INCOMPATIBLE INTERSECTION PROPERTIES

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Abstract. Let $\mathcal{F} \subset 2^{[n]}$ be a family in which any three sets have non-empty intersection and any two sets have at least 38 elements in common. The nearly best possible bound $|\mathcal{F}| \le 2^{n-2}$ is proved. We believe that 38 can be replaced by 3 and provide a simple-looking conjecture that would imply this.

1. Introduction

Let $[n] := \{1, \dots, n\}$ be the standard n-element set and $2^{[n]}$ its power set. Subsets of $2^{[n]}$ are called *families*.

Definition 1. *For positive integers r, t, where $r \ge 2$, a family $\mathcal{F} \subset 2^{[n]}$ is called r-wise t-intersecting, if $|F_1 \cap \ldots \cap F_r| \ge t$ for all $F_1, \dots, F_r \in \mathcal{F}$.*

In the case $t = 1$, instead of 1-intersecting the term *intersecting* is used. Arguably the simplest result in extremal set theory is the following.

Proposition 2. *If $\mathcal{F} \subset 2^{[n]}$ is 2-wise intersecting then*

$$|\mathcal{F}| \le 2^{n-1}. \tag{1}$$

The above result is a small part of the classical Erdős-Ko-Rado paper [2]. Since the family $\mathcal{F}_0 := \{F \subset 2^{[n]} : 1 \in F\}$ is r-wise intersecting for every $r \ge 2$, (1) is the best possible bound for $r \ge 3$ as well. The family $\mathcal{F}_0$ is usually called trivially intersecting.

Let us call a family *non-trivial* if $\bigcap_{F \in \mathcal{F}} F = \emptyset$. The following result is one of the early gems in extremal set theory.

Theorem 3 (Brace-Daykin [1]). *Suppose that $\mathcal{F} \subset 2^{[n]}$ is r-wise intersecting and non-trivial. Then*

$$|\mathcal{F}| \le \frac{r+2}{2^r} 2^{n-1}. \tag{2}$$

Since $r + 2 < 2^r$ for $r \ge 3$ and $(r + 2)2^{-r} \to 0$ as r tends to infinity, (2) is much stronger than (1). The following example shows that it is best possible for $n \ge r + 1$.

$$\mathcal{B}(1, r) := \{B \subset [n] : |B \cap [r + 1] \ge r\}.$$

Let us mention that for $n \le r$ there is no non-trivial r-wise intersecting family. For a simple proof of (2) cf. [4].

Definition 4. *For a family $\mathcal{F} \subset 2^{[n]}$ and an arbitrary integer $r \ge 2$ let $t(\mathcal{F}, r)$ denote the largest integer t such that $|F_1 \cap \ldots \cap F_r| \ge t$ for all $F_1, \dots, F_r \in \mathcal{F}$.*

One can easily check that $t(\mathcal{F}, r + 1) \le \max\{0, t(\mathcal{F}, r) - 1\}$ for non-trivial families. Therefore, $t(\mathcal{F}, 2) \ge 2$ for every non-trivial 3-wise intersecting family $\mathcal{F}$. On the other hand, we believe that assuming $t(\mathcal{F}, 2) \ge 3$ leads to stronger bounds on the size of the family.

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Conjecture 1. *Suppose that $\mathcal{F} \subset 2^{[n]}$ is both 3-wise 1-intersecting and 2-wise 3-intersecting. Then*

$$|\mathcal{F}| \le 2^{n-2}. \tag{3}$$

If $\mathcal{F}$ is trivial, e.g., if $1 \in F$ for all $F \in \mathcal{F}$, then the 2-wise 3-intersecting property implies that $\mathcal{F}(1) := \{F \setminus \{1\} : F \in \mathcal{F}\} \subset 2^{[2,n]}$ is 2-wise intersecting. Applying (1) to $\mathcal{F}(1)$ yields

$$|\mathcal{F}| = |\mathcal{F}(1)| \le 2^{n-2}.$$

This shows that in proving (3) one might assume that $\mathcal{F}$ is non-trivial. From (2) we obtain $|\mathcal{F}| \le \frac{5}{16}2^n = \frac{5}{4} \cdot 2^{n-2}$, which falls short of (3).

Example. Let $t \ge 2$ be a fixed integer and suppose for convenience that $n > t$, $n + t$ is odd. Define

$$\mathcal{T}(n,t) := \Big\{\{1\} \cup T : T \subset [2,n], |T| \ge \frac{n-1+t}{2}\Big\}.$$

Claim 5. *The following hold:*

(i) $\mathcal{T}(n,t)$ *is 3-wise intersecting and 2-wise $(t+1)$-intersecting.*
(ii) $|\mathcal{T}(n,t)| = \sum_{i \ge \frac{n-1+t}{2}} \binom{n-1}{i} = (1-o(1))2^{n-2}$ *as $n \to \infty$.*

We leave the easy proof to the reader. This claim shows that even for t large one cannot expect something much smaller than 2^{n-2}.

We were unable to prove Conjecture 1, but established (3) with 3 replaced by 38.

Theorem 6. *Suppose that $\mathcal{F} \subset 2^{[n]}$ is 3-wise intersecting and 2-wise 38-intersecting. Then (3) holds.*

A family $\mathcal{F} \subset 2^{[n]}$ is called an *up-set* if for all $F \in \mathcal{F}$, $F \subset H \subset [n]$ implies $H \in \mathcal{F}$. Every family generates a unique up-set containing it. Moreover, if it is r-wise t-intersecting then the same holds for the corresponding up-set. Therefore, unless otherwise stated, we shall tacitly assume that the families we consider are up-sets.

Let us mention that the Katona Theorem [7] determines the maximum size $k(n,t)$ of 2-wise t-intersecting families for all $n \ge t \ge 1$. The construction is analogous to $\mathcal{T}(n,t)$ and shows

$$k(n,t) = (1-o(1))2^{n-1} \text{ for } t \text{ fixed and } n \to \infty.$$

That is, for each of the two intersecting properties from Theorem 6, we have a lower bound of the form $(1+o(1))2^{n-1}$ for the largest size of the family satisfying the property. By the lemma of Kleitman [8], two up-sets $\mathcal{F}_1, \mathcal{F}_2 \subset 2^n$ of sizes $2^{n-\alpha_1}, 2^{n-\alpha_2}$, respectively, satisfy $|\mathcal{F}_1 \cap \mathcal{F}_2| \ge 2^{n-\alpha_1-\alpha_2}$. This immediately gives us a lower bound of $(1+o(1))2^{n-2}$ for the largest size of the family satisfying the conditions of Theorem 6. Thus, one may say that, in a sense, 3-wise intersecting and 2-wise t-intersecting properties are as incompatible for large families as any two monotone increasing properties may be.

For a family $\mathcal{F}$, let $\partial(\mathcal{F})$ be its *immediate shadow*:

$$\partial\mathcal{F} := \{G : \exists F \in \mathcal{F}, G \subset F, |F \setminus G| = 1\}.$$

Define also $\sigma(\mathcal{F}) := \mathcal{F} \cup \partial\mathcal{F}$.

It is important to note that $[n] \in \mathcal{F}$ for every non-empty up-set $\mathcal{F} \subset 2^{[n]}$. This implies $\binom{[n]}{n-1} \subset \partial\mathcal{F}$ whence both $\partial\mathcal{F}$ and $\sigma(\mathcal{F})$ are non-trivial.

Conjecture 2. *Suppose that $\mathcal{F} \subset 2^{[n]}$ is 3-wise intersecting. Then*

$$|\sigma(\mathcal{F})| \ge 2|\mathcal{F}|. \tag{4}$$

In the next section we show that Conjecture 2 implies Conjecture 1.

2. Preliminaries

There is a natural partial order $A \prec B$ defined for sets of the same size. Suppose that $A = \{a_1, \dots, a_p\}$, $B = \{b_1, \dots, b_p\}$ are distinct sets with $a_1 < \dots < a_p$ and $b_1 < \dots < b_p$. We write $A \prec B$ iff $a_i \le b_i$ for all $1 \le i \le p$.

Definition 7. *The family $\mathcal{F} \subset 2^{[n]}$ is called initial if $A \prec B$ and $B \in \mathcal{F}$ imply $A \in \mathcal{F}$.*

Extend the above partial order to $2^{[n]}$ by putting $A \prec B$ if $B \subset A$. We call this order the *shifting/inclusion order*. Erdős, Ko and Rado [2] defined an operation on families of sets (called *shifting*) that maintains the r-wise t-intersecting property (cf. [4] for the proof). Since repeated application of shifting always produces an initial family, we shall always assume that the families in question are initial.

Proposition 8 ([3]). *If $\mathcal{F} \subset 2^{[n]}$ is 3-wise t-intersecting and initial, then, for every $F \in \mathcal{F}$, there exists an integer $\ell \ge 0$ such that*

$$|F \cap [3\ell + t]| \ge 2\ell + t. \tag{5}$$

The following result is proven in [5].

Theorem 9 ([5]). *Suppose that $\mathcal{F} \subset 2^{[n]}$ is such that for any $F \in \mathcal{F}$ we have $|F \cap [3\ell+2]| \ge 2\ell+2$ for some $\ell \ge 0$. Then*

$$|\partial(\mathcal{F})| \ge 2|\mathcal{F}|. \tag{6}$$

Corollary 10. *Suppose that $\emptyset \ne \mathcal{F} \subset 2^{[n]}$ is 3-wise 2-intersecting. Then $\sigma(\mathcal{F}) > 2|\mathcal{F}|$.*

Proof. Proposition 8 implies that $\mathcal{F}$ satisfies the conditions of Theorem 9. Now the statement follows from (6) and $[n] \notin \partial\mathcal{F}$. □

Definition 11. *Suppose that $\mathcal{A}, \mathcal{B}, \mathcal{C} \subset 2^{[n]}$ satisfy $|A \cap B \cap C| \ge t$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $C \in \mathcal{C}$. Then we say that $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are cross-t-intersecting.*

Let us recall the following recent result.

Theorem 12 ([6]). *Suppose that $\mathcal{A}, \mathcal{B}, \mathcal{C} \subset 2^{[n]}$ are non-trivial and cross-1-intersecting. Then*

$$|\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| < 2^n. \tag{7}$$

The reason for our interest in $\partial\mathcal{F}$ and $\sigma(\mathcal{F})$ is explained by the following simple statement.

Observation 13. *If $\mathcal{A}, \mathcal{B}, \mathcal{C} \subset 2^{[n]}$ are cross-t-intersecting, $t \ge 2$, then $\sigma(\mathcal{A}), \mathcal{B}, \mathcal{C}$ are cross-$(t-1)$-intersecting.*

We finish this section with a short proof of the fact that Conjecture 2 implies Conjecture 1.

Conjecture 2 implies Conjecture 1. Consider $\mathcal{F}$ as in the statement of Conjecture 1. Then $\sigma(\mathcal{F})$ is 2-wise intersecting, and thus $|\sigma(\mathcal{F})| \le 2^{n-1}$. Therefore, by Conjecture 2, $|\mathcal{F}| \le \frac{1}{2}|\sigma(\mathcal{F})| \le 2^{n-2}$. □

3. Proof of Theorem 6

Consider a shifted family $\mathcal{F} \subset 2^{[n]}$ as in the statement of Theorem 6. For $S \subset [s]$, define

$$\mathcal{F}(S, [s]) := \{F \setminus S : F \in \mathcal{F}, F \cap [s] = S\}.$$

We consider two cases depending on whether the subsets not containing 1 have a strong or weak presence in $\mathcal{F}$. As a criterion, let us fix the set

$$H_0 := \{[2, 8] \cup \{10, 11, 13, 14, 16, 17 \dots\} \cup [n].$$

Note that for all t, $3 \le t \le n/3$,

$$|H_0 \cap [3t]| = 2t+1. \tag{8}$$

Case 1. $H_0 \in \mathcal{F}$**.** Put $w := 54$. We are going to partition $\mathcal{F}$ according to $F \cap [w]$. Set $\tilde{H} := H_0 \cap [w]$ and define

$$\mathcal{G}_0 := \{G \subset [w] : |G| \ge 33, G \ne \tilde{H}\}.$$

It is easy to verify by computer-aided computation that

$$|\mathcal{G}_0| < \frac{1}{13} 2^w. \tag{9}$$

Define $T_0 := [w+1, w+7] \cup \{w+9, w+10, w+12, w+13, \dots\} \cap [n]$. Now we can define

$$\mathcal{G}_1 := \{G \subset [w] : G \notin \mathcal{G}_0, T_0 \notin \mathcal{F}(G, [w])\}.$$

Here we invoke an old result of the first author [3, Lemma 2] which asserts that for any $G \in \mathcal{G}_1$

$$|\mathcal{F}(G, [w])| < \Big(\frac{\sqrt{5}-1}{2}\Big)^8 2^{n-w} < \frac{1}{46} 2^{n-w}. \tag{10}$$

Finally, set $\mathcal{G}_2 := 2^{[w]} \setminus (\mathcal{G}_0 \cup \mathcal{G}_1)$. By construction, the 37-element set $\tilde{H}$ is in $\mathcal{G}_2$. Below we are going to prove the following.

Proposition 14. *$\mathcal{G}_2$ is 3-wise intersecting.*

Let us first show how Proposition 14 implies $|\mathcal{F}| < 2^{n-2}$. First note that the pairwise 38-intersecting property and $|G| \le 37$ for all $G \in \mathcal{G}_2$ imply that for any $G \in \mathcal{G}_2$ the family $\mathcal{F}(G, [w])$ is 2-wise intersecting. Consequently, $|\mathcal{F}(G, [w])| \le \frac{1}{2} 2^{n-w}$.

Partition $\mathcal{F}$ according to $F \cap [w]$: $\mathcal{F}_i := \{F \in \mathcal{F} : F \cap [w] \in \mathcal{G}_i\}$. We have

$$|\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}_1| + |\mathcal{F}_2| = |\mathcal{G}_0| \cdot 2^{n-w} + |\mathcal{G}_1| \cdot \frac{1}{46} 2^{n-w} + |\mathcal{G}_2| \cdot \frac{1}{2} 2^{n-w}. \tag{11}$$

By $\tilde{H} \in \mathcal{G}_2$ and Proposition 14, we may apply the Brace–Daykin Theorem and infer

$$|\mathcal{G}_2| \le \frac{5}{16} 2^{n-w}. \tag{12}$$

Since the coefficient in front of $|\mathcal{G}_1|$ is the smallest, we get an upper bound for the RHS for (11) by making $|\mathcal{G}_0| = \frac{1}{13} 2^w$, $|\mathcal{G}_2| = \frac{5}{16} 2^w$ and $|\mathcal{G}_1| = \big(1 - \frac{1}{13} - \frac{5}{16}\big) 2^w$. We obtain

$$|\mathcal{F}| \le \Big(\frac{1}{13} + \frac{5}{32} + \big(1 - \frac{1}{13} - \frac{5}{16}\big)\frac{1}{46}\Big) 2^n < 2^{n-2},$$

as desired.

Proof of Proposition 14. Take first $F, G, H \in \mathcal{G}_2 \setminus \{H_0\}$ and suppose that $F \cap G \cap H = \emptyset$. By definition, $T_0 \in \mathcal{F}(S, [w])$ for $S = F, G$ and H. Using shiftedness, we can obtain that

$$T' := [w+1, w+7] \cup \{w+8, w+10, w+11, w+13, \dots\} \in \mathcal{F}(G, [w]) \quad \text{and}$$
$$T'' := [w+1, w+7] \cup \{w+8, w+9, w+11, w+12, \dots\} \in \mathcal{F}(H, [w]).$$

The intersection of T', T'' and T_0 is $[w+1, w+7]$. Since $|F| + |G| + |H| \le 3 \cdot 33 = 99 < 2 \cdot 54 - 7$, for each $i \in [7]$ we can replace $w+i$ with an element $[w]$ in one of $F \cup T_0, G \cup T', H \cup T''$ and strictly decrease the common intersection of the three sets. Repeating it for each $i \in [7]$, by shiftedness we get that there are three sets in $\mathcal{F}$ that have empty common intersection, a contradiction.

Now suppose that $H = \tilde{H} = H_0 \cap [w]$. Then $\mathcal{F}(H, [w])$ contains $H' := H \cap [w+1, n] = \{w+1, w+2, w+4, w+5, w+7, w+8, \dots\}$. Taking $T_0 \in \mathcal{F}(F, [w])$ and $T'' \in \mathcal{F}(G, [w])$, respectively, we get that $T'' \cap T_0 \cap H' = \{w+1, w+2, w+4, w+5, w+7\}$. To arrive at the same contradiction, we shift these 5 elements into $[w]$, decreasing the intersection of $T_0 \cup F, T'' \cup G$ and H_0 after each shift. Since $|F| + |G| + |\tilde{H}| \leq 33 + 33 + 37 = 2 \cdot 54 - 5$, this is possible. $\square$

Case 2. $H_0 \notin \mathcal{F}$. This condition implies that, for all $S \subset [2,7]$ and $F \in \mathcal{F}(S,[7])$, there exists ℓ such that

$$|F \cap [8, 3\ell + 9]| \geq 2\ell + 2. \tag{13}$$

Indeed, it is true for $S = [2,7]$ since $H_0 \cap [8,n]$ is the unique maximal set in the shifting/inclusion order that does not have this property, and for $S' \subset S$ we have $\mathcal{F}(S,[7]) \supset \mathcal{F}(S',[7])$. The equations (13) and (6), in turn, imply that, for each $S \subset [2,7]$, we have $|\partial(\mathcal{F}(S,[7]))| \geq 2|\mathcal{F}(S,[7])|$.

For a two-element set $\{x_i, y_i\}$, let us consider the following four ordered triplets:

$$\begin{array}{llll}
(\emptyset, & \{x_i\}, & \{x_i, y_i\}), \\
(\{x_i\}, & \{y_i\}, & \{y_i\} &), \\
(\{y_i\}, & \{x_i, y_i\}, & \emptyset &), \\
(\{x_i, y_i\}, & \emptyset, & \{x_i\} &).
\end{array}$$

Note that all four subsets of $\{x_i, y_i\}$ occur once in each position (column). Also, the sum of sizes of the subsets in each triplet is always 3 and the intersection of the subsets is empty. Suppose that $\{x_1, x_2, x_3, y_1, y_2, y_3\} = [2,7]$ and let (A_i, B_i, C_i), $i \in [3]$, be some of the above triples. We associate with them a *big triple*

$$(\{1\} \cup A_1 \cup A_2 \cup A_3, \{1\} \cup B_1 \cup B_2 \cup B_3, C_1 \cup C_2 \cup C_3).$$

Let us note that, for each big triple, the sum of the sizes of the subsets in it is 11. Altogether, we constructed $4 \times 4 \times 4 = 64$ triples, where each subset of [7] containing 1 appears exactly once in the first and second position and each subset of $[2,7]$ appears exactly once in the third position. Moreover, the intersection of the three subsets is empty for each triple.

For a big triple (A, B, C) we consider the three families $\mathcal{F}(D) := \mathcal{F}(D,[7])$, where $D = A, B$, or C. Recall that $\sigma(\mathcal{F}) = \mathcal{F} \cup \partial\mathcal{F}$.

Proposition 15. *The families $\mathcal{F}(A)$, $\mathcal{F}(B)$, $\mathcal{F}(C)$ are cross 3-wise 4-intersecting. The families $\sigma(\mathcal{F}(A))$, $\sigma(\mathcal{F}(B))$, $\sigma(\mathcal{F}(C))$ are cross 3-wise intersecting.*

Proof. For each triple (A, B, C), either there are three elements in $[2,7]$ that are contained in only one set among A, B, C, or one such element and one element which is not contained in $A \cup B \cup C$. In either case, if $F \in \mathcal{F}(A)$, $G \in \mathcal{F}(B)$, $H \in \mathcal{F}(C)$ satisfy $|F \cap G \cap H| \leq 3$, then we can do (at most) three shifts and replace each element that belongs to the intersection in one set with one of the "low-degree" elements, thus not creating new common intersection. By shiftedness, we will get F', G', H' that belong to $\mathcal{F}$ but whose common intersection is empty.

The second statement obviously follows from the first one. $\square$

Now, if $\mathcal{F}(D)$ is non-empty then $\sigma(\mathcal{F}(D)$ is non-trivial, where $D = A, B, C$. In that case, by (7)

$$|\sigma(\mathcal{F}(A)| + |\sigma(\mathcal{F}(B))| + |\sigma(\mathcal{F}(C)| \leq 2^{n-7}. \tag{14}$$

On the other hand, if one of the families above is empty, the sum of cardinalities of the two remaining ones is at most 2^{n-7} since they are cross-intersecting (due to the 2-wise 38-intersecting property). Note that $1 \notin C$ implies that $\mathcal{F}(C)$ is 3-wise 2-intersecting. In view of Corollary 10, we infer $|\sigma(\mathcal{F}(C))| \geq 2|\mathcal{F}(C)|$. Consequently, in all cases we have

$$|\mathcal{F}(A)| + |\mathcal{F}(B)| + 2|\mathcal{F}(C)| \leq 2^{n-7}.$$

Summing over the 64 big triples gives $2|\mathcal{F}| = 2\sum_{D \subset [7]} |\mathcal{F}(D)| \leq 64 \cdot 2^{n-7}$, that is, $|\mathcal{F}| \leq 2^{n-2}$.

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