# On almost $k$-covers of hypercubes 

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#### Abstract

In this paper, we consider the following problem: what is the minimum number of affine hyperplanes in $\mathbb{R}^{n}$, such that all the vertices of $\{0,1\}^{n} \backslash\{\overrightarrow{0}\}$ are covered at least $k$ times, and $\overrightarrow{0}$ is uncovered? The $k=1$ case is the well-known Alon-Füredi theorem which says a minimum of $n$ affine hyperplanes is required, proved by the Combinatorial Nullstellensatz.

We develop an analogue of the Lubell-Yamamoto-Meshalkin inequality for subset sums, and completely solve the fractional version of this problem, which also provides an asymptotic answer to the integral version for fixed $n$ and $k \rightarrow \infty$. We also use a Punctured Combinatorial Nullstellensatz developed by Ball and Serra, to show that a minimum of $n+3$ affine hyperplanes is needed for $k=3$, and pose a conjecture for arbitrary $k$ and large $n$.


## 1 Introduction

Alon's Combinatorial Nullstellensatz [1] is one of the most powerful algebraic tools in modern combinatorics. Alon and Füredi [2] used this method to prove the following elegant result: any set of affine hyperplanes that covers all the vertices of the $n$-cube $Q^{n}=\{0,1\}^{n}$ but one contains at least $n$ affine hyperplanes. There are many generalizations and analogues of this theorem: for rectangular boxes [2], Desarguesian affine and projective planes [6, 7], quadratic surfaces and Hermitian varieties in $\operatorname{PG}(n, q)$ [4]. The common theme of these results are: in many point-line (point-surface) geometries, to cover all the points except one, more lines are needed than to cover all points.

[^0]In this paper, we consider the following generalization of the Alon-Füredi theorem. Let $f(n, k)$ be the minimum number of affine hyperplanes needed to cover every vertex of $Q^{n}$ at least $k$ times (except for $\overrightarrow{0}=(0, \cdots, 0)$ which is not covered at all). For convenience, from now on we call such a cover an almost $k$-cover of the $n$-cube. The Alon-Füredi theorem gives $f(n, 1)=n$ since the affine hyperplanes $x_{i}=1$, for $i=$ $1, \cdots, n$ covers $Q^{n} \backslash\{\overrightarrow{0}\}$. Their result also leads to $f(n, 2)=n+1$. The lower bound follows from observing that when removing one hyperplane from an almost 2-cover, the remaining hyperplanes form an almost 1 -cover. On the other hand, the $n$ affine hyperplanes $x_{i}=1$, together with $x_{1}+\cdots+x_{n}=1$ cover every vertex of $Q^{n} \backslash\{\overrightarrow{0}\}$ at least twice.

These observations immediately lead to a lower bound $f(n, k) \geqslant n+k-1$ by removing $k-1$ affine hyperplanes, and an upper bound $f(n, k) \leqslant n+\binom{k}{2}$ by considering the following almost $k$-cover: $x_{i}=1$ for $i=1, \cdots, n$, together with $k-t$ copies of $\sum_{i=1}^{n} x_{i}=t$, for $t=1, \cdots, k-1$. In this construction, every binary vector with $t$ 1 -coordinates is covered $t$ times by $\left\{x_{i}=1\right\}$, and $k-t$ times by $x_{1}+\cdots+x_{n}=t$. The total number of hyperplanes is $n+\sum_{t=1}^{k-1}(k-t)=n+\binom{k}{2}$.

Note that for $k=3$, the inequalities above give $n+2 \leqslant f(n, 3) \leqslant n+3$. We used a punctured version of the Combinatorial Nullstellensatz, developed by Ball and Serra [3] to show that the upper bound is tight in this case. We also improve the lower bound for $k \geqslant 4$.

Theorem 1.1. For $n \geqslant 2$,

$$
f(n, 3)=n+3 .
$$

For $k \geqslant 4$ and $n \geqslant 3$,

$$
n+k+1 \leqslant f(n, k) \leqslant n+\binom{k}{2}
$$

Our second result shows that for fixed $n$ and the multiplicity $k \rightarrow \infty$, the aforementioned upper bound $f(n, k) \leqslant n+\binom{k}{2}$ is indeed far from being tight. Indeed $f(n, k) \sim c_{n} k$ when $k \rightarrow \infty$. Note that $f(n, k)$ is the optimum of an integer program. We consider the following linear relaxation of it: we would like to assign to every affine hyperplane $H$ in $\mathbb{R}^{n}$ a non-negative weight $w(H)$, with the constraints

$$
\sum_{\vec{v} \in H} w(H) \geqslant k, \quad \text { for every } \vec{v} \in Q^{n} \backslash\{\overrightarrow{0}\},
$$

and

$$
\sum_{\overrightarrow{0} \in H} w(H)=0
$$

such that $\sum_{H} w(H)$ is minimized. Such an assignment $w$ of weights is called a fractional almost $k$-cover of $Q^{n}$. Denote by $f^{*}(n, k)$ the minimum of $\sum_{H} w(H)$, i.e. the minimum size of a fractional almost $k$-cover. We are able to determine the precise value of $f^{*}(n, k)$ for every value of $n$ and $k$.

Theorem 1.2. For every $n$ and $k$,

$$
f^{*}(n, k)=\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) k .
$$

It implies that for fixed $n$ and $k \rightarrow \infty$,

$$
f(n, k)=\left(1+\frac{1}{2}+\cdots+\frac{1}{n}+o(1)\right) k
$$

which grows linearly in $k$.
As an intermediate step of proving Theorem [1.2, we proved the following theorem, which can be viewed as an analogue of the well-known Lubell-Yamamoto-Meshalkin inequality [5, 8, 9, 10] for subset sums. Moreover the inequality is tight for all non-zero binary vectors $\vec{a}=\left(a_{1}, \cdots, a_{n}\right)$.

Theorem 1.3. Given $n$ real numbers $a_{1}, \cdots, a_{n}$, let

$$
\mathcal{A}=\left\{S: \varnothing \neq S \subset[n], \sum_{i \in S} a_{i}=1\right\} .
$$

Then

$$
\sum_{S \in \mathcal{A}} \frac{1}{|S|\binom{n}{|S|}} \leqslant 1 .
$$

Equivalently, let $\mathcal{A}_{t}=\{S: S \in \mathcal{A},|S|=t\}$, then

$$
\sum_{t=1}^{n} \frac{\left|\mathcal{A}_{t}\right|}{t\binom{n}{t}} \leqslant 1 .
$$

The rest of the paper is organized as follows. In the next section, we resolve the almost 3 -cover case, and show that the answer to the almost 4 -cover problem has only two possible values, thus proving Theorem 1.1. Section 3 contains the proofs of Theorems 1.2 and 1.3. The final section contains some concluding remarks and open problems.

## 2 Almost 3-covers of the $n$-cube

The following Punctured Combinatorial Nullstellensatz was proven by Ball and Serra (Theorem 4.1 in [3]). Let $\mathbb{F}$ be a field and $f$ be a non-zero polynomial in $\mathbb{F}\left[x_{1}, \cdots, x_{n}\right]$. We say $\vec{a}=\left(a_{1}, \cdots, a_{n}\right)$ is a zero of multiplicity $t$ of $f$, if $t$ is the minimum degree of the terms that occur in $f\left(x_{1}+a_{1}, \cdots, x_{n}+a_{n}\right)$.

Lemma 2.1. For $i=1, \cdots, n$, let $D_{i} \subset S_{i} \subset \mathbb{F}$ and $g_{i}=\prod_{s \in S_{i}}\left(x_{i}-s\right)$ and $\ell_{i}=$ $\prod_{d \in D_{i}}\left(x_{i}-d\right)$. If $f$ has a zero of multiplicity at least $t$ at all the common zeros of $g_{1}, \cdots, g_{n}$, except at least one point of $D_{1} \times \cdots \times D_{n}$ where it has a zero of multiplicity
less than $t$, then there are polynomials $h_{\tau}$ satisfying $\operatorname{deg}\left(h_{\tau}\right) \leqslant \operatorname{deg}(f)-\sum_{i \in \tau} \operatorname{deg}\left(g_{i}\right)$, and a non-zero polynomial $u$ satisfying $\operatorname{deg}(u) \leqslant \operatorname{deg}(f)-\sum_{i=1}^{n}\left(\operatorname{deg}\left(g_{i}\right)-\operatorname{deg}\left(\ell_{i}\right)\right)$, such that

$$
f=\sum_{\tau \in T(n, t)} g_{\tau(1)} \cdots g_{\tau(t)} h_{\tau}+u \prod_{i=1}^{n} \frac{g_{i}}{\ell_{i}} .
$$

Here $T(n, t)$ indicates the set of all non-decreasing sequences of length $t$ on $[n]$.
This punctured Nullstellensatz will be our main tool in proving Theorem 1.1. We start with the $k=3$ case.

Theorem 2.2. For $n \geqslant 2, f(n, 3)=n+3$.
Proof. To show that $f(n, 3)=n+3$, it suffices to establish the lower bound. We prove by contradiction. Suppose $H_{1}, \cdots, H_{n+2}$ are $n+2$ affine hyperplanes that form an almost 3-cover of $Q^{n}$. Without loss of generality, assume the equation defining $H_{i}$ is $\left\langle\vec{b}_{i}, \vec{x}\right\rangle=1$, for some non-zero vector $\vec{b}_{i} \in \mathbb{R}^{n}$. Define $P_{i}=\left\langle\vec{b}_{i}, \vec{x}\right\rangle-1$, and let

$$
f=P_{1} P_{2} \cdots P_{n+2} .
$$

Since $H_{1}, \cdots, H_{n+2}$ form an almost 3-cover of $Q^{n}$, every binary vector $\vec{x} \in Q^{n} \backslash\{\overrightarrow{0}\}$ is a zero of multiplicity at least 3 of the polynomial $f$. We apply Lemma 2.1 with

$$
D_{i}=\{0\}, \quad S_{i}=\{0,1\}, \quad g_{i}=x_{i}\left(x_{i}-1\right), \quad \ell_{i}=x_{i},
$$

and write $f$ in the following form:

$$
f=\sum_{1 \leqslant i \leqslant j \leqslant k \leqslant n} x_{i}\left(x_{i}-1\right) x_{j}\left(x_{j}-1\right) x_{k}\left(x_{k}-1\right) h_{i j k}+u \prod_{i=1}^{n}\left(x_{i}-1\right),
$$

with $\operatorname{deg}(u) \leqslant \operatorname{deg}(f)-n=2$.
Note that $f=0$ on $Q^{n} \backslash\{\overrightarrow{0}\}$. Moreover,

$$
\frac{\partial f}{\partial x_{i}}=\sum_{j=1}^{n+2} P_{1} \cdots P_{j-1} \cdot \frac{\partial P_{j}}{\partial x_{i}} \cdot P_{j+1} \cdots P_{n+2}
$$

Recall that $P_{j}$ is a polynomial of degree 1 , thus $\partial f / \partial x_{i}$ is just a linear combination of $P_{1} \cdots \hat{P}_{j} \cdots P_{n+2}$. Note that removing a single hyperplane still gives an almost 2-cover. Therefore $\partial f / \partial x_{i}$ vanishes on $Q^{n} \backslash\{\overrightarrow{0}\}$. One can similarly show that all the second order partial derivatives of $f$ vanish on $Q^{n} \backslash\{\overrightarrow{0}\}$ as well. More generally, if $f$ is the product of equations of the affine hyperplanes from an almost $k$-cover, then all the $j$-th order derivatives of $f$ vanish on $Q^{n} \backslash\{\overrightarrow{0}\}$, for $j=0, \cdots, k-1$. It is not hard to observe that $x_{i}\left(x_{i}-1\right) x_{j}\left(x_{j}-1\right) x_{k}\left(x_{k}-1\right) h_{i j k}=g_{i} g_{j} g_{k} h_{i j k}$ also has its $t$-th order partial derivatives
vanishing on the entire cube $Q^{n}$, for $t \in\{0,1,2\}$, since $x_{i}\left(x_{i}-1\right)=0$ on $Q^{n}$. Therefore the following polynomial

$$
h=u \prod_{i=1}^{n}\left(x_{i}-1\right)
$$

has $j$-th order partial derivatives vanishing on $Q^{n} \backslash\{\overrightarrow{0}\}$, for $j=0,1,2$.
We denote by $e_{i}$ the $n$-dimensional unit vector with the $i$-th coordinate being 1 . By calculations,

$$
\frac{\partial h}{\partial x_{i}}=\frac{\partial u}{\partial x_{i}} \prod_{j=1}^{n}\left(x_{j}-1\right)+u \prod_{j \neq i}\left(x_{j}-1\right)
$$

Therefore

$$
0=\frac{\partial h}{\partial x_{i}}\left(e_{i}\right)=(-1)^{n-1} u\left(e_{i}\right),
$$

and this implies

$$
u\left(e_{i}\right)=0 \text { for } i=1, \cdots, n
$$

Furthermore,

$$
\frac{\partial^{2} h}{\partial x_{i}^{2}}=\frac{\partial^{2} u}{\partial x_{i}^{2}} \prod_{j=1}^{n}\left(x_{j}-1\right)+2 \frac{\partial u}{\partial x_{i}} \prod_{j \neq i}\left(x_{j}-1\right) .
$$

Therefore

$$
0=\frac{\partial^{2} h}{\partial x_{i}^{2}}\left(e_{i}\right)=(-1)^{n-1} \cdot 2 \frac{\partial u}{\partial x_{i}}\left(e_{i}\right),
$$

and this implies

$$
\frac{\partial u}{\partial x_{i}}\left(e_{i}\right)=0 \text { for } i=1, \cdots, n .
$$

Finally,

$$
\frac{\partial^{2} h}{\partial x_{i} x_{j}}=\frac{\partial^{2} u}{\partial x_{i} x_{j}} \prod_{k=1}^{n}\left(x_{k}-1\right)+\frac{\partial u}{\partial x_{i}} \prod_{k \neq j}\left(x_{k}-1\right)+\frac{\partial u}{\partial x_{j}} \prod_{k \neq i}\left(x_{k}-1\right)+u \prod_{k \neq i, j}\left(x_{k}-1\right)
$$

By evaluating it on $e_{i}$ and $e_{i}+e_{j}$, we have

$$
\frac{\partial u}{\partial x_{j}}\left(e_{i}\right)=u\left(e_{i}\right)=0, \quad \text { and } \quad u\left(e_{i}+e_{j}\right)=0
$$

Summarizing the above results $u$ is a polynomial of degree at most 2, satisfying: (i) $u=0$ at $e_{i}$ and $e_{i}+e_{j}$; (ii) $\partial u / \partial x_{i}=0$ at $e_{j}$ (possible to have $i=j$ ). We define a new single-variable polynomial $w$,

$$
w(x)=u\left(x \cdot e_{i}+e_{j}\right)
$$

Then $\operatorname{deg}(w) \leqslant 2$, and $w(0)=w(1)=w^{\prime}(0)=0$, which implies $w \equiv 0$. Let

$$
u=\sum_{i} a_{i i} x_{i}^{2}+\sum_{i<j} a_{i j} x_{i} x_{j}+\sum_{i} b_{i} x_{i}+c .
$$

This gives for all $i \neq j$,

$$
a_{i i}=0, \quad a_{i j}+b_{i}=0, \quad a_{i i}+b_{i}+c=0 .
$$

On other other hand $\partial u / \partial x_{i}=0$ at $e_{i}$ gives

$$
2 a_{i i}+b_{i}=0 .
$$

It is not hard to derive from these equalities that

$$
a_{i i}=a_{i j}=b_{i}=c=0 .
$$

Therefore $u \equiv 0$. But then we have $f(\overrightarrow{0})=0$, which contradicts the assumption that $\overrightarrow{0}$ is not covered by any of the $n+2$ affine hyperplanes. Therefore $f(n, 3)=n+3$ for $n \geqslant 2$. Note that $f(1,3)=3$ and the proof does not work for $n=1$ because $e_{i}+e_{j}$ does not exist in a 1 -dimensional space.

Note that Theorem [2.2] already implies $f(n, 4) \geqslant f(n, 3)+1=n+4$ for all $n \geqslant 2$. For $n=2$, it is straightforward to check that $f(2,4)=6$, with an optimal almost 4 -cover $x_{1}=1$ (twice), $x_{2}=1$ (twice), and $x_{1}+x_{2}=1$ (twice). However for $n \geqslant 3$, we can improve this lower bound by 1 .

Theorem 2.3. For $n \geqslant 3, f(n, 4) \in\{n+5, n+6\}$. Moreover, for $3 \leqslant n \leqslant 5$, $f(n, 4)=n+5$.

Proof. Suppose $n \geqslant 3$, we would like to prove by contradiction that $n+4$ affine hyperplanes cannot form an almost 4 -cover of $Q^{n}$. Following the notations in the previous proof, we have

$$
P_{1} \cdots P_{n+4}=f=\sum_{1 \leqslant i \leqslant j \leqslant k \leqslant l \leqslant n} g_{i} g_{j} g_{k} g_{l} h_{i j k l}+u \prod_{i=1}^{n}\left(x_{i}-1\right),
$$

with $\operatorname{deg}(u) \leqslant 4$. Following similar calculations, $u$ satisfies the following relations: (i) $u=0$ at $e_{i}, e_{i}+e_{j}$ and $e_{i}+e_{j}+e_{k}$ for distinct $i, j, k$; (ii) $\partial u / \partial x_{i}=0$ at $e_{j}$ and $e_{j}+e_{k}$ for distinct $j, k$ ( $i=j$ or $i=k$ possible); (iii) $\partial^{2} u / \partial x_{i}^{2}=0$ at $e_{j}(i=j$ possible); (iv) $\partial^{2} u / \partial x_{i} \partial x_{j}=0$ at $e_{k}$ ( $i=k$ or $j=k$ possible). Suppose

$$
u=\sum a_{i i i i} x_{i}^{4}+\sum a_{i i i j} x_{i}^{3} x_{j}+\cdots+\sum b_{i i i} x_{i}^{3}+\cdots+\sum c_{i i} x_{i}^{2}+\cdots+\sum d_{i} x_{i}+e .
$$

Since $f(\overrightarrow{0})=(-1)^{n+4}=(-1)^{n}$, we know that $u(\overrightarrow{0})=1$ and thus $e=1$.

Let $w(x)=u\left(x \cdot e_{i}+e_{j}\right)$. Then $w(0)=w(1)=w^{\prime}(0)=w^{\prime}(1)=w^{\prime \prime}(0)=0$. Since $w(x)$ has degree at most 4 , we immediately have $w \equiv 0$. This gives

$$
\begin{gather*}
a_{i i i i}=0,  \tag{1}\\
a_{i i i j}+b_{i i i}=0 .  \tag{2}\\
a_{i i j j}+b_{i i j}+c_{i i}=0 .  \tag{3}\\
a_{i j j j}+b_{i j j}+c_{i j}+d_{i}=0 .  \tag{4}\\
a_{j j j j}+b_{j j j}+c_{j j}+d_{j}+1=0 \tag{5}
\end{gather*}
$$

Using $u\left(e_{i}\right)=0, \partial u / \partial x_{i}\left(e_{i}\right)=0$ and $\partial^{2} u / \partial x_{i}^{2}\left(e_{i}\right)=0$, we have

$$
\begin{gathered}
a_{i i i i}+b_{i i i}+c_{i i}+d_{i}+1=0, \\
4 a_{i i i i}+3 b_{i i i}+2 c_{i i}+d_{i}=0, \\
12 a_{i i i i}+6 b_{i i i}+2 c_{i i}=0 .
\end{gathered}
$$

Using $a_{i i i i}=0$, we can solve this system of linear equations and get $b_{i i i}=-1$, $c_{i i}=3, d_{i}=-3$. This implies $a_{i i i j}=1$. Plugged into the equations (3) and (4), we have:

$$
\begin{aligned}
a_{i i j j}+b_{i i j} & =-3, \\
b_{i i j}+c_{i j} & =2 .
\end{aligned}
$$

Now using $\partial^{2} u / \partial x_{i} \partial x_{j}\left(e_{i}\right)=0$, we have $3 a_{i i i j}+2 b_{i i j}+c_{i j}=0$, which gives

$$
2 b_{i i j}+c_{i j}=-3 .
$$

The three linear equations above give $b_{i i j}=-5, c_{i j}=7, a_{i i j j}=2$.
For $n \geqslant 3$, we can also utilize the relation $\partial^{2} u /\left(\partial x_{i} \partial x_{j}\right)=0$ at $e_{k}$. This gives $a_{i j k k}+b_{i j k}+c_{i j}=0$, hence

$$
a_{i j k k}+b_{i j k}=-7 .
$$

Also $\partial u /\left(\partial x_{i}\right)=0$ at $e_{j}+e_{k}$ simplifies to

$$
a_{i j k k}+a_{i j j k}+b_{i j k}+3=0 .
$$

Together they give $b_{i j k}=-11$ and $a_{i j k k}=4$. Finally, by calculations

$$
\begin{aligned}
u\left(e_{i}+e_{j}+e_{k}\right) & =3 a_{i i i i}+6 a_{i i i j}+3 a_{i i j j}+3 a_{i i j k}+3 b_{i i i}+6 b_{i i j}+b_{i j k}+3 c_{i i}+3 c_{i j}+3 d_{i}+e \\
& =2 \neq 0
\end{aligned}
$$

This gives a contradiction. Therefore for $n \geqslant 3$, there is no $u$ of degree at most 4 satisfying the aforementioned relations. This shows for $n \geqslant 3, f(n, 4) \geqslant n+5$. The proof does not work for $n<3$ because $e_{i}+e_{j}+e_{k}$ does not exist in a 1 -dimensional
or 2-dimensional space. Since $f(n, 4) \leqslant n+\binom{4}{2}=n+6$, it can only be either $n+5$ or $n+6$, proving the first claim in Theorem [2.3,

To show that $f(n, 4)=n+5$ for $3 \leqslant n \leqslant 5$, we only need to construct almost 4 -covers of $Q^{n}$ using $n+5$ affine hyperplanes. For $Q^{3}$, note that $x_{1}=1, x_{2}=1$, $x_{3}=1$, and $x_{1}+x_{2}+x_{3}=1$ form an almost 2-cover. Doubling it gives an almost 4 -cover of $Q^{3}$ with 8 affine hyperplanes. For $Q^{4}$, the following 9 affine hyperplanes form an almost 4-cover: $x_{1}=1, x_{2}=1, x_{3}=1, x_{4}=1, x_{1}+x_{4}=1, x_{2}+x_{4}=1$, $x_{3}+x_{4}=1, x_{1}+x_{2}+x_{3}=1, x_{1}+x_{2}+x_{3}+x_{4}=1$. For $Q^{5}$, one can take $x_{i}=1$ for $i=1, \cdots 5$, together with $x_{i}+x_{i+1}+x_{i+2}=1$ for $i=1, \cdots, 5$, where the addition is in $\mathbb{Z}_{5}$.

Now we can combine these two results we just obtained to prove Theorem 1.1.
Proof of Theorem 1.1. The $k=3$ case has been resolved by Theorem 2.2. On the other hand we have

$$
f(n, k) \geqslant f(n, k-1)+1
$$

since removing an affine hyperplane from an almost $k$-cover gives an almost ( $k-1$ )cover. Therefore for $k \geqslant 4$ and $n \geqslant 3$,

$$
f(n, k) \geqslant f(n, 4)+(k-4) \geqslant n+5+(k-4)=n+k+1 .
$$

The upper bound follows from the construction in the introduction.

## 3 Fractional almost $k$-covers of the $n$-cube

In this section, we determine $f^{*}(n, k)$ precisely and prove Theorem 1.2. We first establish an upper bound by an explicit construction of almost $k$-covers.

Lemma 3.1. (i) For every $n, k$,

$$
f^{*}(n, k) \leqslant\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) k .
$$

(ii) When $k$ is divisible by $n x$, with $x=\operatorname{lcm}\left(\binom{n-1}{0},\binom{n-1}{1}, \cdots,\binom{n-1}{n-1}\right.$, we have

$$
f(n, k) \leqslant\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) k .
$$

Proof. For (ii), it suffices to show that when $k=n x$, we can find an almost $k$-cover of $Q^{n}$, using $k(1+1 / 2+\cdots+1 / n)$ hyperplanes. We can then replicate this process to upper bound $f(n, k)$ where $k$ is any multiple of $n x$.

For $j=1, \cdots, n$, we will use every affine hyperplane of the form $x_{i_{1}}+x_{i_{2}}+\cdots+x_{i_{j}}=$ 1 a total of $\frac{n x}{j\binom{n}{j}}$ times. This number is actually an integer since it is equal to $\frac{x}{\binom{n-1}{j-1}}$, and by definition, $x$ is divisible by all $\binom{n-1}{j-1}$.

There are $\binom{n}{j}$ affine hyperplanes in this form, so the total number of being used is

$$
\sum_{j=1}^{n} \frac{n x}{j\binom{n}{j}} \cdot\binom{n}{j}=\sum_{j=1}^{n} \frac{n x}{j}=\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) k
$$

This is the number of hyperplanes claimed. If we could show that they form an almost $n x$-cover of $Q^{n}$, then we can scale the weights by a constant factor to obtain a fractional almost $k$-cover of $Q^{n}$ for every $k$ and (i) follows immediately.

Now we must show that these affine hyperplanes cover each point the appropriate number of times. It is apparent that $(0, \cdots, 0)$ is never covered. Because of the symmetric nature of our construction, we just need to check how many times we have covered a vertex that has $t$ ones as coordinates. It gets covered by $t\binom{n-t}{j-1}$ distinct hyperplanes of the form $x_{i_{1}}+x_{i_{2}}+\cdots+x_{i_{j}}=1$, each of which appears $\frac{n x}{j\binom{n}{j}}$ times. Thus, the total number of times a point with $t$ ones is covered is given by:

$$
\begin{aligned}
\sum_{j=1}^{n} \frac{n x}{j\binom{n}{j}} \cdot t\binom{n-t}{j-1} & =n x t \sum_{j=1}^{n} \frac{\binom{n-t}{j-1}}{j\binom{n}{j}}=n x t \sum_{j=1}^{n} \frac{(n-t)!(n-j)!}{(n-t-j+1)!n!} \\
& =n x t \cdot \frac{(n-t)!}{n!} \cdot \sum_{j=1}^{n} \frac{(n-j)!}{(n-t-j+1)!} \\
& =\frac{n x}{(t-1)!\binom{n}{t}} \sum_{j=1}^{n} \frac{(n-j)!}{(n-t-j+1)!}=\frac{n x}{\binom{n}{t}} \sum_{j=1}^{n}\binom{n-j}{t-1} \\
& =\frac{n x}{\binom{n}{t}}\binom{n}{t}=n x=k
\end{aligned}
$$

To establish the lower bound in Theorem 1.2, first we assign weights to each vertex of $Q^{n}$ we wish to cover. A vertex with $t$ ones as coordinates is given weight $\frac{1}{t\binom{n}{t} \text {. Then }}$ the sum of the weights of all the vertices is:

$$
\sum_{t=1}^{n}\binom{n}{t} \cdot \frac{1}{t\binom{n}{t}}=\sum_{t=1}^{n} \frac{1}{t}
$$

So if we cover each vertex $k$ times, the sum over all affine hyperplanes of the weights of the vertices they cover is $k(1+1 / 2+\cdots+1 / n)$. Thus, if we can show that no hyperplane can cover a set of vertices whose weights sum to more than 1 , we will have proven the lower bound. Given an affine hyperplane $H$ not containing $\overrightarrow{0}$, denote by $\mathcal{A}_{t}$ the set of vertices with $t$ ones covered by $H$. We wish to prove Theorem 1.3, i.e.

$$
\sum_{t=1}^{n} \frac{\left|\mathcal{A}_{t}\right|}{t\binom{n}{t}} \leqslant 1
$$

In general, vertices of $Q^{n} \backslash\{\overrightarrow{0}\}$ correspond to nonempty subsets of [n]. It is worth noting that if the equation of $H$ is $a_{1} x_{1}+\cdots+a_{n} x_{n}=1$, and all coefficients $a_{i}$ are strictly positive, the subsets corresponding to the vertices it covers will form an antichain. By the Lubell-Yamamoto-Meshalkin inequality,

$$
\sum_{t=1}^{n} \frac{\left|\mathcal{A}_{t}\right|}{t\binom{n}{t}} \leqslant \sum_{t=1}^{n} \frac{\left|\mathcal{A}_{t}\right|}{\binom{n}{t}} \leqslant 1 .
$$

However, some coefficients $a_{i}$ may be non-positive. In order to consider a more general hyperplane, we will associate each vertex it covers to some permutations of [ $n$ ]. Consider the vertex $\left(c_{1}, c_{2}, \cdots, c_{n}\right) \in Q^{n}$ where the coordinates which are ones are $c_{i_{1}}, \cdots, c_{i_{t}}$. We will associate this vertex to the permutations, $\left(d_{1}, d_{2}, \cdots, d_{n}\right)$ of $[n]$ which begin with $\left\{i_{1}, i_{2}, \cdots, i_{t}\right\}$ in some order and also have $\sum_{k=1}^{j} a_{d_{k}}<1$ for $1 \leqslant j<t$.

Lemma 3.2. No permutation of $[n]$ is associated to more than one vertex on the same hyperplane.

Proof. Suppose for the sake of contradiction that a permutation is associated to two vertices, $v$ and $w$, of the same hyperplanes. They may have either the same or a different number of ones as coordinates.

Suppose that $v$ and $w$ both have $a$ ones as coordinates. The permutations associated to $v$ have the $a$ indices where $v$ has a 1 as their first $a$ entries and the permutations associated to $v$ will have the $a$ indices where $w$ has a 1 as their first $a$ entries. However, $v$ and $w$ do not have their ones in the exact same places so the set of the first $a$ entries is not the same for any pair of a permutation associated to $v$ and a permutation associated to $w$.

We are left to consider the case where $v$ and $w$ do not have the same number of ones as coordinates. Without loss of generality, $v$ has $a$ ones as coordinates and $w$ has $b$ ones as coordinates where $a>b$. Suppose the permutation associated to both of them begins with $\left(d_{1}, d_{2}, \cdots, d_{b}\right)$. By the restrictions on permutations associated to $v$, we have that $\sum_{j=1}^{b} a_{d_{j}}<1$. However, the conditions on permutations associated to $w$ tell us that $\left(d_{1}, d_{2}, \cdots, d_{b}\right)$ are precisely the indices where $w$ has a 1 coordinate. This implies $\sum_{j=1}^{b} a_{d_{j}}=1$, giving a contradiction.

Lemma 3.3. The total number of permutations associated to a vertex with $t$ ones as coordinates is at least $(t-1)!(n-t)$ !

Proof. There are $(n-t)$ ! ways to arrange the indices other than $\left\{i_{1}, i_{2}, \cdots, i_{t}\right\}$, so it suffices to show that there exist at least $(t-1)$ ! ways to order $\left\{i_{1}, i_{2}, \cdots, i_{t}\right\}$ as $\left(d_{1}, d_{2}, \cdots, d_{t}\right)$ such that we have $\sum_{k=1}^{j} a_{d_{k}}<1$ for $1 \leqslant j<t$. We notice that $(t-1)$ ! is the number of ways to order $\left\{i_{1}, i_{2}, \cdots, i_{t}\right\}$ around a circle (up to rotations, but not
reflections). Thus it suffices to show that for each circular ordering of $\left\{i_{1}, i_{2}, \cdots, i_{t}\right\}$, we can choose a starting place from which we may continue clockwise and label the elements as $\left(d_{1}, d_{2}, \cdots, d_{t}\right)$ in such a way that $\sum_{k=1}^{j} a_{d_{k}}<1$ for all $1 \leqslant j<t$.

Equivalently, the values of $a_{i_{k}}$, which happen to sum to 1 , have been listed around a circle for $1 \leqslant k \leqslant t$. We wish to find some starting point from which all the partial sums of up to $t-1$ terms from that point are less than 1 . We can subtract $1 / t$ from each to give the equivalent problem of $t$ numbers, which sum to 0 , written around a circle and needing to find a starting point from which all the partial sums of $1 \leqslant j \leqslant t-1$ terms are less than $1-\frac{j}{t}$. It suffices to find a starting point for which the aforementioned partial sums are at most 0 .

Consider all possible sums of any number of consecutive terms along the circle and choose the largest. We will label the terms in this sum as $e_{1}, e_{2}, \cdots, e_{m}$ and continue to order clockwise around the circle $e_{m+1}, e_{m+2}, \cdots, e_{t}$. Choose the starting point to be $e_{m+1}$. If any of the partial sums $e_{m+1}+e_{m+2}+\cdots+e_{m+j}$ exceeds 0 , for $m+j \leqslant t$, we could have simply chosen $e_{1}, e_{2}, \cdots, e_{m+j}$ to get a larger sum than $e_{1}+e_{2}+\cdots+e_{m}$. Similarly, if $e_{m+1}+e_{m+2}+\cdots+e_{t}+e_{1}+e_{2}+\cdots+e_{j}>0$ for some $1 \leqslant j<m$, then we can note that $\left(e_{1}+e_{2}+\cdots+e_{t}\right)+\left(e_{1}+e_{2}+\cdots+e_{j}\right)$ exceeds $e_{1}+e_{2}+\cdots+e_{m}$, and since $e_{1}+e_{2}+\cdots+e_{t}=0$, we have that $e_{1}+e_{2}+\cdots+e_{j}>e_{1}+e_{2}+\cdots+e_{m}$, a contradiction. Thus, if we start at $e_{m+1}$ and move clockwise around the circle, the first $t-1$ partial sums will be at most 0 , as desired.

Combining the previous results, we prove Theorem 1.3, which can be viewed as an analogue of the LYM inequality for partial sums.

Proof of Theorem 1.3. By definition, sets in $\mathcal{A}$ correspond to vertices of $Q^{n}$ covered by the hyperplane $H$ with equation $a_{1} x_{1}+\cdots+a_{n} x_{n}=1$. From Lemma 3.2 and 3.3, these vertices define disjoint collections of permutations of length $n$. Moreover if $S \in \mathcal{A}$ has size $t$ then there are at least $(t-1)$ ! $(n-t)$ ! permutations associated to it. Since in total there are at most $n$ ! permutations, we get

$$
\sum_{S \in \mathcal{A}}(|S|-1)!(n-|S|)!\leqslant n!,
$$

which implies

$$
\sum_{S \in \mathcal{A}} \frac{1}{|S|\binom{n}{|S|}} \leqslant 1
$$

as desired.
Now we are ready to prove our main theorem in this section.

Proof of Theorem 1.2. As mentioned before, we assign weight $\frac{1}{t\binom{n}{t}}$ to a vertex of $Q^{n} \backslash\{\overrightarrow{0}\}$ with $t$ ones as coordinates. By Lemma 1.3, every affine hyperplane covers a set of vertices whose weights sum to at most 1 . Therefore in an optimal fractional almost $k$-cover $\{w(H)\}$,

$$
f^{*}(n, k)=\sum_{H} w(H) \geqslant k \cdot \sum_{t=1}^{n} \frac{\binom{n}{t}}{t\binom{n}{t}}=\left(\sum_{i=1}^{n} \frac{1}{i}\right) k .
$$

With the upper bound proved in Lemma 3.1, we have

$$
f^{*}(n, k)=\left(\sum_{i=1}^{n} \frac{1}{i}\right) k
$$

For integral almost $k$-covers, note that $f(n, k) \geqslant f^{*}(n, k)$. Using Lemma 3.1 again,

$$
f(n, k)=f^{*}(n, k)=\left(\sum_{i=1}^{n} \frac{1}{i}\right) k
$$

whenever $n x$ divides $k$. For fixed $n$ and $k \rightarrow \infty$, note that $f(n, k)$ is monotone in $k$, which immediately implies

$$
f(n, k)=\left(1+\frac{1}{2}+\cdots+\frac{1}{n}+o(1)\right) k
$$

For small values of $n$, we can actually determine the value of $f(n, k)$ for every $k$. It seems that for large $k, f(n, k)$ is not far from its lower bound $\left\lceil f^{*}(n, k)\right\rceil$. Trivially $f(1, k)=k$.

Theorem 3.4. The following statements are true:
(i) $f(2, k)=\left\lceil\frac{3 k}{2}\right\rceil$ for $k \geqslant 1$.
(ii) $f(3, k)=\left\lceil\frac{11 k}{6}\right\rceil$ for $k \geqslant 2$ and $f(3,1)=3$.

Proof. (i) From previous discussions, there exists an almost 2-cover of $Q^{2}$ using 3 affine hyperplanes. Therefore $f(2, k+2) \leqslant f(2, k)+3$, and it suffices to check $f(2,1)=2$ and $f(2,2)=3$ which are both obvious.
(ii) There exists an almost 6-cover of $Q^{3}$ using 11 affine hyperplanes. Therefore $f(3, k+$ $6) \leqslant f(3, k)+11$. It suffices to check $f(3, k) \leqslant\left\lceil\frac{11 k}{6}\right\rceil$ for $k=2, \cdots, 5$ and $k=7$. From $f(n, 2)=n+1$, we have $f(3,2)=4 . f(3,3) \leqslant 6$ follows from Theorem 1.1. $f(3,4) \leqslant 8$ since $f(3,4) \leqslant 2 f(3,2) . f(3,5) \leqslant 10$ by taking each of $x_{i}=1$ twice, $x_{1}+x_{2}+x_{3}=1$ three times, and $x_{1}+x_{2}+x_{3}=2$ once. $f(3,7) \leqslant 13$ follows from taking each of $x_{1}=1$, $x_{2}=1, x_{3}=1, x_{1}+x_{2}=1, x_{1}+x_{3}=1$ twice, and $x_{2}+x_{3}=1, x_{2}+x_{3}-x_{1}=1$, $x_{1}+x_{2}+x_{3}=1$ once.

With the assistance of a computer program, we also checked that $f(4, k)=\left\lceil\frac{25 k}{12}\right\rceil$ for $k \geqslant 2$. $f(5, k)=\left\lceil\frac{137}{60} k\right\rceil$ for $k \geqslant 15$ except when $k \equiv 7(\bmod 60)$ where $f(5, k)=$ $\left\lceil\frac{137}{60} k\right\rceil+1$. The following question is natural.

Question 3.5. Does there exist an absolute constant $C>0$ which does not depend on $n$, such that for a fixed integer $n$, there exists $M_{n}$, so that whenever $k \geqslant M_{n}$,

$$
f(n, k) \leqslant\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) k+C ?
$$

If so, it would show that $f(n, k)$ and $f^{*}(n, k)$ differ by at most a constant when $k$ is large.

## 4 Concluding Remarks

In this paper, we determine the minimum size of a fractional almost $k$-cover of $Q^{n}$, and find the minimum size of an integral almost $k$-cover of $Q^{n}$, for $k \leqslant 3$. Note that $f(n, 1)=n$ for $n \geqslant 1, f(n, 2)=n+1$ for $n \geqslant 1$, and $f(n, 3)=n+3$ for $n \geqslant 2$. All of them attain the upper bound $f(n, k) \leqslant n+\binom{k}{2}$ whenever $n$ is not too small. For larger $k$ the following conjecture seems plausible.

Conjecture 4.1. For an arbitrary fixed integer $k \geqslant 1$ and sufficiently large $n$,

$$
f(n, k)=n+\binom{k}{2} .
$$

In other words, for large $n$, an almost $k$-cover of $Q^{n}$ contains at least $n+\binom{k}{2}$ affine hyperplanes.

In particular, for $k=4$, although $f(n, k) \leqslant n+5$ for $n \leqslant 5$, we suspect that for $n \geqslant 6, n+6$ affine hyperplanes are necessary for an almost 4 -cover of $Q^{n}$. If we restrict our attention to almost $k$-covers of $Q^{n}$ which use each of the affine hyperplanes $x_{i}=1$ for $i=1, \cdots, n$, we see that Conjecture 4.1, if true, will imply the following weaker conjecture:

Conjecture 4.2. For fixed $k \geqslant 1$ and sufficiently large $n$, suppose $H_{1}, \cdots, H_{m}$ are affine hyperplanes in $\mathbb{R}^{n}$ not containing $\overrightarrow{0}$, and they cover all the vectors with $t$ ones as coordinates at least $k-t$ times, for $t=1, \cdots, k-1$. Then $m \geqslant\binom{ k}{2}$.

If this conjecture is true, then the $\binom{k}{2}$ bound is the best possible, since one can take $k-t$ copies of $x_{1}+\cdots+x_{n}=t$ for $t=1, \cdots, k-1$. We note that using our weights from earlier, and the fact that a hyperplane cannot cover vertices whose weights sum to more than 1 , we require:

$$
m \geqslant \sum_{t=1}^{k-1}(k-t)\binom{n}{t} \frac{1}{t\binom{n}{t}}=1-k+\sum_{t=1}^{k-1} \frac{k}{t}=(1-o(1)) k \ln k
$$

Remark added. Alon communicated to us that Conjecture 4.2 is true. With his permission, we include his proof using Ramsey-type arguments below. Let $n$ be huge, and let $S$ be a collection of $m$ affine hyperplanes $H_{1}, \cdots, H_{m}$ satisfying the assumptions in Conjecture 4.2 and $N=[n]$. Color each subset of size $k-1$ by the index of the first hyperplane that covers it ( $m$ colors), by Ramsey there is a large subset $N_{1}$ of $N$ so that all $(k-1)$-subsets of it are covered by the same hyperplane. Without loss of generality, the equation of this hyperplane is $\sum_{i} w_{i} x_{i}=1$ and it follows that for all $j \in N_{1}$, all $w_{j}$ are equal and hence all are equal $1 /(k-1)$. Therefore this hyperplane cannot cover any $k-t$ subset of $N_{1}$ for $t \geqslant 2$. Now throw away this hyperplane and repeat the argument for subsets of size $k-2$ of $N_{1}$. Coloring each such subset by the pair of smallest two indices of the hyperplanes that cover it ( $\binom{m}{2}$ colors), we get a monochromatic subset $N_{2}$ of $N_{1}$ and observe that here too each of these two hyperplanes whose equation is $\sum_{i} w_{i} x_{i}=1$ has $w_{j}=1 /(k-2)$ for all $j \in N_{2}$. So these cannot be useful for covering smaller subsets of $N_{2}$, throw them away and repeat this process. After dealing with all subsets including those of size 1 we get the assertion of the conjecture.

Alon and Füredi [2] proved the following result using induction on $n-m$ : for $n \geqslant m \geqslant 1$, then $m$ hyperplanes that do not cover all vertices of $Q^{n}$ miss at least $2^{n-m}$ vertices. Let $g(n, m, k)$ be the minimum number of vertices covered less than $k$ times by $m$ affine hyperplanes not passing through $\overrightarrow{0}$. The Alon-Füredi theorem shows $g(n, m, 1)=2^{n-m}$ for $m=1, \cdots, n$. For $k=2$, it is straightforward to show that for $m=1, \cdots, n+1$, we have:

$$
\begin{equation*}
g(n, m, 2)=2^{n-m+1} \tag{6}
\end{equation*}
$$

This is because $m-1$ hyperplanes leave at least $2^{n-m+1}$ vertices uncovered, and with one more hyperplane, these vertices cannot be covered twice. Similarly, for $k \geqslant 3$, we can obtain a trivial lower bound $g(n, m, k) \geqslant 2^{n-m+k-1}$. On the other hand, suppose $f(d, k)=t$ for $d \leqslant n$, then take the affine hyperplanes $H_{1}, \cdots, H_{t}$ in an almost $k$-cover of $Q^{d}$. Observe that $H_{i} \times \mathbb{R}^{n-d}$ is an affine hyperplane in $Q^{n}$ not containing $\overrightarrow{0}$. It is easy to see that $\left\{H_{i} \times \mathbb{R}^{n-d}\right\}$ covers all the vertices of $Q^{n}$ but those of the form $\overrightarrow{0} \times\{0,1\}^{n-d}$ at least $k$ times. Therefore $g(n, t, k) \leqslant 2^{n-d}$. Theorem 1.1 shows $f(d, 3)=d+3$ for $d \geqslant 2$, therefore $g(n, d+3,3) \leqslant 2^{n-d}$ or $g(n, m, 3) \leqslant 2^{n-m+3}$ for $m \geqslant 5$. We believe that this upper bound is tight. Note that the trivial lower bound is $g(n, m, 3) \geqslant 2^{n-m+2}$.

## Conjecture 4.3.

$$
g(n, m, 3)= \begin{cases}2^{n}, & m=1,2 \\ 2^{n-1}, & m=3 \\ 2^{n-m+3}, & m=4, \cdots, n+3\end{cases}
$$

One can further ask the following question for arbitrary $k$.
Question 4.4. Is it true that for all $n, m, k$,

$$
g(n, m, k)=2^{n-d}
$$

where $d$ is the maximum integer such that $f(d, k) \leqslant m$ ?

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