3. (A) does not hold.

Suppose that (A) holds. As $M$ is internally 4 -connected, $Y \cap V$ is a triangle, or a triad, or contains at most two elements. Clearly, this set is not a triad. Suppose $\lambda(X \cap V) \geq 4$. Then, by submodularity again, $\lambda(Y \cap U) \leq 2$, so $|Y \cap U| \leq 3$. Then $|Y| \leq 6$, so $Y$ contains and so is a cocircuit. As this cocircuit cannot contain a triangle, it follows that $|Y \cap V| \leq 2$, so $|Y| \leq 5$; a contradiction. Thus $\lambda(X \cap V) \leq 3$. If $\lambda(X \cap V) \leq 2$, then $X \cap V$ is contained in a triangle, so $V$ is contained in the union of two lines; a contradiction since $V$ contains a cocircuit that must have six elements and so contain a triangle. We deduce that $\lambda(X \cap V)=3$. Hence $X \cap V \subseteq \operatorname{fcl}(Y \cup U)$. Lemma 4.3 implies that $X \cap V$ has rank at most three. Thus $V$ is contained in the union of a line $L$ and a plane $P$. It now follows by Lemma 4.8 that 5.1 .3 holds.

Next we show that

### 5.1.4. (B) does not hold.

Assume that (B) holds. Since $\lambda(X \cap U) \leq 3$ and $X \cap U$ is properly contained in $X$, either $X \cap U \subseteq \operatorname{fcl}(Y \cup V)$ ), or $\lambda(X \cap U) \leq 2$. It follows using Lemma 4.3 that $r(X \cap U) \leq 3$. Thus, by 5.1.1, $r(X \cap V) \leq 3$. If $r(X \cap V) \leq 2$, then $X$ is contained in the union of a plane and a line. Then, arguing as in (A), it follows that $|X|=6$ or $|X|=9$ and $\operatorname{si}(M / x)$ is internally 4-connected for all $x$ in $X$. Each alternative gives a contradiction. Thus, by 5.1.1, $r(X \cap V)=3=r(X \cap U)$ and $|X \cap V|<|X \cap U| \leq 7$. Hence $4 \leq r(X) \leq 6$.

Now view $M$ as a restriction of $Q=P G(r-1,2)$, where $r=r(M)$. As ( $X, Y$ ) is an exact 4-separation, $\mathrm{cl}_{Q}(X) \cap \mathrm{cl}_{Q}(Y)$ is a plane $P$ of $Q$. Because $Y$ is fully closed, no element of $X$ is in $P$. It follows by orthogonality, since $X$ is a union of cocircuits of $M$, that each triangle that meets an element of $X$ is either contained in $X$ or contains exactly two elements of $X$ with the third element being in $P$.

We show that

### 5.1.5. $r(X) \in\{5,6\}$.

Suppose not. Then $r(X)=4$ and $X \subseteq \operatorname{cl}_{Q}(X)-P$. So $X$ is contained in an $A G(3,2)$-restriction of $M$. As $X$ is a cocircuit, $|X|=6$ or $|X|=8$. Since $|X \cap U| \neq|X \cap V|$ and each is at least three, it follows that $|X|=8$. To have a triangle meeting $X$, there must be an element $y$ of $Y$ in $P$. But $y$ is the tip of a binary spike in $X \cup y$ so it is in at least four triangles. This contradiction proves 5.1.5.

We show next that
5.1.6. $r(X)=5$.

Suppose not. Then $r(X)=6$. As $r(X \cap U)=r(X \cap V)=3$, we deduce that $\operatorname{cl}_{Q}(X \cap U) \cap \operatorname{cl}_{Q}(X \cap V)=\emptyset$, where we recall that $Q=P G(r-1,2)$ and $P=\operatorname{cl}_{Q}(X) \cap \operatorname{cl}_{Q}(Y)$.

Suppose $\mathrm{cl}_{Q}(X \cap V)$ meets $P$. As $3=\lambda(X)=r(X)+r(Y)-r(M)$, we know that $r(Y)=r(M)-3$. Then $\operatorname{cl}_{M}(Y \cup(X \cap V))$ is a flat with rank at most $r(M)-1$. Hence its complement, which is contained in $X \cap U$, contains a cocircuit. But this cocircuit contains at least six elements by Lemma 4.1, so it contains a triangle in $X \cap U$. We deduce that $\mathrm{cl}_{Q}(X \cap V)$ avoids $P$. By symmetry, so does $\operatorname{cl}_{Q}(X \cap U)$. It follows that each triangle that meets $X$ is either contained in $X \cap U$ or $X \cap V$, or contains an element of each of $X \cap U, X \cap V$, and $P$. If $|X \cap U|=7$, then $M \mid(X \cap U) \cong F_{7}$-restriction, so each element in $X \cap U$ is in three triangles contained in $X \cap U$. Then each element in $X \cap V$ is contained in three triangles in $X \cap V$, so $M \mid(X \cap V) \cong F_{7}$, and $|X \cap U|=|X \cap V|$; a contradiction to 5.1.1. Thus $|X \cap U| \leq 6$ and 5.1.1 implies that $|X \cap V| \leq 5$. Thus $X \cap V$ contains an element $v$ that is in at most one triangle in $X \cap V$. Hence $v$ is in triangles $\left\{v, u_{1}, p_{1}\right\}$ and $\left\{v, u_{2}, p_{2}\right\}$ for some $u_{1}$ and $u_{2}$ in $X \cap U$, and $p_{1}$ and $p_{2}$ in $P$. Take $u_{3}$ in $X \cap U$ such that $\left\{u_{1}, u_{2}, u_{3}\right\}$ is a basis for $X \cap U$. Then $\operatorname{cl}\left(Y \cup\left\{v, u_{3}\right\}\right)$ is a flat of rank at most $r(M)-1$ whose complement, which is contained in $X \cap V$, contains a cocircuit. This cocircuit has at most five elements; a contradiction to Lemma 4.1. Hence 5.1 .6 holds.

We now know that $r(X)=5$. It follows, since $r(X \cap U)=r(X \cap V)=3$, that $\mathrm{cl}_{Q}(X \cap U) \cap \mathrm{cl}_{Q}(X \cap V)$ is a point $p$ of $Q$. Moreover, $r(Y)=r(M)-2$, so $r\left(\operatorname{cl}_{Q}(Y) \cap \operatorname{cl}_{Q}(X \cap U)\right)=1$ since $r\left(\operatorname{cl}_{Q}(Y \cup(X \cap U))\right)=r(M)$, otherwise $X \cap V$ contains a cocircuit of $M$ that either has fewer than six elements or contains a triangle. Similarly, $r\left(\mathrm{cl}_{Q}(Y) \cap \mathrm{cl}_{Q}(X \cap V)\right)=1$.

The following is an immediate consequence of the fact that $U$ is closed.
5.1.7. If $p \in X$, then $p \in X \cap U$.

Let $\operatorname{cl}_{Q}(Y) \cap \operatorname{cl}_{Q}(X \cap U)=\{s\}$ and $\operatorname{cl}_{Q}(Y) \cap \operatorname{cl}_{Q}(X \cap V)=\{t\}$. Neither $s$ nor $t$ is in $X$, so

$$
|X \cap U| \leq 6
$$

Hence $|X \cap V| \leq|X \cap U|-1 \leq 5$. Recall that $|X| \geq 9$. As $|X \cap U| \geq|X \cap V|$, it follows that $|X \cap U| \geq 5$. Hence
5.1.8. $|X \cap U| \in\{5,6\}$.

Call a triangle of $M$ special if it contains an element of $X \cap U$, an element of $X \cap V$, and an element of $P$. Construct a bipartite graph $H$ with vertex classes $X \cap U$ and $X \cap V$ with $u v$ being an edge, where $u \in X \cap U$ and $v \in X \cap V$, precisely when $\{u, v\}$ is contained in a special triangle. Clearly

$$
\begin{equation*}
\sum_{u \in X \cap U} d_{H}(u)=\sum_{v \in X \cap V} d_{H}(v) . \tag{1}
\end{equation*}
$$

Next we show the following.
5.1.9. Every vertex $x$ of $V(H)-\{p\}$ has its degree in $\{1,2\}$.

Let $\left\{X^{\prime}, X^{\prime \prime}\right\}=\{X \cap U, X \cap V\}$ and take $x \in X^{\prime}$ such that $x \neq p$. Let $x^{\prime \prime}$ be the element of $\operatorname{cl}_{Q}\left(X^{\prime \prime}\right) \cap P$. Thus $x^{\prime \prime} \in\{s, t\}$. Clearly $d_{H}(x) \leq 3$. Assume $d_{H}(x)=3$. Then $\operatorname{cl}_{Q}(Y \cup x)$ contains $x^{\prime}$, at least three distinct elements of $X^{\prime \prime}$, and $x^{\prime \prime}$. Thus $\mathrm{cl}_{Q}(Y \cup x)$ contains $X^{\prime \prime}$. Hence $E(M)-\operatorname{cl}_{M}(Y \cup x)$ contains at most five elements of $M$; a contradiction to the fact that every cocircuit of $M$ has at least six elements. Thus $d_{H}(x)<3$.

Next suppose that $d_{H}(x)=0$. Then all three triangles containing $x$ are contained in $\operatorname{cl}_{M}\left(X^{\prime}\right)$. Thus $M \mid \mathrm{cl}_{M}\left(X^{\prime}\right) \cong F_{7}$. Hence, for $z \in X^{\prime \prime}-$ $\operatorname{cl}_{M}\left(X^{\prime}\right)$, the three triangles containing $z$ are contained in $\operatorname{cl}_{M}\left(X^{\prime \prime}\right)$. Thus $M \mid \mathrm{cl}_{M}\left(X^{\prime \prime}\right) \cong F_{7}$. Hence $\mathrm{cl}_{M}\left(X^{\prime}\right) \cap \operatorname{cl}_{M}\left(X^{\prime \prime}\right)$ contains a point of $M$ that is in six triangles; a contradiction. Thus 5.1.9 holds.

Now either
(i) $s=t=p$; or
(ii) $s, t$, and $p$ are distinct.

Suppose that (i) holds. Assume first that $p \notin Y$. By 5.1.9, for $W \in$ $\{U, V\}$, every element of $M \mid(X \cap W)$ is in a triangle contained in $X \cap W$. Thus either $M \mid(X \cap W) \cong M\left(K_{4}\right)$ and $\sum_{w \in X \cap W} d_{H}(w)=6$; or $M \mid(X \cap W) \cong$ $M\left(K_{4} \backslash e\right)$ and $\sum_{w \in X \cap W} d_{H}(w)=9$. Since $|X \cap U|>|X \cap V|$, we obtain a contradiction using (1). Thus $p \in Y$.

As $|X \cap U| \in\{5,6\}$ by 5.1.8, we see that $|X \cap U|=5$, otherwise $M \mid((X \cap$ $U) \cup p) \cong F_{7}$, and $d_{H}(x)=0$ for every $x \in X \cap V$; a contradiction to 5.1.9. We deduce that $M \mid((X \cap U) \cup p) \cong M\left(K_{4}\right)$, and $5=\sum_{u \in X \cap U} d_{H}(u)$. Now $p$ is in two triangles in $(X \cap U) \cup p$. Thus, of the three triangles in $\mathrm{cl}_{Q}(X \cap V)$ containing $p$, at most one contains two elements of $X \cap V$. Hence, using 5.1.9, we see that $M \mid \mathrm{cl}_{M}(X \cap V)$ comprises two triangles with a single element,
not $p$, in common. Thus $\sum_{v \in X \cap V} d_{H}(v)=7$; a contradiction to Equation 1 . We conclude that (i) does not hold.

We now know that $s, t$, and $p$ are distinct. We show next that
5.1.10. $p \in X$.

Suppose $p \notin X$. Then $|X \cap U|=5$ so $|X \cap V|=4$. Thus $\sum_{u \in X \cap U} d_{H}(u)$ is five when $s \in Y$ and nine otherwise. As $d_{H}(v)<3$ for each $v \in X \cap V$ by 5.1.9. it follows that $t \in Y$. Then $\sum_{v \in X \cap V} d_{H}(v)$ is eight or seven depending on whether $M \mid(X \cap V)$ is $U_{3,4}$ or $U_{2,3} \oplus U_{1,1}$. Thus, by (1), we have a contradiction. Hence 5.1.10 holds.

Suppose $|X \cap U|=6$. Then $s \notin Y$, otherwise there is an element of $(X \cap U)-p$ with degree zero in $H$; a contradiction to 5.1.9. Then $\sum_{u \in X \cap U} d_{H}(u)=6$. Suppose $t \in Y$. If the line through $\{p, t\}$ contains a third point of $M$, say $q$, then each of the other two lines through $p$ in $\operatorname{cl}_{Q}(X \cap V)$ contains at most one point of $M$. Thus $|X \cap V|=3$ and, as $r(X \cap V)=3$, we see that $\{p, q, t\}$ is the unique triangle in $M \mid \operatorname{cl}_{M}(X \cap V)$ containing $q$. As this triangle is special, it follows that $d_{H}(q)=3$; a contradiction to 5.1.9. Evidently the line through $\{p, t\}$ does not contain a third point of $M$. We deduce that $M \mid \mathrm{c}_{M}(X \cap V)$ comprises two triangles that have one element, not $p$ or $t$, in common. Then $\sum_{v \in X \cap V} d_{H}(v)=5$; a contradiction. We deduce that $t \notin Y$. Then exactly one of the lines in $\operatorname{cl}_{M}(X \cap V)$ through $p$ contains exactly three points. Since no point of $X \cap V$ has degree three in $H$, it follows that $M \mid \mathrm{cl}_{M}(X \cap V)$ comprises two triangles with a point, not $p$, in common. As $p \notin X \cap V$, it follows that $\sum_{v \in X \cap V} d_{H}(v)=7$; a contradiction. We conclude that $|X \cap U| \neq 6$.

It remains to consider the case that $|X \cap U|=5$ and $|X \cap V|=4$. Then $\sum_{u \in X \cap U} d_{H}(u)$ is five or nine depending on whether or not $s$ is in $Y$. From 5.1.10, $p \in X$. Thus $M \mid[(X \cap V) \cup p]$ consists of two three-point lines meeting in a point $z$. If $z=p$, then $\sum_{v \in X \cap V} d_{H}(v)$ is four or eight, depending on whether or not $t$ is in $Y$; a contradiction. Hence $z \neq p$. Thus the third element on the line containing $\{p, t\}$ is in $X$. Again $\sum_{v \in X \cap V} d_{H}(v)$ is seven, if $t \notin Y$, or four, if $t \in Y$; a contradiction to (1). We conclude that 5.1.4 holds and the lemma follows.

It is now straightforward to complete the proof of our main result.
Proof of Theorem 1.1. If $M$ has a 4 -cocircuit, then the result follows by Lemma 4.1. If $M$ has no 4 -cocircuits, then the theorem follows by Lemma 5.1 .

## 6. A (non)-EXTENSION

It is natural to ask whether, for an internally 4 -connected binary matroid $M$ with every element in exactly three triangles, $\operatorname{si}(M / e)$ is internally 4 connected for every element $e$. We now describe an example where this is not the case.

Begin with $K_{3,3}$ having vertex classes $\left\{a_{1}, a_{2}, a_{3}\right\}$ and $\left\{b_{1}, b_{2}, b_{3}\right\}$. Form the graph $G$ by adjoining three new vertices $u, v$, and $w$, each adjacent to all of $a_{1}, a_{2}, a_{3}, b_{1}, b_{2}$, and $b_{3}$ but not to each other. The vertex-edge incidence matrix of $G$ is the matrix $A$ shown below.
$a_{1}$
$a_{2}$
$a_{3}$
$b_{1}$
$b_{2}$
$b_{3}$
$u$
$v$
$w$
$w$$\left(\begin{array}{llllllllllllllllllllllllll}1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0\end{array}\right)\left(\begin{array}{llllllll}0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1\end{array}\right)$

Then $M(G)$ is an internally 4 -connected matroid in which every element is in exactly three triangles. Now adjoin the matrix $B$ to $A$ where $B$ is shown below.

$$
\left(\begin{array}{llllll}
a & b & c & d & e & f \\
1 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1
\end{array}\right)
$$

The matroid $N$ that is represented by $[A \mid B]$ has each element in $M(G)$ in exactly three triangles, and each element of $\{a, b, c, d, e, f\}$ is in exactly two triangles. To see this, observe that $N \mid\{a, b, c, d, e, f\} \cong M\left(K_{4}\right)$. Moreover, no element of $M(G)$ lies on a line with two elements of $\{a, b, c, d, e, f\}$ and it is straightforward to check that no element of $\{a, b, c, d, e, f\}$ is in a triangle with two elements of $M(G)$.

Now take the generalized parallel connection of $M\left(K_{5}\right)$ and $N$ across $\{a, b, c, d, e, f\}$ to get an internally 4 -connected binary matroid $M$ in which every element is in exactly three triangles. Evidently $\operatorname{si}(M / z)$ is not internally 4 -connected for all $z$ in $\{a, b, c, d, e, f\}$.

## References

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