# SEYMOUR'S CONJECTURE ON 2-CONNECTED GRAPHS OF LARGE PATHWIDTH 

TONY HUYNH, GWENAËL JORET, PIOTR MICEK, AND DAVID R. WOOD


#### Abstract

We prove a conjecture of Seymour (1993) stating that for every apexforest $H_{1}$ and outerplanar graph $H_{2}$ there is an integer $p$ such that every 2-connected graph of pathwidth at least $p$ contains $H_{1}$ or $H_{2}$ as a minor. An independent proof was recently obtained by Dang and Thomas (arXiv:1712.04549).


## 1. Introduction

Pathwidth is a graph parameter of fundamental importance, especially in graph structure theory. The pathwidth of a graph $G$ is the minimum integer $k$ for which there is a sequence of sets $B_{1}, \ldots, B_{n} \subseteq V(G)$ such that $\left|B_{i}\right| \leqslant k+1$ for each $i \in[n]$, for every vertex $v$ of $G$, the set $\left\{i \in[n]: v \in B_{i}\right\}$ is a non-empty interval, and for each edge $v w$ of $G$, some $B_{i}$ contains both $v$ and $w$.

In the first paper of their graph minors series, Robertson and Seymour [7] proved the following theorem.
1.1. For every forest $F$, there exists a constant $p$ such that every graph with pathwidth at least $p$ contains $F$ as a minor.

The constant $p$ was later improved to $|V(F)|-1$ (which is best possible) by Bienstock, Robertson, Seymour, and Thomas [1]. A simpler proof of this result was later found by Diestel [5].

[^0]Since forests have unbounded pathwidth, 1.1 implies that a minor-closed class of graphs has unbounded pathwidth if and only if it includes all forests. However, these certificates of large pathwidth are not 2-connected, so it is natural to ask for which minor-closed classes $\mathcal{C}$, does every 2-connected graph in $\mathcal{C}$ have bounded pathwidth?

In 1993, Paul Seymour proposed the following answer (see [4]). A graph $H$ is an apexforest if $H-v$ is a forest for some $v \in V(H)$. A graph $H$ is outerplanar if it has an embedding in the plane with all the vertices on the outerface. These classes are relevant since they both contain 2 -connected graphs with arbitrarily large pathwidth. Seymour conjectured the following converse holds.
1.2. For every apex-forest $H_{1}$ and outerplanar graph $H_{2}$ there is an integer $p$ such that every 2-connected graph of pathwidth at least p contains $H_{1}$ or $H_{2}$ as a minor.

Equivalently, 1.2 says that for a minor-closed class $\mathcal{C}$, every 2 -connected graph in $\mathcal{C}$ has bounded pathwidth if and only if some apex-forest and some outerplanar graph are not in $\mathcal{C}$.

The original motivation for conjecturing 1.2 was to seek a version of 1.1 for matroids (see [3]). Observe that apex-forests and outerplanar graphs are planar duals (see 2.1). Since a matroid and its dual have the same pathwidth (see [6] for the definition of matroid pathwidth), 1.2 provides some evidence for a matroid version of 1.1.

In this paper we prove 1.2. An independent proof was recently obtained by Dang and Thomas [3].

We actually prove a slightly different, but equivalent version of 1.2. Namely, we prove that there are two unavoidable families of minors for 2-connected graphs of large pathwidth. We now describe our two unavoidable families.

A binary tree is a rooted tree such that every vertex has at most two children. For $\ell \geqslant 0$, the complete binary tree of height $\ell$, denoted $\Gamma_{\ell}$, is the binary tree with $2^{\ell}$ leaves such that each root to leaf path has $\ell$ edges. It is well known that $\Gamma_{\ell}$ has pathwidth $\lceil\ell / 2\rceil$. Let $\Gamma_{\ell}^{+}$be the graph obtained from $\Gamma_{\ell}$ by adding a new vertex adjacent to all the leaves of $\Gamma_{\ell}$. See Figure 1. Note that $\Gamma_{\ell}^{+}$is a 2-connected apex-forest, and its pathwidth grows as $\ell$ grows (since it contains $\Gamma_{\ell}$ ).


Figure 1. Complete binary trees with an extra vertex adjacent to all the leaves.
Our second set of unavoidable minors is defined recursively as follows. Let $\nabla_{1}$ be a triangle with a root edge $e$. Let $H_{1}$ and $H_{2}$ be copies of $\nabla_{\ell}$ with root edges $e_{1}$ and
$e_{2}$. Let $\nabla$ be a triangle with edges $e_{1}, e_{2}$ and $e_{3}$. Define $\nabla_{\ell+1}$ by gluing each $H_{i}$ to $\nabla$ along $e_{i}$ and then declaring $e_{3}$ as the new root edge. See Figure 2. Note that $\nabla_{\ell}$ is a 2 -connected outerplanar graph, and its pathwidth grows as $\ell$ grows (since it contains $\left.\Gamma_{\ell-1}\right)$.


Figure 2. Universal outerplanar graphs. The root edges are dashed.

The following is our main theorem.
1.3. For every integer $\ell \geqslant 1$ there is an integer $p$ such that every 2-connected graph of pathwidth at least $p$ contains $\Gamma_{\ell}^{+}$or $\nabla_{\ell}$ as a minor.

In Section 2, we prove that every apex-forest is a minor of a sufficiently large $\Gamma_{\ell}^{+}$and every outerplanar graph is a minor of a sufficiently large $\nabla_{\ell}$. Thus, Theorem 1.3 implies Seymour's conjecture.

We actually prove the following theorem, which by 1.1 , implies 1.3 .
1.4. For all integers $\ell \geqslant 1$, there exists an integer $k$ such that every 2 -connected graph $G$ with a $\Gamma_{k}$ minor contains $\Gamma_{\ell}^{+}$or $\nabla_{\ell}$ as a minor.

Our approach is different from that of Dang and Thomas [3], who instead observe that by the Grid Minor Theorem [8], one may assume that $G$ has bounded treewidth but large pathwidth. Dang and Thomas then apply their machinery of 'non-branching tree decompositions' to prove 1.2.

The rest of the paper is organized as follows. Section 2 proves the universality of our two families. In Sections 3 and 4, we define 'special' ear decompositions and prove that special ear decompositions always yield $\Gamma_{\ell}^{+}$or $\nabla_{\ell}$ minors. In Section 5, we prove that a minimal counterexample to 1.4 always contains a special ear decomposition. Section 6 concludes with short derivations of our main results.

## 2. Universality

This section proves some elementary (and possibly well-known) results. We include the proofs for completeness.
2.1. Outerplanar graphs and apex-forests are planar duals.

Proof. Let $G$ be an apex-forest, where $G-v$ is a forest. Consider an arbitrary planar embedding of $G$. Note that every face of $G$ includes $v$ (otherwise $G-v$ would contain a cycle). Let $G^{*}$ be the planar dual of $G$. Let $f$ be the face of $G^{*}$ corresponding to $v$. Since every face of $G$ includes $v$, every vertex of $G^{*}$ is on $f$. So $G^{*}$ is outerplanar.

Conversely, let $G$ be an outerplanar graph. Consider a planar embedding of $G$, in which every vertex is on the outerface $f$. Let $G^{*}$ be the planar dual of $G$. Let $v$ be the vertex of $G^{*}$ corresponding to $f$. If $G^{*}-v$ contained a cycle $C$, then a face of $G^{*}-v$ 'inside' $C$ would correspond to a vertex of $G$ that is not on $f$. Thus $G^{*}-v$ is a forest, and $G^{*}$ is an apex-forest.

We now show that Theorem 1.3 implies Seymour's conjecture, by proving two universality results.
2.2. Every apex-forest on $n \geqslant 2$ vertices is a minor of $\Gamma_{n-1}^{+}$.

If $H$ is a minor of $G$ and $v \in V(H)$, the branch set of $v$ is the set of vertices of $G$ that are contracted to $v .2 .2$ is a corollary of the following.
2.3. Every tree with $n \geqslant 1$ vertices is a minor of $\Gamma_{n-1}$, such that each branch set includes a leaf of $\Gamma_{n-1}$.

Proof. We proceed by induction on $n$. The base case $n=1$ is trivial. Let $T$ be a tree with $n \geqslant 2$ vertices. Let $v$ be a leaf of $T$. Let $w$ be the neighbour of $v$ in $T$. By induction, $T-v$ is a minor of $\Gamma_{n-2}$, such that each branch set includes a leaf of $\Gamma_{n-2}$. In particular, the branch set for $w$ includes some leaf $x$ of $\Gamma_{n-2}$. Note that $\Gamma_{n-1}$ is obtained from $\Gamma_{n}$ by adding two new leaf vertices adjacent to each leaf of $\Gamma_{n-2}$. Let $y$ and $z$ be the leaf vertices of $\Gamma_{n-1}$ adjacent to $x$. Extend the branch set for $w$ to include $y$ and let $\{z\}$ be the branch set of $v$. For each leaf $u \neq x$ of $\Gamma_{n-2}$, if $u$ is in the branch set of some vertex of $T-v$, then extend this branch set to include one of the new leaves in $\Gamma_{n-1}$ adjacent to $u$. Now $T$ is a minor of $\Gamma_{n-1}$, such that each branch set includes a leaf of $\Gamma_{n-1}$.

Our second universality result is for outerplanar graphs.
2.4. Every outerplanar graph on $n \geqslant 2$ vertices is a minor of $\nabla_{n-1}$.
2.4 is a corollary of the following.
2.5. Every outerplanar triangulation $G$ on $n \geqslant 3$ vertices is a minor of $\nabla_{n-1}$, such that for every edge $v w$ on the outerface of $G$, there is a non-root edge on the outerface of $\nabla_{n-1}$ joining the branch sets of $v$ and $w$.

Proof. We proceed by induction on $n$. The base case, $G=K_{3}$, is easily handled as illustrated in Figure 3. Let $G$ be an outerplanar triangulation with $n \geqslant 4$ vertices. Every such graph has a vertex $u$ of degree 2 , such that if $\alpha$ and $\beta$ are the neighbours
of $u$, then $G-u$ is an outerplanar triangulation and $\alpha \beta$ is an edge on the outerface of $G-u$. By induction, $G-u$ is a minor of $\nabla_{n-2}$, such that for every edge $v w$ on the outerface of $G-u$, there is a non-root edge $v^{\prime} w^{\prime}$ on the outerface of $\nabla_{n-2}$ joining the branch sets of $v$ and $w$. In particular, there is a non-root edge $\alpha^{\prime} \beta^{\prime}$ of $\nabla_{n-2}$ joining the branch sets of $\alpha$ and $\beta$. Note that $\nabla_{n-1}$ is obtained from $\nabla_{n-2}$ by adding, for each non-root edge $p q$ on the outerface of $\nabla_{n-2}$, a new vertex adjacent to $p$ and $q$. Let the branch set of $u$ be the vertex $u^{\prime}$ of $\nabla_{n-1}-V\left(\nabla_{n-2}\right)$ adjacent to $\alpha^{\prime}$ and $\beta^{\prime}$. Thus $\nabla_{n-1}$ contains $G$ as a minor. Every edge on the outerface of $G$ is one of $u \alpha$ or $u \beta$, or is on the outerface of $G-u$. By construction, $u^{\prime} \alpha^{\prime}$ is a non-root edge on the outerface of $\nabla_{n-1}$ joining the branch sets of $u$ and $\alpha$. Similarly, $u^{\prime} \beta^{\prime}$ is a non-root edge on the outerface of $\nabla_{n-1}$ joining the branch sets of $u$ and $\beta$. For every edge $v w$ on the outerface of $G$, where $v w \notin\{u \alpha, u \beta\}$, if $z$ is the vertex in $\nabla_{n-1}-V\left(\nabla_{n-2}\right)$ adjacent to $v^{\prime}$ and $w^{\prime}$, extend the branch set of $v$ to include $z$. Now $z w^{\prime}$ is an edge on the outerface of $\nabla_{n-1}$ joining the branch sets for $v$ and $w$. Thus for every edge $v w$ on the outerface of $G$, there is a non-root edge of $\nabla_{n-1}$ joining the branch sets of $v$ and $w$.


Figure 3. Proof of 2.5 in the base case.

## 3. Binary Ear Trees

Henceforth, all graphs in this paper are finite and simple. In particular, after contracting an edge, we suppress parallel edges and loops. Let $H$ and $G$ be graphs. We write $H \simeq G$ if $H$ and $G$ are isomorphic. Let $H \cup G$ be the graph with $V(H \cup G)=V(H) \cup V(G)$ and $E(H \cup G)=E(H) \cup E(G)$. If $H$ is a subgraph of $G$, then an $H$-ear is a path in $G$ with its two ends in $V(H)$ but with no internal vertex in $V(H)$. The length of a path is its number of edges.

For a vertex $v$ in a rooted tree $T$, let $T_{v}$ be the subtree of $T$ rooted at $v$. A vertex $v$ of $T$ is said to be branching if $v$ has at least two children.

A binary ear tree in a graph $G$ is a pair $(T, \mathcal{P})$, where $T$ is a binary tree, and $\mathcal{P}=\left\{P_{x}\right.$ : $x \in V(T)\}$ is a collection of paths in $G$ of length at least 2 such that, for every non-root vertex $x$ of $T$ the following holds:
(i) $P_{x}$ is a $P_{y}$-ear, where $y$ is the parent of $x$ in $T$, and
(ii) no internal vertex of $P_{x}$ is in $\bigcup_{z \in V(T) \backslash V\left(T_{x}\right)} V\left(P_{z}\right)$.

A binary ear tree $(T, \mathcal{P})$ is clean if for every non-leaf vertex $y$ of $T$, there is an end of $P_{y}$ that is not contained in any $P_{x}$ where $x$ is a child of $y$.

The main result of this section is the following.
3.1. For every integer $\ell \geqslant 1$, if $G$ has a clean binary ear $\operatorname{tree}(T, \mathcal{P})$ such that $T \simeq \Gamma_{3 \ell-2}$, then $G$ contains $\Gamma_{\ell}^{+}$or $\nabla_{\ell}$ as a minor.

Before starting the proof, we first set up notation for a Ramsey-type result that we will need.

If $p$ and $q$ are vertices of a tree $T$, then let $p T q$ denote the unique $p q$-path in $T$. If $T^{\prime}$ is a subdivision of a tree $T$, the vertices of $T^{\prime}$ coming from $T$ are called original vertices and the other vertices of $T^{\prime}$ are called subdivision vertices. Given a colouring of the vertices of $T=\Gamma_{n}$ with colours \{red, blue\}, we say that $T$ contains a red subdivision of $\Gamma_{k}$, if it contains a subdivision $T^{\prime}$ of $\Gamma_{k}$ such that all the original vertices of $T^{\prime}$ are red, and for all $a, b \in V\left(T^{\prime}\right)$ with $b$ a descendant of $a$, the path $a T b$ is descending. (Here a path is descending if it is contained in a path that starts at the root.) Define $R(k, \ell)$ to be the minimum integer $n$ such that every colouring of $\Gamma_{n}$ with colours \{red, blue\} contains a red subdivision of $\Gamma_{k}$ or a blue subdivision of $\Gamma_{\ell}$. We will use the following easy result.
3.2. $R(k, \ell) \leqslant k+\ell$ for all integers $k, \ell \geqslant 0$.

Proof. We proceed by induction on $k+\ell$. As base cases, it is clear that $R(k, 0)=k$ and $R(0, \ell)=\ell$ for all $k, \ell$. For the inductive step, assume $k, \ell \geqslant 1$ and let $T$ be a \{red, blue\}-coloured copy of $\Gamma_{k+\ell}$. By symmetry, we may assume that the root $r$ of $T$ is coloured red. Let $T_{1}$ and $T_{2}$ be the components of $T-r$, both of which are copies of $\Gamma_{k+\ell-1}$. If $T_{1}$ or $T_{2}$ contains a blue subdivision of $\Gamma_{\ell}$, then so does $T$ and we are done. By induction, $R(k-1, \ell) \leqslant k-1+\ell$, so both $T_{1}$ and $T_{2}$ contain a red subdivision of $\Gamma_{k-1}$. Add the paths from $r$ to the roots of these red subdivisions. We obtain a red subdivision of $\Gamma_{k}$, as desired.

The following observation will be helpful when considering subdivision vertices.
3.3. Let $G$ be a graph having a clean binary ear tree $(T, \mathcal{P})$ with $\mathcal{P}=\left\{P_{v}: v \in V(T)\right\}$. Suppose that $y$ is a degree-2 vertex in $T$ with parent $x$ and child $z$. Then there is a clean binary ear tree $\left(T / y z, \mathcal{P}^{\prime}\right)$ of $G$, with $\mathcal{P}^{\prime}=\left\{P_{v}^{\prime}: v \in V(T / y z)\right\}$ where $P_{v}^{\prime}=P_{v}$ for all $v \in V(T) \backslash\{y, z\}$, and $P_{y z}^{\prime}$ is the unique $P_{x}$-ear contained in $P_{y} \cup P_{z}$ that contains $P_{z}$, where the vertex resulting from the contraction of edge $y z$ is denoted $y z$ as well.

Proof. Property (i) of the definition of binary ear trees holds for vertex $y z$ of $T / y z$ by our choice of $P_{y z}^{\prime}$. Property (ii) holds for $y z$ because it held for $y$ and for $z$ in $(T, \mathcal{P})$. Also, these two properties hold for children of $y z$ in $T / y z$ (if any) because they held for $z$ before. Thus, $\left(T / y z, \mathcal{P}^{\prime}\right)$ is a binary ear tree. Finally, note that cleanliness of the binary ear tree $\left(T / y z, \mathcal{P}^{\prime}\right)$ follows from that of $(T, \mathcal{P})$, and the fact that the ends of $P_{y z}^{\prime}$ are the same as the ones of $P_{y}$.

We now prove 3.1.

Proof of 3.1. Let $t$ be a non-leaf vertex of $T$. Let $u$ and $v$ be the children of $t$. Let $u_{1}$ and $u_{2}$ be the ends of $P_{u}$. Let $v_{1}$ and $v_{2}$ be the ends of $P_{v}$. We say that $t$ is nested if $u_{1} P_{t} u_{2} \subseteq v_{1} P_{t} v_{2}$ or $v_{1} P_{t} v_{2} \subseteq u_{1} P_{t} u_{2}$. If $t$ is not nested, then $t$ is split. See Figures 4 and 5 . Regarding split and nested as colours, we apply 3.2 to the tree $T$ with the leaves removed, and obtain a tree $T^{*}$ which is a split subdivision of $\Gamma_{\ell-1}$ or a nested subdivision of $\Gamma_{2 \ell-2}$. For each leaf of $T^{*}$, add back its two children in $T$. This way, we deduce that $T$ contains either a subdivision of $\Gamma_{\ell}$ with all branching vertices split, or a subdivision of $\Gamma_{2 \ell-1}$ with all branching vertices nested. In the first case, we will find a $\nabla_{\ell}$ minor, while in the second we will find a $\Gamma_{\ell}^{+}$minor. The two cases are covered by 3.4 and 3.5.


Figure 4. Examples of a nested vertex $t$ with a path $P_{t}$ in a clean binary ear tree.


Figure 5. Examples of a split vertex $t$ with a path $P_{t}$ in a clean binary ear tree.
3.4. If $T$ contains a subdivision $T^{1}$ of $\Gamma_{\ell}$ such that every branching vertex is split, then $\bigcup_{t \in V\left(T^{1}\right)} P_{t}$ contains $\nabla_{\ell}$ as a minor.

Subproof. Consider the clean binary ear tree 'induced by' the subtree $T^{1}$, that is, the pair $\left(T^{1}, \mathcal{P}^{1}\right)$ where $\mathcal{P}^{1}=\left\{P_{t}: t \in V\left(T^{1}\right)\right\}$. First, for every subdivision vertex $y$ of $T^{1}$ with child $z$, we apply 3.3 to ( $T^{1}, \mathcal{P}^{1}$ ) in order to suppress vertex $y$. Note that every branching vertex of $T^{1}$ stays split. In particular, this is true if $z$ is branching. Hence, we may assume from now on that $T^{1}$ has no subdivision vertices.

Let $P$ be a path in a graph $G$. Let $\nabla_{\ell}^{-}$be the graph obtained from $\nabla_{\ell}$ by deleting its root edge $x y$. We say that a $\nabla_{\ell}^{-}$minor in $G$ is rooted on $P$ if the two roots of the $\nabla_{\ell}^{-}$ minor are the ends of $P$. (By 'roots' we mean the ends of the root edge.)

We prove the following technical statement. Let $m \geqslant 0$ be an integer, and let $T^{\prime}$ be a subtree of $T^{1}$ isomorphic to $\Gamma_{m}$ such that all branching vertices of $T^{\prime}$ are split, then $\bigcup_{t \in V\left(T^{\prime}\right)} P_{t}$ contains a $\nabla_{m+1}^{-}$minor rooted on $P_{r}$, where $r$ is the root of $T^{\prime}$.

This proves 3.4 for $\ell \geqslant 2$, since $\nabla_{\ell+1}^{-}$contains a $\nabla_{\ell}$ minor. For $\ell=1,3.4$ is straightforward.

We prove the above technical statement by induction on $m$. The case $m=0$ is clear since then $T^{\prime}$ is a single vertex $v$ and $\nabla_{1}^{-}$is just a path with three vertices. (Here we use that $\left|V\left(P_{v}\right)\right| \geqslant 3$.)

For the inductive step, let $a$ and $b$ be the children of $r$. By induction, $G_{a}:=\bigcup_{t \in V\left(T_{a}^{\prime}\right)} P_{t}$ contains a $\nabla_{m}^{-}$minor $H_{a}$ rooted on $P_{a}$, and $G_{b}:=\bigcup_{t \in V\left(T_{b}^{\prime}\right)} P_{t}$ contains a $\nabla_{m}^{-}$minor $H_{b}$ rooted on $P_{b}$.

We prove that $G_{a}$ and $G_{b}$ are vertex-disjoint, except possibly at a vertex of $V\left(P_{a}\right) \cap V\left(P_{b}\right)$ (there is at most one such vertex since $r$ is split). Suppose $v$ is a vertex appearing in both $G_{a}$ and $G_{b}$. Let $x$ be the vertex in $T_{a}^{\prime}$ closest to the root such that $v \in V\left(P_{x}\right)$ and let $y$ be the vertex in $T_{b}^{\prime}$ closest to the root such that $v \in V\left(P_{y}\right)$. By property (ii) of binary ear trees we know that no internal vertex of $P_{x}$ lies in $\bigcup_{z \in V\left(T^{1}\right) \backslash V\left(T_{x}^{\prime}\right)} V\left(P_{z}\right)$. Since $y \in V\left(T^{1}\right) \backslash V\left(T_{x}^{\prime}\right)$ and $v \in V\left(P_{y}\right)$, we conclude that $v$ is an end of $P_{x}$. This means that $v$ lies in $T_{p}^{\prime}$ where $p$ is the parent of $x$ in $T^{\prime}$. By the choice of $x$ this is only possible when $x=a$. Thus, $v$ is an end of $P_{a}$ and lies in $P_{r}$. By a symmetric argument we conclude that $v$ is an end of $P_{b}$ as well, as desired.

Let $a_{1}$ and $a_{2}$ be the ends of $P_{a}, b_{1}$ and $b_{2}$ be the ends of $P_{b}$, and $r_{1}$ and $r_{2}$ be the ends of $P_{r}$. By symmetry, we may assume that the ordering of these points along $P_{r}$ is either $r_{1}, a_{1}, b_{1}, a_{2}, b_{2}, r_{2}$ or $r_{1}, a_{1}, a_{2}, b_{1}, b_{2}, r_{2}$. (Note that some vertices may coincide.) Using the observation from the previous paragraph, we obtain a $\nabla_{m+1}^{-}$minor rooted on $P_{r}$ by considering the union of the $\nabla_{m}^{-}$minor rooted on $P_{a}$ and the $\nabla_{m}^{-}$minor rooted on $P_{b}$ that we were given, and contracting the following three subpaths of $P_{r}: r_{1} P_{r} a_{1}, a_{2} P_{r} b_{1}$, and $b_{2} P_{r} r_{2}$. Notice that if $G_{a}$ and $G_{b}$ have a vertex $v$ in common, then $v=a_{2}=b_{1}$. See Figure 6 for an illustration of the construction.
3.5. If $T$ contains a subdivision $T^{2}$ of $\Gamma_{2 \ell-1}$ such that every branching vertex is nested, then $\bigcup_{t \in V\left(T^{2}\right)} P_{t}$ contains $\Gamma_{\ell}^{+}$as a minor.

Subproof. Consider the clean binary ear tree $\left(T^{2}, \mathcal{P}^{2}\right)$ where $\mathcal{P}^{2}=\left\{P_{t}: t \in V\left(T^{2}\right)\right\}$. First, for every subdivision vertex $y$ of $T^{2}$ with child $z$, we apply 3.3 to $\left(T^{2}, \mathcal{P}^{2}\right)$ in order to suppress vertex $y$. Note that every branching vertex of $T^{2}$ stays nested. In particular, this is true if $z$ is branching. Hence, we may assume from now on that $T^{2}$ has no subdivision vertices.

Orient each path in $\mathcal{P}^{2}$ inductively as follows. Let $r$ be the root of $T^{2}$ and orient $P_{r}$ arbitrarily. If $P_{s}$ has already been oriented and $t$ is a child of $s$ in $T^{2}$, then orient $P_{t}$ so that $P_{s} \cup P_{t}$ does not contain a directed cycle. Consider each path in $\mathcal{P}^{2}$ to be oriented from left to right, and thus with left and right ends.

Let $t$ be a non-leaf vertex of $T^{2}$ and let $u$ and $v$ be the children of $t$. Define $t$ to be left-good if the left end of $P_{t}$ is not in $P_{u}$ nor $P_{v}$. Define $t$ to be right-good if the right


Figure 6. Inductively constructing a $\nabla_{3}^{-}$minor.
end of $P_{t}$ is not in $P_{u}$ nor $P_{v}$. Since $\left(T^{2}, \mathcal{P}^{2}\right)$ is clean we know that every non-leaf vertex $t$ of $T^{2}$ is left-good or right-good. We colour the non-leaf vertices of $T^{2}$ with left and right in such a way that when a vertex is coloured left (right), then it is left-good (right-good). Applying 3.2 on the tree $T^{2}$ with branching vertices coloured this way in which we remove all the leaves, we obtain a subdivision $T^{*}$ of $\Gamma_{\ell-1}$ such that all original vertices are coloured left, or all are coloured right, say without loss of generality left. For every leaf of $T^{*}$, add back to $T^{*}$ its two children in $T^{2}$, and denote by $T^{3}$ the resulting tree. Note that $T^{3}$ is a subdivision of $\Gamma_{\ell}$ and all branching vertices of $T^{3}$ are left-good.

We focus on the clean binary ear tree $\left(T^{3}, \mathcal{P}^{3}\right)$ induced by $T^{3}$, where $\mathcal{P}^{3}=\left\{P_{t}: t \in\right.$ $\left.V\left(T^{3}\right)\right\}$. Then, for every subdivision vertex $y$ of $T^{3}$ with child $z$, we apply 3.3 to $\left(T^{3}, \mathcal{P}^{3}\right)$ in order to suppress vertex $y$, as before. Note that every branching vertex of $T^{3}$ stays nested and left-good. Hence, we may assume from now on that $T^{3}$ has no subdivision vertices.

Let $t$ be a non-leaf vertex of $T^{3}$ and $u$ and $v$ be the children of $t$ in $T^{3}$. Let $f(t)$ be the first vertex of $P_{t}$ that is a left end of either $P_{u}$ or of $P_{v}$. Note that $f(t)$ is not the left end of $P_{t}$, since $t$ is left-good. Let $e(t)$ be the last edge of $P_{t}$ incident to a left end of either $P_{u}$ or $P_{v}$. If $t$ is a leaf of $T^{3}$, we define $f(t)$ to be any internal vertex of $P_{t}$ and $e(t)$ to be the last edge of $P_{t}$ incident to $f(t)$.

Let $H:=\bigcup_{t \in V\left(T^{3}\right)} P_{t}$ and $M:=\left\{e(t): t \in V\left(T^{3}\right)\right\}$. Since every branching vertex of $T^{3}$ is nested, $H \backslash M$ contains two components $H_{\text {left }}$ and $H_{\text {right }}$ such that $H_{\text {left }}$ contains all left ends of $\left\{P_{t}: t \in V\left(T^{3}\right)\right\}$ and $H_{\text {right }}$ contains all right ends of $\left\{P_{t}: t \in V\left(T^{3}\right)\right\}$. Using that every branching vertex of $T^{3}$ is left-good, it is easy to see that $H_{\text {left }}$ contains a subdivision $T^{4}$ of $\Gamma_{\ell}$ whose set of original vertices is $\left\{f(t): t \in V\left(T^{3}\right)\right\}$; see Figure 7 . By construction, each leaf of $T^{4}$ is incident to an edge in $M$. Also, $H_{\text {right }}$ is clearly connected. Therefore, after contracting all edges of $H_{\text {right }}, T^{4} \cup M \cup H_{\text {right }}$ contains a $\Gamma_{\ell}^{+}$minor.

This ends the proof of 3.1.


Figure 7. A $\Gamma_{3}$ minor in $H_{\text {left }}$.

## 4. Binary Pear Trees

In order to prove our main theorem, we need something slightly more general than binary ear trees, which we now define. A binary pear tree in a graph $G$ is a pair $(T, \mathcal{B})$, where $T$ is a binary tree, and $\mathcal{B}=\left\{\left(P_{x}, Q_{x}\right): x \in V(T)\right\}$ is a collection of pairs of paths of $G$ of length at least 2 such that $P_{x} \subseteq Q_{x}$ for all $x \in V(T)$, and the following properties are satisfied for each non-root vertex $x \in V(T)$.
(i) $Q_{x}$ is a $P_{y}$-ear, where $y$ is the parent of $x$ in $T$;
(ii) if $x$ has no sibling then no internal vertex of $Q_{x}$ is in $\bigcup_{z \in V(T) \backslash V\left(T_{x}\right)} V\left(Q_{z}\right)$;
(iii) if $x$ has a sibling $x^{\prime}$ then

- no internal vertex of $Q_{x}$ is in $\bigcup_{z \in V(T) \backslash\left(V\left(T_{x}\right) \cup V\left(T_{x^{\prime}}\right)\right)} V\left(Q_{z}\right)$, and
- no internal vertex of $P_{x}$ is in $Q_{x^{\prime}}$.

Furthermore, the binary pear tree is clean if for every non-leaf vertex $y$ of $T$, there is an end of $P_{y}$ that is not contained in any $Q_{x}$ where $x$ is a child of $y$.

Note that if $\left(T,\left\{P_{x}: x \in V(T)\right\}\right)$ is a clean binary ear tree, then $\left(T,\left\{\left(P_{x}, P_{x}\right): x \in\right.\right.$ $V(T)\})$ is a clean binary pear tree. We now prove the following converse.
4.1. If $G$ has a clean binary pear tree $(T, \mathcal{B})$, then $G$ has a minor $H$ such that $H$ has a clean binary ear tree $(T, \mathcal{P})$.

Proof. Say $\mathcal{B}=\left\{\left(P_{v}, Q_{v}\right): v \in V(T)\right\}$. We prove the stronger result that there exist $H$ and $\left(T,\left\{P_{v}^{\prime}: v \in V(T)\right\}\right)$ such that $H$ is a minor of $G,\left(T,\left\{P_{v}^{\prime}: v \in V(T)\right\}\right)$ is a clean binary ear tree in $H$, and $P_{v} \subseteq P_{v}^{\prime}$ for all leaves $v$ of $T$. This last property will be referred to as the leaf property; note that this is a property of $\left(T,\left\{P_{v}^{\prime}: v \in V(T)\right\}\right)$ w.r.t. the pair $(T, \mathcal{B})$ (which is fixed). Arguing by contradiction, suppose that this result is not true. Among all counterexamples, choose $(G,(T, \mathcal{B}))$ such that $|E(G)|$ is minimum. This clearly implies that $|V(T)|>1$.

Let $y$ be a deepest leaf in $T$. If $y$ has a sibling, let $z$ denote this sibling, which is also a leaf of $T$. Let $x$ be the parent of $y$ in $T$. Delete from $G$ the internal vertices of $Q_{y}$ and $Q_{z}$ (if $z$ exists), and denote by $G^{-}$the resulting graph. Note that $\left|E\left(G^{-}\right)\right|<|E(G)|$ since $Q_{y}$ has length at least 2. Let $T^{-}$be the tree obtained from $T$ by removing $y$ and $z$ (if $z$ exists). Notice that no internal vertex of $Q_{y}$ or $Q_{z}$ appears in a path $Q_{v}$ with $v \in V\left(T^{-}\right)$, by properties (ii) and (iii) of the definition of binary pear trees. Thus $\left(T^{-},\left\{\left(P_{v}, Q_{v}\right): v \in V\left(T^{-}\right)\right\}\right)$is a clean binary pear tree. By minimality, $G^{-}$has a minor $H^{-}$such that $H^{-}$has a clean binary ear tree $\left(T^{-},\left\{P_{v}^{-}: v \in V\left(T^{-}\right)\right\}\right)$such that $P_{v} \subseteq P_{v}^{-}$for all leaves $v$ of $T^{-}$. Since $x$ is a leaf of $T^{-}$, we have $P_{x} \subseteq P_{x}^{-}$.

Notice that $Q_{y}$ and $Q_{z}$ (if $z$ exists) are $P_{x}^{-}$-ears. If $z$ does not exist, then let $P_{y}^{-}:=Q_{y}$ and observe that $\left(T,\left\{P_{v}^{-}: v \in V(T)\right\}\right)$ is a clean binary ear tree satisfying the leaf property, contradicting the fact that $(G,(T, \mathcal{B}))$ is a counterexample. Thus, $z$ must exist.

Consider an internal vertex $v$ of $Q_{y}$. If $v$ is included in $Q_{z}$ then $v$ cannot be an end of $Q_{z}$, because ends of $Q_{z}$ are in $P_{x}$, which would imply that $v$ is an end of $Q_{y}$ as well. Thus, if $Q_{y}$ and $Q_{z}$ have a vertex in common, either this vertex is a common end of both paths, or it is internal to both paths.

If $Q_{y}$ and $Q_{z}$ have no internal vertex in common, let $P_{y}^{-}:=Q_{y}$ and $P_{z}^{-}:=Q_{z}$. Note that $\left(T,\left\{P_{v}^{-}: v \in V(T)\right\}\right)$ is a clean binary ear tree satisfying the leaf property, a contradiction. Hence, $Q_{y}$ and $Q_{z}$ must have at least one internal vertex in common.

Next, given an edge $e \in E(G)$ and a path $P$ in $G$, define $P / / e$ to be $P$ if $e \notin E(P)$ and $P / e$ if $e \in E(P)$, and let $\mathcal{B} / e:=\left\{\left(P_{v} / / e, Q_{v} / / e\right): v \in V(T)\right\}$. Suppose that there is an edge $e \in E\left(Q_{y}\right) \cap E\left(Q_{z}\right)$. Since $\left|E\left(P_{y}\right)\right| \geqslant 2$ and $\left|E\left(P_{z}\right)\right| \geqslant 2$, property (iii) of the definition of binary pear trees implies that $e \notin E\left(P_{y}\right) \cup E\left(P_{z}\right)$. Thus $P_{y} / / e=P_{y}$ and $P_{z} / / e=P_{z}$. It follows that $(T, \mathcal{B} / e)$ is a clean binary pear tree of $G / e$, which contradicts the minimality of the counterexample. Hence, no such edge e exists.

So far we established that the two paths $Q_{y}$ and $Q_{z}$ have at least one internal vertex in common and are edge-disjoint. The rest of the proof is split into a number of cases. In each case, we show that either there is an edge $e$ of $G$ such that $G \backslash e$ still has a clean binary pear tree which is indexed by the same tree $T$, or that there is a way to modify $(T, \mathcal{B})$ so that it remains a clean binary pear tree of $G$, and after the modification the two paths $Q_{y}$ and $Q_{z}$ have at least one edge in common. Note that each outcome contradicts the minimality of our counterexample; in the latter case, this is because we can then apply the argument of the previous paragraph and obtain a smaller counterexample.


Figure 8. Cases in the proof of 4.1. $P_{x}$ is drawn in black, $Q_{y}$ in red, and $Q_{z}$ in blue. The bold subpaths of $Q_{y}$ and $Q_{z}$ denote respectively $P_{y}$ and $P_{z}$. The dotted lines illustrate the modifications of the paths $P_{x}, Q_{y}, Q_{z}$.

Let us now proceed with the case analysis, see Figure 8 for an illustration of the different cases. Choose an orientation of $P_{x}$ from left to right, let $x_{1}$ denote its left end and $x_{2}$ denote its right end, and let $y_{1}, y_{2}$ and $z_{1}, z_{2}$ be the two ends of respectively $Q_{y}$ and $Q_{z}$ on $P_{x}$, ordered from left to right. Given two vertices $u, v$ of $P_{x}$, let us simply write $u \leqslant v$ if $u=v$ or $u$ is to the left of $v$ on $P_{x}$. Without loss of generality, we may assume that $y_{1} \leqslant z_{1}$.

Recalling that $Q_{y}$ and $Q_{z}$ have an internal vertex in common, let $v_{1}$ be the first such vertex on the path $Q_{y}$ starting from $y_{1}$. Note that either $P_{y} \subseteq y_{1} Q_{y} v_{1}$ or $P_{y} \subseteq v_{1} Q_{y} y_{2}$, and similarly either $P_{z} \subseteq z_{1} Q_{z} v_{1}$ or $P_{z} \subseteq v_{1} Q_{z} z_{2}$, by property (iii) of the definition of binary pear trees.

First suppose that $P_{y} \subseteq y_{1} Q_{y} v_{1}$ and $P_{z} \subseteq z_{1} Q_{z} v_{1}$. Let $Q_{y}^{1}:=y_{1} Q_{y} v_{1} Q_{z} z_{2}$. (The superscript denotes the case number.) It is easily checked that replacing $Q_{y}$ with $Q_{y}^{1}$ in
$(T, \mathcal{B})$ gives another clean binary pear tree of $G$. Moreover, $Q_{y}^{1}$ and $Q_{z}$ have the path $v_{1} Q_{z} z_{2}$ in common, which contains at least one edge, as desired.

Next suppose that $P_{y} \subseteq y_{1} Q_{y} v_{1}$ and $P_{z} \subseteq v_{1} Q_{z} z_{2}$. We consider whether some internal vertex of the path $v_{1} Q_{z} z_{1}$ is in $Q_{y}$. If there is one, let $v_{2}$ be the last such vertex that is met when going along $Q_{y}$ from $y_{1}$ to $y_{2}$. Let $Q_{y}^{2}:=y_{1} Q_{y} v_{1} Q_{z} v_{2} Q_{y} y_{2}$, and replace $Q_{y}$ with $Q_{y}^{2}$ in $(T, \mathcal{B})$ as in the previous paragraph. Note that $Q_{y}^{2}$ and $Q_{z}$ have the path $v_{1} Q_{z} v_{2}$ in common, and thus at least one edge in common, as desired.

If no internal vertex of $v_{1} Q_{z} z_{1}$ is in $Q_{y}$, we consider whether $y_{1}<z_{1}$ or $y_{1}=z_{1}$. If $y_{1}<z_{1}$, let $Q_{y}^{3}:=y_{1} Q_{y} v_{1} Q_{z} z_{1}$, and replace $Q_{y}$ with $Q_{y}^{3}$ in $(T, \mathcal{B})$. In particular, $Q_{y}^{3}$ and $Q_{z}$ now have the path $v_{1} Q_{z} z_{1}$ in common, and thus at least one edge in common, as desired.

If $y_{1}=z_{1}$, we adopt a different strategy. Let $P_{x}^{4}:=x_{1} P_{x} y_{1} Q_{z} v_{1} Q_{y} y_{2} P_{x} x_{2}$ and let $Q_{x}^{4}$ be the path obtained from $Q_{x}$ by replacing the $P_{x}$ section with $P_{x}^{4}$. Let $Q_{y}^{4}:=y_{1} Q_{y} v_{1}$. Let $w_{1}$ be the first vertex of $Q_{y}$ that is met when starting in $P_{z}$ and walking along $Q_{z}$ toward $z_{1}$. (Note that possibly $w_{1}=v_{1}$.) Let $w_{2}$ be the first vertex of $Q_{y}$ that is met when starting in $P_{z}$ and walking along $Q_{z}$ toward $z_{2}$, if there is one. Let $Q_{z}^{4}:=w_{1} Q_{z} w_{2}$ if $w_{2}$ exists, otherwise let $Q_{z}^{4}:=w_{1} Q_{z} z_{2} P_{x} y_{2}$. Finally, let $e$ be the edge of $P_{x}$ incident to $z_{1}$ that is to the right of $z_{1}$. Observe that $e$ is not included in any of the three paths $Q_{x}^{4}, Q_{y}^{4}, Q_{z}^{4}$. Now, it can be checked that replacing $P_{x}, Q_{x}, Q_{y}, Q_{z}$ in $(T, \mathcal{B})$ with their newly defined counterparts produces a clean binary pear tree of $G \backslash e$, giving the desired contradiction. This concludes the case that $P_{y} \subseteq y_{1} Q_{y} v_{1}$ and $P_{z} \subseteq v_{1} Q_{z} z_{2}$.

Next suppose that $P_{y} \subseteq v_{1} Q_{y} y_{2}$ and $P_{z} \subseteq v_{1} Q_{z} z_{2}$. Let $Q_{z}^{5}:=y_{1} Q_{y} v_{1} Q_{z} z_{2}$. Replacing $Q_{z}$ with $Q_{z}^{5}$ in $(T, \mathcal{B})$ gives another clean binary pear tree of $G$. Moreover, $Q_{y}$ and $Q_{z}^{5}$ have the path $y_{1} Q_{y} v_{1}$ in common, which contains at least one edge, as desired.

Finally, suppose that $P_{y} \subseteq v_{1} Q_{y} y_{2}$ and $P_{z} \subseteq z_{1} Q_{z} v_{1}$. Let $v_{2}$ be the first common internal vertex of $Q_{y}$ and $Q_{z}$ that is met when starting in $z_{2}$ and walking along $Q_{z}$ toward $v_{1}$. (Note that possibly $v_{2}=v_{1}$.) If $P_{y} \subseteq v_{1} Q_{y} v_{2}$ then let $Q_{y}^{6}:=y_{1} Q_{y} v_{2} Q_{z} z_{2}$. Replacing $Q_{y}$ with $Q_{y}^{6}$ in $(T, \mathcal{B})$ gives another clean binary pear tree of $G$. Moreover, $Q_{y}^{6}$ and $Q_{z}$ have the path $v_{2} Q_{z} z_{2}$ in common, which contains at least one edge, as desired.

If $P_{y} \subseteq v_{2} Q_{y} y_{2}$ then consider whether $y_{2}=z_{2}$. If $y_{2} \neq z_{2}$ then let $Q_{y}^{7}:=y_{2} Q_{y} v_{2} Q_{z} z_{2}$. Replacing $Q_{y}$ with $Q_{y}^{7}$ in $(T, \mathcal{B})$ gives another clean binary pear tree of $G$. Moreover, $Q_{y}^{7}$ and $Q_{z}$ have the path $v_{2} Q_{z} z_{2}$ in common, which contains at least one edge, as desired.

If $y_{2}=z_{2}$, then let $P_{x}^{8}:=x_{1} P_{x} y_{1} Q_{y} v_{2} Q_{z} z_{2} P_{x} x_{2}$ and let $Q_{x}^{8}$ be the path obtained from $Q_{x}$ by replacing the $P_{x}$ section with $P_{x}^{8}$. Let $Q_{y}^{8}:=v_{2} Q_{y} y_{2}$. Let $w_{1}$ be the first vertex of $Q_{y}$ that is met when starting in $P_{z}$ and walking along $Q_{z}$ toward $z_{1}$, if there is one. Let $w_{2}$ be the first vertex of $Q_{y}$ that is met when starting in $P_{z}$ and walking along $Q_{z}$ toward $z_{2}$. (Note that possibly $w_{2}=v_{1}$.) Let $Q_{z}^{8}:=w_{1} Q_{z} w_{2}$ if $w_{1}$ exists, otherwise let $Q_{z}^{8}:=y_{1} P_{x} z_{1} Q_{z} w_{2}$. Let $e$ be the edge of $P_{x}$ incident to $z_{1}$ that is to the right of $z_{1}$. Observe that $e$ is not included in any of the three paths $Q_{x}^{8}, Q_{y}^{8}, Q_{z}^{8}$. Now, it can be checked that replacing $P_{x}, Q_{x}, Q_{y}, Q_{z}$ in $(T, \mathcal{B})$ with their newly defined
counterparts produces a clean binary pear tree of $G \backslash e$, giving the desired contradiction. This concludes the proof.

## 5. Finding Binary Pear Trees

A binary tree is full if every internal vertex has exactly two children. The main result of this section is the following.
5.1. For all integers $\ell \geqslant 1$ and $k \geqslant 9 \ell^{2}-3 \ell+1$, if $G$ is a minor-minimal 2 -connected graph containing a subdivision of $\Gamma_{k}$ and $T^{1}$ is a full binary tree of height at most $3 \ell-2$, then either $G$ contains $\Gamma_{\ell}^{+}$as a minor, or $G$ contains a clean binary pear tree $\left(T^{1}, \mathcal{B}\right)$.

We proceed via a sequence of lemmas.
5.2. If $G$ is a minor-minimal 2-connected graph containing a subdivision of $\Gamma_{k}$, then every subdivision of $\Gamma_{k}$ in $G$ is a spanning tree.

Proof. Let $T$ be a subdivision of $\Gamma_{k}$ in $G$. We use the well-known fact that for all $e \in E(G)$, at least one of $G \backslash e$ or $G / e$ is 2-connected. Therefore, if some edge $e$ of $G$ has an end not in $V(T)$, then $G \backslash e$ or $G / e$ is a 2-connected graph containing a subdivision of $\Gamma_{k}$, which contradicts the minor-minimality of $G$.
5.3. Let $1 \leqslant \ell \leqslant k$ and let $T$ be a tree isomorphic to $\Gamma_{k}$ with root $r$. Suppose that $a$ non-empty subset of vertices of $T$ are marked. Then
(i) $T$ contains a subdivision of $\Gamma_{\ell}$, all of whose leaves are marked, or
(ii) there exist a vertex $v \in V(T)$ and a child $w$ of $v$ such that $T_{v}$ has at least one marked vertex but $T_{w}$ has none, and $w$ is at distance at most $\ell$ from $r$.

Proof. A vertex $v$ in $T$ is good if there is a marked vertex in $T_{v}$, and is bad otherwise. Let $T^{\prime}$ be the subtree of $T$ induced by vertices at distance at most $\ell$ from $r$ in $T$. If each leaf of $T^{\prime}$ is good, then for each such leaf $u$ we can find a marked vertex $m_{u}$ in $T_{u}$, and $T^{\prime} \cup \bigcup\left\{u T m_{u}: u\right.$ leaf of $\left.T^{\prime}\right\}$ is a $\Gamma_{\ell}$ subdivision with all leaves marked, as required by (i). Now assume that some leaf $u$ of $T^{\prime}$ is bad. Let $w$ be the bad vertex closest to $r$ on the $r T u$ path. Since some vertex in $T$ is marked, $r$ is good. Thus $w \neq r$. Moreover, the parent $v$ of $w$ is good, by our choice of $w$. Also, $w$ is at distance at most $\ell$ from $r$. Therefore, $v$ and $w$ satisfy (ii).

Our main technical tools are 5.4 and 5.5 below, which are lemmas about 2-connected graphs $G$ containing a subdivision $T$ of $\Gamma_{k}$ as a spanning tree. In order to state them, we need to introduce some definitions and notation.

For the next two paragraphs, let $G$ be a 2-connected graph containing a subdivision $T$ of $\Gamma_{k}$ as a spanning tree. For each vertex $v \in V(G)$, let $\mathrm{h}(v)$ be the number of original non-leaf vertices on the path $v T w$, where $w$ is any leaf of $T_{v}$. We stress the fact that
subdivision vertices are not counted when computing $\mathrm{h}(v)$. Since the length of a path in $\Gamma_{k}$ from a fixed vertex to any leaf is the same, $\mathrm{h}(v)$ is independent of the choice of $w$. We also use the shorthand notation $\operatorname{Out}(v):=V(G) \backslash V\left(T_{v}\right)$ when $G$ and $T$ are clear from the context. For $X, Y \subseteq V(G)$, we say that $X$ sees $Y$ if $x y \in E(G)$ for some $x \in X$ and $y \in Y$. If $P$ is a path with ends $x$ and $y$, and $Q$ is a path with ends $y$ and $z$, then let $P Q$ be the walk that follows $P$ from $x$ to $y$ and then follows $Q$ from $y$ to $z$.

A path $P$ of $G$ is $(x, a, y)$-special if $|V(P)| \geqslant 3$, and $x, y$ are the ends of $P$, and $a$ is a child of $x$ such that $V(P) \backslash\{x, y\} \subseteq V\left(T_{a}\right)$ and $y \notin V\left(T_{a}\right)$. A vertex $w$ is safe for an ( $x, a, y$ )-special path $P$ if $w$ satisfies the following properties:

- the parent $v$ of $w$ is in $V(P) \backslash\{x, y\}$;
- $\mathrm{h}(v) \geqslant \mathrm{h}(x)-2 \ell$;
- $V(P) \cap V\left(T_{w}\right)=\emptyset$;
- $V\left(T_{w}\right)$ does not see $\operatorname{Out}(a) \backslash\{x\}$, and
- if $v$ is an original vertex and $u$ is its child distinct from $w$, then either $V(P) \cap V\left(T_{u}\right) \neq$ $\emptyset$ or $V\left(T_{u}\right)$ does not see $\operatorname{Out}(a) \backslash\{x\}$.
5.4. Let $1 \leqslant \ell \leqslant k$. Let $G$ be a minor-minimal 2 -connected graph containing a subdivision of $\Gamma_{k}$. Let $T$ be a subdivision of $\Gamma_{k}$ in $G, v \in V(T)$ with $\mathrm{h}(v) \geqslant 3 \ell+1$, and $w$ be a child of $v$. Then, either $G$ contains a $\Gamma_{\ell}^{+}$minor, or there is a $\left(v_{0}, w_{0}, v_{0}^{\prime}\right)$-special path $P$ and two distinct safe vertices for $P$ such that:
- $V(P) \subseteq V\left(T_{w}\right)$,
- $\mathrm{h}\left(v_{0}\right) \geqslant \mathrm{h}(v)-\ell$,
- $V\left(T_{v_{0}}\right)$ sees Out $(w) \backslash\{v\}$,
- $V\left(T_{w_{0}}\right)$ does not see $\operatorname{Out}(w) \backslash\{v\}$, and
- $V\left(T_{u_{0}}\right)$ sees $\operatorname{Out}\left(v_{0}\right)$ if $v_{0}$ is an original vertex and $u_{0}$ is its child distinct from $w_{0}$.

Proof. By 5.2, $T$ is a spanning tree of $G$. Colour red each vertex of $T_{w}$ that sees a vertex in $\operatorname{Out}(w) \backslash\{v\}$. Observe that there is at least one red vertex. Indeed, $V\left(T_{w}\right)$ must see Out $(w) \backslash\{v\}$, for otherwise $v$ would be a cut vertex separating $V\left(T_{w}\right)$ from $\operatorname{Out}(w) \backslash\{v\}$ in $G$.

Let $\tilde{T}_{w}$ be the complete binary tree obtained from $T_{w}$ by iteratively contracting each edge of the form $p q$ with $p$ a subdivision vertex and $q$ the child of $p$ into vertex $q$. Declare $q$ to be coloured red after the edge contraction if at least one of $p, q$ was coloured red beforehand. Since $\mathbf{h}(w) \geqslant \mathbf{h}(v)-1 \geqslant 3 \ell$, the tree $\tilde{T}_{w}$ has height at least $3 \ell$.

If $\tilde{T}_{w}$ contains a subdivision of $\Gamma_{\ell}$ with all leaves coloured red, then so does $T_{w}$. Therefore, $G$ contains $\Gamma_{\ell}^{+}$as a minor, because $\operatorname{Out}(w)$ induces a connected subgraph of $G$ which is vertex-disjoint from $V\left(T_{w}\right)$ and which sees all the leaves of $T_{w}$. Thus, by 5.3, we may assume there is a vertex $\tilde{v}_{0}$ of $\tilde{T}_{w}$ and a child $\tilde{w}_{0}$ of $\tilde{v}_{0}$ with $\mathrm{h}\left(\tilde{w}_{0}\right) \geqslant \mathrm{h}(w)-\ell$ such that $T_{\tilde{v}_{0}}$ has at least one red vertex but $T_{\tilde{w}_{0}}$ has none. Going back to $T_{w}$, we deduce that there is a vertex $v_{0}$ of $T_{w}$ and a child $w_{0}$ of $v_{0}$ with $\mathrm{h}\left(w_{0}\right) \geqslant \mathrm{h}(w)-\ell$ such that $T_{v_{0}}$ has at
least one red vertex but $T_{w_{0}}$ has none. To see this, choose $v_{0}$ as the deepest red vertex in the preimage of $\tilde{v}_{0}$. Note that $v_{0}$ or $w_{0}$ could be subdivision vertices.

If $v_{0}$ is an original vertex, let $u_{0}$ denote the child of $v_{0}$ distinct from $w_{0}$. Since $v_{0}$ is not a cut vertex of $G$, one of the two subtrees $T_{u_{0}}$ and $T_{w_{0}}$ sees $\operatorname{Out}\left(v_{0}\right)$. If $T_{u_{0}}$ does not see Out $\left(v_{0}\right)$, then $T_{u_{0}}$ has no red vertex and $T_{w_{0}}$ sees Out $\left(v_{0}\right)$. Therefore, by exchanging $u_{0}$ and $w_{0}$ if necessary, we guarantee that the following two properties hold when $u_{0}$ exists.

$$
\begin{equation*}
T_{u_{0}} \text { sees } \operatorname{Out}\left(v_{0}\right) \quad \text { and } \quad T_{w_{0}} \text { has no red vertex. } \tag{1}
\end{equation*}
$$

We iterate this process in $T_{w_{0}}$. Colour blue each vertex of $T_{w_{0}}$ that sees a vertex in $\operatorname{Out}\left(w_{0}\right) \backslash\left\{v_{0}\right\}$. There is at least one blue vertex, since otherwise $v_{0}$ would be a cut vertex of $G$ separating $V\left(T_{w_{0}}\right)$ from $\operatorname{Out}\left(w_{0}\right) \backslash\left\{v_{0}\right\}$. Defining $\tilde{T}_{w_{0}}$ similarly as above, if $\tilde{T}_{w_{0}}$ contains a subdivision of $\Gamma_{\ell}$ with all leaves coloured blue, then $G$ has a $\Gamma_{\ell}^{+}$minor. Applying 5.3 and going back to $T_{w_{0}}$, we may assume there is a vertex $v_{1}$ of $T_{w_{0}}$ and a child $w_{1}$ of $v_{1}$ with $\mathbf{h}\left(w_{1}\right) \geqslant \mathrm{h}\left(w_{0}\right)-\ell$ such that $T_{v_{1}}$ has at least one blue vertex but $T_{w_{1}}$ has none.

We now define the $\left(v_{0}, w_{0}, v_{0}^{\prime}\right)$-special path $P$, and identify two distinct safe vertices for $P$. To do so, we will need to consider different cases. In all cases, the end $v_{0}^{\prime}$ will be a vertex of $\operatorname{Out}\left(w_{0}\right) \backslash\left\{v_{0}\right\}$ seen by a (carefully chosen) blue vertex in $T_{v_{1}}$, thus $v_{0}^{\prime} \notin V\left(T_{w_{0}}\right)$, and the path $P$ will be such that $V(P) \backslash\left\{v_{0}, v_{0}^{\prime}\right\} \subseteq V\left(T_{w_{0}}\right)$. Note that the end $v_{0}$ of $P$ satisfies $\mathrm{h}\left(v_{0}\right) \geqslant \mathrm{h}(v)-\ell$, as desired.

Before proceeding with the case analysis, we point out the following property of $G$. If $s t$ is an edge of $G$ such that $G / s t$ contains a subdivision of $\Gamma_{k}$, then $G / s t$ is not 2-connected by the minor-minimality of $G$, and it follows that $\{s, t\}$ is a cutset of $G$. Note that this applies if $s t$ is an edge of $T$ such that at least one of $s, t$ is a subdivision vertex, or if $s t$ is an edge of $E(G) \backslash E(T)$ linking two subdivision vertices of $T$ that are on the same subdivided path of $T$. This will be used below.

Case 1. $v_{1}$ is a subdivision vertex:
In this case, $v_{1}$ is the unique blue vertex in $T_{v_{1}}$. Let $v_{0}^{\prime}$ be a vertex of $\operatorname{Out}\left(w_{0}\right) \backslash\left\{v_{0}\right\}$ seen by the blue vertex $v_{1}$. Since $v_{1}$ is not a cut vertex of $G$, there is an edge st with $s \in V\left(T_{w_{1}}\right)$ and $t \in \operatorname{Out}\left(v_{1}\right)$. Note that $t \in V\left(T_{w_{0}}\right) \cup\left\{v_{0}\right\}$, since $T_{w_{1}}$ has no blue vertex.

Case 1.1. There is at least one original vertex on the path $v_{1} T s$ :
Let $q$ be the first original vertex on the path $v_{1} T s$. Let $s_{1}$ denote a child of $q$ not on the $q T s$ path. Let $q^{\prime}$ be the first original vertex distinct from $q$ on the $q T s$ path if any, and otherwise let $q^{\prime}:=s$ (note that possibly $q^{\prime}=q=s$ ). Let $s_{2}$ be a child of $q^{\prime}$ not on the $q T s$ path, and distinct from $s_{1}$ if $q^{\prime}=q$. As illustrated in Figure 9, define

$$
P:=v_{0} T t s T v_{1} v_{0}^{\prime} .
$$

Observe that $V(P) \backslash\left\{v_{0}, v_{0}^{\prime}\right\} \subseteq V\left(T_{w_{0}}\right)$, by construction. Observe also that the parent $q^{\prime}$ of $s_{2}$ satisfies $\mathrm{h}\left(q^{\prime}\right) \geqslant \mathrm{h}(q)-1=\mathrm{h}\left(v_{1}\right)-1 \geqslant \mathrm{~h}\left(v_{0}\right)-\ell-1 \geqslant \mathrm{~h}\left(v_{0}\right)-2 \ell$. It can be checked that $s_{1}, s_{2}$ are two distinct safe vertices for $P$, as desired.

Case 1.2. All vertices of the path $v_{1} T s$ are subdivision vertices:
In particular, $w_{1}$ is a subdivision vertex. We show that the unique child $q$ of $w_{1}$ is an original vertex, and therefore $s=w_{1}$. Indeed, assume not, and let $q^{\prime}$ denote the child of $q$. Since $v_{1}$ is not a cut vertex of $G$ but $\left\{v_{1}, w_{1}\right\}$ is a cutset of $G$, we deduce that $w_{1}$ sees a vertex $w_{1}^{\prime}$ in $\operatorname{Out}\left(v_{1}\right)$ and that $V\left(T_{q}\right)$ does not see $\operatorname{Out}\left(v_{1}\right)$. Similarly, because $w_{1}$ is not a cut vertex of $G$ but $\left\{w_{1}, q\right\}$ is a cutset of $G$, we deduce that $q v_{1} \in E(G)$ and that $V\left(T_{q^{\prime}}\right)$ does not see Out $\left(w_{1}\right)$. Since $q$ is not a cut vertex, some vertex $q^{\prime \prime} \in V\left(T_{q^{\prime}}\right)$ sees $\operatorname{Out}(q)$, and hence sees $w_{1}$ (since $V\left(T_{q^{\prime}}\right)$ does not see Out $\left(v_{1}\right)$ ). But then, because of the edges $q^{\prime \prime} w_{1}$ and $w_{1} w_{1}^{\prime}$, we see that $\left\{v_{1}, q\right\}$ cannot be a cutset of $G$. It follows that $G / v_{1} q$ is 2-connected and contains a $\Gamma_{k}$ minor, contradicting our assumption on $G$.

Hence, $q$ is an original vertex, and $s=w_{1}$. Since $w_{1}$ is not a cut vertex of $G$, there is an edge linking $V\left(T_{q}\right)$ to $\operatorname{Out}\left(w_{1}\right)$. Since $\left\{v_{1}, w_{1}\right\}$ is a cutset of $G$, this edge links some vertex $s^{\prime} \in V\left(T_{q}\right)$ to $v_{1}$.

Let $s_{1}$ denote a child of $q$ not on the $q T s^{\prime}$ path. Let $q^{\prime}$ be the first original vertex distinct from $q$ on the $q T s^{\prime}$ path if any, and otherwise let $q^{\prime}:=s^{\prime}$ (note that possibly $q^{\prime}=s^{\prime}=q$ ). Let $s_{2}$ be a child of $q^{\prime}$ not on the $q T s^{\prime}$ path, and distinct from $s_{1}$ if $q^{\prime}=q$. As illustrated in Figure 9, define

$$
P:=v_{0} T t w_{1} T s^{\prime} v_{1} v_{0}^{\prime} .
$$

Again, note that $V(P) \backslash\left\{v_{0}, v_{0}^{\prime}\right\} \subseteq V\left(T_{w_{0}}\right)$ by construction. Observe also that the parent $q^{\prime}$ of $s_{2}$ satisfies $\mathbf{h}\left(q^{\prime}\right) \geqslant \mathbf{h}(q)-1=\mathbf{h}\left(v_{1}\right)-1 \geqslant \mathbf{h}\left(v_{0}\right)-\ell-1 \geqslant \mathbf{h}\left(v_{0}\right)-2 \ell$. It is easy to see that $s_{1}, s_{2}$ are two distinct safe vertices for $P$, as desired.


Figure 9. Path $P$ and the safe vertices $s_{1}, s_{2}$. Cases 1.1 and 1.2

Case 2. $v_{1}$ is an original vertex:
Let $u_{1}$ denote the child of $v_{1}$ distinct from $w_{1}$. If $T_{u_{1}}$ has no blue vertex, then $v_{1}$ is the unique blue vertex in $T_{v_{1}}$. Let $v_{0}^{\prime}$ be a vertex of $\operatorname{Out}\left(w_{0}\right) \backslash\left\{v_{0}\right\}$ seen by the blue vertex
$v_{1}$. Define

$$
P:=v_{0} T v_{1} v_{0}^{\prime} .
$$

Clearly, $V(P) \backslash\left\{v_{0}, v_{0}^{\prime}\right\} \subseteq V\left(T_{w_{0}}\right)$, and $u_{1}, w_{1}$ are two distinct safe vertices for $P$.
Next, assume that $T_{u_{1}}$ has a blue vertex. In this case, we need to define an extra pair $v_{2}, w_{2}$ of vertices. Observe that $\mathbf{h}\left(u_{1}\right) \geqslant \mathbf{h}\left(w_{0}\right)-\ell \geqslant \mathbf{h}(w)-2 \ell=\mathbf{h}(v)-2 \ell-1 \geqslant \ell$. Let $\tilde{T}_{u_{1}}$ be the tree obtained from $T_{u_{1}}$, as before. Again, if $\tilde{T}_{u_{1}}$ contains a subdivision of $\Gamma_{\ell}$ all of whose leaves are blue, then $G$ contains an $\Gamma_{\ell}^{+}$minor. Thus, by 5.3 , we may assume there is a vertex $v_{2}$ of $T_{u_{1}}$ and a child $w_{2}$ of $v_{2}$ with $\mathbf{h}\left(w_{2}\right) \geqslant \mathbf{h}\left(u_{1}\right)-\ell=\mathbf{h}\left(w_{1}\right)-\ell$ such that $T_{v_{2}}$ has at least one blue vertex, but $T_{w_{2}}$ has none.

Case 2.1. $v_{2}$ is a subdivision vertex:
Here, $v_{2}$ is the unique blue vertex in $T_{v_{2}}$. Let $v_{0}^{\prime}$ be a vertex of $\operatorname{Out}\left(w_{0}\right) \backslash\left\{v_{0}\right\}$ seen by $v_{2}$. As illustrated in Figure 10, define

$$
P:=v_{0} T v_{2} v_{0}^{\prime} .
$$

Observe that $V(P) \backslash\left\{v_{0}, v_{0}^{\prime}\right\} \subseteq V\left(T_{w_{0}}\right)$ by construction, and that $w_{1}, w_{2}$ are two distinct safe vertices for $P$.

Case 2.2. $v_{2}$ is an original vertex:
Let $u_{2}$ be the child of $v_{2}$ distinct from $w_{2}$. Let $b_{2}$ denote a blue vertex in $V\left(T_{u_{2}}\right) \cup\left\{v_{2}\right\}$, distinct from $v_{2}$ if possible. Let $v_{0}^{\prime}$ be a vertex of $\operatorname{Out}\left(w_{0}\right) \backslash\left\{v_{0}\right\}$ seen by the blue vertex $b_{2}$. Define

$$
P:=v_{0} T b_{2} v_{0}^{\prime} .
$$

Again, $V(P) \backslash\left\{v_{0}, v_{0}^{\prime}\right\} \subseteq V\left(T_{w_{0}}\right)$ by construction.
If $b_{2} \neq v_{2}$, then $P$ intersects $V\left(T_{u_{2}}\right)$. If $b_{2}=v_{2}$, then $P$ avoids $V\left(T_{u_{2}}\right)$, and $V\left(T_{u_{2}}\right)$ has no blue vertex. That is, $V\left(T_{u_{2}}\right)$ does not see $\operatorname{Out}\left(w_{0}\right) \backslash\left\{v_{0}\right\}$. Using these observations, one can check that $w_{1}, w_{2}$ are two distinct safe vertices for $P$ in both cases; see Figure 10.
5.5. Let $1 \leqslant \ell \leqslant k$. Let $G$ be a minor-minimal 2 -connected graph containing a subdivision of $\Gamma_{k}$ and let $T$ be a subdivision of $\Gamma_{k}$ in $G$. Let $S$ be an ( $x, a, y$ )-special path with $\mathrm{h}(x) \geqslant 5 \ell+1$. Let $w$ be a safe vertex for $S$ and let $v \in V(S)$ denote the parent of $w$ in $T$. Then, either $G$ contains a $\Gamma_{\ell}^{+}$minor, or there is a $\left(v_{0}, w_{0}, v_{0}^{\prime}\right)$-special path $P$, two distinct safe vertices $w_{1}, w_{2}$ for $P$, and an $S$-ear $Q$ such that:
(a) $V(P) \subseteq V\left(T_{w}\right)$,
(b) $\mathrm{h}\left(v_{0}\right) \geqslant \mathrm{h}(x)-3 \ell$,
(c) $V\left(T_{w_{0}}\right)$ does not see $\operatorname{Out}(w) \backslash\{v\}$,
(d) $P \subseteq Q$,
(e) $V(Q) \backslash V(P) \subseteq \operatorname{Out}\left(w_{0}\right) \backslash\left\{v_{0}\right\}$,
(f) $V(Q) \subseteq V\left(T_{a}\right) \cup\{x\}$,
(g) $V(Q) \cap V\left(T_{w_{i}}\right)=\emptyset$ for $i=1,2$, and
(h) if $e \in E(Q) \backslash E(T)$ and no end of $e$ is in $V\left(T_{w}\right)$, then $v$ is an original vertex with children $u$, $w$, the path $S$ is disjoint from $V\left(T_{u}\right)$, and e links $V\left(T_{u}\right)$ to $\operatorname{Out}(v)$.


Figure 10. Path $P$ and the safe vertices $w_{1}, w_{2}$. Cases 2.1 and 2.2

Proof. By 5.2, $T$ is a spanning tree. Also, $G$ does not contain $\Gamma_{\ell}^{+}$as a minor (otherwise, we are done). Applying 5.4 on vertex $v$ and its child $w$, we obtain a ( $v_{0}, w_{0}, v_{0}^{\prime}$ )-special path $P$ and two distinct safe vertices $w_{1}, w_{2}$ for $P$ such that $V(P) \subseteq V\left(T_{w}\right) ; \mathrm{h}\left(v_{0}\right) \geqslant$ $\mathrm{h}(v)-\ell \geqslant \mathrm{h}(x)-3 \ell ; V\left(T_{v_{0}}\right)$ sees $\operatorname{Out}(w) \backslash\{v\} ; V\left(T_{w_{0}}\right)$ does not see Out $(w) \backslash\{v\}$; and if $v_{0}$ is an original vertex and $u_{0}$ is the child of $v_{0}$ distinct from $w_{0}$ then $V\left(T_{u_{0}}\right)$ sees $\operatorname{Out}\left(v_{0}\right)$. It remains to extend $P$ into an $S$-ear $Q$ satisfying properties (d)-(h). The proof is split into twelve cases, all of which are illustrated in Figure 11.

If $v$ is an original vertex, let $u$ denote the child of $v$ distinct from $w$. In order to simplify the arguments below, we let $V\left(T_{u}\right)$ be the empty set if $u$ is not defined (same for $u_{0}$ ).

First assume that $v_{0}^{\prime} \notin V\left(T_{u_{0}}\right)$. Then $v_{0}^{\prime} \in \operatorname{Out}\left(v_{0}\right) \cap V\left(T_{w}\right)$. Recall that $V\left(T_{v_{0}}\right) \backslash$ $V\left(T_{w_{0}}\right)=V\left(T_{u_{0}}\right) \cup\left\{v_{0}\right\}$ sees $\operatorname{Out}(w) \backslash\{v\}=V\left(T_{u}\right) \cup \operatorname{Out}(v)$. Suppose that there is an edge st $\in E(G)$ with $s \in V\left(T_{u_{0}}\right) \cup\left\{v_{0}\right\}$ and $t \in \operatorname{Out}(v)$. Note that $t \in V\left(T_{a}\right) \cup\{x\}$, since $w$ is a safe vertex for $S$. Let $v^{\prime}$ be the closest ancestor of $t$ in $T$ that lies on $S$. Note that $v^{\prime} \in V\left(T_{a}\right) \cup\{x\}$. Define

$$
Q_{1}:=v T v_{0}^{\prime} P v_{0} T s t T v^{\prime}
$$

Next, suppose that there is no such edge st. Then, there must be an edge st with $s \in V\left(T_{u_{0}}\right) \cup\left\{v_{0}\right\}$ and $t \in V\left(T_{u}\right)$. In particular, $u$ exists. If the path $S$ intersects $V\left(T_{u}\right)$, then let $v^{\prime}$ be a vertex in $V(S) \cap V\left(T_{u}\right)$ that is closest to $t$ in $T$. Define

$$
Q_{2}:=v T v_{0}^{\prime} P v_{0} T s t T v^{\prime}
$$

Otherwise, we have $V(S) \cap V\left(T_{u}\right)=\emptyset$. Since $w$ is a safe vertex for $S, V\left(T_{u}\right)$ does not see Out $(a) \backslash\{x\}$ in this case. If $V\left(T_{u}\right)$ sees $\operatorname{Out}(v)$, then let $s^{\prime} t^{\prime}$ be an edge with $s^{\prime} \in V\left(T_{u}\right)$ and $t^{\prime} \in \operatorname{Out}(v)$, and let $v^{\prime}$ be the closest ancestor of $t^{\prime}$ in $T$ that lies on $S$. Note that both $t^{\prime}$ and $v^{\prime}$ lie in $V\left(T_{a}\right) \cup\{x\}$. Define

$$
Q_{3}:=v T v_{0}^{\prime} P v_{0} T s t T s^{\prime} t^{\prime} T v^{\prime} .
$$



Figure 11. Definition of $S$-ears $Q_{1}, \ldots, Q_{12}$.

Otherwise, $V\left(T_{u}\right)$ does not see $\operatorname{Out}(v)$. Since $v$ is not a cut vertex in $G$, we deduce that $V\left(T_{w}\right)$ sees $\operatorname{Out}(v)$. As we already know that neither $V\left(T_{w_{0}}\right)$ nor $V\left(T_{u_{0}}\right) \cup\left\{v_{0}\right\}$ sees
$\operatorname{Out}(v)$, there is an edge $s^{\prime \prime} t^{\prime \prime} \in E(G)$ with $s^{\prime \prime} \in V\left(T_{w}\right) \backslash V\left(T_{v_{0}}\right)$ and $t^{\prime \prime} \in \operatorname{Out}(v)$. Again, since $w$ is safe for $S$, we know that $t^{\prime \prime} \in V\left(T_{a}\right) \cup\{x\}$. Let $v^{\prime}$ be the closest ancestor of $t^{\prime \prime}$ in $T$ that lies on $S$. Note that $v^{\prime} \in V\left(T_{a}\right) \cup\{x\}$. Define

$$
Q_{4}:=v T t s T v_{0} P v_{0}^{\prime} T s^{\prime \prime} t^{\prime \prime} T v^{\prime}
$$

Next, assume that $v_{0}^{\prime} \in V\left(T_{u_{0}}\right)$. In particular, $u_{0}$ exists. Recall that $V\left(T_{u_{0}}\right)$ sees $\operatorname{Out}\left(v_{0}\right)$. If $V\left(T_{u_{0}}\right)$ sees Out $(v)$ then let st be an edge with $s \in V\left(T_{u_{0}}\right)$ and $t \in \operatorname{Out}(v)$. Observe that $t \in V\left(T_{a}\right) \cup\{x\}$ since $w$ is safe for $S$. Let $v^{\prime}$ be the closest ancestor of $t$ in $T$ that lies on $S$. Note that $v^{\prime} \in V\left(T_{a}\right) \cup\{x\}$ as well. Define

$$
Q_{5}:=v T v_{0} P v_{0}^{\prime} T s t T v^{\prime} .
$$

Next, suppose that $V\left(T_{u_{0}}\right)$ does not see $\operatorname{Out}(v)$. If $V\left(T_{u_{0}}\right)$ sees $V\left(T_{u}\right)$, then let st be an edge with $s \in V\left(T_{u_{0}}\right)$ and $t \in V\left(T_{u}\right)$. In particular, $u$ exists. If $S$ intersects $V\left(T_{u}\right)$, then let $v^{\prime}$ be a vertex in $V(S) \cap V\left(T_{u}\right)$ that is closest to $t$ in $T$. Define

$$
Q_{6}:=v T v_{0} P v_{0}^{\prime} T s t T v^{\prime}
$$

Otherwise, we have $V(S) \cap V\left(T_{u}\right)=\emptyset$. Since $w$ is a safe vertex for $S, V\left(T_{u}\right)$ does not see Out $(a) \backslash\{x\}$ in this case. If $V\left(T_{u}\right)$ sees Out $(v)$, then let $s^{\prime} t^{\prime}$ be an edge with $s^{\prime} \in V\left(T_{u}\right)$ and $t^{\prime} \in \operatorname{Out}(v)$ and let $v^{\prime}$ be the closest ancestor of $t^{\prime}$ in $T$ that lies on $S$. Note that both $t^{\prime}$ and $v^{\prime}$ lie in $V\left(T_{a}\right) \cup\{x\}$. Define

$$
Q_{7}:=v T v_{0} P v_{0}^{\prime} T s t T s^{\prime} t^{\prime} T v^{\prime} .
$$

Next, suppose that $V\left(T_{u}\right)$ does not see $\operatorname{Out}(v)$. Since $v$ is not a cut vertex in $G$, we deduce that $V\left(T_{w}\right)$ sees Out $(v)$. As we already know that neither $V\left(T_{w_{0}}\right)$ nor $V\left(T_{u_{0}}\right)$ sees $\operatorname{Out}(v)$, there is an edge $s^{\prime \prime} t^{\prime \prime} \in E(G)$ with $s^{\prime \prime} \in\left(V\left(T_{w}\right) \backslash V\left(T_{v_{0}}\right)\right) \cup\left\{v_{0}\right\}$ and $t^{\prime \prime} \in \operatorname{Out}(v)$. Again, since $w$ is safe for $S, t^{\prime \prime} \in V\left(T_{a}\right) \cup\{x\}$. Let $v^{\prime}$ be the closest ancestor of $t^{\prime \prime}$ in $T$ that lies on $S$. Note that $v^{\prime} \in V\left(T_{a}\right) \cup\{x\}$. Define

$$
Q_{8}:=v T t s T v_{0}^{\prime} P v_{0} T s^{\prime \prime} t^{\prime \prime} T v^{\prime}
$$

We are done with the cases where $V\left(T_{u_{0}}\right)$ sees $\operatorname{Out}(v)$ or $V\left(T_{u}\right)$. Next, assume that $V\left(T_{u_{0}}\right)$ sees neither of these two sets. Since $V\left(T_{u_{0}}\right)$ sees Out $\left(v_{0}\right)$, there is an edge st with $s \in V\left(T_{u_{0}}\right)$ and $t \in V\left(T_{w}\right) \backslash V\left(T_{v_{0}}\right)$. Recall that $V\left(T_{v_{0}}\right)$ sees $\operatorname{Out}(w) \backslash\{v\}$. Since neither $V\left(T_{u_{0}}\right)$ nor $V\left(T_{w_{0}}\right)$ sees $\operatorname{Out}(w) \backslash\{v\}$, we conclude that $v_{0}$ sees $\operatorname{Out}(w) \backslash\{v\}$. If $v_{0}$ sees $\operatorname{Out}(v)$, then let $v_{0} t^{\prime}$ be an edge with $t^{\prime} \in \operatorname{Out}(v)$. Let $v^{\prime}$ be the closest ancestor of $t^{\prime}$ in $T$. As before, $\left\{t^{\prime}, v^{\prime}\right\} \subseteq V\left(T_{a}\right) \cup\{x\}$. Define

$$
Q_{9}:=v T t s T v_{0}^{\prime} P v_{0} t^{\prime} T v^{\prime} .
$$

Otherwise, $v_{0}$ sees $V\left(T_{u}\right)$. Let $v_{0} t^{\prime}$ be an edge with $t^{\prime} \in V\left(T_{u}\right)$. If $S$ intersects $V\left(T_{u}\right)$, then let $v^{\prime}$ be a vertex in $V(S) \cap V\left(T_{u}\right)$ that is closest to $t^{\prime}$ in $T$. Define

$$
Q_{10}:=v T t s T v_{0}^{\prime} P v_{0} t^{\prime} T v^{\prime}
$$

Otherwise, $V(S) \cap V\left(T_{u}\right)=\emptyset$. Since $w$ is a safe vertex for $S$, we know that $V\left(T_{u}\right)$ does not see $\operatorname{Out}(a) \backslash\{x\}$ in this case. If $V\left(T_{u}\right)$ sees Out $(v)$, then let $s^{\prime \prime} t^{\prime \prime}$ be an edge with $s^{\prime \prime} \in V\left(T_{u}\right)$ and $t^{\prime \prime} \in \operatorname{Out}(v)$ and let $v^{\prime}$ be the closest ancestor of $t^{\prime \prime}$ in $T$ that lies on $S$. Note that both $t^{\prime \prime}$ and $v^{\prime}$ lie in $V\left(T_{a}\right) \cup\{x\}$. Define

$$
Q_{11}:=v T t s T v_{0}^{\prime} P v_{0} t^{\prime} T s^{\prime \prime} t^{\prime \prime} T v^{\prime}
$$

Otherwise, $V\left(T_{u}\right)$ does not see $\operatorname{Out}(v)$. Since $v$ is not a cut vertex in $G$, we deduce that $V\left(T_{w}\right)$ sees Out $(v)$. As we already know that neither $V\left(T_{w_{0}}\right)$ nor $V\left(T_{u_{0}}\right) \cup\left\{v_{0}\right\}$ sees Out $(v)$, there is an edge $s^{\prime \prime} t^{\prime \prime} \in E(G)$ with $s^{\prime \prime} \in V\left(T_{w}\right) \backslash V\left(T_{v_{0}}\right)$ and $t^{\prime \prime} \in \operatorname{Out}(v)$. Again, since $w$ is safe for $S, t^{\prime \prime} \in V\left(T_{a}\right) \cup\{x\}$. Let $v^{\prime}$ be the closest ancestor of $t^{\prime \prime}$ in $T$ that lies on $S$. Note that $v^{\prime} \in V\left(T_{a}\right) \cup\{x\}$. Define

$$
Q_{12}:=v T t^{\prime} v_{0} P v_{0}^{\prime} T s t T s^{\prime \prime} t^{\prime \prime} T v^{\prime} .
$$

One can check that for all $i \in[12]$, if we set $Q=Q_{i}$, then $Q$ is an $S$-ear satisfying properties (d)-(h).

We now prove 5.1 using 5.4 and 5.5.

Proof of 5.1. Let $T$ be a subdivision of $\Gamma_{k}$ in $G$, which is a spanning tree of $G$ by 5.2. Also, $G$ has no $\Gamma_{\ell}^{+}$minor (otherwise, we are done). As before, for $v \in V(G)$, we let $\mathrm{h}(v)$ be the number of original non-leaf vertices on the path $v T w$, where $w$ is any leaf of $T_{v}$. The depth of $x \in V\left(T^{1}\right)$, denoted $\mathrm{d}(x)$, is the number of edges in $x T^{1} r$, where $r$ is the root of $T^{1}$.

We prove the stronger statement that $G$ contains a clean binary pear tree $\left(T^{1},\left\{\left(P_{x}, Q_{x}\right)\right.\right.$ : $\left.\left.x \in V\left(T^{1}\right)\right\}\right)$ such that:
(1) for all $x \in V\left(T^{1}\right)$, the path $P_{x}$ is a $\left(v_{x}, w_{x}, v_{x}^{\prime}\right)$-special path for some vertices $v_{x}, w_{x}, v_{x}^{\prime}$ of $G$ such that $\mathrm{h}\left(v_{x}\right) \geqslant k-3 \ell \mathrm{~d}(x)-\ell$, and $P_{x}$ has two distinguished safe vertices; moreover, if $x$ is not a leaf we associate these safe vertices with the two children $y, z$ of $x$ and denote them $s_{x y}$ and $s_{x z}$;
(2) for all $x, y \in V\left(T^{1}\right), v_{x}$ is an ancestor of $v_{y}$ in $T$ if and only if $x$ is an ancestor of $y$ in $T^{1}$;
(3) for all $x, y \in V\left(T^{1}\right)$ such that $y$ is a child of $x$, the paths $P_{y}$ and $Q_{y}$ are obtained by applying 5.5 on $P_{x}$ with safe vertex $s_{x y}$;
(4) for all $y, z \in V\left(T^{1}\right)$ such that $y$ and $z$ are siblings, no vertex of $Q_{z}$ meets $T_{w_{y}}$, and no vertex of $Q_{y}$ meets $T_{w_{z}}$;
(5) for all leaves $x$ of $T^{1}, V\left(T_{w_{x}}\right)$ and $\bigcup_{p \in V\left(T^{1}\right) \backslash\{x\}} V\left(Q_{p}\right)$ are disjoint.

The proof is by induction on $\left|V\left(T^{1}\right)\right|$. For the base case $\left|V\left(T^{1}\right)\right|=1$, the tree $T^{1}$ is a single vertex $x$. Applying 5.4 with $v$ the root of $T$ and $w$ a child of $v$ in $T$, we obtain a $\left(v_{x}, w_{x}, v_{x}^{\prime}\right)$-special path $P_{x}$ and two distinct safe vertices for $P_{x}$. Let $Q_{x}:=P_{x}$. Then $\left(T^{1},\left\{\left(P_{x}, Q_{x}\right)\right\}\right)$ is a binary pear tree in $G$. Observe that $\mathrm{d}(x)=0$ and $\mathrm{h}\left(v_{x}\right) \geqslant \mathrm{h}(v)-\ell=$ $k-\ell$, thus (1) holds. Properties (2)-(5) hold vacuously since $x$ is the only vertex of $T^{1}$.

Next, for the inductive case, assume $\left|V\left(T^{1}\right)\right|>1$. Let $x$ be a vertex of $T^{1}$ with two children $y, z$ that are leaves of $T^{1}$. Applying induction on the binary tree $T^{1}-\{y, z\}$, we obtain a binary pear tree $\left(T^{1}-\{y, z\},\left\{\left(P_{p}, Q_{p}\right): p \in V\left(T^{1}-\{y, z\}\right)\right\}\right)$ in $G$ satisfying the claim.

Note that $\mathrm{d}(x) \leqslant 3 \ell-3$, and thus $\mathrm{h}\left(v_{x}\right) \geqslant k-3 \ell \mathrm{~d}(x)-\ell \geqslant\left(9 \ell^{2}-3 \ell+1\right)-3 \ell(3 \ell-3)-\ell \geqslant$ $5 \ell+1$. By (1), the path $P_{x}$ comes with two distinguished safe vertices. Considering now the two children $y, z$ of $x$ in the tree $T$, we associate these safe vertices to $y$ and $z$, as expected, and denote them $s_{x y}$ and $s_{x z}$. Let $v_{x y}$ and $v_{x z}$ denote their respective parents in $T$. First, apply 5.5 with the path $P_{x}$ and safe vertex $s_{x y}$, giving a $\left(v_{y}, w_{y}, v_{y}^{\prime}\right)$-special path $P_{y}$ with two distinct safe vertices, and a $P_{x}$-ear $Q_{y}$ satisfying the properties of 5.5. Next, apply 5.5 with the path $P_{x}$ and safe vertex $s_{x z}$, giving a ( $v_{z}, w_{z}, v_{z}^{\prime}$ )-special path $P_{z}$ with two distinct safe vertices, and a $P_{x}$-ear $Q_{z}$ satisfying the properties of 5.5.

Observe that, by property (b) of $5.5, \mathrm{~h}\left(v_{y}\right) \geqslant \mathrm{h}\left(v_{x}\right)-3 \ell \geqslant k-3 \ell \mathrm{~d}(x)-4 \ell=k-3 \ell \mathrm{~d}(y)-\ell$, and similarly $\mathrm{h}\left(v_{z}\right) \geqslant k-3 \ell \mathrm{~d}(z)-\ell$. Thus, property (1) is satisfied. Clearly, property (2) and property (3) are satisfied as well. To establish property (4), it only remains to show that no vertex of $Q_{z}$ meets $T_{w_{y}}$, and that no vertex of $Q_{y}$ meets $T_{w_{z}}$. By symmetry it is enough to show the former, which we do now.

Arguing by contradiction, assume that $Q_{z}$ meets $T_{w_{y}}$. Since $V\left(T_{w_{y}}\right) \subseteq V\left(T_{s_{x y}}\right)$ and $V\left(Q_{x}\right) \cap V\left(T_{s_{x y}}\right)=\emptyset$ (by property (g) of 5.5), and since the two ends of $Q_{z}$ are on $Q_{x}$, we see that the two ends of $Q_{z}$ are outside $V\left(T_{w_{y}}\right)$. Thus, at least two edges of $Q_{z}$ have exactly one end in $V\left(T_{w_{y}}\right)$, and there is an edge st which is not an edge of $T$ (i.e. $\left.s t \neq v_{y} w_{y}\right)$. By symmetry, $s \in V\left(T_{w_{y}}\right)$ and $t \notin V\left(T_{w_{y}}\right)$.

Clearly, $s \notin V\left(T_{s_{x z}}\right)$ since $V\left(T_{w_{y}}\right) \subseteq V\left(T_{s_{x y}}\right)$, and $V\left(T_{s_{x y}}\right) \cap V\left(T_{s_{x z}}\right)=\emptyset$. Moreover, $t \notin$ $V\left(T_{s_{x z}}\right)$, since $V\left(T_{s_{x z}}\right) \subseteq \operatorname{Out}\left(s_{x y}\right) \backslash\left\{v_{x y}\right\}$ and since $V\left(T_{w_{y}}\right)$ does not see Out $\left(s_{x y}\right) \backslash\left\{v_{x y}\right\}$ by property (c) of 5.5. Since st is an edge of $Q_{z}$ not in $T$ with neither of its ends in $V\left(T_{s_{x z}}\right)$, it follows from property (h) of 5.5 that $v_{x z}$ is an original vertex with children $u_{x z}$ and $s_{x z}$; the path $P_{x}$ avoids $V\left(T_{u_{x z}}\right)$; and the edge st has one end in $V\left(T_{u_{x z}}\right)$ and the other in $\operatorname{Out}\left(v_{x z}\right)$. (We remark that we do not know which end is in which set at this point.)

First, suppose $s_{x y}=u_{x z}$. Then $v_{x y}=v_{x z}$. Since $s \in V\left(T_{w_{y}}\right) \subseteq V\left(T_{s_{x y}}\right)$ and $s_{x y}=u_{x z}$, we deduce that $s \in V\left(T_{u_{x z}}\right)$ and $t \in \operatorname{Out}\left(v_{x z}\right)$ in this case. However, $V\left(T_{w_{y}}\right)$ does not see Out $\left(s_{x y}\right) \backslash\left\{v_{x y}\right\}$ (by property (c) of 5.5), and $t \in \operatorname{Out}\left(v_{x z}\right) \subseteq \operatorname{Out}\left(u_{x z}\right) \backslash\left\{v_{x z}\right\}=$ Out $\left(s_{x y}\right) \backslash\left\{v_{x y}\right\}$, a contradiction.

Next, assume that $s_{x y} \neq u_{x z}$. Then $s_{x y} \notin V\left(T_{u_{x z}}\right)$, because the parent $v_{x y}$ of $s_{x y}$ is on the path $P_{x}$, and $P_{x}$ avoids $V\left(T_{u_{x z}}\right)$. Since $s_{x y} \notin V\left(T_{s_{x z}}\right)$ and $s_{x y} \neq v_{x z}$, it follows that $s_{x y} \in \operatorname{Out}\left(v_{x z}\right)$. Since $s \in V\left(T_{w_{y}}\right) \subseteq V\left(T_{s_{x y}}\right)$ and since $s_{x y}$ is not an ancestor of $v_{x z}$ (otherwise $V\left(T_{s_{x y}}\right)$ would contain $v_{x z}$, which is on the path $P_{x}$ ), we deduce that $V\left(T_{s_{x y}}\right) \subseteq \operatorname{Out}\left(v_{x z}\right)$, and thus $s \in \operatorname{Out}\left(v_{x z}\right)$. It then follows that $t \in V\left(T_{u_{x z}}\right)$. Observe that $u_{x z}$ is neither an ancestor of $v_{x y}$ (otherwise $V\left(T_{u_{x z}}\right)$ would contain $v_{x y}$, which is on the path $P_{x}$ ) nor a descendant of $s_{x y}$ (otherwise $V\left(T_{s_{x y}}\right)$ would contain $v_{x z}$ since $u_{x z} \neq s_{x y}$, which is a vertex of $\left.P_{x}\right)$. Hence, we deduce that $V\left(T_{u_{x z}}\right) \subseteq \operatorname{Out}\left(s_{x y}\right) \backslash\left\{v_{x y}\right\}$. However, the edge $s t$ then contradicts the fact that $V\left(T_{w_{y}}\right)$ does not see Out $\left(s_{x y}\right) \backslash\left\{v_{x y}\right\}$ (c.f. property (c) of 5.5). Therefore, $V\left(Q_{z}\right) \cap V\left(T_{w_{y}}\right)=\emptyset$, as claimed. Property (4) follows.

We now verify property (5). First, we show (5) holds for the leaf $y$ of $T^{1}$. Note that $V\left(T_{w_{y}}\right) \subseteq V\left(T_{s_{x y}}\right) \subseteq V\left(T_{w_{x}}\right)$. Thus, $V\left(T_{w_{y}}\right)$ and $\bigcup_{p \in V\left(T^{1}\right) \backslash\{x, y, z\}} V\left(Q_{p}\right)$ are disjoint by induction and property (5) for the leaf $x$ of $T^{1}-\{y, z\}$. Since $V\left(T_{w_{y}}\right) \subseteq V\left(T_{s_{x y}}\right)$ and $V\left(T_{s_{x y}}\right) \cap V\left(Q_{x}\right)=\emptyset$ (by property (g) of 5.5), we deduce that $V\left(T_{w_{y}}\right) \cap V\left(Q_{x}\right)=\emptyset$. Moreover, $V\left(T_{w_{y}}\right) \cap V\left(Q_{z}\right)=\emptyset$, by property (4) shown above. This proves property (5) for the leaf $y$ of $T^{1}$, and also for the leaf $z$ by symmetry.

Every other leaf $q$ of $T^{1}$ is also a leaf in $T^{1}-\{y, z\}$. By induction, $V\left(T_{w_{q}}\right)$ and $\bigcup_{p \in V\left(T^{1}\right) \backslash\{q, y, z\}} V\left(Q_{p}\right)$ are disjoint. Moreover, $V\left(T_{v_{q}}\right)$ and $V\left(T_{v_{x}}\right)$ are disjoint, by property (2). Since $V\left(Q_{y}\right)$ and $V\left(Q_{z}\right)$ are contained in $V\left(T_{v_{x}}\right)$ (by property (f) of 5.5) and $V\left(T_{w_{q}}\right) \subseteq V\left(T_{v_{q}}\right)$, it follows that $V\left(T_{w_{q}}\right)$ and $V\left(Q_{y}\right) \cup V\left(Q_{z}\right)$ are also disjoint. Property (5) follows.

To conclude the proof, it only remains to verify that ( $\left.T^{1},\left\{\left(P_{p}, Q_{p}\right): p \in V\left(T^{1}\right)\right\}\right)$ is a binary pear tree in $G$, and that it is clean. Recall that $\left(T^{1}-\{y, z\},\left\{\left(P_{p}, Q_{p}\right)\right.\right.$ : $\left.\left.p \in V\left(T^{1}-\{y, z\}\right)\right\}\right)$ is a binary pear tree, by induction. By construction, $P_{y} \subseteq Q_{y}$ and $P_{z} \subseteq Q_{z}, P_{y}$ and $P_{z}$ each have length at least 2 , and both are $P_{x}$-ears. Clearly, property (i) of the definition of binary pear trees holds. Property (ii) holds vacuously, since $T^{1}$ is a full binary tree, and thus every non-root vertex of $T^{1}$ has a sibling. Hence, it only remains to show that property (iii) holds.

Let $p$ be a non-root vertex of $T^{1}$, and let $p^{\prime}$ denote its sibling. First we want to show that no internal vertex of $Q_{p}$ is in $\bigcup_{q \in V\left(T^{1}\right) \backslash\left(V\left(T_{p}^{1}\right) \cup V\left(T_{p^{\prime}}^{1}\right)\right)} V\left(Q_{q}\right)$.

If $p$ is an ancestor of $x$ in $T^{1}$ (including $x$ ) then this holds thanks to property (iii) of the binary pear tree $\left(T^{1}-\{y, z\},\left\{\left(P_{q}, Q_{q}\right): q \in V\left(T^{1}-\{y, z\}\right)\right\}\right)$.

Next, suppose $p$ is not an ancestor of $x$ in $T^{1}$ and $p$ is not $y$ nor $z$. Then we already know that no internal vertex of $Q_{p}$ is in $\bigcup_{q \in V\left(T^{1}-\{y, z\}\right) \backslash\left(V\left(T_{p}^{1}\right) \cup V\left(T_{p^{\prime}}^{1}\right)\right)} V\left(Q_{q}\right)$, again by property (iii) of the binary pear tree $\left(T^{1}-\{y, z\},\left\{\left(P_{q}, Q_{q}\right): q \in V\left(T^{1}-\{y, z\}\right)\right\}\right)$. Thus it only remains to show that if some internal vertex of $Q_{p}$ is in $Q_{y}$ then $y$ is a descendant of $p$ or of $p^{\prime}$, and that the same holds for $Q_{z}$. By symmetry, it is enough to prove this for $Q_{y}$. So let us assume that some internal vertex of $Q_{p}$ is in $Q_{y}$. Note that $V\left(Q_{y}\right) \subseteq V\left(T_{w_{x}}\right) \cup\left\{v_{x}\right\}$, by property (f) of 5.5 . By property (5) of the inductive statement, $V\left(T_{w_{x}}\right)$ is disjoint from $V\left(Q_{p}\right)$. Thus, the only vertex that the paths $Q_{p}$ and $Q_{y}$ can have in common is $v_{x}$. Since $v_{x}$ is an internal vertex of $Q_{p}$ (by our assumption) and since $v_{x} \in V\left(Q_{x}\right)$, from property (iii) of the binary pear tree $\left(T^{1}-\{y, z\},\left\{\left(P_{q}, Q_{q}\right)\right.\right.$ : $\left.\left.q \in V\left(T^{1}-\{y, z\}\right)\right\}\right)$ we deduce that $x$ is a descendant of $p$ or $p^{\prime}$, and hence so is $y$, as desired.

Finally, consider the case where $p$ is $y$ or $z$, say $y$. Recall that $V\left(Q_{y}\right) \subseteq V\left(T_{w_{x}}\right) \cup\left\{v_{x}\right\}$. Note also that $v_{x}$ cannot be an internal vertex of $Q_{y}$, since $v_{x} \in V\left(P_{x}\right)$ and $Q_{y}$ is a $P_{x}$-ear. Hence, all internal vertices of $Q_{y}$ are in $V\left(T_{w_{x}}\right)$. Since $V\left(T_{w_{x}}\right)$ and $V\left(Q_{q}\right)$ are disjoint for all $q \in V\left(T^{1}\right) \backslash\{x, y, z\}$ (by induction, using property (5) on the leaf $x$ of $\left.T^{1}-\{y, z\}\right)$. Thus, it only remains to show that no internal vertex of $Q_{y}$ is in $Q_{x}$.

This is the case, because $Q_{y}$ is a $P_{x}$-ear, and $V\left(Q_{x}\right) \backslash V\left(P_{x}\right) \subseteq \operatorname{Out}\left(w_{x}\right) \backslash\left\{v_{x}\right\}$ (by property (e) of 5.5).

To establish property (iii), it remains to show that no internal vertex of $P_{p}$ is in $Q_{p^{\prime}}$, for every two siblings $p, p^{\prime}$ of $T^{1}$. If $\left\{p, p^{\prime}\right\} \neq\{y, z\}$, this is true by property (iii) of the binary pear tree $\left(T^{1}-\{y, z\},\left\{\left(P_{q}, Q_{q}\right): q \in V\left(T^{1}-\{y, z\}\right)\right\}\right)$. Thus by symmetry, it is enough to show that no internal vertex of $P_{y}$ is in $Q_{z}$. This holds because all internal vertices of $P_{y}$ are in $V\left(T_{w_{y}}\right)$ (since $P_{y}$ is a $\left(v_{y}, w_{y}, v_{y}^{\prime}\right)$-special path) and $V\left(Q_{z}\right) \cap V\left(T_{w_{y}}\right)=\emptyset$ by (4).

This concludes the proof that $\left(T^{1},\left\{\left(P_{p}, Q_{p}\right): p \in V\left(T^{1}\right)\right\}\right)$ is a binary pear tree. Finally, note that it is clean because the binary pear tree $\left(T^{1}-\{y, z\},\left\{\left(P_{q}, Q_{q}\right): q \in V\left(T^{1}-\right.\right.\right.$ $\{y, z\})\}$ ) is clean (by induction), and the end $v_{x}^{\prime}$ of $P_{x}$ is not in $Q_{y}$, since $V\left(Q_{y}\right) \subseteq$ $V\left(T_{w_{x}}\right) \cup\left\{v_{x}\right\}$ (by property (f) of 5.5), and since $v_{x}^{\prime} \notin V\left(T_{w_{x}}\right) \cup\left\{v_{x}\right\}$, and similarly $v_{x}^{\prime}$ is not in $Q_{z}$ either.

## 6. Proof of Main Theorems

We have the following quantitative version of 1.4.
6.1. For all integers $\ell \geqslant 1$ and $k \geqslant 9 \ell^{2}-3 \ell+1$, every 2 -connected graph $G$ with $a \Gamma_{k}$ minor contains $\Gamma_{\ell}^{+}$or $\nabla_{\ell}$ as a minor.

Proof. Among all 2-connected graphs containing $\Gamma_{k}$ as a minor, but containing neither $\Gamma_{\ell}^{+}$nor $\nabla_{\ell}$ as a minor, choose $G$ with $|E(G)|$ minimum. Since $\Gamma_{k}$ has maximum degree 3, $G$ contains a subdivision of $\Gamma_{k}$. Therefore, $G$ is a minor-minimal 2-connected graph containing a subdivision of $\Gamma_{k}$. By 5.1, $G$ has a clean binary pear tree $\left(T^{1}, \mathcal{B}\right)$, with $T^{1} \simeq \Gamma_{3 \ell-2}$. By 4.1, $G$ has a minor $H$ such that $H$ has a clean binary ear tree $\left(T^{1}, \mathcal{P}\right)$, with $T^{1} \simeq \Gamma_{3 \ell-2}$. By 3.1, $H$ contains $\Gamma_{\ell}^{+}$or $\nabla_{\ell}$ as a minor, and hence so does $G$.

We have the following quantitative version of 1.3.
6.2. For every integer $\ell \geqslant 1$, every 2 -connected graph $G$ of pathwidth at least $2^{9 \ell^{2}-3 \ell+2}-2$ contains $\Gamma_{\ell}^{+}$or $\nabla_{\ell}$ as a minor.

Proof. As mentioned in Section 1, Bienstock et al. [1] proved that for every forest F, every graph with pathwidth at least $|V(F)|-1$ contains $F$ as a minor. Let $k:=$ $9 \ell^{2}-3 \ell+1$. Note that $\left|V\left(\Gamma_{k}\right)\right|=2^{k+1}-1$. By assumption, $G$ has pathwidth at least $2^{k+1}-2$. Thus $G$ contains $\Gamma_{k}$ as a minor. The result follows from 6.1.

Finally, we have the following quantitative version of 1.2.
6.3. For every apex-forest $H_{1}$ and outerplanar graph $H_{2}$, if $\ell \quad:=$ $\max \left\{\left|V\left(H_{1}\right)\right|,\left|V\left(H_{2}\right)\right|, 2\right\}-1$ then every 2-connected graph $G$ of pathwidth at least $2^{9 \ell^{2}-3 \ell+2}-2$ contains $H_{1}$ or $H_{2}$ as a minor.

Proof. By 6.2, $G$ contains $\Gamma_{\ell}^{+}$or $\nabla_{\ell}$ as a minor. In the first case, by $2.2, H_{1}$ is a minor of $\Gamma_{\left|V\left(H_{1}\right)\right|-1}^{+}$and thus of $G$ (since $\ell \geqslant\left|V\left(H_{1}\right)\right|-1$ ). In the second case, by 2.4, $H_{2}$ is a minor of $\nabla_{\left|V\left(H_{2}\right)\right|-1}$ and thus of $G$ (since $\ell \geqslant\left|V\left(H_{2}\right)\right|-1$ ).

Acknowledgements. We would like to thank Robin Thomas for informing us of the PhD thesis of Thanh N. Dang [2]. We are also grateful to an anonymous referee for several helpful remarks and suggestions.

## References

[1] Dan Bienstock, Neil Robertson, Paul Seymour, and Robin Thomas. Quickly excluding a forest. J. Combin. Theory Ser. B, 52:274-283, 1991. doi: 10.1016/0095-8956(91)90068-U.
[2] Thanh N. Dang. Minors of graphs of large path-width. Ph.D. thesis, Georgia Institute of Technology, 2018. http://www.aco.gatech.edu/sites/default/files/ documents/dang-thesis.pdf.
[3] Thanh N. Dang and Robin Thomas. Minors of two-connected graphs of large path-width. 2017. arXiv: 1712.04549.
[4] Nathaniel Dean. Open problems. In Neil Robertson and Paul Seymour, eds., Graph structure theory. Proc. of AMS-IMS-SIAM Joint Summer Research Conf. on Graph Minors, vol. 147 of Contemporary Mathematics, pp. 677-688. American Mathematical Society, 1993. doi: 10.1090/conm/147.
[5] Reinhard Diestel. Graph minors I: A short proof of the pathwidth theorem. Combinatorics, Probability and Computing, 4(1):27-30, 1995. doi: 10.1017/S0963548300001450.
[6] Navin Kashyap. Matroid pathwidth and code trellis complexity. SIAM J. Discrete Math., 22(1):256-272, 2008. doi: 10.1137/070691152.
[7] Neil Robertson and Paul Seymour. Graph minors. I. Excluding a forest. J. Combin. Theory Ser. B, 35(1):39-61, 1983. doi:10.1016/0095-8956(83)90079-5.
[8] Neil Robertson and Paul Seymour. Graph minors. V. Excluding a planar graph. J. Combin. Theory Ser. B, 41(1):92-114, 1986. doi: 10.1016/0095-8956(86)90030-4.


[^0]:    (T. Huynh) Département de Mathématique, Université Libre de Bruxelles, Brussels, Belgium
    (G. Joret) Département d’Informatique, Université Libre de Bruxelles, Brussels, BelgiUm
    (P. Micek) Theoretical Computer Science Department, Faculty of Mathematics and Computer Science, Jagiellonian University, Kraków, Poland
    (D. R. Wood) School of Mathematics, Monash University, Melbourne, Australia

    E-mail addresses: tony.bourbaki@gmail.com, gjoret@ulb.ac.be, piotr.micek@tcs.uj.edu.pl, david.wood@monash.edu.
    G. Joret acknowledges support from the Australian Research Council and from an Action de Recherche Concertée grant from the Wallonia-Brussels Federation in Belgium. P. Micek is partially supported by a Polish National Science Center grant (SONATA BIS 5; UMO-2015/18/E/ST6/00299). T. Huynh is supported by ERC Consolidator Grant $615640-F o r E F r o n t . ~ T . ~ H u y n h, ~ G . ~ J o r e t, ~ a n d ~ P . ~$ Micek also acknowledge support from a joint grant funded by the Belgian National Fund for Scientific Research (F.R.S.-FNRS) and the Polish Academy of Sciences (PAN). Research of D. R. Wood is supported by the Australian Research Council.

