# LARGE CLIQUES IN HYPERGRAPHS WITH FORBIDDEN SUBSTRUCTURES 

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#### Abstract

A result due to Gyárfás, Hubenko, and Solymosi (answering a question of Erdős) states that if a graph $G$ on $n$ vertices does not contain $K_{2,2}$ as an induced subgraph yet has at least $c\binom{n}{2}$ edges, then $G$ has a complete subgraph on at least $\frac{c^{2}}{10} n$ vertices. In this paper we suggest a "higher-dimensional" analogue of the notion of an induced $K_{2,2}$ which allows us to generalize their result to $k$ uniform hypergraphs. Our result also has an interesting consequence in discrete geometry. In particular, it implies that the fractional Helly theorem can be derived as a purely combinatorial consequence of the colorful Helly theorem.


## 1. Introduction

Among the classical problems in extremal graph theory are the Turán type extremal problems. They ask for the maximum number of edges ex $(n, H)$ in a graph on $n$ vertices provided it does not contain some fixed graph $H$ as a subgraph. In the case when $H=K_{m}$, the complete graph on $m$ vertices, the answer is given by Turan's theorem [22], which also characterizes the extremal graphs which obtain the maximum ex $\left(n, K_{m}\right)$ for all $n$ and $m$. More generally, if the chromatic number $\chi(H) \geq 3$, we have $\operatorname{ex}(n, H)=\left(1-\frac{1}{\chi(H)-1}\right)\binom{n}{2}+o\left(n^{2}\right)$. This is the fundamental Erdős-Stone-Simonovits theorem [7, 8]. While their result also holds for bipartite $H$, it only tells us that ex $(n, H)=o\left(n^{2}\right)$, which is less satisfactory since stronger estimates exist. For instance, the Kővari-Sós-Turán theorem [19] states that for the complete bipartite graph $K_{s, t}$ we have ex $\left(n, K_{s, t}\right)<c_{s, t} n^{2-\frac{1}{s}}$. There are many long-standing unsolved questions in this area and we refer the reader to the extensive survey [10] for more information and further references.

Recently, Loh, Tait, Timmons, and Zhou [20] introduced a new and natural line of investigations related to the Turán type problems. For a pair of graphs $H$ and $F$, they proposed the problem of determining the maximum number of edges in a graph on $n$ vertices, subject to the condition that we simultaneously forbid $H$ as a subgraph and $F$ as an induced subgraph. One of their main results [20, Theorem 1.1] addresses the case when $H=K_{r}$ and $F=K_{s, t}$, where they obtain the same asymptotic upper bound as in the Kővari-Sós-Turán theorem (with a different constant, now depending on $r, s$, and $t$ ). The case $s=t=2$ is interesting in its own right, and is closely related to the following result due to Gyárfás, Hubenko, and Solymosi [11].

Theorem (Gyárfás-Hubenko-Solymosi). Let $G$ be a graph on $n$ vertices and at least $c\binom{n}{2}$ edges. If $G$ does not contain $K_{2,2}$ as an induced subgraph, then $\omega(G) \geq$ $\frac{c^{2}}{10} n$.

Here $\omega(G)$ denotes the maximum number of vertices in a clique contained in $G$. In the aforementioned paper by Loh et al., they also give an extension the Gyárfás-Hubenko-Solymosi Theorem to the case when the forbidden induced subgraph is $K_{2, t}$ [20, Theorem 1.3].

The goal of this paper is to extend the Gyárfás-Hubenko-Solymosi theorem in another direction, more specifically, for $k$-uniform hypergraphs. Throughout the paper we use the following standard notation and terminology. For a positive integer $k$ we let $[k]$ denote the set $\{1, \ldots, k\}$. For a finite set $S$ we let $\binom{S}{k}$ denote the set of all $k$-tuples (i.e. $k$-element subsets) of $S$. A $k$-uniform hypergraph $H=$ $(V, E)$ consists of a finite set of vertices $V$ and a set of edges $E \subset\binom{V}{k}$. A subset $S \subset V$ forms a clique in $H$ if $\binom{S}{k} \subset E$, and we let $\omega(H)$ denote the maximum number of vertices in a clique in $H$.

In order to avoid using ceiling and floor functions in calculations, we extend the binomial coefficient as the continuous convex function

$$
\binom{x}{k}= \begin{cases}\frac{x(x-1) \cdots(x-k+1)}{k!} & x \geq k-1 \\ 0 & x<k-1\end{cases}
$$

Results. We start by giving a new proof of the Gyárfás-Hubenko-Solymosi theorem which has the advantage of producing a quantitative improvement.
Theorem 1.1. Let $G$ be a graph on $n$ vertices and at least $\alpha\binom{n}{2}$ edges. If $G$ does not contain $K_{2,2}$ as an induced subgraph, then $\omega(G) \geq(1-\sqrt{1-\alpha})^{2} n$.

Remark. It is interesting to note that if $G$ is a chordal graph on $n$ vertices and $\alpha\binom{n}{2}$ edges, then $\omega(G) \geq(1-\sqrt{1-\alpha}) n$, which is best possible. (This is a result due to Katchalski and Abbot [1], and was also shown in [11].) The appearance of the factor $(1-\sqrt{1-\alpha})$ in our new bound seems to be a coincidence, and the problem of determining the optimal linear factor in the general case of no induced $K_{2,2}$, even for specific values of $\alpha$, remains an open, although some progress has been made in [12].

The main advantage of our new proof of the Gyárfás-Hubenko-Solymosi theorem is that it can be extended to $k$-uniform hypergraphs. This is interesting because it has implications in discrete geometry and combinatorial topology, more specifically, with respect to the colorful and fractional versions of Helly's theorem [4, 5, 9, 14, 15, 16, 17]. This connection will be discussed further in Section 4.
Let $H=(V, E)$ be a $k$-uniform hypergraph. We call the set $M=\binom{V}{k} \backslash E$ the set of missing edges of $H$. The following definition extends the notion of an induced $K_{2,2}$ in a graph in several ways.
Definition. Let $H$ be a $k$-uniform hypergraph and $m \geq k$ an integer. A family $\left\{\tau_{1}, \ldots, \tau_{m}\right\} \subset M$ is called a complete $m$-tuple of missing edges if
(1) $\tau_{i} \cap \tau_{j}=\emptyset$ for all $i \neq j$, and
(2) $\left\{t_{1}, \ldots, t_{m}\right\}$ is a clique in $H$ for all $t_{i} \in \tau_{i}$ and all $i \in[m]$.

Remark. Note that for $m=k$, condition (2) simply says that $\left\{t_{1}, \ldots, t_{k}\right\}$ is an edge in $H$ for every choice $t_{1} \in \tau_{1}, \ldots, t_{k} \in \tau_{k}$. In the case of graphs ( $k=2$ ), a complete $m$-tuple of missing edges is equivalent to an induced $K_{2}(m)$, that is, the complete multipartite graph on $m$ vertex classes each of size two. However, for $k>2$, a complete $m$-tuple of missing edges should not be thought of as an induced hypergraph since the definition only speaks about edges containing at most one vertex from each $\tau_{i}$.
For a $k$-uniform hypergraph $H$ and $m \geq k$, let $c_{m}(H)$ denote the number of cliques on $m$ vertices in $H$. In particular, $c_{k}(H)$ denotes the number of edges in $H$. We may now state our main result.

Theorem 1.2. For any $m \geq k \geq 2$ and $\alpha \in(0,1)$, there exists a constant $\beta=$ $\beta(\alpha, k, m)>0$ with the following property: Let $H$ be a $k$-uniform hypergraph on $n$ vertices and $c_{m}(H) \geq \alpha\binom{n}{m}$. If $H$ does not contain a complete $m$-tuple of missing edges, then $\omega(H) \geq \beta$ n.
Outline of paper. In section 2 we give the new proof of the Gyárfás-HubenkoSolymosi theorem. This proof contains all the main ideas needed for establishing Theorem 1.2, which will be done in section 3. Finally, in section 4 we review the (topological) colorful Helly theorem and the (topological) fractional Helly theorem and show how these are related via Theorem 1.2.

## 2. Improving the Gyárfás-Hubenko-Solymosi theorem

Here we prove Theorem 1.1. Let $G=(V, E)$ be a graph with $|V|=n$ and $|E| \geq \alpha\binom{n}{2}$. Recall that the missing edges are the elements of $M=\binom{V}{2} \backslash E$.

Let us suppose $\omega(G) \leq \beta n$, where $\beta=(1-\sqrt{1-\alpha})^{2}$, and that $G$ does not contain $K_{2,2}$ as an induced subgraph. Notice that for our choice of $\beta$, we have

$$
(\alpha-\beta)=2 \sqrt{1-\alpha} \sqrt{\beta}
$$

We start by fixing a vertex $v$ and making some observations about its neighbor$\operatorname{hood} N_{v}=\{u \in V: u v \in E\}$ and the induced subgraph $G_{v}=G\left[N_{v}\right]$. The assumption that $G$ does not contain an induced $K_{2,2}$ implies that for every pair of (vertex) disjoint missing edges $\left\{\bar{e}_{1}, \bar{e}_{2}\right\}$ in $G_{v}$ there exists another missing edge $\bar{e}_{3}$ which has one vertex in common with $\bar{e}_{1}$ and one with $\bar{e}_{2}$. Letting $m_{v}$ denote the total number of missing edges in $G_{v}$, and $\mu_{v}$ denote the maximum number of pairwise disjoint missing edges in $G_{v}$, we obtain

$$
m_{v} \geq \mu_{v}+\binom{\mu_{v}}{2}=\binom{\mu_{v}+1}{2}
$$

Note that the vertices in $G_{v}$ not covered by a maximal mathcing of missing edges must form a clique and that $\omega\left(G_{v}\right) \leq \beta n-1$, which implies

$$
\mu_{v}+1 \geq \frac{\left|N_{v}\right|-\beta n+3}{2}
$$

Summing over all $v \in V$ and using $\sum \frac{\left|N_{v}\right|}{2}=|E| \geq \alpha\binom{n}{2}$, we have

$$
\begin{aligned}
\sum\left(\mu_{v}+1\right) & \geq \alpha\binom{n}{2}-\beta \frac{n^{2}}{2}+\frac{3 n}{2} \\
& =(\alpha-\beta) \frac{n^{2}}{2}+(3-\alpha) \frac{n}{2} \\
& \geq \sqrt{1-\alpha} \sqrt{\beta} n^{2}+n
\end{aligned}
$$

By Jensen's inequality, we get

$$
\begin{aligned}
\sum m_{v} & \geq \sum\binom{\mu_{v}+1}{2} \geq n\binom{\frac{1}{n} \sum\left(\mu_{v}+1\right)}{2} \\
& \geq n\binom{\sqrt{1-\alpha} \sqrt{\beta} n+1}{2} \\
& \geq(1-\alpha)\binom{n}{2} \beta n .
\end{aligned}
$$

Since the total number of missing edges in $G$ is at most $(1-\alpha)\binom{n}{2}$, by the pigeon-hole principle, there is a missing edge $\bar{e}$ and a subset of vertices $S \subset V$, with $|S| \geq \beta n$, such that $\bar{e}$ is in the neigborhood of every vertex in $S$. There can not be a missing edge contained in $S$, since together with $\bar{e}$ this would form an induced $K_{2,2}$. Therefore $S$ forms a clique which implies $\omega(G) \geq \beta n$.

## 3. Extending to hypergraphs

We start by generalizing the two key steps from the proof in the previous section. The case $m=k$ of the following lemma was used implicitly in the proof of [18, Theorem 2.2].
Lemma 3.1. Let $H=(V, E)$ be a $k$-uniform hypergraph and let $m \geq k$. If $H$ does not contain a complete $m$-tuple of missing edges, then any subset $S \subset V$ contains at least $\binom{m}{k}^{-1}\binom{\frac{1}{k}(|S|-\omega(H))}{k}$ missing edges of $H$.
Proof. We may assume $|S|>\omega(H)$, otherwise there is nothing to prove, so therefore $S$ contains at least one missing edge. Let $\tau_{1}, \ldots, \tau_{t}$ be a maximal matching of missing edges contained in $S$. Since $S \backslash\left(\tau_{1} \cup \cdots \cup \tau_{t}\right)$ contains no missing edges, we have

$$
k t=\left|\tau_{1} \cup \cdots \cup \tau_{t}\right| \geq|S|-\omega(H) .
$$

Using the hypothesis that $H$ has no complete $m$-tuple of missing edges, it follows that for every $I \in\binom{[t]}{m}$ there is $J \in\binom{I}{k}$ and a missing edge $\tau \in M$ such that $\left|\tau \cap \tau_{j}\right|=1$ for all $j \in J$. Since this particular missing edge can appear in this way for at most $\binom{t-k}{m-k}$ distinct $m$-tuples of $[t]$, it follows that $S$ contains at least

$$
\frac{\binom{t}{m}}{\binom{t-k}{m-k}}=\frac{\binom{t}{k}}{\binom{m}{k}}
$$

distinct missing edges.

In the proof of Theorem 1.2 we will iteratively build up a set of missing edges (eventually ending up in a complete $m$-tuple of missing edges or a clique). This iterative process is defined by the following.
Lemma 3.2. Let $H=(V, E)$ be a $k$-uniform hypergraph on $n$ vertices with $\omega(H) \leq(c / 2) n$, and suppose $H$ does not contain a complete $m$-tuple of missing edges. The following holds for any $i \geq 2$ and for all sufficiently large $n$ : Given a family $F_{i} \subset\binom{V}{i}$ with $\left|F_{i}\right| \geq c\binom{n}{i}$, there exists a family $F_{i-1} \subset\binom{V}{i-1}$ and a missing edge $\tau$ of $H$ such that
(1) $\left|F_{i-1}\right| \geq\left(\frac{c}{12 k m}\right)^{k}\binom{n}{i-1}$, and
(2) $\sigma \cup\{t\} \in F_{i}$ for all $\sigma \in F_{i-1}$ and all $t \in \tau$.

Proof. For every $\sigma \in\binom{V}{i-1}$ define the set $N_{\sigma}=\left\{x \in V: \sigma \cup\{x\} \in F_{i}\right\}$. We want to lower bound the size of the set

$$
X=\left\{(\sigma, \tau): \sigma \in\binom{V}{i-1}, \tau \in M \cap\binom{N_{\sigma}}{k}\right\} .
$$

By Lemma 3.1 and Jensen's inequality, we get

$$
\begin{aligned}
|X| & \geq\binom{ m}{k}^{-1} \sum_{\sigma \in\left(\begin{array}{l}
V-1
\end{array}\right)}\binom{\frac{1}{k}\left(\left|N_{\sigma}\right|-(c / 2) n\right)}{k} \\
& \geq\binom{ m}{k}^{-1}\binom{n}{i-1}\binom{\binom{n}{i-1}^{-1} \frac{1}{k} \sum_{\sigma \in\binom{V-1}{i-1}}\left(\left|N_{\sigma}\right|-(c / 2) n\right)}{k} .
\end{aligned}
$$

Using $\sum_{\sigma \in\binom{V}{i-1}}\left|N_{\sigma}\right|=i\left|F_{i}\right| \geq i c\binom{n}{i}$ and $\binom{n}{i}>\frac{(n-i)}{i}\binom{n}{i-1}$, we get

$$
|X|>\binom{m}{k}^{-1}\binom{n}{i-1}\binom{\frac{c}{2 k} n-\frac{c i}{k}}{k}
$$

Since the term $\frac{c i}{k}$ is a constant which does not depend on $n$, it follows that for all sufficiently large $n$ (depending only on $c, i$, and $k$ ), we get

$$
|X|>\frac{1}{2}\binom{m}{k}^{-1}\binom{n}{i-1}\binom{\frac{c}{2 k} n}{k} \geq\left(\frac{c}{12 k m}\right)^{k}\binom{n}{i-1}\binom{n}{k}
$$

By averaging, there exists a missing edge $\tau \in M$ such that $\tau \subset N_{\sigma}$ for at least $\frac{|X|}{|M|}$ distinct $\sigma \in\binom{[n]}{i-1}$. The lemma now follows since $|M| \leq\binom{ n}{k}$.

Proof of Theorem 1.2. Let $f(x)=\left(\frac{x}{12 k m}\right)^{k}$. (Note that $0<f(x)<x / 2$ for all $x \in(0,1)$.) Define $\alpha_{0}=\alpha$ and $\alpha_{i}=f\left(\alpha_{i-1}\right)$ for all $i \geq 1$. We will show the theorem holds with $\beta=\beta(\alpha, k, m)=\alpha_{m-1}>0$.

Let $F_{m} \subset\binom{V}{m}$ be the the set of $m$-tuples that form cliques in $H$, and so by hypothesis, we have $\left|F_{m}\right|=c_{m}(H) \geq \alpha_{0}\binom{n}{m}$. Assuming both $\omega(H) \leq \beta n$ and that $H$ does not contain a complete $m$-tuple of missing edges, we can apply Lemma 3.2 iteratively, starting with $F_{m}$, obtaining a family $F_{m-1}$, to which we apply Lemma 3.2 , and so on. Moreover, at each step we pick up a new missing edge.

For every $1 \leq i<m$, we claim that after the $i$ th application of Lemma 3.2 we have obtained a subfamily $F_{m-i} \subset\binom{V}{m-i}$ and pairwise disjoint missing edges $\tau_{1}, \ldots, \tau_{i}$ such that

- $\left|F_{m-i}\right| \geq \alpha_{i}\binom{n}{m-i}$, and
- $\sigma \cup\left\{t_{1}, \ldots, t_{i}\right\} \in F_{m}$ for all $\sigma \in F_{m-i}$ and all $t_{1} \in \tau_{1}, \ldots, t_{i} \in \tau_{i}$.

The claim is true for $i=1$, as this is just the statement of Lemma 3.2. Assuming it is true for some $i$, we now check that it holds for $i+1$.

Applying Lemma 3.2 to $F_{m-i}$, we obtain a family $F_{m-i-1} \subset\binom{V}{m-i-1}$ and a missing edge $\tau_{i+1}$ such that

- $\left|F_{m-i-1}\right| \geq \alpha_{i+1}\binom{n}{m-i-1}$, and
- $\sigma \cup\left\{t_{i+1}\right\} \in F_{m-i}$ for all $\sigma \in F_{m-i-1}$ and all $t_{i+1} \in \tau_{i+1}$.

But our assumption on $F_{m-i}$ therefore implies that $\sigma \cup\left\{t_{1}, \ldots, t_{i+1}\right\} \in F_{m}$ for all $\sigma \in F_{m-i-1}$ and $t_{1} \in \tau_{1}, \ldots, t_{i+1} \in \tau_{i+1}$. Note that this also implies that $\tau_{i+1}$ must be disjoint from every $\tau_{1}, \ldots, \tau_{i}$. This proves the inductive step.

After $m-1$ applications of Lemma 3.2, we end up with a subset $F_{1} \subset V$ and pairwise disjoint missing edges $\tau_{1}, \ldots, \tau_{m-1}$ such that

- $\left|F_{1}\right| \geq \alpha_{m-1} n=\beta n$, and
- $\{t\} \cup\left\{t_{1}, \ldots, t_{m-1}\right\} \in F_{m}$ for all $t \in F_{1}$ and all $t_{1} \in \tau_{1}, \ldots, t_{m-1} \in \tau_{m-1}$.

If $F_{1}$ contains a missing edge $\tau_{m} \in M$, then $\left\{\tau_{1}, \ldots, \tau_{m}\right\}$ would be a complete $m$-tuple of missing edges in $H$. Since we assumed this does not exist, it follows that $F_{1}$ is a clique in $H$, and so $\omega(H) \geq \beta n$.

Remark. In the proof above, Lemma 3.2 was used $m-1$ times, and it follows that for fixed $k$ and $m$ we have $\beta=\Omega\left(\alpha^{k^{m-1}}\right)$. If we consider the optimal function $\beta=\beta(\alpha, k, m)$ for which Theorem 1.2 holds, it is worth noting that $\beta \rightarrow 1$ as $\alpha \rightarrow 1$. This does not follow from our definition of $\beta$ in the proof above, but can be deduced directly from Lemma 3.1 by setting $S=V$. The lemma then tells us that if $\omega(H) \leq(1-\epsilon) n$, then $H$ has at least $\binom{m}{k}^{-1}\binom{\frac{\epsilon}{k} n}{k}$ missing edges. It is easy to show by a simple double-counting argument that this implies $c_{m}(H) \leq(1-\delta)\binom{n}{m}$ for some absolute $\delta>0$.

## 4. Applications

Here we present some applications of Theorem 1.2 related to certain Helly-type theorems and the intersection patterns of convex sets. For more information about these types of results we refer the reader to the surveys [3, 6].

Helly's theorem [14] asserts that if every $d+1$ members of a finite family of convex sets in $\mathbb{R}^{d}$ have a point in common, then there is a point in common to every member of the family. Among the numerous generalizations and extensions of Helly's theorem we focus on two important generalizations. The first one is the Colorful Helly Theorem discovered by Lovász and reported by Bárány in [4].

Theorem (Colorful Helly). Let $F_{1}, \ldots, F_{d+1}$ be finite families of convex sets in $\mathbb{R}^{d}$. Suppose for every choice $K_{1} \in F_{1}, \ldots, K_{d+1} \in F_{d+1}$ we have $\bigcap_{i=1}^{d+1} K_{i} \neq \emptyset$. Then for one of the families $F_{i}$ we have $\bigcap_{K \in F_{i}} K \neq \emptyset$.

Note that we recover Helly's theorem in the case when $F_{1}=\cdots=F_{d+1}$. The second generalization of Helly's theorem we are interested in is the Fractional Helly Theorem due to Katchalski and Liu [1]. (See also [21, Chapter 8].)

Theorem (Fractional Helly). For every $d \geq 1$ and $\alpha \in(0,1)$ there exists a $\beta=$ $\beta(\alpha, d) \in(0,1)$ with the following property: Let $F$ be a family of $n>d+1$ convex sets in $\mathbb{R}^{d}$ and suppose at least $\alpha\binom{n}{d+1}$ of the $(d+1)$-tuples in $F$ have nonempty intersection. Then there exists some $\beta n$ members of $F$ whose intersection is non-empty.

Our first application is a new proof of the Fractional Helly Theorem, which uses the Colorful Helly Theorem and Theorem 1.2.

Proof of the fractional Helly theorem. Define a $(d+1)$-uniform hypergraph $H=$ $(F, E)$ where $E=\left\{\sigma \in\binom{F}{d+1}: \bigcap_{K \in \sigma} K \neq \emptyset\right\}$. By hypothesis, $H$ has at least $\alpha\binom{n}{d+1}$ edges, and by the Colorful Helly Theorem $H$ does not contain a complete $(d+1)$-tuple of missing edges. So by Theorem 1.2 , with $k=m=d+1$, there exists a $\beta>0$ such that $H$ has a clique on $\beta n$ vertices. By Helly's theorem, the members of $F$ contained in this clique have non-empty intersection.

The argument above is general enough to give a proof of a topological generalization of the fractional Helly theorem proved by Kalai [15], and independently by Eckhoff [5] in a slightly restricted setting.

Let $K$ be a finite simplicial complex. For an integer $i \geq 0$, let $f_{i}(K)$ denote the number of $i$-dimensional faces in $K$. We say that $K$ is $d$-Leray if $\tilde{H}_{i}(X)=0$ for all induced subcomplexes $X \subset K$ and all $i \geq d$. (Here $\tilde{H}_{i}(X)$ denotes the $i$-dimensional homology of $X$ with coefficients in $\mathbb{Q}$.)

The following is a consequence of the "upper-bound theorem" for $d$-Leray complexes due to Kalai [15], and implies the Fractional Helly Theorem (via the Nerve theorem, see e.g. [13, Corollary 4G.3]).

Theorem (Topological Fractional Helly). For every $d \geq 1$ and $\alpha \in(0,1)$ there exists $a \beta=\beta(\alpha, d) \in(0,1)$ with the following property: If $K$ is a d-Leray complex with $f_{0}(K)=n$ and $f_{d}(K) \geq \alpha\binom{n}{d+1}$, then $K$ has dimension at least $\beta n-1$.

Kalai's proof of this result relies on his technique of algebraic shifting. (See also [2, Section 6] for other algebraic approaches.) We want to give a proof of the Topological Fractional Helly Theorem using Theorem 1.2, but first we need the following auxiliary result, due to Kalai and Meshulam [16, Theorem 1.6]. (See also [9] for an algebraic generalization.)

Theorem (Topological Colorful Helly). Let X be a d-Leray complex on the vertex set $V$ and let $M$ be a matroidal complex on $V$ such that $M \subset X$. Then there exists a simplex $\tau \in X$ such that $\rho(V \backslash \tau) \leq d$.
(Here $\rho$ denotes the rank function of the matroid $M$.) Let us describe the special case of the Topological Colorful Helly Theorem that we need. Let $V_{1}, \ldots, V_{d+1}$ be distinct finite sets with $\left|V_{1}\right|=\cdots=\left|V_{d+1}\right|=d+1$, and let $V=V_{1} \cup \cdots \cup V_{d+1}$.

Define the simplicial complex $M_{d+1}$ as the join of the $V_{i}$, that is

$$
M_{d+1}=V_{1} * \cdots * V_{d+1}=\left\{\sigma \subset V:\left|\sigma \cap V_{i}\right| \leq 1, \text { for all } i\right\}
$$

Note that $M_{d+1}$ is the matroidal complex of the partition matroid induced by the $V_{i}$, which has rank $d+1$.

Suppose $X$ is a $d$-Leray complex such that $M_{d+1} \subset X$. By the Topological Colorful Helly Theorem, there is face $\tau \in X$ such that $\rho(X \backslash \tau) \leq d$. But this means that for some $i$ we have $V_{i} \subset \tau$, so in particular one of the $V_{i}$ is a face in $X$.

Proof of the Topological Fractional Helly Theorem. Let $H=(V, E)$ be the $(d+$ $1)$-uniform hypergraph where $V$ is the vertex set of $K$ and $E$ is the set of $d$ dimensional faces of $K$. Thus, $H$ has $n$ vertices and at least $\alpha\binom{n}{d+1}$ edges. In order to apply Theorem 1.2, with $k=m=d+1$, we need to show that $H$ does not contain a complete $k$-tuple of missing edges. But this is precisely the special case of the Topological Colorful Helly Theorem we described above.

Now Theorem 1.2 implies that there is a clique in $H$ on at least $\beta n$ vertices, which corresponds to a subcomplex $K^{\prime} \subset K$ on at least $\beta n$ vertices whose $d$ dimensional skeleton is complete. The $d$-Leray property now implies that $K^{\prime}$ is a full simplex.

Remark. It should be noted that Kalai's "upper-bound theorem" actually implies the Topological Fractional Helly Theorem with $\beta=1-(1-\alpha)^{\frac{1}{d+1}}$, which is best possible. Our proof gives a far weaker bound on $\beta$, but this is not surprising since the $d$-Leray property is much stronger than excluding a complete set of missing edges in $H$. It would be interesting to find examples for the hypergraph setting of Theorem 1.2 which give non-trivial upper bounds on $\beta$.

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