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On an Invariance Principle for Differential-Algebraic Equations with Jumps and its Application to Switched Differential-Algebraic Equations

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Abstract We investigate the invariance properties of a class of switched systems where the value of a switching signal determines the current mode of operation (among a finite number of them) and, for each fixed mode, its dynamics are described by a Differential Algebraic Equation (DAE). Motivated by the lack of invariance principles of switched DAE systems, we develop such principles for switched DAE systems under arbitrary and dwell-time switching. By obtaining a hybrid system model that describes the switched DAE system, we build from invariance results for hybrid systems the invariance principles for such switched systems. Examples are included to illustrate the results.

1 Introduction

1.1 Background

We consider a class of hybrid dynamical systems with continuous dynamics that can be modeled as differential-algebraic equations (DAEs) – also known as descriptor systems – and with discrete dynamics given by difference inclusions. These type of systems, which we refer to as *hybrid DAE systems*, arise in several applications in engineering such as robot manipulators, vehicular traffic systems [3], power systems, biological systems, and mechanical systems [9]. The systems of interest in this paper include a logic variable which determines the current mode of the system (among finitely many of them) and that, during flows, the dynamics of the other state components evolve according to a linear DAE. Particularly, we

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also consider a class of switched systems with a state evolving according to a linear DAE in-between switching instants and, due to the algebraic constraints of the DAE in each mode, potentially exhibiting jumps in the state at switching instants. Switched DAE representations naturally appear when modeling electrical circuits where algebraic constraints (e.g., due to Kirchhoff's laws) are entangled with differential equations (e.g., governing the change of current and voltages in capacitors and inductors) as well as elements such as (ideal) switches or diodes [13].

The stability analysis of switched DAE systems has its foundations on the stability analysis of descriptor systems (also referred to as differential-algebraic systems, singular systems or semi-state systems), e.g., a survey of results on linear singular systems is given in [4]. Also, a historical review of linear singular systems is presented in [5,12]. The consistency of initial conditions together with Lyapunov theory is used to study the stability properties of the origin for linear singular systems in [19]. Several authors have analyzed switched DAE systems from many perspectives, with most of the research being focused in establishing asymptotic stability of the origin. In [13], Lyapunov's direct method is used to analyze the asymptotic stability of the origin for switched DAE systems. Sufficient conditions for exponential stability of switched singular system with stable subsystems are presented in [26]. It is important to note that a typical assumption that is enforced in such works is that solutions are given by piecewise (right or left) continuous functions, so as to preclude the presence of impulses in the solutions at the times when switches occur. As stated in [14] in the context of switched systems, to deal with such impulses in the solutions, one approach is to consider distributional solutions or weak solutions; see, e.g., [13,21]. However, unless explicitly assumed, neither concept of solution leads to a set of solutions with the so-called sequential compactness property, which is key in the development of invariance-like results [11] (see [16]).

1.2 Motivation and Contributions

Next, as a motivation, we present an example where the analysis of the asymptotic stability of the origin is not suitable and a notion of invariance is required to analyze the solutions of a fairly simple switched DAE system.

Example 1 (Motivational example) Consider a switched DAE system with two modes of operation determined by $\sigma \in \{1,2\}$ and dynamics

$$E_\sigma \dot{\xi} = A_\sigma \xi \tag{1}$$

where $\xi \in \mathbb{R}^2$ is the state of the system and

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2}$$

Let the switching signal $\sigma : \mathbb{R}_{\geq 0} \to \{1,2\}$ be a piecewise-constant right-continuous function. Consider the function

$$V(\xi,\sigma) = (E_\sigma \xi)^\top E_\sigma \xi \quad \forall (\xi,\sigma) \in \mathbb{R}^2 \times \{1,2\} \tag{3}$$

and note that, when σ remains constant, the change of the function V is given as follows:

- If $\sigma = 1$, then $\dot{\overbrace{V(\xi,\sigma)}} = (E_1\dot{\xi})^\top E_1 \xi + (E_1\xi)^\top E_1 \dot{\xi} = 0$ for each $\xi \in \mathbb{R}^2$.
- If $\sigma = 2$, then $\dot{\overbrace{V(\xi,\sigma)}} = (E_2\dot{\xi})^\top E_2 \xi + (E_2\xi)^\top E_2 \dot{\xi} = -2\xi_2^2$ for each $\xi \in \mathbb{R}^2$.

Since (1) for $\sigma = 2$ reduces to $\dot{\xi}_2 = -\xi_2$ and $\xi_1 = 0$, we have that $\overbrace{V(\xi,\sigma)}^{\cdot}$ implies exponential stability of the origin during that mode. On the other hand, if σ jumps at time t_s, the state ξ is mapped to a point in $\mathbb{R}^2$ given by $\xi(t_s^+) = \Pi_{\sigma(t_s^+)}\xi(t_s)$, where the subsequent algebraic restrictions are fulfilled. (These maps are given by the so-called consistency projectors; see also [13, Definition 3.7].) Using the definitions of (E_σ, A_σ) above, for changes from $\sigma = 2$ to 1 (i.e., $\sigma^+ = 1$), the consistency projector is given by Π_1, while, for changes from $\sigma = 1$ to 2 (i.e., $\sigma^+ = 2$), the projector is given by Π_2. These projectors are given by

$$\Pi_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \Pi_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4}$$

Then, the change of the function $V(\xi, \sigma)$ at mode transitions is as follows:

- If $\sigma = 1$, then $\sigma^+ = 2$ and, for each $\xi \in \mathbb{R}^2$,

$$\begin{aligned} V(\Pi_2\xi, 2) - V(\xi, 1) &= \xi^\top \left(\Pi_2^\top E_2^\top E_2 \Pi_2 - E_1^\top E_1\right)\xi \\ &= -\xi_1^2 \end{aligned}$$

- If $\sigma = 2$, then $\sigma^+ = 1$ and, for each $\xi \in \mathbb{R}^2$,

$$\begin{aligned} V(\Pi_1\xi, 1) - V(\xi, 2) &= \xi^\top \left(\Pi_1^\top E_1^\top E_1 \Pi_1 - E_2^\top E_2\right)\xi \\ &= \xi_1^2 = 0, \end{aligned}$$

  where we used the fact that $\xi_1 = 0$ if $\sigma = 2$.

Denoting by V^+ the value of V after the jump at (ξ, σ), the change of V during flows and jumps is given by

$$\begin{aligned} \overbrace{V(\xi,\sigma)}^{\cdot} &= \begin{cases} 0 & \text{if } \sigma = 1 \\ -2\xi_2^2 & \text{if } \sigma = 2 \end{cases} \\ V(\xi,\sigma)^+ - V(\xi,\sigma) &= \begin{cases} -\xi_1^2 & \text{if } \sigma = 1 \\ 0 & \text{if } \sigma = 2 \end{cases} \end{aligned}$$

Note that V is not strictly decreasing during flows or jumps. Depending on the law triggering the change of σ, solutions can either approach the origin or stay away from it for all time. In fact, for any initial condition away from the origin, if σ eventually remains at 1, then the solution would remain at a level set of V for all future time.

Due to the nonstrict decrease of V during flows, asymptotic stability[1] of the origin of (1) cannot be established using the tools in [5,13,19,23,26] for particular classes of switching signals. The main reason is the lack of a tool to characterize the ω-limit set of bounded and complete solutions to (1). One could be tempted to recast this system as a hybrid inclusion as defined in [8,15] and apply the invariance principle in [8]. The resulting hybrid inclusion is given by

$$\mathcal{H}_s \quad \begin{cases} \dot{\xi} &= f_s(\xi,\sigma) \quad (\xi,\sigma) \in C_s \\ \xi^+ &= g_s(\xi,\sigma) \quad (\xi,\sigma) \in D_s \end{cases} \tag{5}$$

[1] Stability can be inferred from the properties of V, but attractivity requires the application of an invariance principle, which is not available in the literature.

where

$$C_s := \{(\xi,\sigma) \in \mathbb{R}^2 \times \{1,2\} \mid \sigma = 1\} \cup \{(\xi,\sigma) \in \mathbb{R}^2 \times \{1,2\} \mid \sigma = 2, \xi_1 = 0\} \tag{6a}$$

$$f_s(\xi,\sigma) := \begin{cases} \Pi_1^{\text{diff}} A_1 \xi = \begin{bmatrix} \xi_2 \\ -\xi_1 \end{bmatrix} & \text{if } \sigma = 1 \\ \Pi_2^{\text{diff}} A_2 \xi = \begin{bmatrix} 0 \\ -\xi_2 \end{bmatrix} & \text{if } \sigma = 2 \end{cases} \tag{6b}$$

$$D_s := \{(\xi,\sigma) \in \mathbb{R}^2 \times \{1,2\} \mid \sigma = 2, \xi_1 \neq 0\} \tag{6c}$$

$$g_s(\xi,\sigma) := \Pi_\sigma \xi. \tag{6d}$$

Notice that this hybrid inclusion has the same solutions as the original switched DAE system. The next step would be to define an autonomous system generating the switching signal and apply the invariance principle in [8]; however, since the set D_s is open, the assumptions in [8] would not hold. In fact, as we will show in detail in Example 2, the hybrid inclusion model associated to (1) is not nominally well-posed, and therefore the invariance principles in [8] are not applicable. This fact motivates the development of invariance principles for systems of the form (1) with jumps on ξ and σ generated by a state-space model.

□

In this paper, we consider dynamical systems with multiple modes of operation and state jumps. Within each mode, the dynamics are given by linear differential-algebraic equations (DAEs). State jumps can occur when in a fixed mode, when transitioning between modes, or they can be induced by an inconsistent initial condition. We refer to this class of hybrid systems as *hybrid DAEs*. Motivated by the lack of results regarding the invariance properties of switched DAE systems, being perhaps the main reason the difficulty in guaranteeing a sequential compactness property of the solutions to such systems [11], we build from the concept of solution to hybrid systems in [8] (referred here as *hybrid inclusions*) and propose a model for switched DAE systems using the framework of *hybrid DAEs* introduced here. For the proposed hybrid DAE model, we establish that when its data satisfies certain mild conditions, the system has a set of solutions with structural properties enabling the development of invariance results. Also, we propose a model for switched DAE systems under certain classes of switching signals and we establish conditions on its data for an invariance principle to hold. More precisely, the contributions of this paper include the following:

1. Building from results for switched DAE systems [13,26], we propose a dynamical model that allows for jumps when initial conditions are not consistent with the algebraic conditions. In fact, our model uses concepts from the literature of switched DAE systems to keep the special structure of the algebraic restrictions in DAE systems, in particular, the so-called consistency jumps driven by the consistency projectors and inconsistent initial conditions [4,13,21]. Using ideas for hybrid systems [8], the proposed model captures the jumps triggered by state conditions. Our model uses the concept of solution, and the invariance notions and results for the class of hybrid systems in [8].
2. For hybrid DAEs with linear dynamics during flows, we determine the properties of ω-limit sets of bounded and complete solutions, and establish an invariance principle. The invariance principle resembles the classical one for continuous-time systems.
3. Building from the results for hybrid DAE systems in item 2 (see also [16]) and for switched systems in [7], we propose invariance principles for switched DAE systems under arbitrary and dwell-time switching.

It is important to note that tools for the study of invariance properties of hybrid systems in [8] are used in this paper, but the invariance principles in [8] cannot be applied directly to the class of systems of interest, in particular switched DAE systems as in (1). This is mainly because the systems we consider do not fulfill the required conditions in [8]. Another important point to highlight is that unlike [13], the concept of solution used in this paper does not take into account impulses. As we point out in Section 4 and illustrate in Example 4 and Example 5, this notion of solution is required so that the ω-limit set of bounded and complete solutions for switched DAE systems have the needed structural properties, namely, weak invariance. Examples 4 and 5 point out that impulses in the solutions do not necessarily lead to ω-limit sets with such key property.

To the best of our knowledge, invariance properties for DAE systems with discontinuous coefficient matrices and with jumps in its states, such as the hybrid DAE and switched DAE models considered here, are not available in the literature.

1.3 Organization and Notation

The notation used throughout the paper is as follows. Given a set $S \subset \mathbb{R}^n$, the closure of S is the intersection of all closed sets containing S, denoted by $\overline{S}$. We define $\mathbb{R}_{\geq 0} := [0, \infty)$ and $\mathbb{N} := \{0, 1, \ldots\}$. Given vectors $\nu \in \mathbb{R}^n$, $\omega \in \mathbb{R}^m$, $[\nu^\top \; \omega^\top]^\top$ is equivalent to (ν, ω), where $(\cdot)^\top$ denotes the transpose operation. Given a function $f : \mathbb{R}^m \to \mathbb{R}^n$, its domain of definition is denoted by $\operatorname{dom} f$, i.e., $\operatorname{dom} f := \{x \in \mathbb{R}^m \mid f(x) \text{ is defined}\}$. The range of f is denoted by $\operatorname{rge} f$, i.e., $\operatorname{rge} f := \{f(x) \mid x \in \operatorname{dom} f\}$. The right limit of the function f is defined as $f^+(x) := \lim_{\nu \to 0^+} f(x + \nu)$ if it exists. The notation $f^{-1}(r)$ stands for the $r-$level set of f on $\operatorname{dom} f$, i.e., $f^{-1}(r) := \{z \in \operatorname{dom} f \mid f(z) = r\}$. The r-sublevel set of the function $V : \operatorname{dom} V \to \mathbb{R}$, namely, the set of points $\{x \in \operatorname{dom} V \mid V(x) \leq r\}$, is denoted by $L_V(r)$. Given two functions $f : \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^m \to \mathbb{R}^n$, $\langle f, h \rangle$ denotes the inner product. We denote the distance from a vector $y \in \mathbb{R}^n$ to a closed set $\mathcal{A} \subset \mathbb{R}^n$ by $|y|_{\mathcal{A}}$, which is given by $|y|_{\mathcal{A}} := \inf_{x \in \mathcal{A}} |x - y|$. Given a matrix $P \in \mathbb{R}^{n \times n}$, the determinant of P is denoted by $\det P$. The column space of the matrix P, i.e., the set of all possible linear combinations of the column vectors of P, is denoted by $\operatorname{im}(P)$. The span of a set of vectors Q is defined as the set of all finite linear combinations of elements of Q, e.g., $\operatorname{span}(Q) = \left\{\sum_{i=1}^{k} \alpha_i v_i \mid k \in \mathbb{N} \setminus \{0\},\; v_i \in Q, \alpha_i \in \mathbb{R}\right\}$. Given two matrices $P, Q \in \mathbb{R}^{n \times n}$, the set of finite eigenvalues $\{\lambda_i\}_{i \in \{1,2,\ldots,n_1\}}$, where $n_1 \leq n$, is denoted by $\lambda(P, Q) := \{s \mid s \in \mathbb{C},\; s \text{ finite}, \det(sP - Q) = 0\}$. Given $n \in \mathbb{N}$, the matrix $0_n \in \mathbb{R}^{n \times n}$ denotes the zero $n \times n$ matrix, also $I_n \in \mathbb{R}^{n \times n}$ denotes the $n \times n$ identity matrix.

The remainder of this paper is organized as follows. In Section 2, the required modeling background is presented. In Section 3, the main results are presented; namely, a description of hybrid DAE systems and the invariance principle for such systems, an invariance principle for switched DAE systems under arbitrary, and an invariance principle for switched DAE systems under dwell-time switching. In Section 4, we present examples where the definitions and the invariance principles are exercised. The technical prerequisites needed in the proofs of the main results are summarized in Section 5.1, while the proofs of the main results are given in Section 5.2. Auxiliary results are in the Appendix.

2 Preliminaries

2.1 Modeling switched differential-algebraic systems

In this paper, we consider the class of linear switched DAE systems given by

$$E_\sigma \dot{\xi} = A_\sigma \xi, \tag{7}$$

where $\xi \in \mathbb{R}^n$ is the state, $\sigma : \mathbb{R}_{\geq 0} \to \Sigma$ is the switching signal, and Σ is a finite discrete set. Solutions to (7) are typically given by (right or left) continuous functions (see [21] and references therein). Definition 5 below introduces the notion of solution to (7) employed here.

Definition 1 (DAE regularity [4, Definition 1-2.1]) The collection $\{(E_\sigma, A_\sigma)\}_{\sigma\in\Sigma}$ is regular if, for each $\sigma \in \Sigma$, the matrix pencil $sE_\sigma - A_\sigma \in \mathbb{R}^{n\times n}$ is regular, where $s \in \mathbb{C}$. The matrix pencil $sE_\sigma - A_\sigma$ is *regular* if $\det(sE_\sigma - A_\sigma)$ is not the zero polynomial. The matrix pair (E_σ, A_σ) and the corresponding DAE is called regular whenever (E_σ, A_σ) is regular.

To define a switched DAE system as in [13], we recall first some concepts regarding the linear subspaces where solutions to (7) belong. Due to the algebraic constraints in (7), the solutions to (7) evolve within a linear subspace called the *consistency* space. If the initial value is not consistent, that is, $\xi(0)$ does not belong to the consistency space, then, jumps may appear in the solution, similar to the jumps induced by switching between modes.

Definition 2 (Consistency space[2]) Given $\sigma \in \Sigma$, the consistency space for (7) is given by

$$\mathfrak{C}_\sigma := \{\xi_0 \in \mathbb{R}^n \mid \exists \text{ a continuously differentiable function } \xi : [0,\tau) \to \mathbb{R}^n \\ \text{s.t. } E_\sigma\dot{\xi}(t) = A_\sigma\xi(t)\ \forall t \in (0,\tau),\ \xi(0) = \xi_0,\ \tau > 0\}$$

For a linear switched DAE system as in (7), for each $\sigma \in \Sigma$, the consistency space is given by a linear subspace. Moreover, this consistency space can be characterized by a set or by a basis. The consistency spaces can be computed using the quasi-Weierstrass form and the Wong Sequences, which are introduced in [24]. The Wong sequences are used to calculate the consistency and inconsistency spaces ν^* and ω^*, which are calculated from the basis of the linear subspaces (see [13, 10]). Notice that, for each $\sigma \in \Sigma$, the consistency space $\mathfrak{C}_\sigma$ is computed directly from the system data of (7), which is given by the matrix pair (E_σ, A_σ). As a reference for the reader, in Appendix A.1, using the aforementioned Wong sequences the definition of the subspaces ν^* and ω^* are given in Definition 9. For an explicit representation of ν_i and ω_i, see [21, Lemma 4.2.1] and [2, Lemma 2.2]. Also, for a Matlab script to compute these subspaces, see [6]. In Appendix A.1, the well known quasi-Weierstrass form is recalled in Theorem 4. For any full rank matrix $[V_\sigma, W_\sigma] \in \mathbb{R}^{n\times n}$, where $V_\sigma \in \mathbb{R}^{n\times n_1^\sigma}$, $W_\sigma \in \mathbb{R}^{n\times n_2^\sigma}$, $n_1^\sigma + n_2^\sigma = n$, with $\mathrm{im}(V_\sigma) = \nu_\sigma^*$ and $\mathrm{im}(W_\sigma) = \omega_\sigma^*$, the matrices $T_\sigma := [V_\sigma, W_\sigma]$ and $S_\sigma := [E_\sigma V_\sigma, A_\sigma W_\sigma]^{-1}$ are invertible and put the matrix pair (E_σ, A_σ) into the quasi-Weierstrass form given by Theorem 4. According to Theorem 4, from the linear DAE system in (7), an equivalent ODE system and its algebraic restrictions can be obtained using the well-known quasi-Weierstrass transformation for regular matrix pencils (see [2] and references therein). Note that this transformation does not change the solutions of the system, but instead, exposes important features of it.

Remark 1 Explicit forms of matrices S_σ and T_σ are given in Definition 9. With this transformation, for each $\sigma \in \Sigma$, system (7) is equivalent to $\dot{v} = J_\sigma v$, $N_\sigma\dot{w} = w$

[2] This definition is adapted from [13, Definition 2.1].

since

$$\begin{aligned} E_\sigma \dot{\xi} &= A_\sigma \xi \\ E_\sigma T_\sigma \begin{bmatrix} \dot{v} \\ \dot{w} \end{bmatrix} &= A_\sigma T_\sigma \begin{bmatrix} v \\ w \end{bmatrix} \\ S_\sigma E_\sigma T_\sigma \begin{bmatrix} \dot{v} \\ \dot{w} \end{bmatrix} &= S_\sigma A_\sigma T_\sigma \begin{bmatrix} v \\ w \end{bmatrix} \\ \begin{bmatrix} I & 0 \\ 0 & N_\sigma \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{w} \end{bmatrix} &= \begin{bmatrix} J_\sigma & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}, \end{aligned}$$

where $\begin{bmatrix} v \\ w \end{bmatrix} = T_\sigma^{-1}\xi$, and S_σ and T_σ are invertible matrices. Then, the matrix pencil of the equivalent system is given by $(S_\sigma E_\sigma T_\sigma, S_\sigma A_\sigma T_\sigma)$ [2].

For each $\sigma \in \Sigma$, the consistency space for (7) is given by

$$\mathfrak{C}_\sigma := \mathrm{im}(\nu_\sigma^*) = \mathrm{span}(V_\sigma)$$

Even though ν_σ^* is the collection of infinitely many elements, as shown in [2], we have that $\nu_\sigma^* := \bigcap_{i\in\mathbb{N}} \nu_i$ is a subspace of $\mathbb{R}^n$. Additionally, for the linear system (7), ν_σ^* is given by a linear subspace of $\mathbb{R}^n$ (see, e.g., [13, Remark 2.2]). Then, the basis $\mathfrak{C}_\sigma$ is given by a finite set of column vectors. In the following sections, we need to describe the intersection between the consistency space and some sets in $\mathbb{R}^n$. To do so, we recast the consistency space as a set in $\mathbb{R}^n$ as follows.

Definition 3 (Consistency set) Given $\sigma \in \Sigma$, the *consistency* set for the system (7) is given by $\mathfrak{O}_\sigma := \{\xi \mid \xi \in \mathrm{span}(\mathfrak{C}_\sigma)\}$.

Given that $\mathfrak{C}_\sigma$ is a basis with finitely many column vectors, the operator span over $\mathfrak{C}_\sigma$ leads to a closed set $\mathfrak{O}_\sigma$.

Next, the consistency and differential projectors are defined.

Definition 4 (Consistency and differential projectors [22, Definition 6.4.1]) Given the regular collection $\{(E_\sigma, A_\sigma)\}_{\sigma\in\Sigma}$ as in Definition 1, for each $\sigma \in \Sigma$, let the matrix pencil $(S_\sigma E_\sigma T_\sigma, S_\sigma A_\sigma T_\sigma)$ and n_1^σ and n_2^σ be given by Definition 9 and Theorem 4. Then, for each $\sigma \in \Sigma$, we define the following:

– Consistency projector: $\Pi_\sigma := T_\sigma \begin{bmatrix} I_{n_1^\sigma} & 0 \\ 0 & 0_{n_2^\sigma} \end{bmatrix} T_\sigma^{-1}$
– Differential projector: $\Pi_\sigma^{\mathrm{diff}} := T_\sigma \begin{bmatrix} I_{n_1^\sigma} & 0 \\ 0 & 0_{n_2^\sigma} \end{bmatrix} S_\sigma$

Now we can define a solution to a switched DAE system.

Definition 5 (Solution to a switched DAE system) A solution $\phi = (\phi_\xi, \sigma)$ to the switched DAE system (7) consists of a piecewise constant function $t \mapsto \sigma(t) \in \Sigma$ and a piecewise continuously differentiable function $t \mapsto \phi_\xi(t) \in \mathfrak{O}_{\sigma(t)}$, both right continuous, such that $E_{\sigma(t)}\dot{\phi}_\xi(t) = A_{\sigma(t)}\phi_\xi(t)$ for almost all $t \in \mathrm{dom}\,\phi_\xi$, with $\mathrm{dom}\,\phi = \mathrm{dom}\,\phi_\xi = \mathrm{dom}\,\sigma$.

Let the switching instants be denoted as t_i, which satisfy $0 = t_0 < t_1 < t_2 < \ldots < t_i < t_{i+1}$. The function $\sigma : \mathbb{R}_{\geq 0} \to \Sigma$ is constant over the intervals $[t_i, t_{i+1})$ for each $i \in \mathbb{N}$. Thus, at switching instants, the jumps in the state of the system between consistency sets are given by the consistency projectors in Definition 4, namely,

$$\phi_\xi^+(t_i) = \Pi_{\sigma(t_i)}\phi_\xi^-(t_i)$$

where the superscript $^+$ denotes value of the state after an instantaneous change and $\phi_\xi^-(t_i)$ denotes the limit from the left of the value of the state, namely $\phi_\xi^-(t_i) := \lim_{t\to t_i^-}\phi_\xi(t)$. Finally, the derivative of the solution of the DAE $(E_{\sigma(t)}, A_{\sigma(t)})$ is computed in the interior of the aforementioned intervals; therefore, this notion of solution does not consider impulses.

A solution ϕ to the switched DAE system in (7) is *complete* if its domain $\operatorname{dom}\phi_\xi$ is equal to $\mathbb{R}_{\geq 0}$ and *precompact* if the solution itself is complete and bounded, where by bounded we mean that there exists a bounded set K such that $\operatorname{rge}\phi_\xi \subset K$. In Lemma 5 of Appendix A.4, we recall an explicit formula of solution to (7).

2.2 Modeling hybrid systems as hybrid inclusions

The hybrid system modeling framework employed here is the one in [8], where a hybrid system is given by a hybrid inclusion $\mathcal{H}$ of the form

$$\mathcal{H}: \quad x \in \mathbb{R}^p \begin{cases} x \in C & \dot{x} \in F(x) \\ x \in D & x^+ \in G(x) \end{cases}$$

The data of $\mathcal{H}$ is given by a set $C \subset \mathbb{R}^p$, called the *flow set*; a set-valued mapping $F : \mathbb{R}^p \rightrightarrows \mathbb{R}^p$, called the *flow map*; a set $D \subset \mathbb{R}^p$, called the *jump set*; and a set-valued mapping $G : \mathbb{R}^p \rightrightarrows \mathbb{R}^p$, called the *jump map*. The flow map F defines the continuous dynamics on the flow set C, while the jump map G defines the discrete dynamics on the jump set D. At times, the *data* of $\mathcal{H}$ is explicitly denoted as $\mathcal{H} = (C, F, D, G)$.

To guarantee a sequential compactness property of the solutions to hybrid systems, some mild conditions on the data of the hybrid inclusion are required. These conditions are listed in Assumption 1 below. Assumption 1 requires the concepts of outer semicontinuity and local boundedness, which are given in [8, Definition 5.9] and [8, Definition 5.14], respectively.

Assumption 1 *(Hybrid basic conditions [8, Assumption 6.5]) The hybrid inclusion $\mathcal{H} = (C, F, D, G)$ satisfies the hybrid basic conditions if*

(A1) C and D are closed subsets of $\mathbb{R}^p$;
(A2) $F : \mathbb{R}^p \rightrightarrows \mathbb{R}^p$ is outer semicontinuous, locally bounded relative to C, $C \subset \operatorname{dom} F$, and $F(x)$ is convex for every $x \in C$;
(A3) $G : \mathbb{R}^p \rightrightarrows \mathbb{R}^p$ is outer semicontinuous, locally bounded relative to D, and $D \subset \operatorname{dom} G$.

A hybrid inclusion that meets the conditions in Assumption 1 is said to be nominally well-posed, see [8, Definition 6.2].

3 Main results

3.1 An invariance principle for hybrid DAE systems

In this section, we introduce a class of hybrid systems that model DAE systems with state-triggered jumps[3]. We refer to these systems as hybrid DAE systems

[3] A simplified version of this model was introduced in our preliminary work in [16] and [17]. The model in [16] considers single-valued functions φ_σ. In [17], we generalize the hybrid DAE model to allow for set-valued maps φ_σ to make it suitable for the study of switched DAE systems. Note that the models in [16] and [17] do not consider the state component χ.

and denote them as $\mathcal{H}_{DAE}$. In this section, we formulate an invariance principle for hybrid DAE systems.

The state vector of an $\mathcal{H}_{DAE}$ is given by

$$x{:=}(\xi, \chi, \sigma) \in \mathbb{R}^n \times \mathbb{R}^m \times \Sigma,$$

where ξ is the state component associated with the DAE system, χ is the state component driven by a differential/difference inclusion, σ is the state component associated with the switching signal, and Σ is a finite discrete set as defined in Section 2.1. The hybrid DAE system is given by the hybrid inclusion

$$\mathcal{H}_{DAE}\begin{cases}\begin{bmatrix}E_\sigma & 0 & 0\\ 0 & I_m & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\dot{\xi}\\ \dot{\chi}\\ \dot{\sigma}\end{bmatrix} \in \begin{bmatrix}A_\sigma\xi\\ \rho(x)\\ 0\end{bmatrix} =: F(x) & x \in C\\ \begin{bmatrix}\xi^+\\ \chi^+\\ \sigma^+\end{bmatrix} \in \bigcup\limits_{\tilde{\sigma}\in\varphi(x)}\begin{bmatrix}g(x,\tilde{\sigma})\\ \gamma(x)\\ \tilde{\sigma}\end{bmatrix} =: G(x) & x \in D\end{cases} \tag{8a}$$

where

$$C := \bigcup_{\sigma\in\Sigma}\left(C_\sigma \cap (\mathfrak{O}_\sigma \times \mathbb{R}^m \times \{\sigma\})\right) \tag{8b}$$

$$D := \bigcup_{\sigma\in\Sigma}\big((D_\sigma \cap (\mathfrak{O}_\sigma \times \mathbb{R}^m \times \{\sigma\})) \cup ((\mathbb{R}^n \setminus \mathfrak{O}_\sigma) \times \mathbb{R}^m \times \{\sigma\})\big) \tag{8c}$$

$$g(x,\tilde{\sigma}) := g_D(x,\tilde{\sigma}) \cup g_{\mathfrak{O}}(x,\tilde{\sigma}) \quad \forall x \in D, \tilde{\sigma} \in \varphi(x) \tag{8d}$$

and

$$g_D((\xi,\chi,\sigma),\tilde{\sigma}) = g_D(x,\tilde{\sigma}) := \begin{cases}\emptyset & \text{if } x \in (\mathbb{R}^n \setminus \mathfrak{O}_\sigma) \times \mathbb{R}^m \times \{\sigma\}\\ \Pi_{\tilde{\sigma}} g_\sigma(\xi,\chi) & \text{if } x \in D_\sigma \cap (\mathfrak{O}_\sigma \times \mathbb{R}^m \times \{\sigma\}),\end{cases} \tag{8e}$$

$$g_{\mathfrak{O}}((\xi,\chi,\sigma),\tilde{\sigma}) = g_{\mathfrak{O}}(x,\tilde{\sigma}) := \begin{cases}\Pi_{\tilde{\sigma}}\xi & \text{if } x \in (\mathbb{R}^n \setminus \mathfrak{O}_\sigma) \times \mathbb{R}^m \times \{\sigma\}\\ \emptyset & \text{if } x \in D_\sigma \cap (\mathfrak{O}_\sigma \times \mathbb{R}^m \times \{\sigma\}),\end{cases} \tag{8f}$$

$$\gamma((\xi,\chi,\sigma)) = \gamma(x) := \begin{cases}\chi & \text{if } x \in (\mathbb{R}^n \setminus \mathfrak{O}_\sigma) \times \mathbb{R}^m \times \{\sigma\}\\ g_\gamma(x) & \text{if } x \in D_\sigma \cap (\mathfrak{O}_\sigma \times \mathbb{R}^m \times \{\sigma\}),\end{cases} \tag{8g}$$

The elements of data that represent $\mathcal{H}_{DAE}$ on the state space $\mathbb{R}^n \times \mathbb{R}^m \times \Sigma$ are given by $(E_\sigma, C_\sigma, A_\sigma, \rho, D_\sigma, g_\sigma, g_\gamma, \varphi)$ and have the following properties:

- The sets $\mathfrak{O}_\sigma$ are the consistency sets and the matrices Π_σ are the consistency projectors, which are generated using E_σ and A_σ for each $\sigma \in \Sigma$, as in Definition 3 and Definition 4, respectively.
- Given A_σ for each $\sigma \in \Sigma$, and a set valued mapping $\rho : \mathbb{R}^{n+m+1} \rightrightarrows \mathbb{R}^m$, the flow map F is a set valued mapping $F : \mathbb{R}^{n+m+1} \rightrightarrows \mathbb{R}^{n+m+1}$ that defines the continuous evolution of x.
- At jumps, the map g defines the changes of ξ while γ defines the changes of χ. The set-valued map φ determines the changes of σ. The jump map G is a set-valued mapping $G : \mathbb{R}^{n+m+1} \rightrightarrows \mathbb{R}^{n+m+1}$ that defines the changes of x at jumps.
- The map g is given by Equation (8d), where functions $g_{\mathfrak{O}}$ and g_D defines the changes of ξ when ξ does not belong to the consistency set $\mathfrak{O}_\sigma$ and when ξ belongs to the intersection between the consistency set $\mathfrak{O}_\sigma$ and the set D_σ, respectively. Notice that due to the consistency projector $\Pi_{\tilde{\sigma}}$, both maps, g_D and $g_{\mathfrak{O}}$, map any point in $\mathbb{R}^{n+m+2}$ to points in the consistency set.
- For each $\sigma \in \Sigma$, the sets C_σ and D_σ are subsets of $\mathbb{R}^{n+m+1}$ that, together with the consistency sets $\mathfrak{O}_\sigma$, define where the evolution of the system according to F and G are possible, respectively. Namely, the sets C and D model where the state of the system can change according to the differential inclusion or the difference inclusion in (8a), respectively.

– The conditions on the state related to the algebraic equations in the DAE are given by the consistency set $\mathfrak{O}_\sigma$, while conditions not related with such algebraic equations are modeled by C_σ and D_σ.
– The set $C_\sigma \cap (\mathfrak{O}_\sigma \times \mathbb{R}^m \times \{\sigma\})$ is the collection of points in $\mathbb{R}^{n+m+1}$ where the system is allowed to flow (since $\mathfrak{O}_\sigma$ is the set of points where flow is "consistent"). The set $D_\sigma \cap (\mathfrak{O}_\sigma \times \mathbb{R}^m \times \{\sigma\})$ is from where jumps of ξ and χ according to g_σ and g_γ, respectively, are allowed.

For convenience, the dimension m of the state component χ is allowed to be zero, in which case the state of the system becomes $x = (\xi, \sigma)$, and the state variable χ as well as the maps ρ, γ, and g_γ do not play a role in the dynamics of (8). On the other hand, the state component χ is motivated (and becomes very useful) in modeling switched DAEs under dwell-time switching in Section 3.2.2.

As in the framework in [8], we define solutions to hybrid DAEs using hybrid time domains. Therefore, during flows, solutions are parametrized by $t \in \mathbb{R}_{\geq 0}$, while at jumps they are parametrized by $j \in \mathbb{N}$. The reader can find a definition of the hybrid time domains in [8, Definition 2.3] and of hybrid arcs in [8, Definition 2.4].

A hybrid arc is a solution to a hybrid DAE system if it satisfies the system dynamics. This notion is made precise next.

Definition 6 (Solution) A hybrid arc $\phi = (\phi_\xi, \phi_\chi, \phi_\sigma)$ is a solution to $\mathcal{H}_{DAE}$ if $\phi(0,0) \in \overline{C} \cup D$ and

(S1) (Flow condition) for each $j \in \mathbb{N}$ such that $I^j := \{t : (t,j) \in \operatorname{dom}\phi\}$ has nonempty interior

$$\begin{bmatrix}\phi_\xi(t,j)\\ \phi_\chi(t,j)\\ \phi_\sigma(t,j)\end{bmatrix} \in C \text{ for all } t \in \operatorname{int} I^j, \quad t \mapsto \begin{bmatrix}\phi_\xi(t,j)\\ \phi_\chi(t,j)\\ \phi_\sigma(t,j)\end{bmatrix} \text{ satisfies}$$

$$\begin{bmatrix} E_{\phi_\sigma(t,j)} & 0 & 0\\ 0 & I_m & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\dot{\phi}_\xi(t,j)\\ \dot{\phi}_\chi(t,j)\\ \dot{\phi}_\sigma(t,j)\end{bmatrix} = \begin{bmatrix} A_{\phi_\sigma(t,j)}(\phi_\xi(t,j))\\ \rho((\phi_\xi(t,j), \phi_\chi(t,j), \phi_\sigma(t,j)))\\ 0\end{bmatrix}$$

$$\text{for almost all } t \in I^j$$

(S2) (Jump condition) for each $(t,j) \in \operatorname{dom}\phi$ such that $(t, j+1) \in \operatorname{dom}\phi$,

$$\phi(t,j) \in D, \quad \phi(t,j+1) \in G(\phi(t,j))$$

A solution ϕ to $\mathcal{H}_{DAE}$ is maximal if there does not exist another solution ψ such that $\operatorname{dom}\phi$ is a proper subset of $\operatorname{dom}\psi$ and $\phi(t,j) = \psi(t,j)$ for all $(t,j) \in \operatorname{dom}\phi$. A solution ϕ is complete if $\operatorname{dom}\phi$ is unbounded and precompact if it is complete and bounded. Notice that the previous concept of solution is impulse free by definition.

In Section 4, Example 3 illustrates the concept of solution in Definition 6, while Examples 4 and 5 show the effect on the ω-limit set of a given solution that contains impulses.

Before introducing the first main result, define

$$\hat{D} := \bigcup_{\sigma \in \Sigma} (D_\sigma \cap (\mathfrak{O}_\sigma \times \mathbb{R}^m \times \{\sigma\})) \tag{9}$$

$$\hat{G}(x) := \bigcup_{\tilde{\sigma} \in \varphi(x)} \begin{bmatrix} g_D(x, \tilde{\sigma})\\ g_\gamma(x))\\ \tilde{\sigma}\end{bmatrix} \tag{10}$$

Due to the sequential compactness property of solutions required to develop an invariance-like result, the data of $\mathcal{H}_{DAE}$ will have to satisfy the following mild regularity properties.

Assumption 2 *(Hybrid DAE basic conditions) A hybrid DAE* $\mathcal{H}_{DAE} = (E_\sigma, C_\sigma, A_\sigma, \rho, D_\sigma, g_\sigma, g_\gamma, \varphi)$ *satisfies the* hybrid DAE basic conditions *if*

(B1) For each $\sigma \in \Sigma$, C_σ *and* D_σ *are closed sets in* $\mathbb{R}^{n+m+1}$*;*
(B2) The collection $\{(E_\sigma, A_\sigma)\}_{\sigma\in\Sigma}$ *is regular (see Definition 1) and* $\rho : \mathbb{R}^{n+m+1} \rightrightarrows \mathbb{R}^m$ *is outer semicontinuous, locally bounded relative to* C, $C \subset \operatorname{dom}\rho$*, and* $\rho(\xi, \chi, \sigma)$ *is convex for every* $(\xi, \chi, \sigma) \in C$*;*
(B3) The set-valued function $\varphi : \mathbb{R}^n \times \mathbb{R}^m \times \Sigma \rightrightarrows \Sigma$ *is outer semicontinuous relative to* $\hat{D}$*. For each* $\sigma \in \Sigma$, $g_\sigma : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ *is outer semicontinuous and locally bounded at each* $(\xi, \chi) \in \mathbb{R}^n \times \mathbb{R}^m$ *such that* $(\xi, \chi, \sigma) \in \hat{D}$*, and* $\hat{D} \subset \cup_{\sigma\in\Sigma}(\operatorname{dom} g_\sigma \times \{\sigma\})$*. Also,* $g_\gamma : \mathbb{R}^{n+m+1} \rightrightarrows \mathbb{R}^m$ *is outer semicontinuous, locally bounded relative to* $\hat{D}$*, and* $\hat{D} \subset \operatorname{dom} g_\gamma$*.*

Following [8, Definition 6.19], we present the following notion of invariance.

Definition 7 (Weak Invariance) For the hybrid DAE system $\mathcal{H}_{DAE}$, the set $\mathcal{M} \subset \mathbb{R}^n \times \mathbb{R}^m \times \Sigma$ is said to be

- *weakly forward invariant* if for each $x_0 \in \mathcal{M}$, there exists at least one complete solution ϕ to $\mathcal{H}_{DAE}$ from x_0 with $\operatorname{rge}\phi \subset \mathcal{M}$;
- *weakly backward invariant* if for each $x^* \in \mathcal{M}$ and each $N > 0$, there exist $x_0 \in \mathcal{M}$ and at least one solution ϕ to $\mathcal{H}_{DAE}$ from x_0 such that for some $(t^*, j^*) \in \operatorname{dom}\phi$, $t^* + j^* \geq N$, we have $\phi(t^*, j^*) = x^*$ and $\phi(t, j) \in \mathcal{M}$ for all $(t, j) \in \operatorname{dom}\phi$ with $t + j \leq t^* + j^*$;
- *weakly invariant* if it is both weakly forward invariant and weakly backward invariant.

Following [20, Theorem 4.7], we consider locally Lipschitz functions V to locate the ω-limit set of precompact solutions to $\mathcal{H}_{DAE}$. To be able to take derivatives, we employ the generalized directional gradient (in the sense of Clarke) of V at x in the direction v, which is given by $V^\circ(x, v) = \max_{\zeta\in\partial V(x)} \langle\zeta, v\rangle$, where $\partial V(x)$ is a closed, convex, and nonempty set equal to the convex hull of all limits of sequences $\nabla V(x_i)$ with x_i converging to x.

Now, we are ready to present the invariance principle for hybrid DAE systems.

Theorem 1 *(Invariance principle for* $\mathcal{H}_{DAE}$ *with linear flow map) Suppose the hybrid DAE* $\mathcal{H}_{DAE}$ *in* (8) *is such that its data* $(E_\sigma, C_\sigma, A_\sigma, \rho, D_\sigma, g_\sigma, g_\gamma, \varphi)$ *satisfies Assumption 2. Furthermore, suppose there exist a function* $V : \mathbb{R}^{n+m+1} \to \mathbb{R}$ *that is continuous on* $\mathbb{R}^{n+m+1}$ *and locally Lipschitz on an open set containing* C*, and functions* $u_C : \mathbb{R}^{n+m+1} \to \mathbb{R}$ *and* $u_D : \mathbb{R}^{n+m+1} \to \mathbb{R}$ *such that*

$$V^\circ\left(x, \begin{bmatrix} \Pi_\sigma^{\mathit{diff}} A_\sigma \xi \\ \rho(x) \\ 0 \end{bmatrix}\right) \leq u_C(x) \qquad \forall x \in C \tag{11a}$$

$$V(\eta) - V(x) \leq u_D(x) \qquad \forall x \in \hat{D} \text{ and } \forall \eta \in \hat{G}(x) \tag{11b}$$

Let ϕ *be a precompact (complete and bounded) solution to* $\mathcal{H}_{DAE}$ *with initial condition* $\phi(0, 0) = (\xi_0, \chi_0, \sigma_0)$*. Suppose that* $K \subset \mathbb{R}^{n+m+1}$ *is nonempty and* $\overline{\operatorname{rge}\phi} \subset K$*. If*

$$u_C(x) \leq 0, \quad u_D(x) \leq 0 \qquad \forall x \in K,$$

then ϕ *approaches the largest weakly invariant set for* $\mathcal{H}_{DAE}$ *contained in*

$$V^{-1}(r) \cap K \cap \left(u_C^{-1}(0) \cup \left[u_D^{-1}(0) \cap G\left(u_D^{-1}(0)\right)\right]\right) \tag{12}$$

for some constant $r \in V(K)$*.*

In the next section, we propose invariance principles for switched DAE systems given by (7) under certain families of switching signals. In particular, for switched DAE systems, we consider arbitrarily fast switching signals and dwell-time switching signals. A signal $t \mapsto \sigma(t)$ is a *dwell-time switching signal* with dwell-time $\tau_D > 0$ if $t_{j+1} - t_j \geq \tau_D$ for each $j \in \{1, 2, \ldots\}$, where t_j is the $j-$th switching time (see [20] for similar constructions). Next, we discuss a switched DAE system under dwell-time switching signal in Example 2. It revisits Example 1 and explores the possibility of recasting the linear switched DAE in (7) as a hybrid inclusion (as defined in [8,15]) and then try to apply the invariance principles in [8].

Example 2 (Motivational example revisited) Consider the switched DAE system in Example 1. Computing the consistency and differential projectors as in Definition 4, we have that $\Pi_\sigma = \Pi_\sigma^{\text{diff}}$ for each $\sigma \in \Sigma = \{1, 2\}$, where Π_σ is given in (4). Let the switching signal σ be considered as an input to the system of dwell-time type with dwell-time $\tau_D > 0$. We can recast this switched DAE system as a hybrid inclusion with inputs as in [15]. We denote this hybrid inclusion as $\mathcal{H}_s$, which is given by

$$\mathcal{H}_s \begin{cases} \dot{\xi} &= f_s(\xi, \sigma) \quad (\xi, \sigma) \in C_s \\ \xi^+ &= g_s(\xi, \sigma) \quad (\xi, \sigma) \in D_s \end{cases} \tag{13}$$

where C_s, f_s, D_s, and g_s are given in (6). This hybrid inclusion has the same solutions as the original switched DAE system.

The invariance principles in [8] are developed for autonomous hybrid inclusions, namely hybrid inclusions without inputs. One can consider an exosystem that generates the dwell-time switching signal $t \mapsto \sigma(t)$ such that $t \mapsto (\xi(t), \sigma(t))$ is also a solution to the system in (6). To do so, consider the following equivalent hybrid inclusion with the state vector $x = (\xi, \tau, \sigma) \in \mathbb{R}^2 \times \mathbb{R} \times \Sigma$

$$\mathcal{H} \begin{cases} \begin{bmatrix} \dot{\xi} \\ \dot{\tau} \\ \dot{\sigma} \end{bmatrix} \in \begin{bmatrix} f_s(\xi, \sigma) \\ \kappa_{\tau_D}(\tau) \\ 0 \end{bmatrix} & x \in C \\ \begin{bmatrix} \xi^+ \\ \tau^+ \\ \sigma^+ \end{bmatrix} = g(x) & x \in D \end{cases} \tag{14}$$

where, the set valued function $\kappa_{\tau_D}(\tau)$ is given by

$$\kappa(\tau_D)(\tau) := \begin{cases} 1 & \text{if } \tau < \tau_D \\ [0, 1] & \text{if } \tau = \tau_D \\ 0 & \text{if } \tau > \tau_D \end{cases} \qquad \forall \tau \in \mathbb{R} \tag{15}$$

the sets $C \subset \mathbb{R}^3 \times \Sigma$ and $D := D_a \cup D_b \subset \mathbb{R}^3 \times \Sigma$ are given by

$$\begin{aligned} C &= \{x \in \mathbb{R}^3 \times \Sigma \mid \sigma = 1, \tau \in [0, \tau_D]\} \cup \{x \in \mathbb{R}^3 \times \Sigma \mid \sigma = 2, \xi_1 = 0, \tau \in [0, \tau_D]\} \\ D_a &= \{x \in \mathbb{R}^3 \times \Sigma \mid \sigma = 1, \tau \geq \tau_D\} \cup \{x \in \mathbb{R}^3 \times \Sigma \mid \sigma = 2, \xi_1 = 0, \tau \geq \tau_D\} \\ D_b &= \{x \in \mathbb{R}^3 \times \Sigma \mid \sigma = 2, \xi_1 \neq 0\} \end{aligned}$$

and, the jump map $g : \mathbb{R}^3 \times \Sigma \to \mathbb{R}^3 \times \Sigma$ is given by

$$g(x) := \begin{cases} \begin{bmatrix} \Pi_{3-\sigma}\xi \\ 0 \\ 3 - \sigma \end{bmatrix} & x \in D_a \\ \begin{bmatrix} \Pi_\sigma \xi \\ \tau \\ \sigma \end{bmatrix} & x \in D_b \end{cases}$$

Notice that due to D_b, the set D is open. Recalling Lemma 6.9 in [8], which states that a necessary condition for well-posedness of hybrid systems is that the set D is closed, we conclude that the equivalent hybrid inclusion in (14) cannot be nominal well-posed. Therefore, the results in [8] are not applicable. □

3.2 Invariance Principles for Switched DAE systems

In this section, we propose invariance principles for switched DAE systems given by (7) under certain families of switching signals. In particular, we consider arbitrarily fast switching signals and dwell-time switching signals.

Next, we introduce mild regularity properties on the data of the collection $\{(E_\sigma, A_\sigma)\}_{\sigma\in\Sigma}$ that are required in the following sections.

Assumption 3 *(Basic conditions for the collection $\{(E_\sigma, A_\sigma)\}_{\sigma\in\Sigma}$). Given the collection $\{(E_\sigma, A_\sigma)\}_{\sigma\in\Sigma}$, we have that*

(C1) The collection $\{(E_\sigma, A_\sigma)\}_{\sigma\in\Sigma}$ is regular (see Definition 1).

3.2.1 Invariance principle for switched DAE systems under arbitrary switching

Before introducing the second main result, define

$$C = \hat{D} = \bigcup_{\sigma\in\Sigma} \left(\mathfrak{O}_\sigma \times \{\sigma\}\right) \tag{16}$$

Next, we present an invariance principle for this class of switched DAE systems.

Theorem 2 *(Invariance principle for switched DAE systems under arbitrary switching). Given a switched DAE system as in (7), suppose that the collection $\{(E_\sigma, A_\sigma)\}_{\sigma\in\Sigma}$ satisfies Assumption 3. Let $\Pi_\sigma^{\mathit{diff}}$ be given by Definition 4. Furthermore, suppose there exist a function $V:\ \mathbb{R}^{n+1}\to\mathbb{R}$ that is continuous on $\mathbb{R}^{n+1}$ and locally Lipschitz on an open set containing C, and functions $u_C:\ C\to\mathbb{R}$ and $u_D:\ \hat{D}\to\mathbb{R}$ such that, for all $(\xi,\sigma)\in \bigcup_{\sigma\in\Sigma}(\mathfrak{O}_\sigma\times\{\sigma\})$, we have*

$$V^\circ\left((\xi,\sigma), \begin{bmatrix}\Pi_\sigma^{\mathit{diff}} A_\sigma \xi\\ 0\end{bmatrix}\right) \le u_C(\xi,\sigma) \qquad \forall(\xi,\sigma)\in C \tag{17a}$$

$$V(\Pi_{\tilde\sigma}\xi,\tilde\sigma) - V(\xi,\sigma) \le u_D(\xi,\sigma) \qquad \forall\tilde\sigma\in\Sigma\setminus\{\sigma\} \text{ and } \forall(\xi,\sigma)\in\hat{D} \tag{17b}$$

where C and $\hat{D}$ are given in (16). Let ϕ be a precompact solution to the switched DAE system in (7). Suppose that $K\subset\mathbb{R}^{n+1}$ is nonempty and $\overline{\operatorname{rge}\phi}\subset K$. If

$$u_C(\xi,\sigma)\le 0,\quad u_D(\xi,\sigma)\le 0 \qquad \forall(\xi,\sigma)\in K,$$

then ϕ approaches the largest weakly invariant set for the switched DAE system contained in

$$V^{-1}(r)\cap K\cap\left(u_C^{-1}(0)\cup\left[u_D^{-1}(0)\cap G\left(u_D^{-1}(0)\right)\right]\right) \tag{18}$$

for some constant $r\in V(K)$.

Remark 2 Notice that given that σ is an argument of the functions V, u_C, and u_D, it is possible to consider different functions V, u_C, and u_D for each σ. In particular, the results in Section 3.2.2 use functions V explicitly indexed by σ.

3.2.2 Invariance principle for switched DAE systems under dwell-time switching

Next, to locate the ω-limit set of precompact solutions to switched DAE systems an invariance principle for switched DAE systems under dwell-time switching signals is presented following the ideas in [7]. First, let us introduce the required assumptions.

Assumption 4 *For each $\sigma \in \Sigma$, $V_\sigma : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz and $W_\sigma : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a continuous function on an open set O_σ, $\mathfrak{D}_\sigma \subset O_\sigma$, and*

$$V_\sigma^\circ\left(\xi, \Pi_\sigma^{\text{diff}} A_\sigma \xi\right) \leq -W_\sigma(\xi) \quad \forall \xi \in O_\sigma \tag{19}$$

Assumption 5 *Given a dwell-time switching signal $t \mapsto \sigma(t)$, the solution $t \mapsto \phi_\xi(t)$ to (7) is such that for each $p \in \Sigma$, for any two consecutive intervals (t_j, t_{j+1}), (t_k, t_{k+1}) such that $\sigma(t) = p$ for all $t \in (t_j, t_{j+1})$ and all $t \in (t_k, t_{k+1})$, we have*

$$V_p(\phi_\xi(t_{j+1})) - V_p(\phi_\xi(t_k)) \geq 0 \tag{20}$$

Theorem 3 *(Invariance principle for switched DAE systems under dwell-time switching) Let Assumption 3 and Assumption 4 hold, and let ϕ be a precompact* dwell-time solution, *with dwell-time $\tau_D > 0$, to the switched DAE system in (7) satisfying Assumption 5. Then, there exist $r_1, r_2, \ldots, r_{\sigma_{max}} \in \mathbb{R}$ such that ϕ_ξ approaches*

$$\mathcal{M} = \bigcup_{\sigma \in \Sigma} \mathcal{M}_\sigma(r_\sigma, \tau_D), \tag{21}$$

where, for each $\sigma \in \Sigma$, $\mathcal{M}_\sigma(r_\sigma, \tau_D)$ is the largest subset of $V_\sigma^{-1}(r_\sigma) \cap W_\sigma^{-1}(0)$ that is invariant in the following sense: for each constant $\sigma \in \Sigma$ and for each $\xi_0 \in \mathcal{M}_\sigma(r_\sigma, \tau_D)$ there exists a solution[4] $(\phi_{\tilde{\xi}}, \sigma)$ to $E_\sigma \dot{\xi} = A_\sigma \xi$ on $[0, \tau_D/2]$ such that $\phi_{\tilde{\xi}}(t) \in \mathcal{M}_\sigma(r_\sigma, \tau_D)$ for all $t \in [0, \tau_D/2]$ and either $\phi_{\tilde{\xi}}(0) = \xi_0$ or $\phi_{\tilde{\xi}}(\tau_D/2) = \xi_0$.

Similar as in [7], we will say that a pair of functions $(\Pi_\sigma^{\text{diff}} A_\sigma \xi, W_\sigma)$ is observable if, for each $a < b$, the only solution $\phi_\xi : [a, b] \to \mathbb{R}^n$ to $\dot{\xi} = \Pi_\sigma^{\text{diff}} A_\sigma \xi$ with $W_\sigma(\phi_\xi(t)) = 0$ for all $t \in [a, b]$ is $\phi_\xi(t) = 0$ for all $t \in [a, b]$.

Assumption 6 *For each $\sigma \in \Sigma$, the pair $(\Pi_\sigma^{\text{diff}} A_\sigma \xi, W_\sigma)$ is observable.*

Corollary 1 *(Invariance principle for switched DAE systems under observability) Let Assumptions 3, 4, and 6 hold. Then, every precompact dwell-time solution ϕ_ξ to the switched DAE system in (7) satisfying Assumption 5 is such that ϕ_ξ converges to the origin.*

4 Examples

Next, we introduce examples to illustrate the concepts and results in the previous section. Example 3 illustrates the concept of solution to hybrid DAE systems, while Examples 4 and 5 discuss the effect on the ω-limit set of a solution that contains impulses. Examples 6, 7, and 8 portray the usage of Theorems 1, 2, and 3, respectively.

In Examples 3 through 6, the state component $\chi \in \mathbb{R}^m$ with $m = 0$ is omitted; therefore, in those examples we ignore the maps ρ, γ, and g_γ in (8). In the following examples, consistency spaces, consistency projectors, and difference projectors $\mathfrak{C}_\sigma$, Π_σ, and Π_σ^{diff}, respectively, are computed using the algorithm in [6].

[4] The notion of solution of a DAE is given by the component ϕ_ξ as in Definition 5 when σ is constant.

Example 3 (Motivational example revisited) Consider the $\mathcal{H}_{DAE}$ system in (8) with the data given by (2). Consider the flow and jump sets given by $C_1 := \{(\xi,\sigma) \in \mathbb{R}^2 \times \Sigma \mid \xi_1 \geq 0, \sigma = 1\}$, $D_1 := \{(\xi,\sigma) \in \mathbb{R}^2 \times \Sigma | \ \xi_1 \leq 0, \xi_2 \leq 0, \sigma = 1\}$, $C_2 := \{(\xi,\sigma) \in \mathbb{R}^2 \times \Sigma \mid \xi_2 \leq 0, \sigma = 2\}$, $D_2 := \{(\xi,\sigma) \in \mathbb{R}^2 \times \Sigma | \ \xi_2 \geq 0, \sigma = 2\}$. Also, consider $g_1(\xi) = -\xi$, $g_2(\xi) = \xi$, and $\varphi(\xi,\sigma) = 3 - \sigma$. Computing the consistency spaces and consistency projectors, we obtain

$$\mathfrak{C}_1 = \operatorname{im}\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right), \quad \mathfrak{C}_2 = \operatorname{im}\left(\begin{bmatrix}0\\ 1\end{bmatrix}\right), \quad \Pi_1 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad \Pi_2 = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} \tag{22}$$

The resulting consistency sets are given by

$$\mathfrak{O}_1 = \{x \in \mathbb{R}^2 \times \Sigma \mid \sigma = 1\}, \quad \mathfrak{O}_2 = \{x \in \mathbb{R}^2 \times \Sigma \mid \xi_1 = 0, \sigma = 2\} \tag{23}$$

Consider the initial condition $(\xi(0,0), \sigma(0,0)) = ((0,1),1)$, where $\xi(0,0) \in \overline{C}$. Now, consider a hybrid time domain given by $\mathcal{T} := \{([0,\pi],0)\} \cup \{([\pi,\pi],1)\} \cup \{([\pi,\pi],2)\} \cup \{([\pi,2\pi],3)\} \cup \{([2\pi,2\pi],4)\} \cup \ldots$ and a hybrid arc $\phi = (\phi_\xi, \phi_\sigma)$ defined as

- for all $t \in I^0 := [0,\pi]$, $\phi_\xi(t,0) = [\sin(t), \cos(t)]^\top$ and $\phi_\sigma(t,0) = 1$;
- for all $t \in I^1 := [\pi,\pi]$, $\phi_\xi(t,1) = \Pi_2 g_1((0,-1)) = (0,1)$ and $\phi_\sigma(t,1) = \varphi(\phi_\xi(\pi,0),1) = 2$;
- for all $t \in I^2 := [\pi,\pi]$, $\phi_\xi(t,2) = \Pi_1 g_2((0,1)) = (0,1)$ and $\phi_\sigma(t,2) = \varphi(\phi_\xi(\pi,1),2) = 1$;
- for all $t \in I^3 := [\pi,2\pi]$, $\phi_\xi(t,3) = [\sin(t-\pi), \cos(t-\pi)]^\top$ and $\phi_\sigma(t,3) = 1$;
- for all $t \in I^4 := [2\pi,2\pi]$, $\phi_\xi(t,4) = \Pi_2 g_1((0,-1)) = (0,1)$ and $\phi_\sigma(t,4) = \varphi(\phi_\xi(2\pi,3),1) = 2$;
- $\vdots$

This hybrid arc is a solution to the hybrid DAE system since *(S1)* and *(S2)* in Definition 6 hold. □

It is well known that inconsistent initial conditions in a switched DAE system may induce impulses in the solutions. For this reason, it is a common practice to impose certain restrictions on the data (E_σ, A_σ) to ensure impulse free solutions, see, e.g., [13, Theorem 3.8] and [25,26] for descriptor systems. The solution concept in Definition 6 leads to impulse free solutions but allows for jumps on the state at initial time instances and after switching of modes. This is due to the derivative of the solutions being computed in the interior of the intervals I^j's (without empty interior).

Example 4 (A system with no impulse free solutions [13, Example 3.6]) Consider the $\mathcal{H}_{DAE}$ system with the data in (8), where

$$E_1 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \ A_1 = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \ E_2 = \begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}, \ A_2 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$\Sigma = \{1,2\}$, and $g_\sigma(\xi) = \xi$. Now, suppose that C, D, and φ are such that there exists a solution ϕ from $\phi_\xi(0,0) = (1,0,0)$ with σ component $\phi_\sigma(t,0) = 1$ for each $t \in [0,1]$ and $\phi_\sigma(t,1) = 2$ for each $t \in [1,\infty)$. Then, using the definition of solution to a hybrid DAE in Definition 6, the only possibility for the ξ component is $\phi_\xi(t,0) = \xi(0,0)$ for each $t \in [0,1]$ and $\phi_\xi(t,1) = (0,0,0)$ for each $t \in [1,\infty)$. On the other hand, using the concept of solution in [13], a solution $t \mapsto x(t) = (x_1(t), x_2(t), x_3(t))$ to the switched DAE system (7) with data $\{(E_\sigma, A_\sigma)\}_{\sigma \in \Sigma}$ would not be impulse free. In fact, at the jump at $t = 1$, since $x_1(t^+) = 0$, then its derivative, which defines the x_2 component of the solution when $\sigma = 2$, would have a delta Dirac at $t = 1$ and its second derivative, which defines the x_3 component of the solution when $\sigma = 2$, would involve the derivative of the delta Dirac at $t = 1$. Such impulsive solutions are not allowed in the solution concept introduced in Definition 6. Despite the differences in the concepts of solution, notice that over the interior of the intervals, $[0, 1) \times \{0\}$ and $[1, \infty) \times \{1\}$ for ϕ_ξ while $[0, 1)$ and $[1, \infty)$ for x, the solutions ϕ_ξ and x coincide. □

Recall from [16] that an ω-limit point of a solution ϕ to $\mathcal{H}_{DAE}$ is a point $x \in \mathbb{R}^{n+m+1}$ such that there exists a sequence of points $(t_i, j_i) \in \operatorname{dom}\phi$ with $\lim_{i\to\infty}(t_i + j_i) = \infty$ and $\lim_{i\to\infty}\phi(t_i, j_i) = x$. The set of all ω-limit points of ϕ is defined as the ω-limit set of ϕ. We present a formal definition of the ω-limit set of a solution in Definition 8.

Example 5 (Invariant set with no impulse free solutions) Consider the $\mathcal{H}_{DAE}$ system with the data in (8) and

$$\begin{aligned}
&E_1 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\ A_1 = \begin{bmatrix}0&0&-1\\0&0&-1\\0&0&-1\end{bmatrix},\ E_2 = \begin{bmatrix}0&0&0\\1&0&0\\0&0&1\end{bmatrix},\ A_2 = \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\\
&g_1(\xi) = g_2(\xi) = \xi,\ \varphi(\xi, \sigma) = 3 - \sigma,
\end{aligned}$$

where $\xi \in \mathbb{R}^3$ and $\Sigma = \{1, 2\}$. The flow and jump sets are given by $C_1 = \{(\xi, \sigma) \in \mathbb{R}^3 \times \Sigma \mid 1/2 \le \xi_3 \le 1, \sigma = 1\}$, $C_2 = \{(\xi, \sigma) \in \mathbb{R}^3 \times \Sigma \mid 1/2 \le \xi_3 \le 1, \sigma = 2\}$, $D_1 = \{(\xi, \sigma) \in \mathbb{R}^3 \times \Sigma \mid \xi_3 \le 1/2, \sigma = 1\}$, and $D_2 = \{(\xi, \sigma) \in \mathbb{R}^3 \times \Sigma \mid \xi_3 \ge 1, \sigma = 2\}$. Computing the consistency projectors, we have

$$\Pi_1 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \quad \Pi_2 = \begin{bmatrix}0&0&-1\\0&0&-1\\0&0&1\end{bmatrix}$$

Taking into account the concept of solution in [13], a nonzero solution $t \mapsto x(t) = (x_1(t), x_2(t), x_3(t))$ to the switched DAE system (7) with data $\{(E_\sigma, A_\sigma)\}_{\sigma\in\Sigma}$ would have an impulse in the state x_2 when switching from mode 1 to 2. When switching from mode 1 to 2, according to (E_2, A_2), the algebraic equation $0 = x_1 + x_3$ should be fulfilled after the switch, which suggest that, if $0 = x_1 + x_3$ does not hold before the switch, the solution would have a jump in its x_1 component. Also, given that x_2 is given by $\dot{x}_1$, a delta Dirac δ would be induced in the x_2 component of that solution (i.e., if t_s is the switching time, then $\delta(t - t_s)$ appears as an additive term in $x_2(t)$). It is important to mention that (7) is a time-varying system, thus the usual concept of ω-limit of a solution does not apply [11]. Nevertheless, one can consider an auxiliary system, known as the *exosystem*, that generates a given switching signal $t \mapsto \sigma(t)$ such that $t \mapsto (x(t), \sigma(t))$ is also a solution to (8), leading to an autonomous system. Consider a switching signal $t \mapsto \sigma(t)$ for (7) that has a jump at time instants $t_i \in \mathbb{R}_{\ge 0}$, $i \in \mathbb{N}$. Also, suppose that, given an initial condition $\xi(0)$ to (7) (and a initial value $\sigma(0)$ to the exosystem), at each switching instant t_i, $\sigma(t_i^+)$ toggles (by $\sigma(t_i^+) = 3 - \sigma(t_i)$) when $\sigma(t_i) = 1$ and $x_3(t_i) \le \frac{1}{2}$ or when $\sigma(t_i) = 2$ and $x_3(t_i) \ge 1$. Following the exosystem idea, the switching signal is given by

$$\sigma(t) = \begin{cases}\sigma(0) & \text{if } t_{2i} \le t < t_{2i+1}\\ 3 - \sigma(0) & \text{if } t_{2i+1} \le t < t_{2i+2}\end{cases} \qquad \forall t \in \mathbb{R}_{\ge 0}, i \in \mathbb{N}$$

Given the initial condition[5] $x(0) = (1,1,1)$ and $\sigma(0) = 1$, according to the concept of solution in [13], the solution has infinitely many jumps and impulses. It follows that the ω-limit set of such solution is given by $\mathcal{M}_c = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_\delta$, where

$$\begin{aligned}
\mathcal{M}_1 &= \{(x,\sigma) \in \mathbb{R}^3 \times \Sigma \mid x_1 = x_2 = x_3 - 2, 1/2 < x_3 \leq 1, \sigma = 1\} \\
\mathcal{M}_2 &= \{(x,\sigma) \in \mathbb{R}^3 \times \Sigma \mid x_1 = x_2 = -x_3, 1/2 < x_3 < 1, \sigma = 2\} \\
\mathcal{M}_\delta &= \{(x,\sigma) \in \mathbb{R}^3 \times \Sigma \mid x_2 = \delta(0) - 1/2, x_1 = -1/2, x_3 = 1/2, \sigma = 2\}
\end{aligned}$$

Notice that for each $(x_0,\sigma_0) \in \mathcal{M}_1 \cap \mathcal{M}_2$ there exist at least one complete solution (x,σ) to (7) and the exosystem from (x_0,σ_0) with $\operatorname{rge}(x,\sigma) \subset \mathcal{M}_c$. On the other hand, for each $(x_0,\sigma_0) \in \mathcal{M}_\delta$ there is no solution (x,σ) to (7) and the exosystem from (x_0,σ_0) with $\operatorname{rge}(x,\sigma) \subset \mathcal{M}_c$. Therefore, the set $\mathcal{M}_c$ is not weakly forward invariant. It is important to highlight that one would expect the ω-limit set of the solution to be weakly forward invariant, but this is not the case when solutions have impulses.

On the other hand, following the concept of solution in Definition 6, consider the solution $(t,j) \mapsto (\phi_\xi(t,j), \phi_\sigma(t,j)) = ((\phi_1(t,j), \phi_2(t,j), \phi_3(t,j)), \phi_\sigma(t,j))$ to (8) from $(\phi_\xi(0,0), \phi_\sigma(0,0)) = (x(0), \sigma(0)) = ((1,1,1),1)$. Due to the derivative of the solutions being computed in the interior of the intervals I^j's (without empty interior), that concept of solution does not take into account the aforementioned impulses. Even more, notice that disregarding the impulses, the solutions (x,σ) and (ϕ_ξ, ϕ_σ) coincide. Therefore, it is natural to expect that the ω-limit set $\mathcal{M}$ of (ϕ_ξ, ϕ_σ) is such that $\mathcal{M} \subset \overline{\mathcal{M}_c}$. Consider the set $\mathcal{M} = \overline{\mathcal{M}_1} \cup \overline{\mathcal{M}_2}$, and notice that for each $(\xi_0,\sigma_0) \in \mathcal{M}$ there is a solution to (8) from (ξ_0,σ_0) with $\operatorname{rge}(\phi_\xi,\phi_\sigma) \subset \mathcal{M}$; therefore, the set $\mathcal{M}$ is weakly forward invariant with respect to (8). Additionally, following the result in Theorem 1 it is possible to show that the set $\mathcal{M}$ is also weakly backward invariant. Thus, the set $\mathcal{M}$ is weakly invariant for (8) (see Definition 7 for formal definitions to these notions). ◻

To illustrate the use of the invariance principle in Theorem 1 for hybrid DAE systems, we first revisit the example in Section 1.2.

Example 6 (Motivational example revisited) Consider the $\mathcal{H}_{DAE}$ system in (8), with the data given by (2) and the single-valued function $\varphi(\xi,\sigma) = 3 - \sigma$, where $\Sigma = \{1,2\}$. For each $\sigma \in \Sigma$, the flow and jump sets are given by $C_\sigma = D_\sigma = \mathbb{R}^2 \times \{\sigma\}$. The consistency spaces, consistency projectors, and consistency sets are given by (22) and (23), respectively. Also, computing the difference projectors $\Pi_\sigma^{\mathrm{diff}}$, we have $\Pi_\sigma = \Pi_\sigma^{\mathrm{diff}}$.

Notice that, ignoring the requirements on the maps ρ and g_γ due to $m = 0$, the data $(E_\sigma, C_\sigma, A_\sigma, \rho, D_\sigma, g_\sigma, g_\gamma, \varphi)$ fulfills Assumption 2. Considering the same Lyapunov-like function in (3), $K = \mathbb{R}^2 \times \{1,2\}$, and functions $u_C : \mathbb{R}^3 \to \mathbb{R}$ and $u_D : \mathbb{R}^3 \to \mathbb{R}$ given by

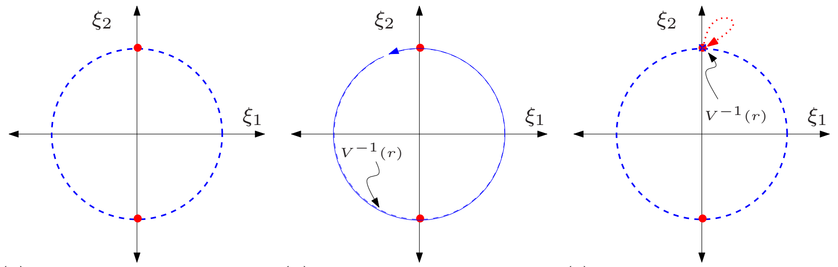
$$u_C(x) := (\sigma - 1)(-2\xi_2^2) \qquad \forall x \in \mathbb{R}^{2+1} \tag{26a}$$
$$u_D(x) := (2 - \sigma)(-\xi_1^2) \qquad \forall x \in \mathbb{R}^{2+1} \tag{26b}$$

we apply Theorem 1.

Let $x \in C$. We have the following cases:

- If $x \in C_1 \cap \mathfrak{D}_1 = \{x \in \mathbb{R}^2 \times \{1,2\} \mid \sigma = 1\}$, then $V^\circ\left(x, \begin{bmatrix} \Pi_1^{\mathrm{diff}} A_1 \xi \\ 0 \end{bmatrix}\right) = 0 = u_C(x)$.

[5] Notice that the initial condition $(x(0),\sigma(0))$ is consistent with (E_1, A_1) since $x(t_0) = \Pi_{\sigma(t_0)} x(t_0)$



(a) Representation of the invariant set $\mathcal{M}'$.

(b) Solution that remains in a level set of V.

(c) Solution that start from initial conditions $\xi(0,0) = (0,\sqrt{r})$ and stay there jumping.

Fig. 1 Representation of the invariant set $\mathcal{M}'$, where the circle in blue (dashed) represents $\{x \in \mathbb{R}^2 \times \{1,2\} \mid \xi_1^2 + \xi_2^2 = r, \sigma = 1\}$ and the points in red represents $\{x \in \mathbb{R}^2 \times \{1,2\} \mid \xi_1 = 0, \xi_2^2 = r, \sigma = 2\}$. The solid (blue) curve indicates flow. The dashed (red) arc indicates jumps.

- If $x \in C_2 \cap \mathfrak{D}_2 = \{x \in \mathbb{R}^2 \times \{1,2\} \mid \xi_1 = 0,\ \sigma = 2\}$, then $V^\circ\left(x, \begin{bmatrix} \Pi_2^{\text{diff}} A_2 \xi \\ 0 \end{bmatrix}\right) = -2\xi_2^2 = u_C(x)$.

Let $x \in D$. We have the following cases:

- If $x \in D_1 \cap \mathfrak{D}_1 = \{x \in \mathbb{R}^2 \times \{1,2\} \mid \sigma = 1\}$, then $V(\Pi_2\xi, 2) - V(\xi, 1) = -\xi_1^2 = u_D(x)$.
- If $x \in D_2 \cap \mathfrak{D}_2 = \{x \in \mathbb{R}^2 \times \{1,2\} \mid \xi_1 = 0,\ \sigma = 2\}$, then $V(\Pi_1\xi, 1) - V(\xi, 2) = 0 = u_D(x)$.

Notice that (11a) and (11b) hold for u_C and u_D in (26a) and (26b) respectively. Computing the sets involved in (12), we have, for each $r \in \mathbb{R}_{\geq 0}$,

$$V^{-1}(r) \cap \left(u_C^{-1}(0) \cup \left(u_D^{-1}(0) \cap G\left(u_D^{-1}(0)\right)\right)\right) = (V^{-1}(r) \cap C_1) \cup (V^{-1}(r) \cap \{x \in \mathbb{R}^2 \times \{1,2\} \mid \xi_1 = 0,\ \sigma = 2\})$$

$$= (V^{-1}(r) \cap C_1) \cup \{x \in \mathbb{R}^2 \times \{1,2\} \mid \xi_1 = 0, \xi_2^2 = r, \sigma = 2\}$$
$$= \{x \in \mathbb{R}^2 \times \{1,2\} \mid \xi_1^2 + \xi_2^2 = r, \sigma = 1\} \cup \{x \in \mathbb{R}^2 \times \{1,2\} \mid \xi_1 = 0, \xi_2^2 = r, \sigma = 2\}$$
$$=: \mathcal{M}'$$

Then, using Theorem 1, every precompact solution to $\mathcal{H}_{DAE}$ converges to the largest weakly invariant set inside $\mathcal{M}'$ for some $r \in \mathbb{R}_{\geq 0}$. This is a tight result as $\mathcal{M}'$ is weakly forward invariant. In Figure 1(a), we present a representation of the invariant set $\mathcal{M}'$. In fact, there are precompact solutions ϕ to $\mathcal{H}_{DAE}$ from $\mathcal{M}'$ that have ϕ_σ equal to one, in which case, ϕ remains in the level set of $V^{-1}(r)$, with $r = V(\phi(0,0))$, as is shown in Figure 1(b). There are also precompact solutions ϕ that start from initial conditions $\xi_1(0,0) = 0$ and $\xi_2(0,0)^2 = r$ and stay there jumping as is shown in Figure 1(c). ◻

Next, we use Theorem 2 to describe the largest weakly invariant set for a switched DAE system under arbitrary switching.

Example 7 (2D switched DAE system) Consider the switched DAE system (as presented in [21, Example 1b]) with the data given by

$$E_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix},\ E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

where $\Sigma = \{1, 2\}$. Notice that the collection $\{(E_\sigma, A_\sigma)\}_{\sigma\in\Sigma}$ is regular; therefore, the switched DAE system fulfills Assumption 3. The consistency spaces and projectors are given by

$$\begin{aligned} \mathfrak{C}_1 &= \operatorname{im}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right),\ \Pi_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},\ \Pi_1^{\text{diff}} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \\ \mathfrak{C}_2 &= \operatorname{im}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right),\ \Pi_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ \Pi_2^{\text{diff}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

The consistency sets are given by $\mathfrak{O}_1 = \{\xi \in \mathbb{R}^2 \mid \xi_1 = \xi_2\}$ and $\mathfrak{O}_2 = \{\xi \in \mathbb{R}^2 \mid \xi_1 = 0\}$.

Now, consider the Lyapunov-like function $V : \mathbb{R}^{2+1} \to \mathbb{R}$ given by

$$V(\xi, \sigma) = \sigma\left(\frac{1}{2}\xi_1^2 + \frac{1}{2}\xi_2^2\right), \tag{27a}$$

and the subset $K \subset \mathbb{R}^3$ given by $K = \bigcup_{p\in\Sigma}\left(\mathfrak{O}_p \times \{p\}\right)$. Also, consider the function $u_D : \hat{D} \to \mathbb{R}$, which for points $(\xi, \sigma) \in \hat{D}$, is given by

$$u_D(\xi, \sigma) := 0, \tag{27b}$$

and the function $u_C : C \to \mathbb{R}$, which for points in $(\xi, \sigma) \in C$, is given by

$$u_C(\xi, \sigma) := -\xi_1^2 - \xi_2^2 \tag{27c}$$

where C and $\hat{D}$ are given by (16). To apply Theorem 2, let $(\xi, \sigma) \in \bigcup_{p\in\Sigma}(\mathfrak{O}_p \times \{p\})$. We have the following cases:

- If $(\xi, \sigma) \in \mathfrak{O}_1 \times \{\sigma\}$, then $V^\circ\left((\xi, \sigma), \begin{bmatrix} \Pi_1^{\text{diff}} A_1 \xi \\ 0 \end{bmatrix}\right) = -\xi_1^2 - \xi_2^2 = u_C(\xi, \sigma)$ and $V(\Pi_{3-\sigma}\xi, 3 - \sigma) - V(\xi, \sigma) = 0 = u_D(\xi, \sigma)$.
- If $(\xi, \sigma) \in \mathfrak{O}_2 \times \{\sigma\}$, then $V^\circ\left((\xi, \sigma), \begin{bmatrix} \Pi_2^{\text{diff}} A_2 \xi \\ 0 \end{bmatrix}\right) \le -\xi_1^2 - \xi_2^2 \le u_C(\xi, \sigma)$ and $V(\Pi_{3-\sigma}\xi, 3 - \sigma) - V(\xi, \sigma) = 0 = u_D(\xi, \sigma)$.

Observe that the assumptions in Theorem 2 hold for V, u_C, and u_D in (27). Computing the sets involved in (18), we have

$$V^{-1}(r) \cap K \cap \left(u_C^{-1}(0) \cup \left(u_D^{-1}(0) \cap G\left(u_D^{-1}(0)\right)\right)\right) = V^{-1}(r) \cap K \cap (\mathfrak{O}_1 \cup \mathfrak{O}_2)$$

Then, using Theorem 2, every precompact solution to (7) converges to the largest weakly invariant subset in (18) for some $r \in \mathbb{R}_{\ge 0}$, where (18) is given by

$$\begin{aligned} \mathcal{M}' = &\left\{(\xi, \sigma) \in \mathbb{R}^2 \times \{1, 2\} \;\middle|\; \frac{\xi_1^2}{2} + \frac{\xi_2^2}{2} = r, \sigma = 1\right\} \cup \\ &\qquad\left\{(\xi, \sigma) \in \mathbb{R}^2 \times \{1, 2\} \mid \xi_1 = 0, \xi_2^2 = r, \sigma = 2\right\} \end{aligned}$$

This is a tight result as $\mathcal{M}'$ is weakly forward invariant. In fact, there are precompact solutions ϕ from $\mathcal{M}'$ that have constant ϕ_σ, in which case, ϕ_ξ approaches the origin. Also, for every solution ϕ with $\operatorname{dom}\phi = \bigcup_{j=0}^{\infty}([t_j, t_{j+1}], j)$ with infinitely many intervals $[t_j, t_{j+1}]$ such that $t_{j+1} - t_j > 0$, due to the strictly decaying rate of V during flows, the largest invariant set is $\{0\} \times \Sigma$.

In fact, consider the switched DAE system recast as a hybrid DAE system denoted as $\mathcal{H}^{\mathcal{SW}}_{DAE}$. Notice that $\mathcal{H}^{\mathcal{SW}}_{DAE}$ has Zeno solutions. In particular, from every point in $\mathcal{M}'$, there is a solution that remains in $\mathcal{M}'$ by jumping (in backward and forward time), making the set $\mathcal{M}'$ weakly forward invariant for some $r > 0$; i.e., there are precompact solutions ϕ to $\mathcal{H}^{\mathcal{SW}}_{DAE}$ that start from consistent initial conditions $\xi_1(0,0)^2/2 + \xi_2(0,0)^2/2 = r$ and $\sigma(0,0) = 1$; or $\xi_1(0,0) = 0$, $\xi_2(0,0)^2 = r$, and $\sigma(0,0) = 2$ and stay jumping in between these two points. However, notice that the notion of solution to (7) in Definition 5 does not allow for Zeno solutions. It should be noted that Zeno solutions to $\mathcal{H}^{\mathcal{SW}}_{DAE}$ are not solutions to the switched DAE system in (7). □

Example 8 (2*D* switched DAE system under dwell-time switching) Consider the same data as in Example 7, but with a variation on the second mode given by

$$E_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

The consistency spaces and projectors for the second mode are given by

$$\mathfrak{C}_2 = \operatorname{im}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right), \; \Pi_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \; \Pi_2^{\text{diff}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

The consistency sets are given by $\mathfrak{D}_1 = \{\xi \in \mathbb{R}^2 \mid \xi_1 = \xi_2\}$ and $\mathfrak{D}_2 = \{\xi \in \mathbb{R}^2 \mid \xi_1 = 0\}$. Also, for each $\sigma \in \{1, 2\}$ consider the functions $V_\sigma : \mathbb{R}^2 \to \mathbb{R}$ and $W_\sigma : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\begin{aligned} V_\sigma(\xi) &= \frac{1}{2}\xi_1^2 + \frac{1}{2}\xi_2^2 \\ W_\sigma(\xi) &= \xi_1^2 + \xi_2^2, \end{aligned}$$

where the consistency sets are given by $\mathfrak{D}_1 = \{\xi \in \mathbb{R}^2 \mid \xi_1 = \xi_2\}$ and $\mathfrak{D}_2 = \{\xi \in \mathbb{R}^2 \mid \xi_1 = 0\}$.

To apply Theorem 3 we have the following cases

- If $(\xi, \sigma) \in (\mathfrak{D}_1 \times \{1\}) = \{(\xi, \sigma) \in \mathbb{R}^2 \times \{1, 2\} \mid \xi_1 = \xi_2, \sigma = 1\}$, then $V_\sigma^\circ\left((\xi, \sigma), \begin{bmatrix} \Pi_1^{\text{diff}} A_1 \xi \\ 0 \end{bmatrix}\right) = -\xi_1^2 - \xi_2^2 \le -W_\sigma((\xi, \sigma))$.
- If $(\xi, \sigma) \in (\mathfrak{D}_2 \times \{2\}) = \{(\xi, \sigma) \in \mathbb{R}^2 \times \{1, 2\} \mid \xi_1 = 0, \sigma = 2\}$, then $V_\sigma^\circ\left((\xi, \sigma), \begin{bmatrix} \Pi_2^{\text{diff}} A_2 \xi \\ 0 \end{bmatrix}\right) = -\xi_2^2 \le -W_\sigma((\xi, \sigma))$.

Notice that Assumption 4 holds for V_σ and W_σ. To satisfy Assumption 5, we need to evaluate V along the solutions. The decrement rate of V_σ during flows is

$$\lambda_\sigma = \frac{-V_\sigma^\circ\left(\xi, \Pi_\sigma^{\text{diff}} A_\sigma \xi\right)}{V_\sigma(\xi)}$$

where

- If $(\xi, \sigma) \in (\mathfrak{D}_1 \times \{1\})$, then $\lambda_1 = \frac{-(-\xi_1^2 - \xi_2^2)}{\frac{1}{2}\xi_1^2 + \frac{1}{2}\xi_2^2} = \frac{2\xi_1^2}{\xi_1^2} = 2$.

– If $(\xi,\sigma) \in (\mathfrak{O}_2 \times \{2\})$, then $\lambda_2 = \frac{-(-\xi_1\xi_2-\xi_2^2)}{\frac{1}{2}\xi_1^2+\frac{1}{2}\xi_2^2} = \frac{\xi_2^2}{\frac{1}{2}\xi_2^2} = 2$.

Thus, $\lambda_1 = \lambda_2 = \lambda = 2$. The decrement of V at jumps is as follows:

– If $(\xi,\sigma) \in (\mathfrak{O}_1 \times \{1\})$, then $V_{3-\sigma}(\Pi_{3-\sigma}\xi, 3-\sigma) - V_\sigma(\xi,\sigma) = \xi_2^2$.
– If $(\xi,\sigma) \in (\mathfrak{O}_2 \times \{2\}))$, then $V_{3-\sigma}(\Pi_{3-\sigma}\xi, 3-\sigma) - V_\sigma(\xi,\sigma) = \frac{1}{2}\xi_2^2$.

where we can deduce that at jumps

$$\begin{aligned}
V_\sigma(\xi(t_{j+1})) - V_\sigma(\xi(t_l)) &\geq 0 \\
V_\sigma(\xi(t_{j+1})) - 2V_{3-\sigma}(\xi(t_{k+1})) &\geq 0 \\
V_\sigma(\xi(t_{j+1})) - 2e^{-\lambda(t_{k+1}-t_k)}V_{3-\sigma}(\xi(t_k)) &\geq 0 \\
V_\sigma(\xi(t_{j+1})) - 2e^{-\lambda(t_{k+1}-t_k)}2V_\sigma(\xi(t_{j+1})) &\geq 0 \\
V_\sigma(\xi(t_{j+1}))(1-4e^{-\lambda\tau_D}) &\geq 0
\end{aligned}$$

for each $\sigma \in \{1,2\}$. Thus, Assumption 5 holds when $\tau_D \geq \ln(4)\frac{1}{2}$.

Consequently, following Theorem 3, ϕ being a precompact dwell-time solution with dwell-time $\tau_D \geq \ln(4)\frac{1}{2}$, ϕ_ξ approaches the largest invariant subset of $\mathcal{M} = \bigcup_{\sigma\in\Sigma} \mathcal{M}_\sigma(r_\sigma,\tau_D)$. Computing the sets involved in (21) with $W_\sigma^{-1}(0) = \{0\}$ for $\sigma \in \{1,2\}$, we have that the only set that is invariant is the origin. In fact that property follows from Corollary 1 since the pair $(\Pi_\sigma^{\text{diff}}A_\sigma\xi, W_\sigma)$ satisfies Assumption 6. Thus, any precompact dwell-time solution ϕ_ξ with dwell-time $\tau_D \geq \ln(4)\frac{1}{2}$ approaches the origin. In Figure 2(a) (left), a numerical approximation of the solution to the system for a given initial condition and dwell-time $\tau_D = \ln(4)\frac{1}{2}$ is shown. ◻

Notice that the bound for τ_D in Example 8 may be not tight enough; actually, the bound on τ_D depends on the selection of V_σ as is shown in the following example.

Example 9 (2D switched DAE system under dwell-time switching) Consider the switched DAE system in Example 8 under dwell-time $\tau_D > 0$ and the functions $V_\sigma : \mathbb{R}^2 \to \mathbb{R}$ and $W_\sigma : \mathbb{R}^2 \to \mathbb{R}$ for $\sigma \in \{1,2\}$ given by the trivial choices

$$V_\sigma(\xi) = 0, \qquad W_\sigma(\xi) = 0$$

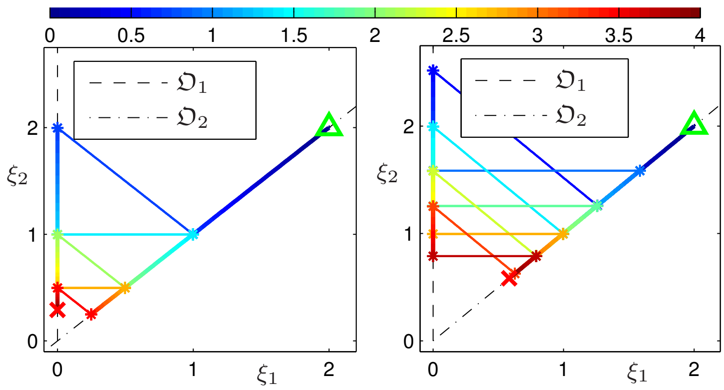
for all $(\xi,\sigma) \in \mathfrak{O}_\sigma \times \{\sigma\}$.

– If $(\xi,\sigma) \in (\mathfrak{O}_1 \times \{1\})$, then $V_1^\circ\left(x, \begin{bmatrix}\Pi_1^{\text{diff}}A_1\xi \\ 0\end{bmatrix}\right) = 0 \leq -W_1(x)$.
– If $(\xi,\sigma) \in (\mathfrak{O}_2 \times \{2\})$, then $V_2^\circ\left(x, \begin{bmatrix}\Pi_2^{\text{diff}}A_2\xi \\ 0\end{bmatrix}\right) = 0 \leq -W_2(x)$.

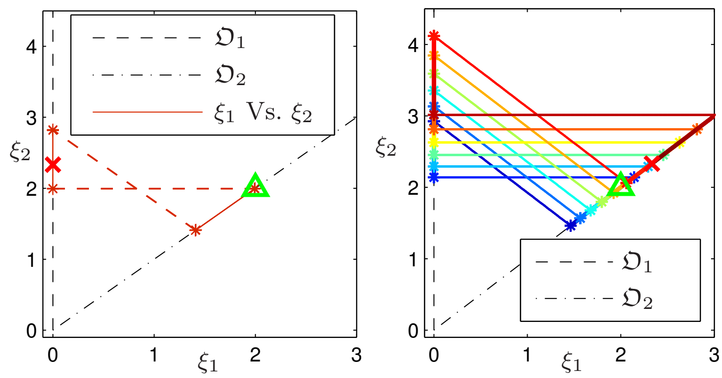
To satisfy the conditions in Theorem 3, we need to ensure that solutions are precompact as in Proposition 2. Thus, we choose an arbitrary compact set K given by a ball of radius r centered in the origin for some $r \in \mathbb{R}$. We use an auxiliary function $\kappa : \mathbb{R}^2 \to \mathbb{R}$ to define K as

$$K := \{\xi \in \mathbb{R}^2 \mid \kappa(\xi) \leq r\}, \tag{28}$$

where $\kappa(\xi) := \frac{\xi_1^2}{2} + \frac{\xi_2^2}{2}$. As in the previous example, it is possible to show that the function κ decays through flowing solutions by the decay rate $\lambda = 2$. Also, it is possible to show that at switching instants the function κ is given by $\kappa(\xi(t_k)) = 2\kappa(\xi(t_{j+1}))$ for all consecutive intervals (t_j, t_{j+1}) and $(t_k, t_{k+1}) \subset \operatorname{dom}\xi$. Then, to



(a) With $\tau_D = \ln(4)\frac{1}{2}$ (left) and $\tau_D = \ln(2)\frac{4}{6}$ (right).



(b) With $\tau_D = \ln(2)\frac{1}{2}$ (left) and $\tau_D = \ln(2)\frac{9}{20}$ (right).

Fig. 2 Solutions of the switched DAE in Example 8 and Example 9 for different dwell-time switching signals using the same initial condition $\xi_1(0) = \xi_2(0) = 2$ and $\sigma(0) = 1$. Color gradient represents time, the green triangle represents the initial condition, and the red $\times$ symbol represents the final condition at $t = 4$.

assure that the solution ξ belongs the compact set K (for some $r > 0$), after each jump the following inequality must hold:

$$\kappa(\xi(t_k)) \le r \tag{29a}$$

$$2\kappa(\xi(t_{j+1})) \le r \tag{29b}$$

$$2e^{-2(t_{j+1}-t_j)}\kappa(\xi(t_j)) \le r \tag{29c}$$

This condition is satisfied taking $t_{j+1} - t_j = \tau_D$, with $\tau_D \ge \ln\left(2\frac{\kappa(\xi(t_j))}{r}\right)\frac{1}{2}$. Thus, $\kappa(\xi(t_j)) = r$ give us the bound $\tau_D \ge \ln(2)\frac{1}{2}$ which makes ξ precompact for K given in (28). With the bound on τ_D it is possible to apply Theorem 3, and using auxiliary function κ we deduce that if $\tau_D = \ln(2)\frac{1}{2}$ then ξ approaches $\mathcal{M} = \bigcup_{\sigma\in\{1,2\}} \mathcal{M}_\sigma(r_\sigma, \tau_D)$ which, since $\operatorname{rge}\xi \subset K$, is a subset of

$$\bigcup_{\sigma\in\{1,2\}} V_\sigma^{-1}(r_\sigma) \cap K \cap W_\sigma^{-1}(0) = K \cap (\mathfrak{D}_1 \cup \mathfrak{D}_2)$$

In Figure 2(b) (left), the solution to the system for a given initial condition and dwell-time $\tau_D = \ln(2)\frac{1}{2}$ is shown. Additionally, given the strict decrease of $\kappa(\xi(t))$ through flows given by

$$\left\langle \nabla\kappa(\xi), \Pi_\sigma^{\text{diff}} A_\sigma \xi \right\rangle < 0$$

for $\sigma \in \{1,2\}$ and for all $\xi \in \mathbb{R}^2 \setminus \{0\}$, and (29), it is easy to see that if $\tau_D > \ln(2)\frac{1}{2}$ the largest invariant subset $\mathcal{M}$ is the origin. In Figure 2(a) (right), for $\tau_D = \ln(2)\frac{4}{6} > \ln(2)\frac{1}{2}$ it is shown how a solution for a given initial condition approaches the origin. Furthermore, if $\tau_D < \ln(2)\frac{1}{2}$ then $\kappa(\xi(t_j)) < r$, and therefore, the sequence $\kappa(\xi(t_k))$ for $k = \{1, 2, \ldots\}$ grows unbounded. In Figure 2(b) (right), it is shown how the solution does not converge when $\tau_D = \ln(2)\frac{9}{20} < \ln(2)\frac{1}{2}$. □

5 Proofs

5.1 Technical prerequisites

As pointed out above, solutions to $\mathcal{H}_{DAE}$ can exist from any point in $\mathbb{R}^n \times \mathbb{R}^m \times \Sigma$. Due to the fact that the set D in (8c) is open. The invariance principles for hybrid inclusions in [8] cannot be applied. This is the reason we introduce the following hybrid DAE system:

$$\hat{\mathcal{H}}_{DAE} \begin{cases} \begin{bmatrix} E_\sigma & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\xi} \\ \dot{\chi} \\ \dot{\sigma} \end{bmatrix} \in \begin{bmatrix} A_\sigma \xi \\ \rho(x) \\ 0 \end{bmatrix} = F(x) & x \in C \\ \begin{bmatrix} \xi^+ \\ \chi^+ \\ \sigma^+ \end{bmatrix} \in \bigcup_{\tilde{\sigma} \in \varphi(x)} \begin{bmatrix} g_D(x, \tilde{\sigma}) \\ g_\gamma(x) \\ \tilde{\sigma} \end{bmatrix} =: \hat{G}(x) & x \in \hat{D} \end{cases} \tag{30a}$$

where g_D is given in (8e), and

$$\hat{D} := \bigcup_{\sigma \in \Sigma} \left(D_\sigma \cap (\mathfrak{O}_\sigma \times \mathbb{R}^m \times \{\sigma\}) \right) \tag{30b}$$

For this system, under proper assumptions, $\hat{D}$ can be a closed set.

Remark 3 By replacing D with $\hat{D}$ and G with $\hat{G}$, Definition 6 also defines the notion of solution $\hat{\phi}$ to the hybrid DAE system $\hat{\mathcal{H}}_{DAE}$ in (30).

Remark 4 Notice that the model in (30) does not allow for inconsistent initial conditions, unlike the nominal hybrid DAE system in (8). Therefore, the system in (30) is not intended to replace the model in (8), but to be used as an intermediate model (simplified model) to carry out the proofs in this section.

5.1.1 Equivalence of solutions between $\hat{\mathcal{H}}_{DAE}$ and $\mathcal{H}_{DAE}$

Next, we present an equivalence between the solutions of $\hat{\mathcal{H}}_{DAE}$ and $\mathcal{H}_{DAE}$.

Lemma 1 *(Equivalence of solutions between $\mathcal{H}_{DAE}$ and $\hat{\mathcal{H}}_{DAE}$ systems) Consider $\mathcal{H}_{DAE}$ and $\hat{\mathcal{H}}_{DAE}$ in* (8) *and* (30) *with data $(E_\sigma, C_\sigma, A_\sigma, \rho, D_\sigma, g_\sigma, g_\gamma, \varphi)$. The following hold:*

i) Consistent initial value: if the initial value $(\xi_0, \chi_0, \sigma_0)$ *belongs to* $(\mathfrak{O}_{\sigma_0} \times \mathbb{R}^m \times \{\sigma_0\}) \cap (C_{\sigma_0} \cup D_{\sigma_0})$*, then* ϕ *is a maximal solution to* $\mathcal{H}_{DAE}$ *if and only if* ϕ *is a maximal solution to* $\hat{\mathcal{H}}_{DAE}$.

ii) Inconsistent initial value: if the initial value $(\xi_0, \chi_0, \sigma_0)$ *belongs to* $(\mathbb{R}^n \setminus \mathfrak{O}_{\sigma_0}) \times \mathbb{R}^m \times \{\sigma_0\}$*, for each maximal solution* ϕ *to* $\mathcal{H}_{DAE}$ *with* $\phi(0,0) = (\xi_0, \chi_0, \sigma_0)$*,* $(0,1) \in \operatorname{dom}\phi$*, and*

$$\phi(0,1) \in \bigcup_{\tilde{\sigma} \in \varphi((\xi_0,\chi_0,\sigma_0))} \left((\mathfrak{O}_{\tilde{\sigma}} \times \mathbb{R}^m \times \{\tilde{\sigma}\}) \cap (C_{\tilde{\sigma}} \cup D_{\tilde{\sigma}}) \right) \tag{31}$$

there exists a maximal solution $\hat{\phi}$ *to* $\hat{\mathcal{H}}_{DAE}$ *such that* $\operatorname{dom}\hat{\phi} \subset \operatorname{dom}\phi$*, where* $\hat{\phi}(t, j-1) = \phi(t,j)$ *for all* $(t,j) \in \operatorname{dom}\phi \setminus \{(0,0)\}$ *and* $\hat{\phi}(0,0) \in G(\xi_0, \chi_0, \sigma_0)$.

Given the construction of the systems in (8) and (30), the statement in this lemma holds immediately (for details, see Appendix A.2). A detailed version of the proofs in this manuscript are available in [18].

Remark 5 Notice that for each precompact (complete and bounded) solution ϕ to $\mathcal{H}_{DAE}$ there exists a precompact solution $\hat{\phi}$ to $\hat{\mathcal{H}}_{DAE}$ such that *i)* or *ii)* in Lemma 1 hold. Particularly, for each precompact solution ϕ to $\mathcal{H}_{DAE}$ from $(\xi_0, \chi_0, \sigma_0) \in (\mathbb{R}^n \setminus \mathfrak{O}_{\sigma_0}) \times \mathbb{R}^m \times \{\sigma_0\}$, we have that $(0,1) \in \operatorname{dom}\phi$ and there exists a precompact solution $\hat{\phi}$ to $\hat{\mathcal{H}}_{DAE}$ from $\hat{\phi}(0,0) = \phi(0,1) \in G(\xi_0, \chi_0, \sigma_0)$ such that (31) holds. Additionally, notice that due to the possibility of several inconsistent initial conditions mapping to the same singleton in the consistency set, a family of precompact solutions to $\mathcal{H}_{DAE}$ may have a single precompact solution to $\hat{\mathcal{H}}_{DAE}$ that satisfy item *ii)* in Lemma 1.

5.1.2 Equivalent $\widetilde{\mathcal{H}}$ and its solutions

The results in the next section require an equivalent hybrid inclusion representation of $\hat{\mathcal{H}}_{DAE}$. This representation is given in the next result.

Lemma 2 *(Equivalence between solutions of* $\hat{\mathcal{H}}_{DAE}$ *and* $\widetilde{\mathcal{H}}$*) Suppose that the data of* $\hat{\mathcal{H}}_{DAE}$ *is such that the collection* $\{(E_\sigma, A_\sigma)\}_{\sigma\in\Sigma}$ *satisfy Assumption 3. From a consistent initial condition* $\phi_0 = (\xi_0, \chi_0, \sigma_0) \in \mathfrak{O}_{\sigma_0} \times \mathbb{R}^m \times \{\sigma_0\}$*,* ϕ *is a solution to* $\hat{\mathcal{H}}_{DAE}$ *if and only if* ϕ *is a solution to the hybrid inclusion* $\widetilde{\mathcal{H}}$ *given by*

$$\widetilde{\mathcal{H}} \begin{cases} \begin{bmatrix} \dot{\xi} \\ \dot{\chi} \\ \dot{\sigma} \end{bmatrix} \in \begin{bmatrix} \Pi^{\text{diff}}_\sigma A_\sigma \xi \\ \rho(x) \\ 0 \end{bmatrix} =: \tilde{F}(x) & x \in C \\ \begin{bmatrix} \xi^+ \\ \chi^+ \\ \sigma^+ \end{bmatrix} \in \bigcup_{\tilde{\sigma}\in\varphi(x)} \begin{bmatrix} g_D(x,\tilde{\sigma}) \\ g_\gamma(x) \\ \tilde{\sigma} \end{bmatrix} = \hat{G}(x) & x \in \hat{D} \end{cases} \tag{32a}$$

where $\tilde{F} : \bigcup_{\sigma\in\Sigma} (\mathfrak{O}_\sigma \times \mathbb{R}^m \times \{\sigma\}) \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m \times \Sigma$.

Proof Since the collection $\{(E_\sigma, A_\sigma)\}_{\sigma\in\Sigma}$ is regular, for each $\sigma \in \Sigma$, the differential projector Π^{diff}_σ can be computed as in Definition 4. Notice that given the construction of $\hat{\mathcal{H}}_{DAE}$, $\hat{G}$ maps to $\cup_{\tilde{\sigma}\in\varphi(x)}(\mathfrak{O}_{\tilde{\sigma}} \times \mathbb{R}^m \times \{\tilde{\sigma}\})$ after each jump.

Furthermore, according to Definitions 2, 3, and [13, Definition 2.1], given a solution ϕ to $\hat{\mathcal{H}}_{DAE}$, we have that $\phi(t,j) = (\phi_\xi(t,j), \phi_\chi(t,j), \phi_\sigma(t,j)) \in \mathfrak{O}_{\sigma(t,j)} \times$

$\mathbb{R}^m \times \{\sigma(t,j)\}$ for all $(t,j) \in \operatorname{dom}\phi$. For each j such that $I^j \times \{j\} \subset \operatorname{dom}\phi$ has a nonempty interior, the solution components ϕ_ξ and ϕ_χ satisfy

$$E_{\phi_\sigma(t,j)}\dot{\phi}_\xi(t,j) = A_{\phi_\sigma(t,j)}\phi_\xi(t,j) \;\forall (t,j) \in \operatorname{int} I^j \times \{j\}, \tag{33a}$$

and

$$\dot{\phi}_\chi(t,j) \in \rho(\phi_\xi(t,j), \phi_\chi(t,j), \phi_\sigma(t,j)) \qquad \forall (t,j) \in \operatorname{int} I^j \times \{j\} \tag{33b}$$

Applying [22, Theorem 6.5.1] to (33a) for each interval $I^j \times \{j\} \in \operatorname{dom}\phi$ with nonempty interior, $t \mapsto \phi_\xi(t,j)$ also satisfies

$$\dot{\phi}_\xi(t,j) = \Pi^{\mathrm{diff}}_{\phi_\sigma(t,j)} A_{\phi_\sigma(t,j)}\phi_\xi(t,j) \tag{34}$$

To establish that ϕ is also a solution to $\widetilde{\mathcal{H}}$, notice that given the construction of $\hat{G}$ in $\hat{\mathcal{H}}_{DAE}$ and $\widetilde{\mathcal{H}}$, the value of $(\phi_\xi, \phi_\chi, \phi_\sigma)$ after each jump at (ξ, χ, σ) belongs to $\cup_{\tilde{\sigma} \in \varphi(x)}(\mathfrak{O}_{\tilde{\sigma}} \times \mathbb{R}^m \times \{\tilde{\sigma}\})$ and that holds for each $(t, j+1) \in \operatorname{dom}\phi$ such that $(t,j) \in \operatorname{dom}\phi$.

The above shows that ϕ is a solution to $\widetilde{\mathcal{H}}$. The other direction follows similarly since, for each interval $I^j \times \{j\} \in \operatorname{dom}\phi$ with nonempty interior, a function $t \mapsto \phi_\xi(t,j)$ is a solution to $\dot{\phi}_\xi(t,j) = \Pi^{\mathrm{diff}}_{\phi_\sigma(t,j)} A_{\phi_\sigma(t,j)}\phi_\xi(t,j)$ if and only if is a solution to $E_{\phi_\sigma(t,j)}\dot{\phi}_\xi(t,j) = A_{\phi_\sigma(t,j)}\phi_\xi(t,j)$. ■

In the following sections, the equivalent hybrid inclusion $\widetilde{\mathcal{H}}$ allows us to use previous results in the analysis of invariance properties of hybrid systems to establish invariance principles for hybrid DAE systems and switched DAE systems.

Recalling the hybrid basic conditions for hybrid inclusions in Assumption 1, and the definition of a nominally well-posed hybrid inclusion in [8, Definition 6.2], we present conditions for $\widetilde{\mathcal{H}}$ to be nominally well posed next.

Lemma 3 *(Well posedness of $\widetilde{\mathcal{H}}$) Let $\mathcal{H}_{DAE}$ satisfy Assumption 2. Then, the hybrid inclusion $\widetilde{\mathcal{H}}$ satisfies the hybrid basic conditions (Assumption 1).*

Proof We show that $(A1)$, $(A2)$, and $(A3)$ from Assumption 1 are satisfied:

- To show that $(A1)$ holds, note that for each $\sigma \in \Sigma$, C_σ, D_σ, and $\mathfrak{O}_\sigma$ are closed (owing to $(B1)$ and that $\mathfrak{O}_\sigma$ is given by the span of a subspace described by a basis $\mathfrak{C}_\sigma$ (see Definition 3)), the sets described by $C_\sigma \cap (\mathfrak{O}_\sigma \times \mathbb{R}^m \times \{\sigma\})$ and $D_\sigma \cap (\mathfrak{O}_\sigma \times \mathbb{R}^m \times \{\sigma\})$ are also closed. Therefore, by the construction in (8b) and (30b), C and $\hat{D}$ are closed.
- To show that $(A2)$ is satisfied, recall $(B2)$ from Assumption 3 and note that the collection $\{(E_\sigma, A_\sigma)\}_{\sigma\in\Sigma}$ is regular; therefore, for each $\sigma \in \Sigma$ (see Theorem 4 and [16, Theorem 3.4]) the differential projector $\Pi^{\mathrm{diff}}_\sigma$ can be computed as in Definition 4. Also, notice that for each $\sigma \in \Sigma$, the function $\xi \mapsto \Pi^{\mathrm{diff}}_\sigma A_\sigma \xi$, which for each $(\xi, \chi, \sigma) \in (\mathfrak{O}_\sigma \times \mathbb{R}^m \times \{\sigma\})$ is a smooth single-valued function, is outer semicontinuous and locally bounded. Further using (B2), the flow map

$$\tilde{F}((\xi,\chi,\sigma)) = \tilde{F}(x) = \begin{bmatrix} \Pi^{\mathrm{diff}}_\sigma A_\sigma \xi \\ \rho(x) \\ 0 \end{bmatrix}$$

  is outer semicontinuous, locally bounded relative to C, and $\tilde{F}(x)$ is convex for every $(\xi, \chi, \sigma) \in C$.

– To show that $(A3)$ holds, recall that for each $(\xi, \chi, \sigma) \in \hat{D}$

$$\hat{G}((\xi, \chi, \sigma)) = \hat{G}(x) = \bigcup_{\tilde{\sigma} \in \varphi(x)} \begin{bmatrix} g_D(x, \tilde{\sigma}) \\ g_\gamma(x) \\ \tilde{\sigma} \end{bmatrix}, \tag{35}$$

Notice that with $(B3)$ and the linear transformation $\Pi_{\tilde{\sigma}}$, the map $(\Pi_{\tilde{\sigma}} g_\sigma, g_\gamma, \tilde{\sigma}) : \mathbb{R}^n \times \mathbb{R}^m \times \{\tilde{\sigma}\} \rightrightarrows \mathfrak{O}_{\tilde{\sigma}} \times \mathbb{R}^m \times \{\tilde{\sigma}\}$ is outer semicontinuous, locally bounded at $\hat{D}$ (see [8, Definition 5.9] and [8, Definition 5.14], respectively). Now, given that $\hat{D}$ is closed and the fact that, for each $\tilde{\sigma} \in \Sigma$, the map $(\Pi_{\tilde{\sigma}} g_\sigma, g_\gamma, \tilde{\sigma})$ is outer semicontinuous and locally bounded at $\hat{D}$, the map $(g_D, g_\gamma, \tilde{\sigma})$ is also outer semicontinuous and locally bounded at $\hat{D}$. Then, $\hat{G} : \mathbb{R}^n \times \mathbb{R}^m \times \{\Sigma\} \rightrightarrows \cup_{\tilde{\sigma} \in \Sigma} (\mathfrak{O}_{\tilde{\sigma}} \times \mathbb{R}^m \times \{\tilde{\sigma}\})$ is outer semicontinuous and locally bounded at $\hat{D}$ due to φ being outer semicontinuous and $\hat{G}$ being given by a finite union of outer semicontinuous maps $(g_D, g_\gamma, \tilde{\sigma})$.

■

Remark 6 Notice that if φ is single-valued and (g_σ, g_γ) is outer semicontinuous, then $\hat{G}$ is already outer semicontinuous.

Remark 7 Notice that the dimension of $\chi \in \mathbb{R}^m$ is allowed to be zero $(m = 0)$, in which case the state of the system becomes $x = (\xi, \sigma)$ and the state χ and the maps ρ, γ, and g_γ do not play a role in the dynamics of (8) (and (30)). In such a case, the result of Lemma 3 holds for the data given by $(E_\sigma, C_\sigma, A_\sigma, D_\sigma, g_\sigma, \varphi)$ and ignoring the requirements on the maps ρ and g_γ in Assumption 2.

Given a complete solution to $\mathcal{H}_{DAE}$, we define its ω-limit set as follows; see [8, Definition 6.17].

Definition 8 (ω-limit set) The ω-limit set of a complete solution $\phi : \operatorname{dom}\phi \to \mathbb{R}^{n+m+1}$, denoted by $\Omega(\phi)$, is the set of all points $x \in \mathbb{R}^{n+m+1}$ for which there exists an increasing sequence[6] $\{(t_i, j_i)\}_{=1}^{\infty}$ of points $(t_i, j_i) \in \operatorname{dom}\phi$ with $\lim_{i\to\infty} t_i + j_i = \infty$ and $\lim_{i\to\infty} \phi(t_i, j_i) = x$. Every such point x is an ω-limit point of ϕ.

Lemma 4 *(ω-limit set for $\mathcal{H}_{DAE}$) Suppose that $\mathcal{H}_{DAE}$ satisfies Assumption 2. Let ϕ be a precompact (i.e., complete and bounded) solution to $\mathcal{H}_{DAE}$. Then, $\Omega(\phi)$ is nonempty, closed, weakly invariant, and* $\lim_{t+j\to\infty, (t,j)\in\operatorname{dom}\phi} |\phi(t, j)|_{\Omega(\phi)} = 0$.

Proof Given that the data of the $\mathcal{H}_{DAE}$ system and, consequently, the data of the $\hat{\mathcal{H}}_{DAE}$ system satisfy Assumption $(B2)$, an equivalent hybrid inclusion $\widetilde{\mathcal{H}}$ can be constructed as in Lemma 2 (Equation (32a)). Moreover, given a precompact solution ϕ to $\mathcal{H}_{DAE}$, by Lemma 1, there exists a precompact solution to $\hat{\mathcal{H}}_{DAE}$ that, at most, is a shift by one jump of ϕ. Additionally, by Lemma 2, such precompact solution is also a solution to $\widetilde{\mathcal{H}}$. With some abuse of notation, we do not relabel that solution. By Lemma 3, if $\mathcal{H}_{DAE}$ satisfies Assumption 2 then $\widetilde{\mathcal{H}}$ satisfies Assumption 1. Furthermore by [8, Theorem 6.8], if a hybrid inclusion meets the conditions in Assumption 1 it is nominally well-posed. Applying [8, Proposition 6.21] to the nominally well-posed hybrid inclusion $\widetilde{\mathcal{H}}$ and the precompact solution ϕ, the ω-limit set $\Omega(\phi)$ is nonempty, closed, and weakly invariant. Then, using the equivalence of solutions between $\hat{\mathcal{H}}_{DAE}$ and $\mathcal{H}_{DAE}$ in Lemma 1, $\Omega(\phi)$ is weakly invariant for $\mathcal{H}_{DAE}$ and $|\phi(t, j)|_{\Omega(\phi)} \to 0$ as $t + j \to \infty$. ■

[6] Given a solution ϕ, the sequence $\{(t_i, j_i)\}_{i=0}^{\infty}$ of points in $\operatorname{dom}\phi$ is increasing if, for each $i \in \mathbb{N}$, $t_i + j_i \le t_{i+1} + j_{i+1}$.

Remark 8 Notice that given ϕ precompact either $i)$ or $ii)$ in Lemma 1 hold. Moreover, given the construction of G in (8) and (30), and the fact that the solution ϕ_ξ to the DAE $E_p\dot{\phi}_\xi(t) = A_p\phi_\xi(t)$ is such that $t \mapsto \phi_\xi(t) \in \mathfrak{O}_p$, if $i)$ (or $ii)$ holds then $\phi(t,j) \in \bigcup_{\tilde{\sigma}\in\Sigma} \big((\mathfrak{O}_{\tilde{\sigma}} \times \mathbb{R}^m \times \{\tilde{\sigma}\}) \cap (C_{\tilde{\sigma}} \cup D_{\tilde{\sigma}})\big)$ for all $(t,j) \in \operatorname{dom}\phi$ (for all $(t,j) \in \operatorname{dom}\phi \setminus \{(0,0)\}$, respectively). Following Definition 7, for each $x^* \in \bigcup_{\tilde{\sigma}\in\Sigma}((\mathbb{R}^n \setminus \mathfrak{O}_{\tilde{\sigma}}) \times \mathbb{R}^m \times \{\tilde{\sigma}\})$ there does not exist $N > 0$, x_0, and a solution ϕ to $\mathcal{H}_{DAE}$ from x_0 such that the conditions for weakly backward invariance in Definition 7 are fulfilled. Thus, $\{x^*\}$ cannot be weakly invariant, namely, $\Omega(\phi) \subset \bigcup_{\tilde{\sigma}\in\Sigma}((\mathfrak{O}_{\tilde{\sigma}} \times \mathbb{R}^m \times \{\tilde{\sigma}\}) \cap (C_{\tilde{\sigma}} \cup D_{\tilde{\sigma}}))$.

Proposition 1 *(Invariance principle for $\hat{\mathcal{H}}_{DAE}$ with linear flow map) Suppose the hybrid DAE $\hat{\mathcal{H}}_{DAE}$ in* (30) *is such that its data $(E_\sigma, C_\sigma, A_\sigma, \rho, D_\sigma, g_\sigma, g_\gamma, \varphi)$ satisfies Assumption 2. For each $\sigma \in \Sigma$, let Π_σ^{diff} be given as in Definition 4. Furthermore, suppose there exist a function $V : \mathbb{R}^{n+m+1} \to \mathbb{R}$ that is continuous on $\mathbb{R}^{n+m+1}$ and locally Lipschitz on an open set containing C, and functions $u_C : \mathbb{R}^{n+m+1} \to \mathbb{R}$ and $u_D : \mathbb{R}^{n+m+1} \to \mathbb{R}$ such that*

$$V^\circ\left(x, \begin{bmatrix} \Pi_\sigma^{diff} A_\sigma \xi \\ \rho(x) \\ 0 \end{bmatrix}\right) \le u_C(x) \qquad \forall x \in C \tag{36a}$$

$$V(\eta) - V(x) \le u_D(x) \qquad \forall x \in \hat{D} \text{ and } \forall \eta \in \hat{G}(x) \tag{36b}$$

Suppose that $K \subset \mathbb{R}^{n+m+1}$ is nonempty and $\hat{\phi}$ is a precompact solution to $\hat{\mathcal{H}}_{DAE}$ with $\overline{\operatorname{rge}\hat{\phi}} \subset K$. If

$$u_C(x) \le 0, \quad u_D(x) \le 0 \qquad \forall x \in K,$$

then $\hat{\phi}$ approaches the largest weakly invariant set for $\hat{\mathcal{H}}_{DAE}$ contained in

$$V^{-1}(r) \cap K \cap \left(u_C^{-1}(0) \cup \left[u_D^{-1}(0) \cap \hat{G}\left(u_D^{-1}(0)\right)\right]\right) \tag{37}$$

for some constant $r \in V(K)$.

Proof Given the precompact solution $\hat{\phi}$ to $\hat{\mathcal{H}}_{DAE}$ satisfying $\overline{\operatorname{rge}\hat{\phi}} \subset K$, by Lemma 2, $\hat{\phi}$ is a solution to $\hat{\mathcal{H}}_{DAE}$ if and only if it is a solution to the hybrid inclusion $\widetilde{\mathcal{H}}$ in (32a). Consider $\mathcal{H}_{DAE}$ with the same data as $\hat{\mathcal{H}}_{DAE}$. Since the data of $\hat{\mathcal{H}}_{DAE}$ fulfills Assumption 2 then the data of $\mathcal{H}_{DAE}$ also fulfills such assumption. By Lemma 3, since $\mathcal{H}_{DAE}$'s data fulfills Assumption 2, then $\widetilde{\mathcal{H}}$ satisfies Assumption 1. Moreover, by construction of $\widetilde{\mathcal{H}}$, for the function V in the assumptions, we have that (36a) and (36b) hold for $\widetilde{\mathcal{H}}$ as well. Then, an application of [20, Theorem 4.7] to $\widetilde{\mathcal{H}}$ implies that $\hat{\phi}$ approaches the largest weakly invariant set in (37) for some $r \in V(K)$. Hence, the claim follows since, by Lemma 2, $\hat{\phi}$ is also a solution to $\hat{\mathcal{H}}_{DAE}$.

■

5.2 Proofs of main results

5.2.1 Proof of Theorem 1

Proof (Proof of Theorem 1) Consider $\hat{\mathcal{H}}_{DAE}$ with the same data as $\mathcal{H}_{DAE}$. Since the data of $\mathcal{H}_{DAE}$ fulfills Assumption 2 then the data of $\hat{\mathcal{H}}_{DAE}$ also fulfills such assumption. Given that $\hat{\mathcal{H}}_{DAE}$ satisfies Assumption 2, the collection $\{(E_p, A_p)\}_{p\in\Sigma}$ is regular (see Definition 1). Define the hybrid inclusion $\widetilde{\mathcal{H}}$ as in Equation (32a) in Lemma 2. Recalling Lemma 1 and Remark 5, since ϕ is a precompact solution to $\mathcal{H}_{DAE}$ Equation (31) holds for inconsistent initial conditions, and there exists a precompact solution to $\hat{\mathcal{H}}_{DAE}$ that, at most, is a shift by one jump of ϕ. With some abuse of notation, we do not relabel that solution. Notice that, by Lemma 2, ϕ is also a solution to $\widetilde{\mathcal{H}}$. Given that, by Lemma 3, $\widetilde{\mathcal{H}}$ satisfies Assumption 1, an application of [8, Theorem 6.8] to $\widetilde{\mathcal{H}}$ shows that the hybrid inclusion $\widetilde{\mathcal{H}}$ is nominally well-posed. Thus, by construction of $\widetilde{\mathcal{H}}$, for the function V in the assumptions, we have that (11a) and (11b) hold for $\widetilde{\mathcal{H}}$ as well. Then, an application of [20, Theorem 4.7] to the nominally well-posed hybrid inclusion $\widetilde{\mathcal{H}}$ and the precompact solution ϕ implies that ϕ approaches the largest weakly invariant set $\mathcal{M}$, which is a subset of (12) for some $r \in V(K)$.

It just remains to show that the largest weakly invariant set $\mathcal{M}$ in (12) for $\widetilde{\mathcal{H}}$ is also the largest weakly invariant set in (12) for $\mathcal{H}_{DAE}$. We will prove this statement by contradiction. So, we assume that there exists $x^* \in \mathcal{M}$ that does not belong to the largest invariant set in (12) for $\mathcal{H}_{DAE}$. Then, every solution ϕ' to $\mathcal{H}_{DAE}$ from x^* eventually leaves (12) (backward or forward in time). However, since $\mathcal{M}$ is weakly invariant for $\widetilde{\mathcal{H}}$, using Lemma 1.i and Lemma 2, ϕ' is a solution to $\widetilde{\mathcal{H}}$ that eventually leaves (12) (backward or forward in time). This is a contradiction to the property established in the previous paragraph. ■

5.2.2 Proof of Theorem 2

Given $E_p, A_p \in \mathbb{R}^{n\times n}$ for each $p \in \Sigma$, a switched DAE system under arbitrary switching signals can be captured by a hybrid DAE system. Particularly, when modeling switched DAE systems under arbitrary switching signals, the state component χ and the maps ρ, γ, and g_γ are not needed and, hence, not taken into account from now on. To every solution to the switching DAE in (7) there corresponds a solution to the hybrid DAE in (8). Given the initial time t_0 and the switching instants $t_1, t_2, \ldots$, one can build a solution to $\mathcal{H}_{DAE}$ on a hybrid time domain $\mathcal{T} = \cup_{j=0}^{\mathcal{I}} ([t_j, t_{j+1}] \times \{j\})$ (with $\mathcal{I}$ finite or infinite) that corresponds to (ξ, σ). Then, a switched DAE system under arbitrary switching signals is given by the hybrid DAE

$$\mathcal{H}_{DAE}^{\mathcal{SW}} \begin{cases} \begin{bmatrix} E_\sigma & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\xi} \\ \dot{\sigma} \end{bmatrix} = \begin{bmatrix} A_\sigma \xi \\ 0 \end{bmatrix} = F(x) & x \in C \\ \begin{bmatrix} \xi^+ \\ \sigma^+ \end{bmatrix} \in \bigcup\limits_{\tilde{\sigma}\in\varphi((\xi,\sigma))} \begin{bmatrix} \Pi_{\tilde{\sigma}\sigma}\xi \\ \tilde{\sigma} \end{bmatrix} = G(x) & x \in D \end{cases} \tag{38a}$$

which has data $(E_\sigma, C_\sigma, A_\sigma, D_\sigma, g_\sigma, \varphi)$ and state vector $x = (\xi, \sigma)$. Recalling (8), for each $\sigma \in \Sigma$, we have $C_\sigma = \mathbb{R}^n \times \{\sigma\}$, $D_\sigma = \mathbb{R}^n \times \{\sigma\}$, $g_\sigma(\xi) = \xi$, and $\varphi(\xi, \sigma) = \Sigma \setminus \{\sigma\}$, leading to

$$C = \bigcup_{\sigma\in\Sigma} \big(\mathfrak{O}_\sigma \times \{\sigma\}\big), \quad D = \bigcup_{\sigma\in\Sigma} \big(\mathbb{R}^n \times \{\sigma\}\big) \tag{38b}$$

Also, consider the set

$$\hat{D} = \bigcup_{\sigma\in\Sigma} \big(\mathfrak{O}_\sigma \times \{\sigma\}\big) \tag{38c}$$

Since the collection $\{(E_\sigma, A_\sigma)\}_{\sigma\in\Sigma}$ only plays a role in the definition of the objects leading to (38), for simplicity, we write it as $\mathcal{H}_{DAE}^{\mathcal{SW}} = ((E_\sigma, A_\sigma), \Sigma)$.

Proof (Proof of Theorem 2) Let $\mathcal{H}_{DAE}^{\mathcal{SW}}$ be the hybrid DAE modeling the given switched DAE system under arbitrary switching signals, where $g_\sigma(\xi) = \xi$, $\varphi(\xi, \sigma) = \Sigma \backslash \{\sigma\}$, and C and $\hat{D}$ are given by (38b). Since $\mathcal{H}_{DAE}^{\mathcal{SW}}$ is a particular case of a hybrid DAE as in (8), and given that the collection $\{(E_\sigma, A_\sigma)\}_{\sigma\in\Sigma}$ fulfills Assumption 3, the data of $\mathcal{H}_{DAE}^{\mathcal{SW}}$ fulfills Assumption 2.

Since ϕ is a precompact solution to (7), passing to a hybrid time domain (see preamble of Section 3.2), it is also a precompact solution to $\mathcal{H}_{DAE}^{\mathcal{SW}}$. Notice that given the domain of definition of u_C and u_D, we can extend those domains for both functions by defining

$$\tilde{u}_C((\xi,\sigma)) := \begin{cases} u_C(\xi,\sigma) & \text{if } (\xi,\sigma) \in \operatorname{dom} u_C \\ -\infty & \text{otherwise} \end{cases}$$

$$\tilde{u}_D((\xi,\sigma)) := \begin{cases} u_D(\xi,\sigma) & \text{if } (\xi,\sigma) \in \operatorname{dom} u_D \\ -\infty & \text{otherwise} \end{cases}$$

Notice that omitting the state component χ, the functions ρ, γ, and g_γ and replacing u_C and u_D by $\tilde{u}_C$ and $\tilde{u}_D$, the bound in Equation (36) holds with the data of $\mathcal{H}_{DAE}^{\mathcal{SW}}$ and is equivalent to the bound (17) given in the assumptions. Then, an application of Theorem 1 to the hybrid DAE $\mathcal{H}_{DAE}^{\mathcal{SW}}$ with the function V in the assumptions, $\tilde{u}_C$, $\tilde{u}_D$, and the precompact solution ϕ implies that ϕ approaches the largest weakly invariant set that is a subset of (18) for some $r \in V(K)$. The claim is proven following similar steps as those in the proof of Theorem 1. ■

5.2.3 Proof of Theorem 3 and Corollary 1

Given matrices $E_\sigma, A_\sigma \in \mathbb{R}^{n\times n}$ for each $\sigma \in \Sigma$, a switched DAE system under dwell-time switching signals can be captured by the following hybrid DAE system:

$$\mathcal{H}_{DAE}^{\tau_D} \begin{cases} \begin{bmatrix} E_\sigma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\xi} \\ \dot{\chi} \\ \dot{\sigma} \end{bmatrix} \in \begin{bmatrix} A_\sigma \xi \\ \rho(\chi) \\ 0 \end{bmatrix} = F(x) & (\xi,\chi,\sigma) \in C \\ \begin{bmatrix} \xi^+ \\ \chi^+ \\ \sigma^+ \end{bmatrix} \in \bigcup_{\tilde{\sigma}\in\varphi(x)} \begin{bmatrix} \Pi_{\tilde{\sigma}} \xi \\ 0 \\ \tilde{\sigma} \end{bmatrix} = G(x) & (\xi,\chi,\sigma) \in D \end{cases} \tag{39a}$$

where

$$C = \bigcup_{\sigma\in\Sigma} \left(C_\sigma \cap (\mathfrak{O}_\sigma \times \mathbb{R} \times \{\sigma\})\right) \tag{39b}$$

$$\rho(\chi) := \begin{cases} 1 & \text{if } \chi < \tau_D \\ [0,1] & \text{if } \chi = \tau_D \\ 0 & \text{if } \chi > \tau_D \end{cases} \qquad \forall \chi \in \mathbb{R} \tag{39c}$$

$$D = \bigcup_{\sigma\in\Sigma} \big((D_\sigma \cap (\mathfrak{O}_\sigma \times \mathbb{R} \times \{\sigma\})) \cup ((\mathbb{R}^n \setminus \mathfrak{O}_\sigma) \times \mathbb{R} \times \{\sigma\}) \big) \tag{39d}$$

where Π_σ and $\mathfrak{O}_\sigma$ are computed using the data (E_σ, A_σ) for each $\sigma \in \Sigma$. The state vector of the hybrid DAE system $\mathcal{H}_{DAE}^{\tau_D}$ is given by $x = (\xi, \chi, \sigma)$, where $\chi \in \mathbb{R}^m$, $m = 1$, and the remainder of its data $(E_\sigma, C_\sigma, A_\sigma, \rho, D_\sigma, g_\sigma, g_\gamma, \varphi)$ is given by

$$C_\sigma = \mathbb{R}^n \times [0, \tau_D] \times \{\sigma\} \tag{39e}$$

$$D_\sigma = \mathbb{R}^n \times [\tau_D, \infty) \times \{\sigma\} \tag{39f}$$

$$g_\sigma(\xi) = \xi \tag{39g}$$

$$g_\gamma(x) = 0 \tag{39h}$$

and $\varphi(\xi,\chi,\sigma) = \Sigma \setminus \{\sigma\}$. Since the family $\{(E_\sigma, A_\sigma)\}_{\sigma\in\Sigma}$ and τ_D only play a role in the definition of the objects leading to (39), for simplicity, we refer to it as $\mathcal{H}^{\tau_D}_{DAE} = ((E_\sigma, A_\sigma), \tau_D, \Sigma)$. Notice that for each dwell-time solution to $\mathcal{H}^{\mathcal{SW}}_{DAE} = ((E_\sigma, A_\sigma), \tau_D, \Sigma)$ in (38) with dwell-time $\tau_D > 0$ there corresponds a solution to the hybrid DAE system $\mathcal{H}^{\tau_D}_{DAE}$ in (39).

Solutions to $\mathcal{H}^{\tau_D}_{DAE}$ can exist from any point in $\mathbb{R}^n \times \mathbb{R}_{\geq 0} \times \Sigma$. At times, it might be desired to not allow for inconsistent initial conditions in the state component ξ. For such situations, consider the hybrid DAE $\hat{\mathcal{H}}_{DAE}$ in (30) with the data of $\mathcal{H}^{\tau_D}_{DAE}$ and relabel it as $\hat{\mathcal{H}}^{\tau_D}_{DAE}$.

The proof of Theorem 3 requires Proposition 2, which is presented first. Note that Assumption 4 and Assumption 5 are given in Section 3.2.2.

Proposition 2 *(Auxiliary system $\mathcal{H}^1_{DAE}$) Given $\sigma \in \Sigma$, consider the DAE system $E_\sigma\dot{\xi} = A_\sigma\xi$, where $(E_\sigma, A_\sigma) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is regular (see Definition 1). Let ρ be as in* (39c)*, and using (E_σ, A_σ), let $\mathfrak{O}_\sigma$ and $(\Pi_\sigma, \Pi^{\text{diff}}_\sigma)$ be generated as in Definition 3 and Definition 4, respectively. Furthermore, suppose there exist a function $V_\sigma : \mathbb{R}^n \to \mathbb{R}$ that is locally Lipschitz on an open set O_σ, $\mathfrak{O}_\sigma \subset O_\sigma$, and a function $W_\sigma : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is continuous on O_σ such that*

$$V^\circ_\sigma\left(\xi, \Pi^{\text{diff}}_\sigma A_\sigma\xi\right) \leq -W_\sigma(\xi) \qquad \forall \xi \in O_\sigma \tag{40}$$

Define the following hybrid DAE system, denoted by $\mathcal{H}^1_{DAE}$, as

$$\mathcal{H}^1_{DAE}\begin{cases} \begin{bmatrix} E_\sigma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{\xi} \\ \dot{\chi} \\ \dot{\sigma} \end{bmatrix} \in \begin{bmatrix} A_\sigma\xi \\ \rho(\chi) \\ 0 \end{bmatrix} & (\xi,\chi,\sigma) \in C^* \\ \begin{bmatrix} \xi^+ \\ \chi^+ \\ \sigma^+ \end{bmatrix} \in \begin{bmatrix} K \\ 0 \\ \sigma \end{bmatrix} & (\xi,\chi,\sigma) \in D^* \end{cases} \tag{41a}$$

where

$$C^* = C_\sigma \cap (\mathfrak{O}_\sigma \times \mathbb{R} \times \{\sigma\}), \tag{41b}$$
$$D^* = D_\sigma \cap (\mathfrak{O}_\sigma \times \mathbb{R} \times \{\sigma\}), \tag{41c}$$

with C_σ and D_σ as given in (39e) *and* (39f)*, respectively, and $K \subset O_\sigma$ is a nonempty and compact set. Let $\phi = (\phi_\xi, \phi_\chi, \phi_\sigma) : \operatorname{dom}\phi \to O_\sigma \times \mathbb{R} \times \{\sigma\}$ be a complete solution to $\mathcal{H}^1_{DAE}$ such that the ξ-component of ϕ satisfies $\overline{\operatorname{rge}\phi_\xi} \subset K$ and such that*

$$V_\sigma(\phi_\xi(t, j+1)) - V_\sigma(\phi_\xi(t,j)) \leq 0 \tag{42}$$

for all $(t,j) \in \operatorname{dom}\phi$ such that $(t, j+1) \in \operatorname{dom}\phi$. Then, for some constant $r \in \mathbb{R}$, the solution component ϕ_ξ of ϕ approaches the largest subset $\mathcal{M}$ contained in

$$V^{-1}_\sigma(r) \cap K \cap W^{-1}_\sigma(0) \tag{43}$$

that is invariant in the following sense: for each $\xi_0 \in \mathcal{M}$ there exists a solution $(\phi_{\tilde{\xi}}, \sigma)$ (given as in Definition 5 when σ is constant and equal to σ) to $E_\sigma\dot{\xi} = A_\sigma\xi$ on $[0, \tau_D/2]$ such that $\phi_{\tilde{\xi}}(t) \in \mathcal{M}$ for all $t \in [0, \tau_D/2]$ and either $\phi_{\tilde{\xi}}(0) = \xi_0$ or $\phi_{\tilde{\xi}}(\tau_D/2) = \xi_0$.

Proof Given the hybrid DAE system $\mathcal{H}^1_{DAE}$ in (41), consider a hybrid DAE system $\hat{\mathcal{H}}_{DAE}$ as in (30) with the data of $\mathcal{H}^1_{DAE}$ in (41) and relabel it as $\hat{\mathcal{H}}^1_{DAE}$, where $g_D((\xi,\chi,\sigma),\tilde{\sigma}) = K$, $g_\gamma((\xi,\chi,\sigma)) = 0$, $\varphi((\xi,\chi,\sigma)) = \sigma$, $\Sigma = \{\sigma\}$, and C^* and D^* given as in (41b) and (41c), respectively. Since the hybrid DAE system $\mathcal{H}^1_{DAE}$ is a particular hybrid DAE system, given that C_σ and D_σ are closed, the matrix pair (E_σ, A_σ) satisfies Assumption 3, the maps ρ, φ, g_σ and g_γ fullfill the conditions in Assumption 2 by the fact that K is nonempty and compact. With some abuse of notation, the data of $\mathcal{H}^1_{DAE}$ fulfills Assumption 2. Given that $\mathcal{H}^1_{DAE}$ satisfies Assumption 2, then $\hat{\mathcal{H}}^1_{DAE}$ satisfies Assumption 2. Since the matrix pair (E_σ, A_σ) is regular, a hybrid inclusion $\widetilde{\mathcal{H}}$ is obtained as in Lemma 2. We relabel that hybrid inclusion as $\widetilde{\mathcal{H}}^{\tau_D}$. Namely, $\widetilde{\mathcal{H}}^{\tau_D}$ is given by

$$\widetilde{\mathcal{H}}^{\tau_D} \begin{cases} \begin{bmatrix} \dot{\xi} \\ \dot{\chi} \\ \dot{\sigma} \end{bmatrix} \in \begin{bmatrix} \Pi_p^{\text{diff}} A_p \xi \\ \rho(\chi) \\ 0 \end{bmatrix} = \tilde{F}((\xi,\chi,\sigma)) & (\xi,\chi,\sigma) \in C^* \\ \begin{bmatrix} \xi^+ \\ \chi^+ \\ \sigma^+ \end{bmatrix} \in \begin{bmatrix} K \\ 0 \\ \sigma \end{bmatrix} = G((\xi,\chi,\sigma)) & (\xi,\chi,\sigma) \in D^* \end{cases} \tag{44}$$

By Lemma 3, since the data of $\mathcal{H}^{\tau_D}_{DAE}$ fulfills Assumption 2, then $\widetilde{\mathcal{H}}^{\tau_D}$ satisfies Assumption 1. Given that $\widetilde{\mathcal{H}}^{\tau_D}$ satisfies Assumption 1, an application of [8, Theorem 6.8] to $\widetilde{\mathcal{H}}^{\tau_D}$ shows that the hybrid inclusion $\widetilde{\mathcal{H}}^{\tau_D}$ is nominally well-posed. Notice that for all (nontrivial) solutions to $\widetilde{\mathcal{H}}^{\tau_D}$, the state σ remains constant. Then, to analyze the components ξ and τ of the solutions, we consider the following hybrid inclusion:

$$\widetilde{\mathcal{H}}^1 \begin{cases} \begin{bmatrix} \dot{\xi} \\ \dot{\chi} \end{bmatrix} \in \begin{bmatrix} \Pi_\sigma^{\text{diff}} A_\sigma \xi \\ \rho(\chi) \end{bmatrix} & (\xi,\chi) \in \mathfrak{D}_\sigma \times [0,\tau_D] \\ \begin{bmatrix} \xi^+ \\ \chi^+ \end{bmatrix} \in \begin{bmatrix} K \\ 0 \end{bmatrix} & (\xi,\chi) \in \mathfrak{D}_\sigma \times [\tau_D,\infty) \end{cases} \tag{45}$$

By the assumptions in this proposition, ϕ is a complete solution to $\mathcal{H}^1_{DAE}$ and ϕ_ξ is bounded by K while ϕ_χ is bounded by the construction of $\mathcal{H}^1_{DAE}$. Thus, the component (ϕ_ξ, ϕ_χ) of the solution is precompact. Recalling Lemma 1 and Remark 5, notice that given ϕ precompact (complete and bounded) to $\mathcal{H}^1_{DAE}$, there exists a precompact solution to $\hat{\mathcal{H}}^1_{DAE}$ that, at most, is a shift by one jump of ϕ. Particularly, notice that $(0,1) \in \operatorname{dom}\phi$ and (31) holds for inconsistent initial conditions. With some abuse of notation, we do not relabel that solution. Notice that, by Lemma 2, ϕ is also a solution to $\widetilde{\mathcal{H}}^{\tau_D}$. Also, notice that the component (ϕ_ξ, ϕ_χ) of ϕ is a solution to $\widetilde{\mathcal{H}}^1$ in (45).

Now, notice that the hybrid inclusion $\widetilde{\mathcal{H}}^1$ satisfies Assumption 1. Given that the component (ϕ_ξ, ϕ_χ) of ϕ is a solution to $\widetilde{\mathcal{H}}^1$, for the functions V_σ and W_σ in the assumptions of this proposition, we have that (40) holds for the data of $\widetilde{\mathcal{H}}^1$ as well. Then, an application of [7, Theorem 4.1][7] to $\widetilde{\mathcal{H}}^1$ implies that ξ approaches the largest subset $\mathcal{M}$ of

$$V_\sigma^{-1}(r) \cap K \cap W_\sigma^{-1}(0)$$

that is invariant in the following sense: for each $\xi_0 \in \mathcal{M}$ there exists a solution $\phi_{\bar{\xi}}$ to $\dot{\xi} = \Pi_\sigma^{\text{diff}} A_\sigma \xi$ on $[0, \tau_D/2]$ such that $\phi_{\bar{\xi}}(t) \in \mathcal{M}$ for all $t \in [0, \tau_D/2]$ and either

[7] Even though [7, Theorem 4.1] requires a continuously differentiable function V, the arguments therein follow similarly for a locally Lipschitz function V_σ. This is due to the invariance principles in [20] allowing for locally Lipschitz Lyapunov functions.

$\phi_{\tilde{\xi}}(0) = \xi_0$ or $\phi_{\tilde{\xi}}(\tau_D/2) = \xi_0$. Hence, the claim follows since (ϕ_ξ, ϕ_χ) is also a component of the solution ϕ to $\mathcal{H}^1_{DAE}$ in (41). ■

Now, we are ready to present the invariance principle that involves only properties of switched DAE systems (7) under dwell-time switching signals. Its proof uses an equivalence between the solutions to such systems and those to $\mathcal{H}^{\tau_D}_{DAE}$, and, for each $p \in \Sigma$, exploits the property of $\mathcal{H}^1_{DAE}$ in Proposition 2.

Proof (Proof of Theorem 3) For each $p \in \Sigma$ for which there is infinitely many disjoint intervals $(t_j, t_j + \Delta t_j)$, $j = 0, 1, \ldots$, $\Delta t_j \geq \tau_D$ on which $\sigma(t) = p$, consider a hybrid arc z with

$$\operatorname{dom} z = \bigcup_{j=0}^{\infty} \left[\sum_{i=0}^{j-1} \Delta t_i, \sum_{i=0}^{j} \Delta t_i \right] \times \{j\} \tag{46}$$

(with the convention that $\sum_{i=0}^{-1} \Delta t_i = 0$) defined by

$$z(t, j) = \phi_\xi \left(t_j + t - \sum_{i=0}^{j-1} \Delta t_i \right) \tag{47}$$

for $t \in \left[\sum_{i=0}^{j-1} \Delta t_i, \sum_{i=0}^{j} \Delta t_i \right]$. Such a hybrid arc is a component of the solution to $\mathcal{H}^1_{DAE}$ of Proposition 2, and meets the assumptions of that theorem, with E_p, A_p, V_p, W_p for all $p \in \Sigma$, and with $K \subset O_p$ being any compact set such that $\phi_\xi(t, j) \in K$ whenever $\sigma(t) = p$ (with K large enough to hold for jumps and flows of the solution component ϕ_ξ). Proposition 2 implies the claim. ■

Proof (Proof of Corollary 1) The claim follows from the proof of Proposition 2. If $(\Pi_p^{\text{diff}} A_p \xi, W_p)$ is observable then the set $\mathcal{M}$ is the origin; therefore, all sets $\mathcal{M}_p(r_p, \tau_D)$ in Corollary 3 are also the origin. ■

6 Conclusion

In this paper, we consider switched DAE systems and hybrid DAE systems, which are dynamical systems with multiple modes of operation and state jumps. The proposed hybrid DAE model borrows the concept of solution from hybrid systems theory, as well as concepts from switched DAE systems to include algebraic restrictions and jumps driven by inconsistent initial conditions. The properties of the ω-limit set of a solution for these systems was characterized and invariance principles for hybrid DAE systems and switched DAE systems under arbitrary and dwell-time switching signals are introduced. While using previous results on invariance principles for switched systems via hybrid systems techniques, we extended the results in [7] to the case of locally Lipschitz Lyapunov functions. The hybrid DAE model and the invariance principle in this document can be extended to the nonlinear case following the ideas for switched DAE systems in [13]. This generalization is part of the current efforts and future directions.

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A Appendix

A.1 Supplementary definitions

Definition 9 (Wong sequences[8]) Consider a regular matrix pair (E_σ, A_σ) where $\sigma \in \Sigma$. The associated Wong sequences are defined by

$$\nu_0 := \mathbb{R}^n, \quad \nu_{i+1} := A_\sigma^{-1}(E_\sigma \nu_i), \quad \nu_\sigma^* := \bigcap_{i\in\mathbb{N}} \nu_i, \ \forall i \in \mathbb{N}$$
$$\omega_0 := \{0\}, \quad \omega_{i+1} := E_\sigma^{-1}(A_\sigma \omega_i), \quad \omega_\sigma^* := \bigcup_{i\in\mathbb{N}} \omega_i, \ \forall i \in \mathbb{N}$$

which are nested subspaces and get stationary after exactly ς_σ steps, namely, $\nu_{i+1} \subseteq \nu_i$, $\omega_{i+1} \supseteq \omega_i$ for each $i \in \mathbb{N}$ and $\nu_{i+1} = \nu_i = \nu_\sigma^*$, $\omega_{i+1} = \omega_i = \omega_\sigma^*$ for all $i \geq \varsigma_\sigma$, where the integer ς_σ is called the index of the σ-th DAE.

Theorem 4 *(The quasi-Weierstrass form[9]) Suppose (E_σ, A_σ) is regular, where $\sigma \in \Sigma$. Then, there exist matrices S_σ and T_σ that transform (E_σ, A_σ) into the quasi-Weierstrass form*

$$(S_\sigma E_\sigma T_\sigma, S_\sigma A_\sigma T_\sigma) = \left(\begin{bmatrix} I_{n_1^\sigma} & 0 \\ 0 & N_\sigma \end{bmatrix}, \begin{bmatrix} J_\sigma & 0 \\ 0 & I_{n_2^\sigma} \end{bmatrix} \right)$$

for some $J_\sigma \in \mathbb{R}^{n_1^\sigma \times n_1^\sigma}$ and a nilpotent matrix $N_\sigma \in \mathbb{R}^{n_2^\sigma \times n_2^\sigma}$, where $(N_\sigma)^{n_2^\sigma} = 0$, $n_1^\sigma + n_2^\sigma = n$, $\varsigma_\sigma \in \mathbb{N}$ is the smallest number such that $(N_\sigma)^{\varsigma_\sigma} = 0$. The identity matrices $I_{n_1^\sigma}$, $I_{n_2^\sigma}$ and zero matrices have the proper dimensions.

A.2 Details of the proof of Lemma 1.

Proof Given the two representations $\hat{\mathcal{H}}_{DAE}$ and $\mathcal{H}_{DAE}$, $\mathcal{S}_{\hat{\mathcal{H}}_{DAE}}$ and $\mathcal{S}_{\mathcal{H}_{DAE}}$ are the set of maximal solutions (according to [8, Definition 2.7]) to $\hat{\mathcal{H}}_{DAE}$ and $\mathcal{H}_{DAE}$, respectively. For all $\phi \in \mathcal{S}_{\mathcal{H}_{DAE}}$ (for all consistent and inconsistent initial conditions) we have:

i) Consistent initial value: following Definition 5, given a consistent initial condition $\xi_0 \in \mathfrak{O}_{\sigma_0}$ the solution ϕ_ξ to the DAE $E_{\sigma_0}\dot{\phi}_\xi(t) = A_{\sigma_0}\phi_\xi(t)$ is such that $t \mapsto \phi_\xi(t) \in \mathfrak{O}_{\sigma_0}$. Also, consider the following cases of consistent initial conditions to (8) and (30):
   - *a)* $x^* = (\xi_{t^*}, \chi_{t^*}, \sigma_{t^*}) \in (\mathfrak{O}_{\sigma_{t^*}} \times \mathbb{R}^m \times \{\sigma_{t^*}\}) \cap C_{\sigma_{t^*}}$: Notice that given the construction of F in (8) and (30), for the time interval $[t^*, t^* + \tau) \times \{j^*\}$ and $\tau > 0$, the solutions $\phi : [t^*, t^* + \tau) \times \{j^*\} \to \mathbb{R}^{n+m+1}$ and $\hat{\phi} : [t^*, t^* + \tau) \times \{j^*\} \to \mathbb{R}^{n+m+1}$ are such that $\phi(t, j) = \hat{\phi}(t, j) \in (\mathfrak{O}_{\phi_\sigma(t,j)} \times \mathbb{R}^m \times \{\phi_\sigma(t, j)\})$ for all $(t, j) \in [t^*, t^* + \tau) \times \{j^*\}$.
   - *b)* $x^* = (\xi_{t^*}, \chi_{t^*}, \sigma_{t^*}) \in (\mathfrak{O}_{\sigma_{t^*}} \times \mathbb{R}^m \times \{\sigma_{t^*}\}) \cap D_{\sigma_{t^*}}$: Given the construction of G and $\hat{G}$ in (8) and (30), respectively,

   $$G(x^*) = \hat{G}(x^*) \subset \bigcup_{\tilde{\sigma} \in \varphi(\xi_{t^*}, \chi_{t^*}, \sigma_{t^*})} (\mathfrak{O}_{\tilde{\sigma}} \times \mathbb{R}^m \times \{\tilde{\sigma}\})$$

   Notice that conditions *a)* and/or *b)* are fulfilled for all (t, j) in $\operatorname{dom}\phi$; therefore, from *a)* and *b)* we conclude that given an initial condition $\phi(0, 0) = \hat{\phi}(0, 0) = (\xi_0, \chi_0, \sigma_0) \in (\mathfrak{O}_{\sigma_0} \times \mathbb{R}^m \times \{\sigma_0\}) \cap (C_{\sigma_0} \cup D_{\sigma_0})$ we have that $\phi(t, j), \hat{\phi}(t, j) \in \bigcup_{\tilde{\sigma}\in\Sigma} \big((\mathfrak{O}_{\tilde{\sigma}} \times \mathbb{R}^m \times \{\tilde{\sigma}\}) \cap (C_{\tilde{\sigma}} \cup D_{\tilde{\sigma}})\big)$ and $\phi(t, j) \equiv \hat{\phi}(t, j)$ for all $(t, j) \in \operatorname{dom}\phi = \operatorname{dom}\hat{\phi}$, namely, at flows and jumps the solutions ϕ and $\hat{\phi}$ match and $\hat{\phi} \in \mathcal{S}_{\hat{\mathcal{H}}_{DAE}}$.

[8] This definition is adapted from [19, Lemma 2.1], [13, Theorem 2.3], [2, Theorem 2.4], and [1, Proposition 1].
[9] This theorem is adapted from [2, Theorem 2.4].

ii) Inconsistent initial value: for all nontrivial solutions ϕ to $\mathcal{H}_{DAE}$, if $\phi(0,0) = (\xi_0, \chi_0, \sigma_0) \in (\mathbb{R}^n \setminus \mathfrak{O}_{\sigma_0}) \times \mathbb{R}^m \times \{\sigma_0\}$ and using the definition of G in (8), we have

$$\begin{aligned}\phi(0,1) \in G(\xi_0, \chi_0, \sigma_0) &= \bigcup_{\tilde{\sigma} \in \varphi(\xi_0, \chi_0, \sigma_0)} \begin{bmatrix} g(\xi_0, \chi_0, \sigma_0, \tilde{\sigma}) \\ \chi_0 \\ \tilde{\sigma} \end{bmatrix} \\ &= \bigcup_{\tilde{\sigma} \in \varphi(\xi_0, \chi_0, \sigma_0)} \begin{bmatrix} \Pi_{\tilde{\sigma}} \xi_0 \\ \chi_0 \\ \tilde{\sigma} \end{bmatrix}\end{aligned}$$

If

$$\phi(0,1) \in \bigcup_{\tilde{\sigma} \in \varphi(\xi_0, \chi_0, \sigma_0)} ((\mathfrak{O}_{\tilde{\sigma}} \times \mathbb{R}^m \times \{\tilde{\sigma}\}) \cap (C_{\tilde{\sigma}} \cup D_{\tilde{\sigma}}))$$

and considering $\phi(0,1)$ as the consistent initial condition required in *i*), then, according to *i*), there exists a solution $\hat{\phi} \in \mathcal{S}_{\hat{\mathcal{H}}_{DAE}}$ from $\hat{\phi}(0,0) = \phi(0,1) \in G(\xi_0, \chi_0, \sigma_0)$ such that $\operatorname{dom} \hat{\phi} \subset \operatorname{dom} \phi$ and $\hat{\phi}(t, j-1) = \phi(t,j)$ for all $(t,j) \in \operatorname{dom} \phi \setminus \{(0,0)\}$.

■

A.3 Details of the proof of Lemma 2.

Since, for each $\sigma \in \Sigma$, the matrix pairs (E_σ, A_σ) are regular (see Definition 1), there exist matrices T_σ and S_σ that put $(S_\sigma E_\sigma T_\sigma, S_\sigma A_\sigma T_\sigma)$ into the quasi-Weierstrass form (see Lemma 5).

Notice that given the construction of $\hat{\mathcal{H}}_{DAE}$, G maps to $\mathfrak{O}_p$ after each jump, where p is the value to which σ is updated to. This is due to $\mathfrak{C}_p = \operatorname{im}(\Pi_p)$ and $\mathfrak{O}_p := \{\xi \mid \xi \in \operatorname{span}(\mathfrak{C}_p),\ \sigma \equiv p\}$, then at a jump at $(\xi, \sigma) \in D_\sigma \cap \mathfrak{O}_\sigma$ thus $(\xi^+, p) \in \mathfrak{O}_p$.

Furthermore (according to Definitions 2, 3, and [13, Definition 2.1 and references therein]), given a consistent initial condition, the "classical" solution to $E_p \dot{\xi}(t) = A_p \xi(t)$ belongs to the consistency space $\mathfrak{C}_p$ for the open interval $(0, \infty)$. Then, given a solution ϕ to $\hat{\mathcal{H}}_{DAE}$, we have that $(\xi(t,j), \sigma(t,j)) \in \mathfrak{O}_{\sigma(t,j)}\ \forall (t,j) \in \operatorname{dom} \phi$. For each j such that $I^j \times \{j\} \subset \operatorname{dom} \phi$ and I^j has a positive Lebesgue measure, the solution component ξ satisfies

$$E_{\sigma(t,j)} \dot{\xi}(t,j) = A_{\sigma(t,j)} \xi(t,j)\ \forall (t,j) \in I^j \times \{j\}, \tag{48}$$

where $I^j \times \{j\} := \{t \in \mathbb{R}_{\geq 0} \ :\ (t,j) \in \operatorname{dom} \phi\} \times \{j\}$. In fact, over the interval $I^j \times \{j\}$, ξ is given by the "classical" solution to the DAE $E_{\sigma(t,j)} \dot{\xi} = A_{\sigma(t,j)} \xi$ (see Appendix A.4). Applying Lemma 5 to (48) for all open intervals $I^j \times \{j\} \in \operatorname{dom} \phi$, for each interval $I^j \times \{j\}$, $\xi(t,j)$ satisfies

$$\dot{\xi}(t,j) = T_{\sigma(t,j)} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S_{\sigma(t,j)} A_{\sigma(t,j)} \xi(t,j) \tag{49}$$

To establish that ϕ is also a solution to $\widetilde{\mathcal{H}}$, notice that given the construction of G (in $\hat{\mathcal{H}}_{DAE}$ and $\widetilde{\mathcal{H}}$), the initial value of ξ after each jump belongs to the consistency set $\mathfrak{O}_\sigma$, in other words $(\xi(t, j+1), \sigma(t, j+1)) \in \mathfrak{O}_{\sigma(t,j+1)}$ holds for each $(t,j) \in \operatorname{dom} \phi$ such that $(t, j+1) \in \operatorname{dom} \phi$.

Also, a function $t \mapsto \xi(t)$ that is a solution to $\dot{\xi} = \Pi_\sigma^{\text{diff}} A_\sigma \xi$ if and only if is a solution to $E_\sigma \dot{\xi} = A_\sigma \xi$; therefore, following same steps above a solution to $\widetilde{\mathcal{H}}$ is also a solution to $\hat{\mathcal{H}}_{DAE}$.

A.4 Classical Solutions for Invariant Regular-DAEs

Lemma 5 *(Explicit solution formula for switched DAE systems [22, Theorem 6.4.4]) Let $I \subset [\underline{t}, \overline{t}] \subset \mathbb{R}_{\geq 0}$ have nonempty interior and the regular matrix pair $(E_{\sigma^*}, A_{\sigma^*})$. Also, let the projectors defined in Definition 4. The smooth functions $\xi, h : I \to \mathbb{R}^n$ satisfy*

$$E_{\sigma^*}\dot{\xi}(t) = A_{\sigma^*}\xi(t) + h(t)$$

if and only if they satisfy,

$$\xi(t) = e^{\Pi_{\sigma^*}^{diff} A_{\sigma^*}(t-\underline{t})}\Pi_{\sigma^*} c + \int_{\underline{t}}^{t} e^{\Pi_{\sigma^*}^{diff} A_{\sigma^*}(t-\tau)}\Pi_{\sigma^*}^{diff} h(\tau) d\tau - \sum_{k=0}^{\varsigma} \left(\Pi_{\sigma^*}^{imp} E_{\sigma^*}\right)^k \Pi_{\sigma^*}^{imp} h^{(k)}(t), \tag{50}$$

for $c \in \mathbb{R}^n$ *and an initial condition* $\xi(\underline{t}) = \xi_0$ *such that*

$$\xi_0 + \sum_{k=0}^{\varsigma} \left(\Pi_{\sigma^*}^{imp} E_{\sigma^*}\right)^k \Pi_{\sigma^*}^{imp} h^{(k)}(\underline{t}) \in \mathfrak{O}_{\sigma^*} \tag{51}$$

Furthermore, if $h \equiv 0$ *then* (50) *can be replaced by* (52).

$$\dot{\xi}(t) = T_{\sigma^*} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S_{\sigma^*} A_{\sigma^*} \xi(t) \tag{52}$$