

Almost global asymptotic stability of a grid-connected synchronous generator

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Abstract. We study the global asymptotic behavior of a grid-connected constant field current synchronous generator (SG). The grid is regarded as an “infinite bus”, i.e. a three-phase AC voltage source. The generator does not include any controller other than the frequency droop loop. This means that the mechanical torque applied to this generator is an affine function of its angular velocity. The negative slope of this function is the frequency droop constant. We derive sufficient conditions on the SG parameters under which there exist exactly two periodic state trajectories for the SG, one stable and another unstable, and for almost all initial states, the state trajectory of the SG converges to the stable periodic trajectory (all the angles are measured modulo 2π). Along both periodic state trajectories, the angular velocity of the SG is equal to the grid frequency. Our sufficient conditions are easy to check computationally. An important tool in our analysis is an integro-differential equation called the *exact swing equation*, which resembles a forced pendulum equation and is equivalent to our fourth order model of the grid-connected SG. Apart from our objective of providing an analytical proof for a global asymptotic behavior observed in a classical dynamical system, a key motivation for this work is the development of synchronverters which are inverters that mimic the behavior of SGs. Understanding the global dynamics of SGs can guide the choice of synchronverter parameters and operation. As an application we find a set of stable nominal parameters for a 500kW synchronverter.

Key words. synchronous machine, infinite bus, almost global asymptotic stability, forced pendulum equation, synchronverter, virtual inductor.

AMS classification. 34D23, 93D20, 94C99.

1. Introduction

Synchronous generators (SGs), once synchronized to the power grid, tend to remain synchronized even without any control unless very strong disturbances destroy the synchronism - this is a feature that enabled the development of the AC electricity grid at the end of the XIX century. We investigate this feature by considering one synchronous generator and analyzing its ability to synchronize when it is connected to a much more powerful grid, so that this one generator has practically no influence on the grid. Thus we model the grid as an “infinite bus”, i.e. a three-phase AC

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voltage source. Following standard practice, the prime mover (the engine that gives the mechanical torque to the generator) is assumed to provide a torque of the form $T_m - D_{p,\text{droop}}\omega$. Here $T_m > 0$ is a mechanical torque constant, $D_{p,\text{droop}} > 0$ is the frequency droop constant (this is used to stabilize the utility grid) and ω is the angular velocity of the rotor. The question we address is: *under what conditions will the state trajectory of a grid-connected SG, driven by a prime mover as above, having a constant field current (rotor current) and starting from an arbitrary initial state, converge to a state of synchronous rotation?* (Synchronous rotation means a constant difference between the grid angle and the SG rotor angle.)

The above question can be reformulated as a question of almost global asymptotic stability of a SG model in a transformed coordinate system. The importance of the stability of a grid-connected generator has been recognized for a long time and this or closely related problems have been studied, for instance, in [4, 12, 13, 19, 25, 26, 31, 35]. A full model of the SG consists of the electrical equations governing the fluxes in the stator, rotor and damper windings, along with the mechanical swing equation governing the rotor dynamics. As far as we know, all the available stability studies are based on some sort of simplification/reduction of the full model obtained by: (i) reducing the full model to a lower order (usually second or third order) nonlinear system by approximating the stator and the damper flux dynamics by static equations and sometimes assuming constant rotor current, or (ii) linearizing the full or the reduced order model around some equilibrium point. Most of the studies that use reduced order models focus on the local stability properties of the generator. A notable exception in this regard is [14], which considers various reduced order SG models and derives sufficient conditions for every state trajectory of the model to converge to an equilibrium point. The paper [3] considers (among other things) a synchronous machine connected to a three-phase AC voltage source having a constant phase difference with respect to the machine angle. Such a dependent voltage source is encountered in “brushless DC motors”. The rotor current is assumed to be constant and there are no damper windings, and in this respect, their setup resembles ours. They prove the global asymptotic stability of this system. This is an interesting problem, but different from the stability of a SG connected to an infinite bus. The paper [10] proves (among other things) the global asymptotic stability of a full (8th order) SG model when it is connected to a linear resistive load (not a grid), using the formalism of port-Hamiltonian systems.

In the present work, we study the global asymptotic stability properties of a grid-connected generator without approximating the stator flux dynamics (analyzing reduced order models that approximate the stator flux dynamics can lead to incorrect conclusions about the full model, see Remark 3.2). But we do restrict our attention to the case where the rotor current is constant and the damper windings are absent. We derive sufficient conditions on the SG parameters under which there exist exactly two periodic state trajectories, one stable and another unstable, and for almost all initial states, the state trajectory of the SG converges to the stable periodic trajectory (all the angles are measured modulo 2π), see Theorem 6.3. Along both the periodic trajectories, the rotor angular velocity is equal to the grid frequency.

To derive the sufficient conditions, a fourth order nonlinear time-invariant model for the grid-connected SG is constructed in a transformed coordinate system using the Park transformation in Section 3. In this coordinate system the two periodic state trajectories of the SG are mapped into two distinct points which are the unique stable and unstable equilibrium points of the fourth order model. If for almost every initial state, the state trajectory of the SG model converges to the stable equilibrium point, then we call the model *almost globally asymptotically stable*. In Section 4 we derive an integro-differential equation called the *exact swing equation* (ESE), which resembles a forced pendulum equation and is equivalent to the fourth order SG model from Section 3. Every trajectory of the fourth order model converges to one of its equilibrium points if and only if every trajectory of the ESE converges to one of two possible limit points. We derive some new estimates for the asymptotic response of a forced pendulum equation driven by a time-varying bounded forcing in Section 5. Applying these estimates to the ESE, we define a nonlinear map $\mathcal{N} : (0, \Gamma] \rightarrow [0, \infty)$ in Section 6 which (along with Γ) depends on the SG parameters. We prove that if $\mathcal{N}(x) < x$ for all $x \in (0, \Gamma]$, then the SG is almost globally asymptotically stable. For any given set of SG parameters, it is easy to plot \mathcal{N} to verify if the above sufficient stability condition is satisfied.

The inherent stability of networks of synchronous generators coupled with various types of loads and power sources (such as inverters) is currently an area of high interest and intense research, see for instance [3, 6, 7, 10, 27]. This is partly due to the proliferation of power sources that are not synchronous generators, which threatens the stability of the power grid. One approach to addressing this threat has been the introduction of synchronverters, see [1, 2, 5, 8, 32, 33, 34]. A synchronverter consists of an inverter (i.e. a DC to three-phase AC switched power converter) together with a passive filter (inductors and capacitors) that behave towards the power grid like a SG. A synchronverter has a rotor with inertia, a field coil with inductance and three stator coils with inductance and resistance, like a SG. But the field coils and the rotor in a synchronverter are virtual, i.e. they are implemented in software, while the stator coils are realized using the filter inductors. The dynamical equations governing the SG and the synchronverter are the same. Thus the synchronverter can be controlled like a SG, employing droop control loops and other controllers. This makes the power grid with inverters implemented as synchronverters easier to control using well established algorithms developed for SGs.

One motivation for our study comes from the development of synchronverters. In [32] an initial synchronization algorithm was proposed that can be run (typically for some seconds) before connecting the synchronverter to the grid. The purpose of this algorithm is to ensure that the voltages generated by the inverter are practically equal to the grid voltages. During this initial synchronization stage, the filter inductors are not used. Instead, the control algorithm creates virtual stator coils between the synchronous internal voltage and the grid, which carry virtual currents, and the initial synchronization is carried out using these virtual currents instead of real currents. Thus, even very high virtual currents that may arise as a transient phenomenon, do not cause any damage. A natural question is: will this initial

synchronization stage always succeed? If we simplify this question by assuming a constant field current and a constant grid frequency, then this question reduces to the one addressed in this paper. We remark that it is possible to construct an initial synchronization algorithm, using the results in this work, that is guaranteed to succeed (the details of such an algorithm are not included in this paper).

Our conclusions are relevant not only for the initial synchronization stage, but also for finding a good choice of parameters for the synchronverter. Indeed, our study shows that it is beneficial to have stator coils with large inductance in a synchronverter. We shall indicate in Section 7 how to realize the effect of a large inductor in the control algorithm of the synchronverter, without actually using a large and expensive filter inductor in the hardware. As an application, we find a set of stable nominal parameters for a 500kW synchronverter in Example 7.1.

The motivation for formulating the question of stability of a grid-connected SG in a global setting (i.e. for arbitrary initial states) comes from intensive simulations which indicate that for a range of parameters the SG could be almost globally asymptotically stable. We wanted to develop a rigorous analytical proof for this numerical observation about a classical dynamical system, which turned out to be very challenging. Our sufficient conditions for almost global asymptotic stability seem to be conservative: according to simulations, there are grid-connected SGs that do not satisfy our conditions, but nevertheless appear to be almost globally asymptotically stable. Also, it is easy to find such systems that have a locally stable equilibrium point but are not almost globally asymptotically stable. It is more difficult, but still possible, to find such systems whose equilibrium points are all unstable. Examples of systems described above are in Section 7.

2. Model of a SG connected to an infinite bus

Detailed mathematical models for synchronous machines can be found in [11, 13, 18, 19, 30]. In this section we will briefly derive the equations for a grid connected synchronous generator, as required in this work, using the notation and sign conventions in [22, 34]. We consider a SG with round (non-salient pole) rotor and, for the sake of simplicity, assume that the generator has one pair of field poles. The generator is “perfectly built”, meaning that in each stator winding, the flux caused by the rotor is a sinusoidal function of the rotor angle θ (with shifts of $\pm 2\pi/3$ between the phases of course). The rotor current $i_f > 0$ is assumed to be constant (or equivalently, the rotor is a permanent magnet). The stator windings are connected in star, with no neutral connection, and there are no damper windings.

Figure 1 shows the structure of the SG being considered. The stator windings have self-inductance $L > 0$, mutual inductance $-M < 0$ and resistance $R_s > 0$. (The typical value for M is $L/2$.) We define $L_s = L + M$. A current in a stator winding is considered positive if it flows outwards (see Figure 1). The vectors $e = [e_a \ e_b \ e_c]^\top$, $v = [v_a \ v_b \ v_c]^\top$ and $i = [i_a \ i_b \ i_c]^\top$ are the electromotive force (also called the synchronous internal voltage), stator terminal voltage and stator current,

respectively. The voltage at the (unconnected) center of the star is denoted by v_s . Let $v^n = [v_s \ v_s \ v_s]^\top$. Then, using $i_a + i_b + i_c = 0$ (there is no neutral line), we have

$$L_s \dot{i} + R_s i = e - v + v^n. \quad (2.1)$$

Note that if the synchronous generator is connected to the infinite bus via an impedance that consists of a resistor and an inductor in series, then these can be regarded as being parts of R_s and L_s , respectively.

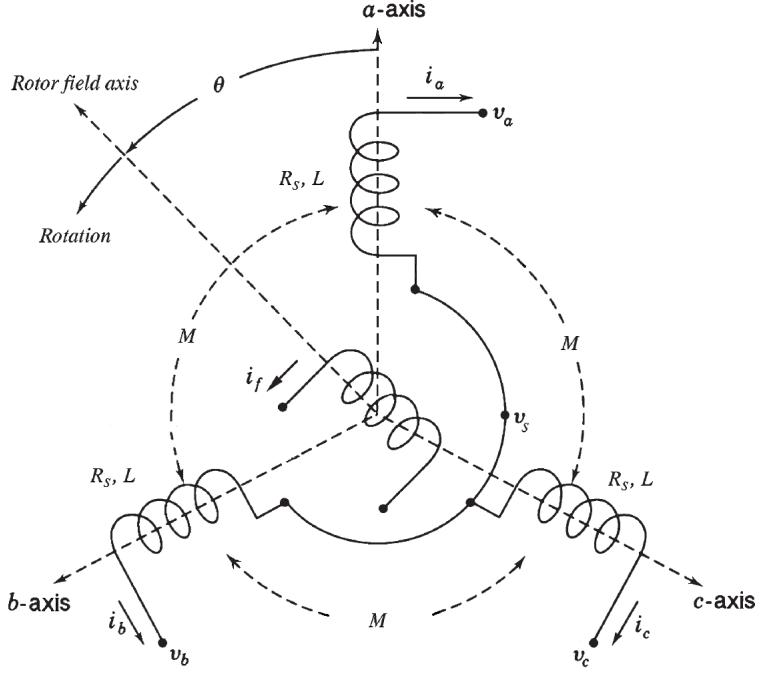


Figure 1. Structure of an idealized three-phase round-rotor SG, modified from [13, Fig. 3.4]. The rotor angle is θ and the field current is i_f .

Denote the rotor angle by θ and the angular velocity by ω . The power invariant version of the Park transformation is the unitary matrix

$$U(\theta) = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos \theta & \cos(\theta - \frac{2\pi}{3}) & \cos(\theta + \frac{2\pi}{3}) \\ -\sin \theta & -\sin(\theta - \frac{2\pi}{3}) & -\sin(\theta + \frac{2\pi}{3}) \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

With the notation $e_{dq} = U(\theta)e$, $v_{dq} = U(\theta)v$, $v_{dq}^n = U(\theta)v^n$ and $i_{dq} = U(\theta)i$, (2.1) can be written as

$$L_s U(\theta) \dot{i} + R_s i_{dq} = e_{dq} - v_{dq} + v_{dq}^n. \quad (2.2)$$

Let $e_{dq} = [e_d \ e_q \ e_0]^\top$, $v_{dq} = [v_d \ v_q \ v_0]^\top$ and $i_{dq} = [i_d \ i_q \ i_0]^\top$. It is easy to check that if $x_{dq} = [x_d \ x_q \ x_0]^\top = U(\theta)x$, then regardless of the physical meaning of x

$$\frac{d}{dt} \begin{bmatrix} x_d \\ x_q \\ x_0 \end{bmatrix} = U(\theta) \dot{x} + \omega \begin{bmatrix} x_q \\ -x_d \\ 0 \end{bmatrix}.$$

This, the easily verifiable expression $v_{dq}^n = [0 \ 0 \ \sqrt{3}v_s]^\top$ and (2.2) yield

$$L_s \dot{i}_d = -R_s i_d + \omega L_s i_q + e_d - v_d, \quad L_s \dot{i}_q = -\omega L_s i_d - R_s i_q + e_q - v_q. \quad (2.3)$$

Note that $i_0 = 0$ since $i_a + i_b + i_c = 0$ and hence $e_0 = v_0 - \sqrt{3}v_s$. Since the rotor current i_f is constant, it can be shown that

$$e = M_f i_f \omega \begin{bmatrix} \sin \theta \\ \sin(\theta - \frac{2\pi}{3}) \\ \sin(\theta + \frac{2\pi}{3}) \end{bmatrix}, \quad (2.4)$$

where $M_f > 0$ is the peak mutual inductance between the rotor winding and any one stator winding (see [34, equation (4)]). This, by a short computation, gives

$$e_d = 0, \quad e_q = -m\omega i_f, \quad (2.5)$$

where $m = \sqrt{\frac{3}{2}}M_f$. The rotational dynamics of the generator is governed by the equation

$$J\dot{\omega} = T_m - T_e - D_p\omega, \quad (2.6)$$

where $J > 0$ is the moment of inertia of all the parts rotating with the rotor, $T_m > 0$ is a mechanical torque constant (see the explanations further below), T_e is the electromagnetic torque developed by the generator (which normally opposes the movement) and $D_p > 0$ is a damping factor. T_e can be found from energy considerations, see for instance [34, equation (7)]:

$$T_e = -m i_f i_q.$$

The constant D_p is a sum of $D_{p,\text{fric}} > 0$ which accounts for the viscous friction acting on the rotor and $D_{p,\text{droop}} > 0$ which is created by a feedback, called the *frequency droop*, from ω to the mechanical torque of the prime mover (as explained in the cited references). The frequency droop increases the active power in response to a drop of the grid frequency. Normally, $D_{p,\text{droop}}$ is much larger than $D_{p,\text{fric}}$. The actual active mechanical torque T_a coming from the prime mover is $T_m - D_{p,\text{droop}}\omega$. Substituting the expression for T_e into (2.6), we obtain

$$J\dot{\omega} = m i_f i_q - D_p\omega + T_m. \quad (2.7)$$

The stator terminals are connected to the grid. Denote the grid voltage magnitude and angle by V and θ_g , respectively. By this we mean that the components of v are

$$v_a = \sqrt{\frac{2}{3}}V \sin \theta_g, \quad v_b = \sqrt{\frac{2}{3}}V \sin(\theta_g - \frac{2\pi}{3}), \quad v_c = \sqrt{\frac{2}{3}}V \sin(\theta_g + \frac{2\pi}{3}).$$

Define the angle difference δ , called *the power angle*, as $\delta = \theta - \theta_g$. Applying the Park transformation to v , we get

$$v_d = -V \sin \delta, \quad v_q = -V \cos \delta.$$

Substituting this and (2.5) into (2.3) gives

$$L_s \dot{i}_d = -R_s i_d + \omega L_s i_q + V \sin \delta, \quad L_s \dot{i}_q = -\omega L_s i_d - R_s i_q - m\omega i_f + V \cos \delta.$$

Denoting $\omega_g = \dot{\theta}_g$ (the grid frequency), it is clear from the definition of δ that

$$\dot{\delta} = \omega - \omega_g. \quad (2.8)$$

The last three equations together with (2.7) can be written in matrix form:

$$\begin{bmatrix} L_s \dot{i}_d \\ L_s \dot{i}_q \\ J \dot{\omega} \\ \dot{\delta} \end{bmatrix} = \begin{bmatrix} -R_s & \omega L_s & 0 & 0 \\ -\omega L_s & -R_s & -mi_f & 0 \\ 0 & mi_f & -D_p & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_d \\ i_q \\ \omega \\ \delta \end{bmatrix} + \begin{bmatrix} V \sin \delta \\ V \cos \delta \\ T_m \\ -\omega_g \end{bmatrix}. \quad (2.9)$$

The above fourth order nonlinear dynamical system, with state variables i_d, i_q, ω and δ is our model for a grid connected synchronous generator. In a synchronous generator we may control i_f indirectly via the rotor voltage (this adds i_f as one more state variable to the system) and we may control also D_p and T_m (though not instantly). In a synchronverter we may control i_f, D_p, T_m and even J instantly, but in this study they are considered to be positive constants.

3. Equilibrium points of the SG model

The right side of the SG model (2.9) is a locally Lipschitz function on its state space \mathbb{R}^4 . For any $(i_{d0}, i_{q0}, \omega_0, \delta_0) \in \mathbb{R}^4$, it follows from standard wellposedness results (see for instance [17, Ch. 3]) that there exists a unique solution $(i_d, i_q, \omega, \delta)$ for (2.9) defined on a maximal time interval $[0, T_{\max})$, with $T_{\max} > 0$, such that $(i_d(0), i_q(0), \omega(0), \delta(0)) = (i_{d0}, i_{q0}, \omega_0, \delta_0)$. We will show, via contradiction, that $T_{\max} = \infty$. To this end, suppose that T_{\max} is finite. For each $t \in [0, T_{\max})$ define $W(t) = (L_s i_d^2(t) + L_s i_q^2(t) + J \omega^2(t))/2$. Then

$$\dot{W}(t) = -R_s(i_d^2(t) + i_q^2(t)) - D_p \omega^2(t) + V i_d(t) \sin \delta(t) + V i_q(t) \cos \delta(t) + T_m \omega(t)$$

for all $t \in [0, T_{\max})$. Define $C = V^2/(2R_s) + T_m^2/(4D_p)$. Clearly

$$\dot{W}(t) \leq -R_s \left(|i_d(t)| - \frac{V}{2R_s} \right)^2 - R_s \left(|i_q(t)| - \frac{V}{2R_s} \right)^2 - D_p \left(|\omega(t)| - \frac{T_m}{2D_p} \right)^2 + C$$

which shows that if either $|i_d(t)|$, $|i_q(t)|$ or $|\omega(t)|$ is sufficiently large, then $\dot{W}(t) < 0$. In other words, if $W(t)$ is sufficiently large, then $\dot{W}(t) < 0$. Therefore W (and hence also i_d, i_q and ω) are bounded on $[0, T_{\max})$. Since T_{\max} is finite, it follows from (2.8) that δ must also be bounded on $[0, T_{\max})$. Hence $(i_d, i_q, \omega, \delta)$ are bounded functions on $[0, T_{\max})$, which contradicts [16, Corollary II.3]. Therefore $T_{\max} = \infty$. So for all initial conditions there exists a unique global (in time) solution for (2.9).

Denote $p = R_s/L_s$. Let the angle $\phi \in (0, \pi/2)$ be determined by the equations

$$\sin \phi = \frac{\omega_g}{\sqrt{p^2 + \omega_g^2}}, \quad \cos \phi = \frac{p}{\sqrt{p^2 + \omega_g^2}}. \quad (3.1)$$

Any equilibrium point $(i_d^e, i_q^e, \omega^e, \delta^e)$ of (2.9) must satisfy

$$\omega^e = \omega_g, \quad i_q^e = \frac{D_p \omega_g - T_m}{mi_f}, \quad i_d^e = \frac{\omega_g(D_p \omega_g - T_m)}{mi_f p} + \frac{V \sin \delta^e}{R_s}, \quad (3.2)$$

$$\cos(\delta^e + \phi) = \frac{(D_p \omega_g - T_m) L_s \sqrt{p^2 + \omega_g^2}}{m i_f} + \frac{m i_f \omega_g p}{V \sqrt{p^2 + \omega_g^2}}. \quad (3.3)$$

Denote the right side of (3.3) by Λ . Depending on $|\Lambda|$, (3.3) has either zero, one or two solutions, modulo 2π . For typical sets of SG parameters $|\Lambda| < 1$ and (3.3) has two solutions $\delta^{e,1} = \lambda - \phi$ and $\delta^{e,2} = -\lambda - \phi$. Here $\lambda \in (0, \pi)$ is such that $\cos \lambda = \Lambda$. Corresponding to these two solutions, two equilibrium points $(i_d^{e,1}, i_q^e, \omega_g, \delta^{e,1})$ and $(i_d^{e,2}, i_q^e, \omega_g, \delta^{e,2})$ for (2.9) can be determined using (3.2). If $(i_d^e, i_q^e, \omega_g, \delta^e)$ is an equilibrium point for (2.9), then so is $(i_d^e, i_q^e, \omega_g, \delta^e + 2k\pi)$ for any integer k . Therefore, when $|\Lambda| < 1$ there are in fact two sequences of equilibrium points for (2.9) in \mathbb{R}^4 . In general depending on $|\Lambda|$, like the pendulum equation with constant forcing, (2.9) has either zero, one or two sequences of equilibrium points and in any such sequence the last component δ differs by an integer multiple of 2π .

An equilibrium point of (2.9) is called *locally exponentially stable* (in short: stable) if the linearization of the system around this point is exponentially stable.

The linearization of (2.9) around an equilibrium point $(i_d^e, i_q^e, \omega_g, \delta^e)$ is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -p & \omega_g & i_q^e & (V \cos \delta^e)/L_s \\ -\omega_g & -p & -i_d^e - m i_f/L_s & -(V \sin \delta^e)/L_s \\ 0 & m i_f/J & -D_p/J & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad (3.4)$$

where $x_1 = i_d - i_d^e$, $x_2 = i_q - i_q^e$, $x_3 = \omega - \omega_g$ and $x_4 = \delta - \delta^e$. The characteristic polynomial of the matrix in (3.4) is $s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$, where

$$a_0 = \frac{m i_f V \sqrt{p^2 + \omega_g^2}}{J L_s} \sin(\delta^e + \phi). \quad (3.5)$$

The equilibrium point $(i_d^e, i_q^e, \omega_g, \delta^e)$ is stable if all the roots of the above characteristic polynomial are in the open left half complex plane. For this to occur it is necessary (but not sufficient) that $a_0, a_1, a_2, a_3 > 0$. Note that if $(i_d^e, i_q^e, \omega_g, \delta^e)$ is a stable (or unstable) equilibrium point, then so is $(i_d^e, i_q^e, \omega_g, \delta^e + 2k\pi)$ for any $k \in \mathbb{Z}$. When $|\Lambda| < 1$, the sign of a_0 when $\delta^e = \delta^{e,1}$ is opposite the sign of a_0 when $\delta^e = \delta^{e,2}$, so that (2.9) has at least one sequence of unstable equilibrium points. It is also possible that (2.9) has two sequences of unstable equilibrium points and no stable equilibrium point (see end of Section 7 for an example). Apart from equilibrium points, simulations show that when $|\Lambda| < 1$, (2.9) can have attracting periodic orbits (see Section 7). Hence the global phase portrait of (2.9) can be quite complicated.

Remark 3.1. Recall the currents $i = [i_a \ i_b \ i_c]^\top$ and $i_{dq} = [i_d \ i_q \ 0]^\top$ and the Park transformation $U(\theta)$ from Section 2. Since $i = U(\theta)^\top i_{dq}$ and $\theta = \delta + \theta_g$, it follows that each equilibrium point of (2.9) corresponds to a periodic state trajectory of the grid-connected SG, if we use the state variables $(i_a, i_b, \omega, \theta)$ (with $i_c = -i_a - i_b$). This periodic trajectory is stable (unstable) if the equilibrium point of (2.9) is stable (unstable). From the earlier discussion we get that when $|\Lambda| < 1$, if we measure all the angles modulo 2π , then the grid-connected SG with state variables $(i_a, i_b, \omega, \theta)$ has two unique periodic state trajectories and at least one of them is unstable.

Remark 3.2. Often in the literature on the control of power systems, the stator currents i_d and i_q are viewed as fast variables and (using singular perturbation theory) algebraic expressions are derived for them. If we follow this approach then, by substituting an algebraic expression for i_q in the differential equation (2.7), we get a second order nonlinear differential equation in δ as a reduced order approximation for the SG model (2.9). When $|\Lambda| < 1$, unlike the SG model, this nonlinear equation always has one sequence of stable equilibrium points and one sequence of unstable equilibrium points. So the SG model and its second order approximation can exhibit fundamentally different local and global dynamics for some SG parameters. This suggests that any controller designed using a reduced order model that approximates i_d and i_q must be validated for its performance on the full model.

Definition 3.3. The SG model (2.9) is *almost globally asymptotically stable* if all its state trajectories, except those starting from a set of measure zero and converging to an unstable equilibrium point, converge to a stable equilibrium point.

Note that this definition allows multiple stable and unstable equilibrium points, but it does not allow limit cycles or unbounded state trajectories.

Extensive simulations suggest that for a range of SG parameters (2.9) is almost globally asymptotically stable (aGAS). Our objective is to develop a practical test for verifying if for a given set of SG parameters (2.9) is aGAS. In this regard, our main result is Theorem 6.3 (also see Remark 6.5). Clearly if (2.9) is aGAS, then irrespective of initial conditions the SG rotor eventually synchronizes with the grid.

Definition 3.4. An equilibrium point $(i_d^e, i_q^e, \omega_g, \delta^e)$ of (2.9) is *hyperbolic* if all the eigenvalues of the matrix on the right side of (3.4) have non-zero real part.

For typical SG parameters, all the equilibrium points of (2.9) are hyperbolic.

Lemma 3.5. *If all the equilibrium points of (2.9) are hyperbolic and every trajectory of (2.9) converges to some equilibrium point, then (2.9) is aGAS.*

Proof. By assumption (2.9) has equilibrium points and so $|\Lambda| \leq 1$ ($|\Lambda|$ is defined below (3.3)). If $|\Lambda| = 1$, then for each equilibrium point $(i_d^e, i_q^e, \omega_g, \delta^e)$ of (2.9) we have $a_0 = 0$ (a_0 is introduced below (3.4)) meaning that the equilibrium point is not hyperbolic, contradicting the assumption in the lemma. Thus we can conclude that $|\Lambda| < 1$. From our earlier discussion, we get that (2.9) has a sequence of unstable equilibrium points. Let $z^e = (i_d^e, i_q^e, \omega_g, \delta^e)$ be an unstable equilibrium point. It then follows from the stable manifold theorem [29, Theorem 1.7.2] that the set of initial conditions for which the trajectory of (2.9) converges to z^e is the image of a C^1 injective map from $\mathbb{R}^k \rightarrow \mathbb{R}^4$, with $k < 4$. Using Sard's theorem [28, Theorem 4.1] we conclude that this set, called the stable manifold of z^e , has Lebesgue measure zero. Let \mathcal{M} be the union of the stable manifolds of all the unstable equilibrium points of (2.9). Since the set of unstable equilibrium points is countable, \mathcal{M} has measure zero. Since every trajectory of (2.9) converges to an equilibrium point, it follows that (2.9) must have a sequence of stable equilibrium points and all trajectories of (2.9) except those starting from \mathcal{M} converge to these stable equilibrium points. \square

If $(i_d, i_q, \omega, \delta)$ is the solution of (2.9) for the initial state $(i_d(0), i_q(0), \omega(0), \delta(0))$, then $(i_d, i_q, \omega, \delta + 2\pi)$ is the solution for the initial state $(i_d(0), i_q(0), \omega(0), \delta(0) + 2\pi)$. Thus, in the terminology of [20, Definition 2.4.1], (2.9) is a pendulum-like system.

4. An exact swing equation for the SG

In this section, starting from (2.9) we will derive an integro-differential equation governing the power angle δ , that resembles the nonlinear pendulum equation with forcing. It is a version of the classical swing equation (see [19, 35]) obtained by using the precise expressions for the mechanical torque and the electrical torque.

Recall $p = R_s/L_s$. The first two equations in (2.9) can then be written as

$$\begin{bmatrix} \dot{i}_d \\ \dot{i}_q \end{bmatrix} = \begin{bmatrix} -p & \omega \\ -\omega & -p \end{bmatrix} \begin{bmatrix} i_d \\ i_q \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{mi_f \omega}{L_s} \end{bmatrix} + \frac{V}{L_s} \begin{bmatrix} \sin \delta \\ \cos \delta \end{bmatrix}. \quad (4.1)$$

We regard ω and δ as continuous exogenous signals in (4.1). Therefore (4.1) is a linear time-varying system with state matrix

$$A(t) = \begin{bmatrix} -p & \omega(t) \\ -\omega(t) & -p \end{bmatrix}.$$

Clearly $A(t_1)A(t_2) = A(t_2)A(t_1)$ for all $t_1, t_2 \geq 0$. So an explicit expression for the state transition matrix $\Phi(t, \tau)$ generated by A can be computed to be

$$\begin{aligned} \Phi(t, \tau) &= e^{\int_\tau^t A(\sigma) d\sigma} = e^{\begin{bmatrix} -p(t-\tau) & \int_\tau^t \omega(\sigma) d\sigma \\ -\int_\tau^t \omega(\sigma) d\sigma & -p(t-\tau) \end{bmatrix}} \\ &= e^{-p(t-\tau)} \begin{bmatrix} \cos\left(\int_\tau^t \omega(\sigma) d\sigma\right) & \sin\left(\int_\tau^t \omega(\sigma) d\sigma\right) \\ -\sin\left(\int_\tau^t \omega(\sigma) d\sigma\right) & \cos\left(\int_\tau^t \omega(\sigma) d\sigma\right) \end{bmatrix} \quad \forall t, \tau \geq 0. \end{aligned}$$

For any initial state $[i_d(0) \ i_q(0)]^\top$ and some functions δ and ω , the unique solution of (4.1) is given by the expression

$$\begin{bmatrix} i_d(t) \\ i_q(t) \end{bmatrix} = \Phi(t, 0) \begin{bmatrix} i_d(0) \\ i_q(0) \end{bmatrix} + \int_0^t \Phi(t, \tau) \left(\begin{bmatrix} 0 \\ -\frac{mi_f \omega(\tau)}{L_s} \end{bmatrix} + \frac{V}{L_s} \begin{bmatrix} \sin(\delta(\tau)) \\ \cos(\delta(\tau)) \end{bmatrix} \right) d\tau \quad (4.2)$$

for each $t \geq 0$. The first term under the integral in (4.2), sans the constant $\frac{-mi_f}{L_s}$, can be written as

$$\begin{aligned} \int_0^t \Phi(t, \tau) \begin{bmatrix} 0 \\ \omega(\tau) \end{bmatrix} d\tau &= \int_0^t e^{-p(t-\tau)} \begin{bmatrix} \sin\left(\int_\tau^t \omega(\sigma) d\sigma\right) \omega(\tau) \\ \cos\left(\int_\tau^t \omega(\sigma) d\sigma\right) \omega(\tau) \end{bmatrix} d\tau \\ &= e^{-pt} \left[\begin{bmatrix} e^{p\tau} \cos\left(\int_\tau^t \omega(\sigma) d\sigma\right) \\ -e^{p\tau} \sin\left(\int_\tau^t \omega(\sigma) d\sigma\right) \end{bmatrix} \right]_{\tau=0}^{\tau=t} + p \int_0^t e^{-p(t-\tau)} \begin{bmatrix} -\cos\left(\int_\tau^t \omega(\sigma) d\sigma\right) \\ \sin\left(\int_\tau^t \omega(\sigma) d\sigma\right) \end{bmatrix} d\tau. \quad (4.3) \end{aligned}$$

Using the expression $\delta(\tau) = \delta(0) + \int_0^\tau \omega(\sigma) d\sigma - \omega_g \tau$ for all $\tau \geq 0$, the second term under the integral in (4.2), sans the constant $\frac{V}{L_s}$, can be written as

$$\begin{aligned} \int_0^t \Phi(t, \tau) \begin{bmatrix} \sin(\delta(\tau)) \\ \cos(\delta(\tau)) \end{bmatrix} d\tau &= \int_0^t e^{-p(t-\tau)} \begin{bmatrix} \sin\left(\int_\tau^t \omega(\sigma) d\sigma + \delta(\tau)\right) \\ \cos\left(\int_\tau^t \omega(\sigma) d\sigma + \delta(\tau)\right) \end{bmatrix} d\tau \\ &= \int_0^t e^{-p(t-\tau)} \begin{bmatrix} \sin\left(\int_0^t \omega(\sigma) d\sigma + \delta(0) - \omega_g \tau\right) \\ \cos\left(\int_0^t \omega(\sigma) d\sigma + \delta(0) - \omega_g \tau\right) \end{bmatrix} d\tau \\ &= \frac{pe^{-pt}}{(p^2 + \omega_g^2)} \begin{bmatrix} e^{p\tau} \sin\left(\int_0^t \omega(\sigma) d\sigma + \delta(0) - \omega_g \tau\right) \\ e^{p\tau} \cos\left(\int_0^t \omega(\sigma) d\sigma + \delta(0) - \omega_g \tau\right) \end{bmatrix}_{\tau=0}^{\tau=t} \\ &\quad + \frac{\omega_g e^{-pt}}{(p^2 + \omega_g^2)} \begin{bmatrix} e^{p\tau} \cos\left(\int_0^t \omega(\sigma) d\sigma + \delta(0) - \omega_g \tau\right) \\ -e^{p\tau} \sin\left(\int_0^t \omega(\sigma) d\sigma + \delta(0) - \omega_g \tau\right) \end{bmatrix}_{\tau=0}^{\tau=t}. \end{aligned}$$

Using the angle ϕ introduced in (3.1), the above equation can be written as

$$\int_0^t \Phi(t, \tau) \begin{bmatrix} \sin(\delta(\tau)) \\ \cos(\delta(\tau)) \end{bmatrix} d\tau = \frac{e^{-pt}}{\sqrt{p^2 + \omega_g^2}} \begin{bmatrix} e^{p\tau} \sin\left(\int_0^t \omega(\sigma) d\sigma + \delta(0) - \omega_g \tau + \phi\right) \\ e^{p\tau} \cos\left(\int_0^t \omega(\sigma) d\sigma + \delta(0) - \omega_g \tau + \phi\right) \end{bmatrix}_{\tau=0}^{\tau=t}.$$

Putting together (4.2), (4.3) and the last equation, and using the notation

$$i_v = \frac{V}{L_s \sqrt{p^2 + \omega_g^2}}, \quad (4.4)$$

we obtain that for all $t \geq 0$

$$\begin{aligned} i_q(t) &= i_v \cos\left(\int_0^t \omega(\sigma) d\sigma + \delta(0) - \omega_g t + \phi\right) \\ &\quad - \frac{mi_f p}{L_s} \int_0^t e^{-p(t-\tau)} \sin\left(\int_\tau^t \omega(\sigma) d\sigma\right) d\tau + e^{-pt} f(t), \end{aligned}$$

where

$$\begin{aligned} f(t) &= -\sin\left(\int_0^t \omega(\sigma) d\sigma\right) i_d(0) + \cos\left(\int_0^t \omega(\sigma) d\sigma\right) i_q(0) \\ &\quad - \frac{mi_f}{L_s} \sin\left(\int_0^t \omega(\sigma) d\sigma\right) - i_v \cos\left(\int_0^t \omega(\sigma) d\sigma + \delta(0) + \phi\right). \end{aligned} \quad (4.5)$$

Clearly f is a continuous function of time that depends on ω , but nevertheless can be bounded with a constant independent of ω . Substituting for $i_q(t)$ in the equations for ω and δ in (2.9) we obtain the following integro-differential equation for $\delta(t)$:

$$\begin{aligned} J\ddot{\delta}(t) + D_p \dot{\delta}(t) - mi_f i_v \cos(\delta(t) + \phi) &= T_m - D_p \omega_g \\ &\quad - \frac{m^2 i_f^2 p}{L_s} \int_0^t e^{-p(t-\tau)} \sin\left(\int_\tau^t \omega(\sigma) d\sigma\right) d\tau + mi_f e^{-pt} f(t). \end{aligned}$$

If we introduce the new variable η by

$$\eta(t) = \frac{3\pi}{2} + \delta(t) + \phi \quad (4.6)$$

so that $\dot{\eta}(t) = \omega(t) - \omega_g$, then the above equation becomes

$$J\ddot{\eta}(t) + D_p\dot{\eta}(t) + m i_f i_v \sin \eta(t) = T_m - D_p \omega_g + m i_f e^{-pt} f(t) - \frac{m^2 i_f^2 p}{L_s} \int_0^t e^{-p(t-\tau)} \sin [\eta(t) - \eta(\tau) + \omega_g(t - \tau)] d\tau. \quad (4.7)$$

We will refer to (4.7) as the *exact swing equation* (ESE). For all initial conditions $(\eta(0), \dot{\eta}(0))$ and every function f given by (4.5) for some $i_d(0)$ and $i_q(0)$, there exists a unique global solution $(\eta, \dot{\eta})$ for ESE. Indeed $(\eta, \dot{\eta}) = (3\pi/2 + \delta + \phi, \dot{\delta})$, where δ is such that $(i_d, i_q, \omega, \delta)$ is the unique solution of (2.9) for the initial condition $(i_d(0), i_q(0), \dot{\eta}(0) + \omega_g, \eta(0) - \phi - 3\pi/2)$. Clearly there is a 1-1 correspondence between the solutions of (2.9) and the solutions of (4.7) when f is given by (4.5).

The integral in (4.7) may be regarded as the output of a first order low-pass filter (with corner frequency p) driven by a bounded input, so that it is bounded. If we regard the right side of (4.7) as a bounded exogenous function, then (4.7) is a forced pendulum equation. In the next section, we derive certain bounds to quantify the asymptotic response of forced pendulum equations. These bounds are applied to (4.7) in Section 6 to establish the main result of this paper.

5. Asymptotic response of a forced pendulum

Consider the forced pendulum equation

$$\ddot{\psi}(t) + \alpha \dot{\psi}(t) + \sin \psi(t) = \beta + \gamma(t) \quad \forall t \geq 0, \quad (5.1)$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are constants and $\gamma \in L^\infty([0, \infty); \mathbb{R})$ is a continuous function of the time t satisfying $\|\gamma\|_{L^\infty} < d$ for some $d \in \mathbb{R}$. We assume that $|\beta| + d < 1$. Define the angles $\psi_1, \psi_2 \in (-\pi/2, \pi/2)$ so that

$$\sin \psi_1 = \beta + d, \quad \sin \psi_2 = \beta - d. \quad (5.2)$$

For any initial state $(\psi(0), \dot{\psi}(0))$, there is a unique solution ψ to (5.1) on a maximal time interval $[0, t_{\max})$, according to standard results on ordinary differential equations (ODEs), see for instance [17, Ch. 3]. Since $|\beta + \gamma(t) - \sin \psi| < |\beta| + d + 1$ for all $t \geq 0$ and $\alpha > 0$, we get from (5.1) (by looking at the linear ODE $\dot{z} + \alpha z = u$, with $z = \dot{\psi}$) that

$$\sup_{t \in [0, t_{\max})} |\dot{\psi}(t)| < |\dot{\psi}(0)| + \frac{|\beta| + d + 1}{\alpha}. \quad (5.3)$$

Hence $|\dot{\psi}(t)|$ cannot blow up to infinity in a finite time, and hence the same holds for $|\psi(t)|$. From [16, Corollary II.3] it follows that $t_{\max} = \infty$. Since γ is a continuous function of time, we get from (5.1) that the function ψ is of class C^2 .

The aim of this section is to show that if α is sufficiently large, then the solutions ψ of (5.1) are eventually confined to a narrow interval, see Theorem 5.14.

We will often regard the solution $(\psi, \dot{\psi})$ of (5.1) as a curve in the phase plane. Recall that in the phase plane the angle ψ is on the x -axis and the angular velocity $\dot{\psi}$ is on the y -axis. The curve corresponding to $(\psi, \dot{\psi})$ satisfies the ODE

$$\frac{d\dot{\psi}(t)}{d\psi(t)} = -\alpha + \frac{\beta + \gamma(t) - \sin \psi(t)}{\dot{\psi}(t)} \quad \text{whenever} \quad \dot{\psi}(t) \neq 0. \quad (5.4)$$

Suppose that the curve $(\psi, \dot{\psi})$ passes through a point $(\psi_0, \dot{\psi}_0)$ in the phase plane. We use the notation $\dot{\psi}|_{\psi=\psi_0}$ to denote $\dot{\psi}_0$ provided there is no ambiguity. We refer to Figures 2 and 3 for typical state trajectory curves in the phase plane.

The following two lemmas establish a monotonicity in the behavior of the solutions to (5.1) with respect to the infinity norm of the forcing term.

Lemma 5.1. *Consider the pendulum equation*

$$\ddot{\psi}_p(t) + \alpha \dot{\psi}_p(t) + \sin \psi_p(t) = \beta + d \quad \forall t \geq 0. \quad (5.5)$$

Let ψ and ψ_p be the solutions of (5.1) and (5.5), respectively, for the initial conditions $\psi(0) = \psi_p(0) = \psi_0$ and $\dot{\psi}(0) = \dot{\psi}_p(0) = \dot{\psi}_0$ and recall that $\|\gamma\|_{L^\infty} < d$. Suppose that $\dot{\psi}(t) \geq 0$ for all $t \in [0, \tau]$ and $\psi(\tau) \neq \psi(0)$ for some $\tau > 0$. Then the curve $(\psi_p, \dot{\psi}_p)$ lies above the curve $(\psi, \dot{\psi})$ in the phase plane on the angle interval $(\psi(0), \psi(\tau))$, i.e. for each $\varphi \in (\psi(0), \psi(\tau))$ we have $\dot{\psi}_p|_{\psi_p=\varphi} > \dot{\psi}|_{\psi=\varphi}$.

Proof. The curve $(\psi_p, \dot{\psi}_p)$ satisfies the following ODE in the variable $\psi_p(t)$:

$$\frac{d\dot{\psi}_p(t)}{d\psi_p(t)} = -\alpha + \frac{\beta + d - \sin \psi_p(t)}{\dot{\psi}_p(t)} \quad \text{whenever} \quad \dot{\psi}_p(t) \neq 0. \quad (5.6)$$

First we claim that there exists $\sigma \in (0, \tau)$ such that $\psi_p(\sigma) < \psi(\tau)$ and

$$\dot{\psi}_p(t) > 0 \quad \forall t \in (0, \sigma].$$

If $\dot{\psi}_0 > 0$ then this is obvious. If $\dot{\psi}_0 = 0$ then, since $\dot{\psi}(t) \geq 0$ for all $t \in [0, \tau]$, $\ddot{\psi}(0) \geq 0$. Using this and $\|\gamma\|_{L^\infty} < d$, (5.1) and (5.5) give that $\ddot{\psi}_p(0) > 0$, which together with $\dot{\psi}_p(0) \geq 0$ implies the existence of $\sigma \in (0, \tau)$ with the desired properties.

Our second claim is that for each $\varphi_1 \in (\psi_0, \psi_p(\sigma))$, there exists a $\varphi \in (\psi_0, \varphi_1)$ such that

$$\dot{\psi}_p|_{\psi_p=\varphi} > \dot{\psi}|_{\psi=\varphi}. \quad (5.7)$$

Indeed, if this claim were false, then

$$\dot{\psi}_p|_{\psi_p=\varphi} \leq \dot{\psi}|_{\psi=\varphi} \quad \forall \varphi \in (\psi_0, \varphi_1) \quad (5.8)$$

which using (5.4) and (5.6) gives that $\frac{d\dot{\psi}_p(t)}{d\psi_p(t)}|_{\psi_p(t)=\varphi} > \frac{d\dot{\psi}(t)}{d\psi(t)}|_{\psi(t)=\varphi}$ for all $\varphi \in (\psi_0, \varphi_1)$.

This contradicts (5.8) since $\dot{\psi}_p|_{\psi_p=\psi_0} = \dot{\psi}|_{\psi=\psi_0} = \dot{\psi}_0$.

So far we have shown that we can find points $\varphi \in (\psi_0, \psi(\tau))$ arbitrarily close to ψ_0 such that (5.7) holds. To complete the proof of this lemma, it is sufficient to establish the following claim: if (5.7) holds for some $\varphi \in (\psi_0, \psi(\tau))$, then (5.7) holds for all $\tilde{\varphi} \in (\varphi, \psi(\tau))$ (with $\tilde{\varphi}$ in place of φ).

To prove the above claim, suppose that it is not true for some φ . Then define

$$\varphi_{bad} = \inf \left\{ \tilde{\varphi} \in (\varphi, \psi(\tau)) \mid \dot{\psi}_p|_{\psi_p=\tilde{\varphi}} = \dot{\psi}|_{\psi=\tilde{\varphi}} \right\}. \quad (5.9)$$

Let $t_{bad} \in (0, \tau)$ be such that $\psi(t_{bad}) = \varphi_{bad}$, so that $\dot{\psi}_p(t_{bad}) = \dot{\psi}(t_{bad})$. We will first show by contradiction that $\dot{\psi}(t_{bad}) > 0$. To this end, suppose that $\dot{\psi}(t_{bad}) = 0$. This implies that $\dot{\psi}(t_{bad})$ is a local minimum for $\dot{\psi}$ and so $\ddot{\psi}(t_{bad}) = 0$. From (5.1) we get that $-\sin \varphi_{bad} + \beta + \gamma(t_{bad}) = 0$, hence $-\sin \varphi_{bad} + \beta + d > 0$. From (5.5) we get that $\ddot{\psi}_p(t_{bad}) > 0$, so that for $t < t_{bad}$ very close to t_{bad} and satisfying $\psi(t) \in (\varphi, \psi(\tau))$, $\dot{\psi}_p(t) < \dot{\psi}_p(t_{bad}) = 0$. But (5.9) gives that $\dot{\psi}(t) < \dot{\psi}_p(t)$ and so $\dot{\psi}(t) < 0$, which contradicts the assumption $\dot{\psi} \geq 0$ in the lemma. Thus $\dot{\psi}_p(t_{bad}) = \dot{\psi}(t_{bad}) > 0$. Now (5.4) and (5.6) give that for some $\mu > 0$

$$\left. \frac{d\dot{\psi}_p(t)}{d\psi_p(t)} \right|_{\psi_p(t)=\tilde{\varphi}} > \left. \frac{d\dot{\psi}(t)}{d\psi(t)} \right|_{\psi(t)=\tilde{\varphi}} \quad \forall \tilde{\varphi} \in (\varphi_{bad} - \mu, \varphi_{bad}) \quad (5.10)$$

and $\varphi_{bad} - \mu > \varphi$. This is because the above inequality holds when $\tilde{\varphi} = \varphi_{bad}$. By assumption $\dot{\psi}_p|_{\psi_p=\varphi_{bad}-\mu} > \dot{\psi}|_{\psi=\varphi_{bad}-\mu}$ which, along with (5.10), gives the contradiction $\dot{\psi}_p|_{\psi_p=\varphi_{bad}} > \dot{\psi}|_{\psi=\varphi_{bad}}$. This proves the claim above (5.9). \square

Lemma 5.2. *Consider the pendulum equation*

$$\ddot{\psi}_n(t) + \alpha \dot{\psi}_n(t) + \sin \psi_n(t) = \beta - d \quad \forall t \geq 0. \quad (5.11)$$

Let ψ and ψ_n be the solutions of (5.1) and (5.11), respectively, for the initial conditions $\psi(0) = \psi_n(0) = \psi_0$ and $\dot{\psi}(0) = \dot{\psi}_n(0) = \dot{\psi}_0$. Suppose that $\dot{\psi}(t) \leq 0$ for all $t \in [0, \tau]$ and $\psi(\tau) \neq \psi(0)$ for some $\tau > 0$. Then the curve $(\psi_n, \dot{\psi}_n)$ lies below the curve $(\psi, \dot{\psi})$ in the phase plane on the angle interval $(\psi(\tau), \psi(0))$, i.e. for each $\varphi \in (\psi(\tau), \psi(0))$ we have $\dot{\psi}_n|_{\psi_n=\varphi} < \dot{\psi}|_{\psi=\varphi}$.

Proof. Apply the change of variables $\psi \mapsto -\psi$ and $\psi_n \mapsto -\psi_p$ to (5.1) and (5.11), respectively. Now apply Lemma 5.1 to the resulting equations (instead of (5.1) and (5.5)) after redefining β and γ to be $-\beta$ and $-\gamma$, respectively. \square

The following result on the nonexistence of non-constant periodic solutions to the pendulum equation with a constant forcing term has been established in [15].

Theorem 5.3. *Consider the pendulum equation*

$$\ddot{\psi}_h(t) + \alpha \dot{\psi}_h(t) + \sin \psi_h(t) = \sin \lambda \quad \forall t \geq 0, \quad (5.12)$$

where $\alpha > 0$ and $\lambda \in (0, \pi/2)$. If $\alpha > 2 \sin(\lambda/2)$ and ψ_h is a solution of (5.12) such that $\dot{\psi}_h$ is non-negative and periodic, then ψ_h is constant.

The locally asymptotically stable equilibrium points of (5.12) are located at $\psi_h = \lambda + 2k\pi$ ($k \in \mathbb{Z}$) and $\dot{\psi}_h = 0$ while the unstable equilibria are at $\psi_h = \pi - \lambda + 2k\pi$ and $\dot{\psi}_h = 0$, regardless of the size of the damping factor $\alpha > 0$. Figure 2 shows the typical shape of the curves $(\psi_h, \dot{\psi}_h)$ in the phase plane for a sufficiently large (but

not too large) $\alpha > 0$, so that we are in the case considered in Theorem 5.3. In this case every state trajectory converges to an equilibrium point. If α gets even larger, then of course we are still in the case considered in Theorem 5.3, but the curves do not spiral around the stable equilibria. This distinction may be visually remarkable, but is not important for our analysis, so we do not discuss it further.

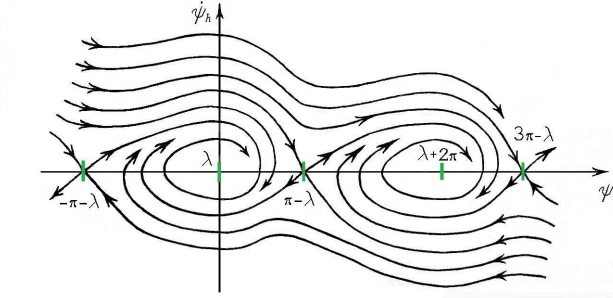


Figure 2. Phase plane curves for the damped pendulum from (5.12) for moderately large damping factor $\alpha > 2 \sin(\lambda/2)$ (not to scale).

When $\alpha > 0$ is small, then the curves $(\psi_h, \dot{\psi}_h)$ look fundamentally different: some state trajectories still converge to one of the same equilibrium points. Other state trajectories approach a curve that is a stable limit cycle if we measure the angle ψ_h modulo 2π , see Figure 3 and the note after the proof of Proposition 5.4. The critical value of α that separates between these two types of behavior (which depends on λ) is estimated in Theorem 5.3 due to W. Hayes in 1953 [15]. We are not aware of any better estimate available now (other than by simulation experiments). Our interest is in the forced pendulum (5.1), and for us (5.12) is only a tool for comparison. Much material about systems related to (5.12) can be found in [20, Ch. 3].

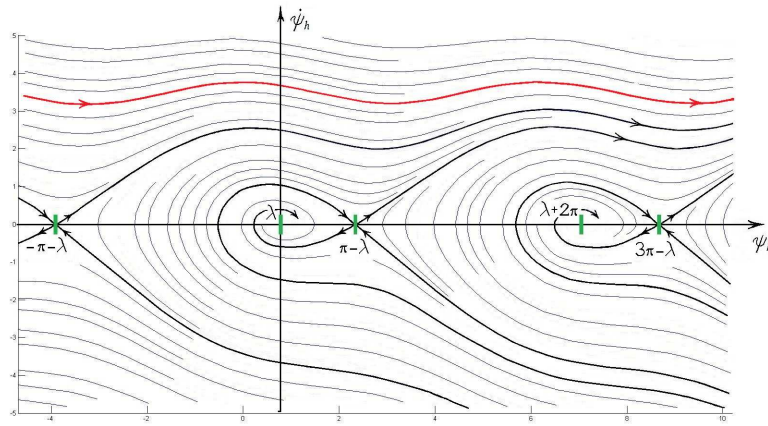


Figure 3. Phase plane curves for the damped pendulum from (5.12) for small α ($\alpha = 0.2$, $\sin \lambda = 0.7$). The limit cycle is shown as a red curve.

For the pendulum system (5.1), we define the energy function E as

$$E(t) = \frac{1}{2} \dot{\psi}(t)^2 + (1 - \cos \psi(t)) \quad \forall t \geq 0. \quad (5.13)$$

The time derivative of E along the trajectories of (5.1) is given by

$$\dot{E}(t) = -\alpha\dot{\psi}(t)^2 + (\beta + \gamma(t))\dot{\psi}(t). \quad (5.14)$$

Therefore for any $t_2 > t_1 \geq 0$

$$E(t_2) - E(t_1) = \int_{t_1}^{t_2} [-\alpha\dot{\psi}(s) + \beta + \gamma(s)]\dot{\psi}(s)ds.$$

If $\dot{\psi}(t) \neq 0$ for all $t \in [t_1, t_2]$, then using the change of variables $s \mapsto \psi(s)$ we get

$$E(t_2) - E(t_1) = \int_{\psi(t_1)}^{\psi(t_2)} \left[-\alpha\dot{\psi}(\psi^{-1}(\varphi)) + \beta + \gamma(\psi^{-1}(\varphi)) \right] d\varphi. \quad (5.15)$$

Using Theorem 5.3, the next proposition shows that if α is sufficiently large, then each solution $(\psi, \dot{\psi})$ of (5.1) must either converge to a limit point or its velocity must change sign at least once after any given time $t \geq 0$.

Proposition 5.4. *Recall the angles ψ_1 and ψ_2 from (5.2). Assume that*

$$\alpha > 2 \sin \frac{|\psi_1|}{2}, \quad \alpha > 2 \sin \frac{|\psi_2|}{2}. \quad (5.16)$$

Then there exists no solution ψ of (5.1) such that ψ is unbounded and $\dot{\psi}(t)$ is either non-negative or non-positive for all $t \geq 0$.

Proof. In the first part of this proof we assume that ψ is a solution of (5.1) such that $\dot{\psi}(t) \geq 0$ for all $t \geq 0$. Our first claim is that if ψ is unbounded, then

$$\beta + d > 0, \quad \text{hence } \psi_1 > 0.$$

Indeed, if not, then the right-hand side of (5.14) is $\leq -\mu\dot{\psi}(t)$ for some $\mu > 0$, forcing E to become eventually negative, which is impossible.

Our second claim is that for any $\tau > 0$,

$$\text{if } \sin \psi(\tau) > \beta + \|\gamma\|_{L^\infty}, \text{ then } \ddot{\psi}(\tau) < 0 \text{ and } \dot{\psi}(\tau) > 0. \quad (5.17)$$

If $\sin \psi(\tau) > \beta + \|\gamma\|_{L^\infty}$ then clearly $\sin \psi(\tau) > \beta + \gamma(\tau)$, hence from (5.1) we get $\ddot{\psi}(\tau) < 0$. It follows that $\dot{\psi}(\tau) > 0$ since otherwise (if it is zero) then for $t > \tau$ close to τ we would have $\dot{\psi}(t) < 0$, contradicting our assumption that $\dot{\psi} \geq 0$.

In the sequel, we assume that ψ is unbounded (which will lead to a contradiction). Let $t_0 > 0$ be such that $\psi(t_0) = 2m\pi + \psi_1$, $m \in \mathbb{Z}$. Our third claim is that

$$\inf \left\{ \dot{\psi}(t) \mid \psi(t) \in [2k\pi + \psi_1, (2k+1)\pi - \psi_1], \quad k \in \mathbb{Z}, k \geq m \right\} \geq \varepsilon > 0. \quad (5.18)$$

Define $\psi_\gamma \in (-\pi/2, \pi/2)$ so that $\sin \psi_\gamma = \beta + \|\gamma\|_{L^\infty}$. Let $t_1, t_2 > 0$ be such that $\psi(t_1) = (2m+1)\pi - \psi_1$ and $\psi(t_2) = (2m+1)\pi - \psi_\gamma$, so that $t_0 < t_1 < t_2$. Then since $\ddot{\psi}(t) < 0$ for all $t \in [t_0, t_1]$ it follows that for all $t \in [t_0, t_1]$ we have $\dot{\psi}(t) > \dot{\psi}(t_1) > 0$. To prove (5.18), we have to find a lower bound on $\dot{\psi}(t_1)$ that is independent of m . If we regard $(\psi, \dot{\psi})$ as a curve in the phase plane, then from (5.4) and (5.17) we see that

for all $t \in [t_0, t_2)$, $\frac{d\psi(t)}{dt} < -\alpha$. From here, by integration, $\dot{\psi}(t_1) > \dot{\psi}(t_2) + \alpha(\psi_1 - \psi_\gamma)$. Using again (5.17) we see that we can take $\varepsilon = \alpha(\psi_1 - \psi_\gamma) > 0$ in (5.18).

Let ψ_p be the solution of (5.5) with $\psi_p(0) = \psi(t_0)$ and $\dot{\psi}_p(0) = \dot{\psi}(t_0)$. Then $\dot{\psi}_p$ is bounded (by the argument in (5.3)) and ψ_p is defined on $[0, \infty)$. Our fourth claim is that $\dot{\psi}_p|_{\psi_p=\varphi} > \dot{\psi}|_{\psi=\varphi}$ for all $\varphi > \psi(t_0)$. Indeed, from Lemma 5.1 it follows that the curve $(\psi_p, \dot{\psi}_p)$ lies above the curve $(\psi, \dot{\psi})$ on the angle interval $(\psi(t_0), \infty)$. Hence $\dot{\psi}_p$ is a strictly positive function and (5.18) holds if we replace ψ and $\dot{\psi}$ with ψ_p and $\dot{\psi}_p$.

The fifth claim is that there is a solution ψ_p^f of (5.5) such that $\dot{\psi}_p^f$ is a periodic and strictly positive function of time. For this, first we find the function f which represents one period of ψ_p^f in the phase plane. Consider the sequence of strictly positive continuous functions $(f_k)_{k=m}^\infty$ defined on the angle interval $I = [\psi_1, \psi_1 + 2\pi]$ as follows: $f_k(\varphi) = \dot{\psi}_p|_{\psi_p=\varphi+2k\pi}$ for each $\varphi \in I$, where ψ_p is as defined in the previous paragraph. Clearly for each $k \geq m$, the curve defined by the graph of f_k in the phase plane is a segment of the curve $(\psi_p, \dot{\psi}_p)$ and so it follows from (5.6) that

$$\frac{df_k(\varphi)}{d\varphi} = -\alpha + \frac{\beta + d - \sin \varphi}{f_k(\varphi)} \quad \forall \varphi \in [\psi_1, \psi_1 + 2\pi]. \quad (5.19)$$

Since no two curves corresponding to two distinct solutions of (5.5) can intersect in the phase plane, for $k_1 \neq k_2$ the curves defined by f_{k_1} and f_{k_2} must either be the same or do not intersect at all. This, along with the fact that $f_k(\psi_1 + 2\pi) = f_{k+1}(\psi_1)$ for each $k \geq m$, implies that $f_{k+1} - f_k$ is either a non-negative function for all k or it is a non-positive function for all k . Therefore the sequence $(f_k)_{k=m}^\infty$ converges to f which is a non-negative continuous function defined on I satisfying $f(\psi_1) = f(\psi_1 + 2\pi)$ (here we have used the fact that the functions f_k are uniformly bounded). By using the version of (5.18) with ψ_p in place of ψ , for each $\varphi \in [\psi_1, \pi - \psi_1]$ we get that $f_k(\varphi) \geq \varepsilon$ for all $k \geq m$ and so $f(\varphi) \geq \varepsilon$. This means that for all $\varphi \in [\psi_1, \pi - \psi_1]$ and all $k \geq m$, the right side of (5.19) is bounded in absolute value by $(\beta + d + 1)/\varepsilon + \alpha$. Using this, we can take the limit as $k \rightarrow \infty$ on both sides of (5.19) to conclude that f satisfies (5.19) on the interval $\varphi \in [\psi_1, \pi - \psi_1]$.

To complete the proof of the above claim, let ψ_p^f be the solution of (5.5) for the initial state $\psi_p^f(0) = \psi_1$, $\dot{\psi}_p^f(0) = f(\psi_1)$. Since the curve $(\psi_p^f, \dot{\psi}_p^f)$ satisfies (5.6) and f satisfies (5.19) (which is the same ODE as (5.6)), it follows that $f(\varphi) = \dot{\psi}_p^f|_{\psi_p^f=\varphi}$ for all $\varphi \in [\psi_1, \pi - \psi_1]$. In particular $\dot{\psi}_p^f(t) \geq \varepsilon$ as long as $\psi_p^f(t) \leq \pi - \psi_1$. We now show that $\dot{\psi}_p^f(t) > 0$ as long as $\psi_p^f(t) \in (\pi - \psi_1, \psi_1 + 2\pi)$. Indeed, if $\dot{\psi}_p^f(t) < (\beta + d - \sin \varphi)/\alpha$ (which is a positive number) and $\psi_p^f(t) \in (\pi - \psi_1, \psi_1 + 2\pi)$, then (5.5) gives that $\ddot{\psi}_p^f(t) > 0$, so that $\dot{\psi}_p^f$ is increasing and hence it cannot become ≤ 0 . Therefore $\lim_{t \rightarrow \infty} \psi_p^f(t) \geq \psi_1 + 2\pi$. By the same argument as used earlier for $\varphi \in [\psi_1, \pi - \psi_1]$, $f(\varphi) = \dot{\psi}_p^f|_{\psi_p^f=\varphi}$ for all $\varphi \in [\psi_1, \psi_1 + 2\pi]$. Therefore $\dot{\psi}_p^f|_{\psi_p^f=\psi_1} = \dot{\psi}_p^f|_{\psi_p^f=\psi_1+2\pi}$, so that $\dot{\psi}_p^f$ is a periodic and strictly positive function.

The fifth claim (that we proved) together with the first inequality in (5.16) contradict Theorem 5.3, because ψ_p^f is a solution of (5.12) when $\lambda = \psi_1$ (here we have used the first claim). Thus if $\dot{\psi}(t) \geq 0$ for all $t \geq 0$, then ψ must be bounded.

Next assume that ψ is unbounded and $\dot{\psi}(t) \leq 0$ for all $t \geq 0$. Then $-\psi$ is unbounded and $-\dot{\psi}(t) \geq 0$ and $-\psi$ is a solution of (5.1) when $\beta + \gamma(t)$ on the right side is replaced with $-\beta - \gamma(t)$. The above proof, using $-\beta$ in place of β , and the second inequality in (5.16) will again give rise to a contradiction implying that if $\dot{\psi}(t) \leq 0$ for all $t \geq 0$, then ψ must be bounded. \square

Note that the above proof also contains (around the fifth claim) the main ingredients of the proof of the following fact: If $\beta + d > 0$, then any unbounded solution ψ_p of (5.5) converges to a solution ψ_p^f such that $\dot{\psi}_p^f$ is positive and periodic (both as a function of time and as a function of ψ_p^f) (shown as the red curve in Figure 3). A similar statement holds for $\beta + d < 0$, in which case $\dot{\psi}_p^f$ is negative and periodic.

Definition 5.5. A point $(\varphi, 0)$ in the phase plane is called a *positive acceleration point* if $\beta + d - \sin \varphi > 0$, i.e. $\ddot{\psi}_p(0) > 0$ according to (5.5) when $\psi_p(0) = \varphi$ and $\dot{\psi}_p(0) = 0$. A point $(\varphi, 0)$ in the phase plane is called a *negative acceleration point* if $\beta - d - \sin \varphi < 0$, which has a similar interpretation as $\ddot{\psi}_n(0) < 0$ using (5.11).

Using the notation (5.2), the set of positive acceleration points is

$$\{(\varphi, 0) \mid \varphi \in ((2k-1)\pi - \psi_1, 2k\pi + \psi_1), k \in \mathbb{Z}\}$$

and the set of negative acceleration points is

$$\{(\varphi, 0) \mid \varphi \in (2k\pi + \psi_2, (2k+1)\pi - \psi_2), k \in \mathbb{Z}\}.$$

Lemma 5.6. Suppose that $\alpha > 2 \sin(|\psi_1|/2)$ and ψ_p is the solution of (5.5) when

$$(\psi_p(0), \dot{\psi}_p(0)) = (\varphi^0, 0), \quad (2m-1)\pi - \psi_1 < \varphi^0 < 2m\pi + \psi_1, \quad m \in \mathbb{Z}$$

(so that $(\varphi^0, 0)$ is a positive acceleration point). Denote

$$\tau = \sup \{T > 0 \mid \dot{\psi}_p(t) > 0 \text{ for all } t \in (0, T)\}.$$

If $\tau < \infty$ then $(\psi_p(\tau), \dot{\psi}_p(\tau)) = (\varphi^1, 0)$, where $\varphi^1 \in (2m\pi + \psi_1, (2m+1)\pi - |\psi_1|)$.

If $\tau = \infty$ then $\lim_{t \rightarrow \infty} (\psi_p(t), \dot{\psi}_p(t)) = (\varphi^1, 0)$, where $\varphi^1 = 2m\pi + \psi_1$.

Note that in both cases listed above, $(\varphi^1, 0)$ is a negative acceleration point.

Proof. Choose $\lambda \in (|\psi_1|, \pi/2)$ such that $\alpha > 2 \sin(\lambda/2)$. Suppose that $\lim_{t \rightarrow \tau} \psi_p(t) > (2m+1)\pi - \lambda$ (which will lead to a contradiction). Let ψ_h be a solution of (5.12) with $\psi_h(0) = \varphi^0$ and $\dot{\psi}_h(0) \geq 0$. Since $\sin \lambda > \sin \psi_1$ it can be shown (like in the proof of Lemma 5.1) that the curve $(\psi_h, \dot{\psi}_h)$ is above the curve $(\psi_p, \dot{\psi}_p)$ in the phase plane on the angle interval $(\varphi^0, (2m+1)\pi - \lambda)$. Therefore $\dot{\psi}_h|_{\psi_h=(2m+1)\pi-\lambda} > 0$. This implies that there exists $\tau_1 > 0$ such that $\psi_h(\tau_1) = 2(m+1)\pi + \lambda$ and

$$\dot{\psi}_h|_{\psi_h=\varphi} > 0 \quad \forall \varphi \in ((2m+1)\pi - \lambda, 2(m+1)\pi + \lambda).$$

This is because if $\dot{\psi}_h|_{\psi_h=\varphi} < (\sin \lambda - \sin \varphi)/\alpha$ (which is > 0), then (5.12) gives that $\ddot{\psi}_h|_{\psi_h=\varphi} > 0$. It now follows from the above discussion that

$$\dot{\psi}_h|_{\psi_h=\varphi} > 0 \quad \forall \varphi \in (\varphi^0, \varphi^0 + 2\pi]. \quad (5.20)$$

Let ψ_h^1 be the solution of (5.12) with $(\psi_h^1(0), \dot{\psi}_h^1(0)) = (\varphi^0, \dot{\psi}_h|_{\psi_h=\varphi^0+2\pi})$. Then repeating the above argument we obtain that (5.20) holds with ψ_h^1 in place of ψ_h . By concatenating ψ_h with ψ_h^1 , we obtain that the solution ψ_h of (5.12) actually advances by at least two full circles from its initial angle φ^0 , and we have

$$\dot{\psi}_h|_{\psi_h=\varphi} > 0 \quad \forall \varphi \in (\varphi^0, \varphi^0 + 4\pi].$$

Continuing by induction, we obtain that ψ_h is unbounded and $\dot{\psi}_h > 0$ all the time. This contradicts Proposition 5.4 in which we replace ψ_1 with $\tilde{\psi}_1 > \lambda$ such that $\alpha > 2 \sin(\tilde{\psi}_1/2)$ still holds. Indeed, then ψ_h is a solution of (5.1) when $\beta = 0$ and $\gamma(t) = \sin \lambda$ for all $t \geq 0$. Hence the assumption at the start of our proof is false, which means that

$$\lim_{t \rightarrow \tau} \psi_p(t) \leq (2m+1)\pi - \lambda.$$

Consider the case when $\tau < \infty$, so that $\dot{\psi}_p(\tau) = 0$. We claim that $\varphi^1 = \psi_p(\tau) \geq 2m\pi + \psi_1$. Indeed, $\dot{\psi}_p(t)$ cannot reach 0 for a time t when $\varphi^0 < \psi_p(t) < 2m\pi + \psi_1$, because (5.5) would imply that $\ddot{\psi}_p(t) > 0$. Thus, $\varphi^1 \in [2m\pi + \psi_1, (2m+1)\pi - \lambda]$. Next we claim that $\varphi^1 > 2m\pi + \psi_1$. Indeed, if $\varphi^1 = 2m\pi + \psi_1$, then $(\varphi^1, 0)$ is an equilibrium point of the system (5.5) and $\mathbf{x} = (\psi_p, \dot{\psi}_p) - (\varphi^1, 0)$ satisfies an ODE of the form $\dot{\mathbf{x}} = f(\mathbf{x})$, where $f \in C^1$ and $f(0) = 0$. It is well known that for such an ODE, any trajectory starting from $\mathbf{x}(0) \neq 0$ cannot reach the point $(0, 0)$ in a finite time. Thus, we have $\varphi^1 > 2m\pi + \psi_1$. Combining this with the fact that $\lambda > |\psi_1|$, we get that $\varphi^1 \in (2m\pi + \psi_1, (2m+1)\pi - |\psi_1|)$, as stated in the lemma.

Now consider the case when $\tau = \infty$. Since ψ_p is increasing and bounded, clearly $\dot{\psi}_p \in L^1[0, \infty)$. Since $\dot{\psi}_p$ is bounded (by the argument at (5.3)), it follows from (5.5) that $\ddot{\psi}_p$ is also bounded, so that $\dot{\psi}_p$ is uniformly continuous. Now applying Barbălat's lemma (see [17, Lemma 8.2] or see [9, 21] for nice presentations with a more general perspective), we get that $\lim_{t \rightarrow \infty} \dot{\psi}_p(t) = 0$. We have $\varphi^1 = \lim_{t \rightarrow \infty} \psi_p(t) \in [2m\pi + \psi_1, (2m+1)\pi - |\psi_1|)$, for similar reasons as in the case $\tau < \infty$. By differentiating (5.5), we see that $\dot{\psi}_p$ is also bounded. Since the expressions $\int_0^t \ddot{\psi}_p(\sigma) d\sigma = \dot{\psi}_p(t)$ are uniformly bounded (with respect to t), we can apply again Barbălat's lemma, this time to $\ddot{\psi}_p$, to show that $\lim_{t \rightarrow \infty} \ddot{\psi}_p(t) = 0$. Looking at (5.5), it follows that $\lim_{t \rightarrow \infty} \sin \psi_p(t) = \sin \psi_1$, so that $\sin \varphi^1 = \sin \psi_1$. Looking at the range of possible values of φ^1 , we conclude that it has indeed the value stated in the lemma. \square

With the notation of the last lemma, we call $(\varphi^1, 0)$ the *first negative acceleration point* for ψ_p . We remark that $\tau = \infty$ for sufficiently large α , regardless of φ^0 .

The following lemma concerning solutions of (5.11) is similar to Lemma 5.6.

Lemma 5.7. *Suppose that $\alpha > 2 \sin(|\psi_2|/2)$ and ψ_n is the solution of (5.11) when*

$$(\psi_n(0), \dot{\psi}_n(0)) = (\varphi^1, 0), \quad 2m\pi + \psi_2 < \varphi^1 < (2m+1)\pi - \psi_2, \quad m \in \mathbb{Z}$$

(so that $(\varphi^1, 0)$ is a negative acceleration point). Denote

$$\tau = \sup \{T > 0 \mid \dot{\psi}_n(t) < 0 \text{ for all } t \in (0, T)\}.$$

If $\tau < \infty$ then $(\psi_n(\tau), \dot{\psi}_n(\tau)) = (\varphi^2, 0)$, where $\varphi^2 \in ((2m-1)\pi + |\psi_2|, 2m\pi + \psi_2)$.

If $\tau = \infty$ then $\lim_{t \rightarrow \infty} (\psi_n(t), \dot{\psi}_n(t)) = (\varphi^2, 0)$, where $\varphi^2 = 2m\pi + \psi_2$.

Note that in both cases listed above, $(\varphi^2, 0)$ is a positive acceleration point.

Proof. Define $\tilde{\psi}_p = -\psi_n$ and $\tilde{\beta} = -\beta$, then $\tilde{\psi}_p$ satisfies (5.5) with $\tilde{\beta}$ in place of β . Define $\tilde{\psi}_1 = -\psi_2$, then the first expression in (5.2) holds with $\tilde{\psi}_1$ and $\tilde{\beta}$ in place of ψ_1 and β , and of course $\alpha > 2 \sin(|\tilde{\psi}_1|/2)$. Define $\tilde{\varphi}^0 = -\varphi^1$ and $\tilde{m} = -m$, then these satisfy the assumption on initial conditions in Lemma 5.6. Thus, we can apply Lemma 5.6 with the tilde variables in place of the original ones, and we get exactly the conclusions of the lemma that we are now proving, with $\varphi^2 = -\tilde{\varphi}^1$. \square

With the notation of the last lemma, we call $(\varphi^2, 0)$ the *first positive acceleration point* for ψ_n . We remark that $\tau = \infty$ for sufficiently large α , regardless of φ^1 .

Next we define a family of continuous curves in the phase plane referred to as *spiral curves*. These curves have the structure of an inward spiral.

Definition 5.8. Suppose that α satisfies (5.16). Let $\varphi^0 = (2m - 1)\pi - \psi_2$ for some integer m . Construct a sequence $(\varphi^k)_{k=0}^\infty$ as follows: for each odd k , $(\varphi^k, 0)$ is the first negative acceleration point for the solution ψ_p^k of (5.5) with initial conditions $\psi_p^k(0) = \varphi^{k-1}$, $\dot{\psi}_p^k(0) = 0$. For each even $k > 0$, $(\varphi^k, 0)$ is the first positive acceleration point for the solution ψ_n^k of (5.11) with initial conditions $\psi_n^k(0) = \varphi^{k-1}$, $\dot{\psi}_n^k(0) = 0$. For $k \in \mathbb{N}$, denote the segment of the curve $(\psi_p^k, \dot{\psi}_p^k)$ (or $(\psi_n^k, \dot{\psi}_n^k)$) between $(\varphi^{k-1}, 0)$ and $(\varphi^k, 0)$ by Γ_k . A *spiral curve* Γ starting from $(\varphi^0, 0)$ is a continuous curve in the phase plane obtained by concatenating all Γ_k ($k \in \mathbb{N}$).

Lemmas 5.6 and 5.7 ensure that the points φ^k introduced in Definition 5.8 in fact exist for all $k \in \mathbb{N}$. The spiral curve can be interpreted as the phase plane trajectory of the solution of (5.1) with $\psi(0) = \varphi^0$, $\dot{\psi}(0) = 0$ and

$$\gamma(t) = d \operatorname{sign}(\dot{\psi}(t)), \quad (5.21)$$

where the trajectory is continued even if it happens that a segment Γ_k takes an infinite amount of time. The above expression for $\gamma(t)$ is like a static friction torque acting on a pendulum, but with the wrong sign. We remark (but will not use) that for any sufficiently large damping coefficient α the sequence $(\varphi^k)_{k=1}^\infty$ is such that

$$\varphi^1 = \varphi^3 = \varphi^5 = \dots = 2m\pi + \psi_1 \quad \text{and} \quad \varphi^2 = \varphi^4 = \varphi^6 = \dots = 2m\pi + \psi_2. \quad (5.22)$$

For smaller α only a part of the equalities in (5.22) hold, possibly none. Figure 4 shows possible shapes of Γ_1 , Γ_2 and some limit curves Γ_a , Γ_b that will be introduced later, in the case when none of the equalities in (5.22) holds.

Lemma 5.9. Suppose that α satisfies (5.16). Fix an integer m and consider the spiral curve Γ starting from $(\varphi^0, 0)$ with $\varphi^0 = (2m - 1)\pi - \psi_2$. There exists a simple closed curve Γ_c in the phase plane to which Γ converges, i.e. for any $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ with $n \geq N_\varepsilon$,

$$\mathbf{d}(\mathbf{x}, \Gamma_c) < \varepsilon \quad \forall \mathbf{x} \in \Gamma_n. \quad (5.23)$$

Here \mathbf{d} is the Euclidean distance in \mathbb{R}^2 .

Proof. Let $(\varphi^k)_{k=0}^\infty$ be the sequence introduced in Definition 5.8. It follows directly from Lemmas 5.6 and 5.7 that $\varphi^2 > \varphi^0$. Using the fact that two curves corresponding

to two distinct solutions of (5.5) cannot intersect, we conclude that $\varphi^3 \leq \varphi^1$. (We remark that equality can only occur if $\varphi^3 = \varphi^1 = 2m\pi + \psi_1$, since distinct solutions may meet at a common limit point, which is an equilibrium point of (5.5). In this case we have (5.22) except possibly the last equality.) Using the fact that no two curves corresponding to two distinct solutions of (5.11) can intersect, we conclude that $\varphi^4 \geq \varphi^2$. (We remark that if $\varphi^3 < \varphi^1$, then $\varphi^4 = \varphi^2$ can only occur if $\varphi^4 = \varphi^2 = 2m\pi + \psi_2$, since distinct solutions may meet at a common limit point, which is an equilibrium point of (5.11). In this case we have (5.22) except for the first and possibly the last equality from the first string.) Continuing like this, we get that the sequence $(\varphi^{2k+1})_{k=0}^\infty$ is nonincreasing and bounded from below by $2m\pi + \psi_1$, while the sequence $(\varphi^{2k})_{k=0}^\infty$ is nondecreasing and bounded from above by $2m\pi + \psi_2$. Let

$$\varphi_{low} = \lim_{k \rightarrow \infty} \varphi^{2k}, \quad \varphi_{high} = \lim_{k \rightarrow \infty} \varphi^{2k+1}. \quad (5.24)$$

Clearly these are positive and negative acceleration points, respectively.

For the remainder of this proof, for any $\varphi \in \mathbb{R}$ we denote by $\psi_p(\cdot, \varphi)$ the solution ψ_p of (5.5) satisfying $\psi_p(0) = \varphi$ and $\dot{\psi}_p(0) = 0$. Let $(\tilde{\varphi}_{high}, 0)$ be the first negative acceleration point for $\psi_p(\cdot, \varphi_{low})$. By Lemma 5.6 we have $\tilde{\varphi}_{high} \geq 2m\pi + \psi_1$. Since for any $k \in \mathbb{N}$ we have $\varphi_{low} \geq \varphi^{2k}$ and no two curves corresponding to two distinct solutions of (5.5) can intersect in the phase plane, we have $\tilde{\varphi}_{high} \leq \varphi^{2k+1}$. Taking limits, we obtain that $\tilde{\varphi}_{high} \leq \varphi_{high}$. Thus, using also Lemma 5.6,

$$2m\pi + \psi_1 \leq \tilde{\varphi}_{high} \leq \varphi_{high} < (2m+1)\pi - |\psi_1|. \quad (5.25)$$

Denote the segment of the curve $(\psi_p(\cdot, \varphi_{low}), \dot{\psi}_p(\cdot, \varphi_{low}))$ between $(\varphi_{low}, 0)$ and $(\tilde{\varphi}_{high}, 0)$ by Γ_a . We claim that for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\text{if } 2n \geq N_\varepsilon, \quad \text{then } \mathbf{d}(\mathbf{x}, \Gamma_a) < \varepsilon \quad \forall \mathbf{x} \in \Gamma_{2n+1}. \quad (5.26)$$

Note that this implies (by an easy argument that we omit) that $\tilde{\varphi}_{high} = \varphi_{high}$.

To prove (5.26), we have to consider two cases:

Case 1: $\tilde{\varphi}_{high} > 2m\pi + \psi_1$ (this is the easier case). According to Lemma 5.6, there exists a smallest $\tau > 0$ such that $(\psi_p(\tau, \varphi_{low}), \dot{\psi}_p(\tau, \varphi_{low})) = (\tilde{\varphi}_{high}, 0)$. It is easy to see that there exists $T > \tau$ such that

$$2m\pi + \psi_1 < \psi_p(t, \varphi_{low}) < \tilde{\varphi}_{high} \quad \text{and} \quad \dot{\psi}_p(t, \varphi_{low}) < 0 \quad \forall t \in (\tau, T].$$

According to the standard result on the continuous dependence of solutions of differential equations (satisfying a Lipschitz condition) on their initial conditions, for any $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $2n \geq N_\varepsilon$,

$$|\psi_p(t, \varphi_{low}) - \psi_p(t, \varphi^{2n})| + |\dot{\psi}_p(t, \varphi_{low}) - \dot{\psi}_p(t, \varphi^{2n})| < \varepsilon \quad \forall t \in [0, T]. \quad (5.27)$$

For each such n , let τ_{2n} be the smallest positive number such that $(\psi_p(\tau_{2n}, \varphi^{2n}), 0)$ is the first negative acceleration point of $\psi_p(\cdot, \varphi^{2n})$. Then it is easy to verify, using (5.27), that $\varepsilon < |\dot{\psi}_p(T, \varphi_{low})|$ implies $\dot{\psi}_p(T, \varphi^{2n}) < 0$, so that

$$\text{if } \varepsilon < |\dot{\psi}_p(T, \varphi_{low})| \quad \text{and} \quad 2n \geq N_\varepsilon, \quad \text{then } \tau_{2n} < T.$$

From here, by an easy argument using (5.27) we obtain that for ε and n as above, $\mathbf{d}(\mathbf{x}, \Gamma_a) < \varepsilon$ for all $\mathbf{x} \in \Gamma_{2n+1}$. Clearly this implies (5.26).

Case 2: $\tilde{\varphi}_{high} = 2m\pi + \psi_1$, so that $(\tilde{\varphi}_{high}, 0)$ is a locally asymptotically stable (in particular, Lyapunov stable) equilibrium point of (5.5). From the Lyapunov stability, for every $\varepsilon > 0$ there exists $\delta_\varepsilon \in (0, \varepsilon)$ such that the following holds: if, for some $\varphi \in \mathbb{R}$ and $T > 0$,

$$|\psi_p(T, \varphi) - \tilde{\varphi}_{high}| + |\dot{\psi}_p(T, \varphi)| < \delta_\varepsilon, \quad (5.28)$$

then $|\psi_p(t, \varphi) - \tilde{\varphi}_{high}| + |\dot{\psi}_p(t, \varphi)| < \varepsilon$ for all $t \geq T$. For some $\varepsilon > 0$, let $T > 0$ be such that (5.28) holds, with φ_{low} in place of φ and $\delta_\varepsilon/2$ in place of δ_ε . Using again the standard result on the continuous dependence of solutions of ODEs on their initial conditions, there exists an $N_\varepsilon \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $2n \geq N_\varepsilon$, (5.27) holds with $\delta_\varepsilon/2$ in place of ε . Using the Lyapunov stability, this implies that $|\psi_p(t, \varphi^{2n}) - \tilde{\varphi}_{high}| + |\dot{\psi}_p(t, \varphi^{2n})| < \varepsilon$ for all $t \geq T$. This implies that $\mathbf{d}(\mathbf{x}, \Gamma_a) < \varepsilon$ for all \mathbf{x} in the phase plane curve of $\psi_p(\cdot, \varphi^{2n})$ (for positive time), and in particular for all $\mathbf{x} \in \Gamma_{2n+1}$. Thus, we have proved (5.26) also in the second case.

For the remainder of this proof, for any $\varphi \in \mathbb{R}$ we denote by $\psi_n(\cdot, \varphi)$ the solution ψ_n of (5.11) satisfying $\psi_n(0) = \varphi$ and $\dot{\psi}_n(0) = 0$. Denote the segment of the curve $(\psi_n(\cdot, \varphi_{high}), \dot{\psi}_n(\cdot, \varphi_{high}))$ between $(\varphi_{high}, 0)$ and its first positive acceleration point $(\tilde{\varphi}_{low}, 0)$ by Γ_b . We claim that for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\text{if } 2n \geq N_\varepsilon, \text{ then } \mathbf{d}(\mathbf{x}, \Gamma_b) < \varepsilon \quad \forall \mathbf{x} \in \Gamma_{2n}. \quad (5.29)$$

This implies that $\tilde{\varphi}_{low} = \varphi_{low}$. The proof of these facts is similar to the proof of (5.26), by replacing everywhere ψ_p with $-\psi_n$, (5.5) with (5.11), m with $-m$, φ^{2n} with $-\varphi^{2n+1}$, φ_{low} with $-\varphi_{high}$ and viceversa, $\tilde{\varphi}_{high}$ with $-\tilde{\varphi}_{low}$ and Γ_a with $-\Gamma_b$.

It follows from $\tilde{\varphi}_{high} = \varphi_{high}$ and $\tilde{\varphi}_{low} = \varphi_{low}$ that the union of the curves Γ_a and Γ_b defined above is a simple closed curve in the phase plane (see Figure 4), which we denote by Γ_c . It follows from (5.26) and (5.29) that (5.23) holds. \square

Remark 5.10. Putting together (5.25), the obvious $\varphi^2 \leq \varphi_{low}$ and the lower estimate for φ^2 from Lemma 5.7, we have (with the notation of the last proof)

$$(2m-1)\pi + |\psi_2| < \varphi_{low} \leq 2m\pi + \psi_2 < 2m\pi + \psi_1 \leq \varphi_{high} < (2m+1)\pi - |\psi_1|.$$

Recall the curves Γ_a and Γ_b introduced in the last proof. We now show that, under some conditions, the regions in the phase plane enclosed by Γ_a and the horizontal axis (and by Γ_b and the horizontal axis) are convex, as illustrated in Figure 4. Using this, in Lemma 5.12 we derive upper bounds for the heights of Γ_a and Γ_b . These bounds are then used to derive an estimate for $\varphi_{high} - \varphi_{low}$.

Lemma 5.11. *Let α satisfy (5.16) and $m \in \mathbb{Z}$. Recall $\varphi_{low}, \varphi_{high}, \Gamma_a, \Gamma_b$ introduced in the last proof. Denote the closed subsets of the phase plane enclosed by the curve Γ_a and the horizontal axis by Δ_a , and by the curve Γ_b and the horizontal axis by Δ_b . If $\varphi_{high} \neq 2m\pi + \psi_1$, then Δ_a is convex. If $\varphi_{low} \neq 2m\pi + \psi_2$, then Δ_b is convex.*

Proof. In this proof, we denote by ψ_p the solution of (5.5) corresponding to the initial condition $(\psi_p(0), \dot{\psi}_p(0)) = (\varphi_{low}, 0)$ and let τ be the time that it takes $(\psi_p, \dot{\psi}_p)$ to

reach $(\varphi_{high}, 0)$ (while moving along Γ_a). We assume that $\varphi_{high} \neq 2m\pi + \psi_1$. Hence by Lemma 5.6, $\tau < \infty$ and $\dot{\psi}_p(t) > 0$ for all $t \in (0, \tau)$. We consider the function ψ_p only on the interval $[0, \tau]$. For $\varphi \in [\varphi_{low}, \varphi_{high}]$ we denote $f(\varphi) = \dot{\psi}_p(t)|_{\psi_p(t)=\varphi}$, so that Γ_a is the graph of f . The slope of f is, according to (5.6),

$$f'(\varphi) = \frac{d\dot{\psi}_p(t)}{d\psi_p(t)} \Big|_{\psi_p(t)=\varphi} = -\alpha + \frac{\beta + d - \sin \varphi}{f(\varphi)} \quad \forall \varphi \in (\varphi_{low}, \varphi_{high}). \quad (5.30)$$

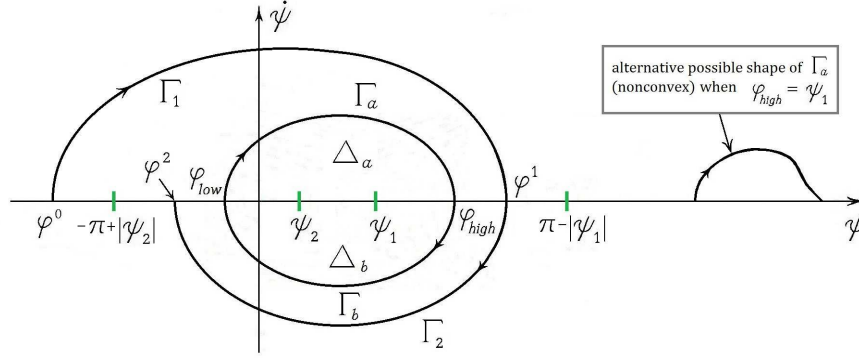


Figure 4. Possible shape of the curves Γ_1 , Γ_2 , Γ_a and Γ_b in the phase plane, and of the sets Δ_a and Δ_b , when (5.16) holds and $m = 0$. We have shown the case when none of the equalities in (5.22) holds and $\psi_2 > 0$.

We claim that $f''(\varphi) \leq 0$ for all $\varphi \in (\varphi_{low}, \varphi_{high})$. It can be shown by a somewhat tedious computation that for every $\varphi \in (\varphi_{low}, \varphi_{high})$,

$$f''(\varphi) = -\frac{f(\varphi) \cos \varphi + (\beta + d - \sin \varphi) f'(\varphi)}{f(\varphi)^2}, \quad (5.31)$$

$$f'''(\varphi) = -\frac{3f'(\varphi) + \alpha}{f(\varphi)} \cdot f''(\varphi) + \frac{\sin \varphi}{f(\varphi)}. \quad (5.32)$$

Suppose that our claim is false. Then $f''(\varphi_0) > 0$ for some $\varphi_0 \in (\varphi_{low}, \varphi_{high})$. Using (5.31) and the facts that $f(\varphi_{low}) = 0$, $\varphi_{low} < 2m\pi + \psi_2$ (see Lemma 5.7) and $\psi_2 < \psi_1$, is easy to verify that for a sufficiently small $\varepsilon > 0$, $f''(\varphi) < 0$ for each $\varphi \in (\varphi_{low}, \varphi_{low} + \varepsilon)$. Hence there exists $\eta_{low} \in (\varphi_{low}, \varphi_0)$ such that

$$f''(\eta_{low}) = 0, \quad f'''(\eta_{low}) \geq 0.$$

This, using (5.32), implies that $\sin \eta_{low} \geq 0$. Since $\varphi_{low} \geq (2m-1)\pi + |\psi_2|$ (see Lemma 5.7), we conclude that $\eta_{low} \in [2m\pi, \varphi_0)$ and hence $\varphi_0 > 2m\pi$.

According to (5.25) we have $\varphi_{high} \in (2m\pi + \psi_1, (2m+1)\pi - |\psi_1|)$. It follows from (5.31) (using $f(\varphi_{high}) = 0$) that for a sufficiently small $\varepsilon > 0$ we have $f''(\varphi) < 0$ for all $\varphi \in (\varphi_{high} - \varepsilon, \varphi_{high})$. Therefore there exists $\eta_{high} \in (\varphi_0, \varphi_{high})$ such that

$$f''(\eta_{high}) = 0, \quad f'''(\eta_{high}) \leq 0.$$

Using (5.32), this gives us that $\sin \eta_{high} \leq 0$. Since, according to our earlier steps, $\eta_{high} \in (2m\pi, (2m+1)\pi)$, this is a contradiction, proving our claim. By a well known fact in analysis, our claim implies that Δ_a is a convex set.

When $\varphi_{low} \neq 2m\pi + \psi_2$, the convexity of Δ_b can be established similarly. \square

Lemma 5.12. *Let $m, \alpha, \Gamma_a, \Gamma_b, \varphi_{low}$ and φ_{high} be as in Lemma 5.11. Define*

$$v_a = \max_{(\psi_p, \dot{\psi}_p) \in \Gamma_a} \dot{\psi}_p, \quad v_b = \max_{(\psi_n, \dot{\psi}_n) \in \Gamma_b} |\dot{\psi}_n|$$

Recall d, ψ_1 and ψ_2 from (5.2). Then the following relations hold:

$$\varphi_{high} \neq 2m\pi + \psi_1 \implies v_a < \frac{4d}{\alpha}, \quad (5.33)$$

$$\varphi_{low} \neq 2m\pi + \psi_2 \implies v_b < \frac{4d}{\alpha}, \quad (5.34)$$

$$\varphi_{high} \neq 2m\pi + \psi_1, \quad \varphi_{low} \neq 2m\pi + \psi_2 \implies v_a + v_b < \frac{4d}{\alpha}. \quad (5.35)$$

Furthermore

$$\varphi_{high} - \varphi_{low} < \psi_1 - \psi_2 + \frac{4d}{\alpha^2}, \quad (5.36)$$

$$\varphi_{high} < \psi_1 + 2m\pi + \frac{4d}{\alpha^2}, \quad \varphi_{low} > 2m\pi + \psi_2 - \frac{4d}{\alpha^2}. \quad (5.37)$$

Proof. First assume that $\varphi_{high} \neq 2m\pi + \psi_1$. Then Γ_c , the closed curve that is the union of Γ_a and Γ_b (introduced in Lemma 5.9), can be regarded as a segment of the curve corresponding to the solution $(\psi, \dot{\psi})$ of (5.1) when $(\psi(0), \dot{\psi}(0)) = (\varphi_{low}, 0)$ and γ is given by (5.21). This solution takes a finite time τ to reach $(\varphi_{high}, 0)$ and then a possibly infinite amount of time to return to $(\varphi_{low}, 0)$. We denote by τ_c the (possibly infinite) time that it takes for $(\psi, \dot{\psi})$ to go around the closed curve Γ_c .

Recall the function $f : [\varphi_{low}, \varphi_{high}] \rightarrow [0, \infty)$ introduced before (5.30), so that Γ_a is the graph of f . Similarly, we introduce $g : [\varphi_{low}, \varphi_{high}] \rightarrow (-\infty, 0]$ so that Γ_b is the graph of g . Since $E(0) = \lim_{t \rightarrow \tau_c} E(t)$, it follows from (5.15) that

$$\int_{\varphi_{low}}^{\varphi_{high}} \alpha[f(\varphi) - g(\varphi)] d\varphi = 2d(\varphi_{high} - \varphi_{low}). \quad (5.38)$$

Let $\psi_+ \in (\varphi_{low}, \varphi_{high})$ be the angle at which f reaches its maximum v_a . From Lemma 5.11 we know that Δ_a is convex. Hence, the triangle in the phase plane with vertices $(\varphi_{low}, 0)$, $(\varphi_{high}, 0)$ and (ψ_+, v_a) lies inside Δ_a . Therefore

$$\int_{\varphi_{low}}^{\varphi_{high}} f(\varphi) d\varphi \geq \frac{v_a}{2}(\varphi_{high} - \varphi_{low}). \quad (5.39)$$

This and (5.38) imply that $2d(\varphi_{high} - \varphi_{low}) > \frac{\alpha v_a}{2}(\varphi_{high} - \varphi_{low})$, whence (5.33).

Now replace the assumption $\varphi_{high} \neq 2m\pi + \psi_1$ with $\varphi_{low} \neq 2m\pi + \psi_2$. By repeating the above arguments after (5.38), but using the function g and the convex set Δ_b , we get that

$$\int_{\varphi_{low}}^{\varphi_{high}} -g(\varphi) d\varphi \geq \frac{v_b}{2}(\varphi_{high} - \varphi_{low}) \quad (5.40)$$

which, together with (5.38), implies (5.34). Finally if $\varphi_{high} \neq 2m\pi + \psi_1$ and $\varphi_{low} \neq 2m\pi + \psi_2$, then both (5.39) and (5.40) hold, which together with (5.38) imply (5.35).

Next we will derive (5.36). First assume that $\varphi_{high} \neq 2m\pi + \psi_1$. Let ψ_p and τ be as at the beginning of the proof of Lemma 5.11 (so that $\psi_p : [0, \tau] \rightarrow [0, v_a]$,

$\psi_p(0) = \varphi_{low}$ and $\psi_p(\tau) = \varphi_{high}$. Let $\tau_1 \in [0, \tau]$ be such that $\psi_p(\tau_1) = 2m\pi + \psi_1$. Using the energy from (5.13) (and (5.15) with d in place of γ) we get

$$\begin{aligned} \frac{\dot{\psi}_p(\tau_1)^2}{2} &= E(\tau_1) - E(\tau) + \cos \psi_1 - \cos \varphi_{high} \\ &= \int_{2m\pi + \psi_1}^{\varphi_{high}} \alpha \dot{\psi}_p|_{\psi_p = \varphi} d\varphi + \int_{2m\pi + \psi_1}^{\varphi_{high}} (\sin \varphi - \beta - d) d\varphi \\ &\geq \frac{\alpha}{2} (\varphi_{high} - \psi_1 - 2m\pi) \dot{\psi}_p(\tau_1) + \int_{2m\pi + \psi_1}^{\varphi_{high}} (\sin \varphi - \beta - d) d\varphi. \end{aligned} \quad (5.41)$$

To derive the last inequality, we have used the convexity of the set Δ_a . Since the integral term in (5.41) is positive it follows that if $\varphi_{high} \neq 2m\pi + \psi_1$ then

$$\varphi_{high} - \psi_1 - 2m\pi < \frac{\dot{\psi}_p(\tau_1)}{\alpha} \leq \frac{v_a}{\alpha}. \quad (5.42)$$

Next assume that $\varphi_{low} \neq 2m\pi + \psi_2$. Doing a similar argument as we did to derive (5.42), but now working on the curve Γ_b instead of Γ_a , and using the convexity of Δ_b , we get that

$$2m\pi + \psi_2 - \varphi_{low} < \frac{v_b}{\alpha}. \quad (5.43)$$

Finally, (5.36) follows by adding (5.42) and (5.43), using (5.33)-(5.35). The inequalities (5.37) follow immediately from (5.42) and (5.43), using (5.33)-(5.34). \square

Lemma 5.13. *We use the assumptions and the notation of Definition 5.8. For each even $k \in \mathbb{N}$, let Δ_k be the closure of the set encircled by the curves Γ_{k-1} , Γ_k and the line L_k joining the points $(\varphi^{k-2}, 0)$ and $(\varphi^k, 0)$. Then for any solution ψ of (5.1), if $(\psi(0), \dot{\psi}(0)) \in \Delta_k$, then $(\psi(t), \dot{\psi}(t)) \in \Delta_k$ for all $t \geq 0$.*

Proof. Fix k and let ψ be a solution of (5.1) with $(\psi(0), \dot{\psi}(0)) \in \Delta_k$. It follows from Lemma 5.1 that the curve $(\psi, \dot{\psi})$ cannot go out of Δ_k by crossing the curve Γ_{k-1} and from Lemma 5.2 that it cannot go out of Δ_k by crossing Γ_k . It is easy to check that the curve $(\psi, \dot{\psi})$ cannot escape through L_k , because if it is on L_k , then the velocity $\dot{\psi}$ along the curve starts to increase, forcing the curve to stay within Δ_k . \square

Theorem 5.14. *Using the notation from (5.2), suppose that α satisfies (5.16). Then for every solution ψ of (5.1), there exists a $T > 0$ such that for any $t_1, t_2 > T$,*

$$|\psi(t_1) - \psi(t_2)| < \psi_1 - \psi_2 + \frac{4d}{\alpha^2}. \quad (5.44)$$

Moreover, for some integer m and each $t > T$, one of the following expressions hold:

$$2m\pi + \psi_2 - \frac{4d}{\alpha^2} < \psi(t) < 2m\pi + \psi_1 + \frac{4d}{\alpha^2}, \quad (5.45)$$

$$(2m+1)\pi - \psi_1 < \psi(t) < (2m+1)\pi - \psi_2. \quad (5.46)$$

Proof. We call a solution ψ of (5.1) *oscillating* if for each $T > 0$ there exists $t_1, t_2 > T$ such that $\dot{\psi}(t_1) > 0$ and $\dot{\psi}(t_2) < 0$.

First consider the case when the solution ψ is not oscillating (hence it is eventually non-increasing or non-decreasing). It then follows from Proposition 5.4 that ψ must

remain bounded and hence it converges to a finite limit, $\lim_{t \rightarrow \infty} \psi(t) = \psi_\infty$, which trivially implies (5.44). It follows from (5.1) and (5.3) that $\ddot{\psi}$ is a continuous bounded function of time. So we can apply Barbălat's lemma (see [9, 17, 21]) to $\dot{\psi}$ to conclude that $\lim_{t \rightarrow \infty} \dot{\psi}(t) = 0$. Using this and taking upper and lower limits in (5.1), we get

$$\begin{aligned}\limsup_{t \rightarrow \infty} \ddot{\psi}(t) &= \beta + \limsup_{t \rightarrow \infty} \gamma(t) - \sin \psi_\infty \geq 0, \\ \liminf_{t \rightarrow \infty} \ddot{\psi}(t) &= \beta + \liminf_{t \rightarrow \infty} \gamma(t) - \sin \psi_\infty \leq 0,\end{aligned}$$

whence $\beta - d < \beta + \liminf_{t \rightarrow \infty} \gamma(t) \leq \sin \psi_\infty \leq \beta + \limsup_{t \rightarrow \infty} \gamma(t) < \beta + d$. In short, $\sin \psi_1 < \sin \psi_\infty < \sin \psi_2$. This implies that either (5.45) or (5.46) hold.

Next suppose that ψ is oscillating. We claim that ψ will eventually be captured in one of two types of bounded intervals. Specifically, one of the following holds:

- (i) For some $m \in \mathbb{Z}$ and all t large enough, $\psi(t) \in ((2m-1)\pi - \psi_2, (2m+1)\pi - \psi_1)$ and moreover, $(\psi(t), \dot{\psi}(t)) \in \Delta_2$.
- (ii) For some $m \in \mathbb{Z}$ and all t large enough, $\psi(t) \in [(2m-1)\pi - \psi_1, (2m-1)\pi - \psi_2]$.

To prove the above claim, suppose that (ii) does not hold. Then for any $T > 0$ there must exist times $\tau_2 > \tau_1 > T$ such that $\dot{\psi}(\tau_1) = 0$, $\dot{\psi}(t) \neq 0$ for $t \in (\tau_1, \tau_2]$ and $\psi(\tau_2) \notin [(2m-1)\pi - \psi_1, (2m-1)\pi - \psi_2]$, $\forall m \in \mathbb{Z}$. Without loss of generality, we may assume that $\dot{\psi}(\tau_2) > 0$, since for $\dot{\psi}(\tau_2) < 0$ the argument is similar. Then $\dot{\psi}(t) > 0$ for all $t \in (\tau_1, \tau_2]$, whence $\ddot{\psi}(\tau_1) \geq 0$ and $\psi(\tau_2) > \psi(\tau_1)$.

Since $\dot{\psi}(\tau_1) = 0$ and $\ddot{\psi}(\tau_1) \geq 0$, from (5.1) we get $\sin \psi_1 > \sin(\psi(\tau_1))$, so that $\psi(\tau_1) \in ((2m-1)\pi - \psi_1, 2m\pi + \psi_1)$ for some $m \in \mathbb{Z}$. Therefore $\psi(\tau_2) > (2m-1)\pi - \psi_2$ and $(\psi(\tau_1), 0)$ is a positive acceleration point. Let $\tau_3 = \min\{t > \tau_1 \mid \dot{\psi}(t) = 0\}$. Then $\ddot{\psi}(\tau_3) \leq 0$, which using (5.1) implies that $\sin(\psi(\tau_3)) > \sin \psi_2$. Since $\psi(\tau_3) > \psi(\tau_2)$, this implies that $\psi(\tau_3) > 2m\pi + \psi_2$. Let ψ_p be the solution of (5.5) when $(\psi_p(0), \dot{\psi}_p(0)) = (\psi(\tau_1), 0)$. From Lemma 5.1 we have $\dot{\psi}_p|_{\psi_p=\psi(t)} > \dot{\psi}(t)$ for each $t \in (\tau_1, \tau_3)$. This and the fact that the first negative acceleration point $(\varphi, 0)$ for $(\psi_p, \dot{\psi}_p)$ is such that $\varphi < (2m+1)\pi - |\psi_1|$ (see Lemma 5.6) imply that $\psi(\tau_3) \in (2m\pi + \psi_2, (2m+1)\pi - |\psi_1|)$. We now have to consider two cases:

Case (a) If $\psi(\tau_3) \in (2m\pi + \psi_2, 2m\pi + \psi_1]$ then, since the line joining the points $(2m\pi + \psi_2, 0)$ and $(2m\pi + \psi_1, 0)$ in the phase plane is contained in Δ_2 , we get that $(\psi(\tau_3), 0) \in \Delta_2$. According to Lemma 5.13, $(\psi(t), \dot{\psi}(t))$ remains in Δ_2 for all $t \geq \tau_3$, and moreover $(\psi(t), \dot{\psi}(t))$ cannot reach the corner $(\varphi^0, 0) \in \Delta_2$, so that (i) holds.

Case (b) If $\psi(\tau_3) \in (2m\pi + \psi_1, (2m+1)\pi - |\psi_1|)$, then from (5.1) we get that $\ddot{\psi}(\tau_3) < 0$. In this case, we have to do one more iteration: Denote $\tau_4 = \min\{t > \tau_3 \mid \dot{\psi}(t) = 0\}$, then $\dot{\psi}(t) < 0$ for all $t \in (\tau_3, \tau_4)$. By a reasoning similar to the one used before case (a), using Lemmas 5.2 and 5.7, we get that $\psi(\tau_4) \in ((2m-1)\pi + |\psi_2|, 2m\pi + \psi_1)$, so that $(\psi(\tau_4), 0) \in \Delta_2$. By the same argument as employed in case (a), this implies that (i) holds. Thus, we have proved our claim.

First we consider the case when (ii) holds. Then actually $\psi(t) \in ((2m-1)\pi - \psi_1, (2m-1)\pi - \psi_2)$ for some $m \in \mathbb{Z}$ and all t large enough. Indeed, for t large enough, $\psi(t)$ can no longer reach the endpoints of the interval in (ii), because at the

endpoints we would have $\dot{\psi}(t) = 0$ and from (5.1) we see that the acceleration $\ddot{\psi}(t)$ would force $\psi(t)$ to leave the interval. Now (5.44) and (5.46) follow trivially.

Now we consider the case when (i) holds. We show by induction that given any even $k \in \mathbb{N}$, there exists a $\tau_k > 0$ such that $(\psi(t), \dot{\psi}(t)) \in \Delta_k$ for all $t \geq \tau_k$. This, along with Lemma 5.9 and 5.12 will imply that (5.44) and (5.45) hold.

Assume that for some even $k \in \mathbb{N}$ and some $\tau_k > 0$, $(\psi(t), \dot{\psi}(t)) \in \Delta_k$ for all $t \geq \tau_k$. Let $T > \tau_k$ be such that $\dot{\psi}(T) < 0$. Define $\tau_{k+2} = \min\{t > T \mid \dot{\psi}(t) = 0\}$. Then $\ddot{\psi}(\tau_{k+2}) \geq 0$ and since $(\psi(\tau_{k+2}), \dot{\psi}(\tau_{k+2})) \in \Delta_k$, it is easy to see that $\psi(\tau_{k+2}) \in [\varphi^k, 2m\pi + \psi_1]$ (the upper bound follows from (5.1)). So $(\psi(\tau_{k+2}), \dot{\psi}(\tau_{k+2})) \in \Delta_{k+2}$ and (from Lemma 5.13) $(\psi(t), \dot{\psi}(t)) \in \Delta_{k+2}$ for all $t \geq \tau_{k+2}$. \square

Remark 5.15. The bounds for $|\psi(t_1) - \psi(t_2)|$ and $\psi(t)$ in (5.44), (5.45) and (5.46) depend only on α , β and d and they are independent of the initial state $(\psi(0), \dot{\psi}(0))$ of (5.1). Therefore, even if d satisfies the less restrictive inequality $\limsup |\gamma(t)| < d$ (instead of $\|\gamma\|_{L^\infty} < d$), Theorem 5.14 continues to hold.

Remark 5.16. In the pendulum equation (5.1), γ can be viewed as a bounded disturbance. Often, bounds like (5.45) that characterize the asymptotic response of dynamical systems driven by bounded disturbances are derived using Lyapunov functions. Using Proposition 5.4 and Lemmas 5.6 and 5.7 it can be shown that all the solutions of (5.1) (with the angles measured modulo 2π) are eventually in a bounded region Ω of the phase plane. Then a Lyapunov function V for (5.1) which is positive-definite on Ω and for which \dot{V} , evaluated along the solutions of (5.1) with $\gamma = 0$, is negative-definite on Ω can be constructed (see [17, Example 4.4] for a Lyapunov function that can be used when $\beta = 0$). Using V a bound like (5.45) can be derived for a given α , β and d . The main problem with this approach is that it is hard to express the bounds thus derived as simple functions of α , β and d . This makes it difficult, not only to state the main result of this paper concisely using them, but also to verify the sufficient conditions in the main result.

6. Stability of the SG connected to the bus

In this section we derive sufficient conditions for the SG parameters under which the system (2.9) is almost globally asymptotically stable. We obtain these conditions by applying the asymptotic bounds derived in Section 5 for the forced pendulum equation to the exact swing equation (ESE) in (4.7). To this end, we first write the ESE in the standard form for forced pendulum equations shown in (5.1). Recall i_v from (4.4) and $p = R_s/L_s$. Define V_r, ρ and P_∞ (all > 0) as follows:

$$V_r = \frac{mi_f}{L_s i_v}, \quad \rho = \sqrt{\frac{J}{mi_f i_v}}, \quad P_\infty = \frac{p\omega_g}{\omega_g^2 + p^2}. \quad (6.1)$$

It will be useful to note that

$$p\rho \lim_{s \rightarrow \infty} \int_0^s e^{-p\rho(s-\tau)} \sin(\rho\omega_g(s-\tau)) d\tau = p\rho \int_0^\infty e^{-p\rho\sigma} \sin(\rho\omega_g\sigma) d\sigma = P_\infty.$$

We also introduce the constants

$$\alpha = \frac{D_p}{\sqrt{m i_f i_v J}}, \quad \beta = \frac{T_m - D_p \omega_g}{m i_f i_v} - V_r P_\infty. \quad (6.2)$$

Consider the new time variable $s = t/\rho$ and new angle variable $\psi(s) = \eta(\rho s)$. In terms of these variables, ESE has the following representation, equivalent to (4.7):

$$\psi''(s) + \alpha \psi'(s) + \sin \psi(s) = \beta + \gamma(s), \quad (6.3)$$

$$\gamma(s) = \frac{e^{-p\rho s} f(\rho s)}{i_v} + V_r P_\infty - V_r P(s), \quad (6.4)$$

$$P(s) = p\rho \int_0^s e^{-p\rho(s-\tau)} \sin[\psi(s) - \psi(\tau) + \rho\omega_g(s-\tau)] d\tau, \quad (6.5)$$

where f is the bounded function defined in (4.5) using the function ω and the initial conditions $i_d(0)$, $i_q(0)$ and $\delta(0)$. In turn, ω depends on ψ as $\omega(t) = (1/\rho)\psi'(t/\rho) + \omega_g$ (according to (2.8) and (4.6)). The global existence of a unique solution to the system of integro-differential equations (6.3)-(6.5) together with (4.5), for any initial conditions $\psi(0)$, $\psi'(0)$, $i_d(0)$ and $i_q(0)$, follows from the global existence of unique solutions to (4.7) (see also the discussion below (4.7)).

Consider a solution $(i_d, i_q, \omega, \delta)$ of (2.9) and the corresponding solution $\psi(s) = (3\pi/2) + \delta(\rho s) + \phi$ of (6.3), with f as in (4.5), γ as in (6.4) and P as in (6.5). Clearly f is bounded and $|P(s)| < 1$, and both f and P are continuous functions. Therefore γ is a bounded continuous function and ψ is the corresponding solution of (6.3) regarded as a forced pendulum equation. So the bounds in Theorem 5.14, developed for solutions of forced pendulum equations, can be used to obtain asymptotic bounds for ψ in terms of any $d > 0$ that satisfies $\limsup |\gamma(s)| < d$ (see Remark 5.15) and $|\beta| + d < 1$. Using these asymptotic bounds, Theorem 6.3 shows that under some conditions on the SG parameters $\limsup |\gamma(s)| = 0$. This implies that (ψ, ψ') converges to a limit point, using which we can conclude that $(i_d, i_q, \omega, \delta)$ converges to an equilibrium point of (2.9). Since the stability conditions in Theorem 6.3 are independent of the initial state of (2.9) we get that, whenever these conditions hold, every solution of (2.9) converges to an equilibrium point.

We briefly explain the idea behind the stability conditions in Theorem 6.3. Let

$$\Gamma = (1 + P_\infty)V_r. \quad (6.6)$$

It is easy to see from (6.4) and (6.5) that $\limsup |\gamma(s)| \leq \Gamma$. From (6.5) we get that

$$\begin{aligned} P(s) &= p\rho \int_0^s e^{-p\rho(s-\tau)} \sin\left(\int_\tau^s \omega_\rho(\sigma) d\sigma\right) d\tau \\ \implies P(s) &= p\rho \int_0^s e^{-p\rho\tau} \sin\left(\int_0^\tau \omega_\rho(s-\sigma) d\sigma\right) d\tau, \end{aligned} \quad (6.7)$$

where $\omega_\rho(s) = \rho\omega(\rho s)$, so that $\omega_\rho(\sigma) = \psi'(\sigma) + \rho\omega_g$. We define a function $\mathcal{N} : (0, \Gamma] \rightarrow [0, \infty)$ as follows. Fix $d \in (0, \Gamma]$. Suppose that $\limsup |\gamma(s)| < d$. Using

the asymptotic bounds in Section 5 and Lemma 6.1 below, we derive upper and lower bounds for $\psi'(s)$ that are valid for large s . Using these bounds, the expression $\omega_\rho(s) = \psi'(s) + \rho\omega_g$ and (6.7), we derive an upper bound P_u^d and a lower bound P_l^d for $P(s)$, which is again valid for large s (this step uses Lemma 6.2). Finally we define $\mathcal{N}(d) = V_r \max\{P_u^d - P_\infty, P_\infty - P_l^d\}$. It is clear from (6.4) that $\limsup |\gamma(s)| \leq \mathcal{N}(d)$. Our stability condition is $\mathcal{N}(d) < d$ for all $d \in (0, \Gamma]$, from which we can conclude that $\limsup |\gamma(s)| = 0$ (see Theorem 6.3 for details).

Lemma 6.1. *Suppose $\beta \in \mathbb{R}$, $d > 0$ and $\alpha > 0$. If $|\beta| + d < 1$ and (5.16) holds, where ψ_1, ψ_2 are as in (5.2), then define $\phi_1, \phi_2 \in [-\pi/2, \pi/2]$ by*

$$\phi_1 = \min \left\{ \frac{\pi}{2}, \psi_1 + \frac{4d}{\alpha^2} \right\}, \quad \phi_2 = \max \left\{ -\frac{\pi}{2}, \psi_2 - \frac{4d}{\alpha^2} \right\}$$

and let $S_n = -\sin \phi_1$ and $S_p = -\sin \phi_2$. In all other cases let $S_n = -1$ and $S_p = 1$. Define

$$\omega_n = \frac{S_n + \beta - d}{\alpha}, \quad \omega_p = \frac{S_p + \beta + d}{\alpha}. \quad (6.8)$$

Assume that $\gamma : [0, \infty) \rightarrow \mathbb{R}$ is continuous, $\limsup |\gamma(s)| < d$ and ψ is a corresponding solution of (6.3). Then for some $T > 0$ we have

$$\omega_n < \psi'(s) < \omega_p \quad \forall s \geq T. \quad (6.9)$$

Proof. We claim that there exists $T_0 > 0$ such that

$$S_n + \beta - d < -\sin \psi(s) + \beta + \gamma(s) < S_p + \beta + d \quad \forall s \geq T_0. \quad (6.10)$$

Let $\tau > 0$ be such that $|\gamma(s)| < d$ for all $s > \tau$. When $|\beta| + d \geq 1$ or when (5.16) is false, then it is easy to see that (6.10) holds with $T_0 = \tau$. When $|\beta| + d < 1$ and (5.16) holds, then it follows by applying Theorem 5.14 and Remark 5.15 to (6.3) that there exists $T_1 > 0$ such that for all $s \geq T_1$ either (5.45) or (5.46) holds (with s in place of t). This implies that (6.10) holds with $T_0 = \max\{\tau, T_1\}$. By regarding (6.3) as a stable first order linear dynamical system with state ψ' and external forcing $-\sin \psi(s) + \beta + \gamma(s)$, it can be easily verified using (6.10) that (6.9) holds. \square

The next lemma derives an upper bound and a lower bound for $P(s)$, given an upper and lower bound for ω_ρ .

Lemma 6.2. *Suppose that there exist constants $\omega_{\max} > 0$ and $\omega_{\min} > 0$ such that*

$$\omega_{\min} \leq \omega_{\max} \leq 2\omega_{\min} \quad \text{and} \quad \omega_{\min} \leq \omega_\rho(\sigma) \leq \omega_{\max} \quad (6.11)$$

for all $\sigma \geq 0$. Let $T_{\max} = 2\pi/\omega_{\max}$ and $T_{\min} = 2\pi/\omega_{\min}$. Define the function g on the interval $[0, T_{\max}]$ and the function h on the interval $[0, T_{\min}]$ as follows:

$$g(\tau) = \begin{cases} \sin(\omega_{\max}\tau) & \text{if } 0 \leq \tau < \frac{\pi}{2\omega_{\max}} \\ 1 & \text{if } \frac{\pi}{2\omega_{\max}} \leq \tau < \frac{\pi}{2\omega_{\min}} \\ \sin(\omega_{\min}\tau) & \text{if } \frac{\pi}{2\omega_{\min}} \leq \tau < \frac{3\pi}{\omega_{\min} + \omega_{\max}} \\ \sin(\omega_{\max}\tau) & \text{if } \frac{3\pi}{\omega_{\min} + \omega_{\max}} \leq \tau \leq \frac{2\pi}{\omega_{\max}} \end{cases}, \quad (6.12)$$

$$h(\tau) = \begin{cases} \sin(\omega_{\min}\tau) & \text{if } 0 \leq \tau < \frac{\pi}{\omega_{\min} + \omega_{\max}} \\ \sin(\omega_{\max}\tau) & \text{if } \frac{\pi}{\omega_{\min} + \omega_{\max}} \leq \tau < \frac{3\pi}{2\omega_{\max}} \\ -1 & \text{if } \frac{3\pi}{2\omega_{\max}} \leq \tau < \frac{3\pi}{2\omega_{\min}} \\ \sin(\omega_{\min}\tau) & \text{if } \frac{3\pi}{2\omega_{\min}} \leq \tau \leq \frac{2\pi}{\omega_{\min}} \end{cases}. \quad (6.13)$$

Define $T = T_{\max}$ if $\int_0^{T_{\min}} e^{-p\rho\tau} h(\tau) d\tau < 0$ and $T = T_{\min}$ otherwise. Then for any $s \geq 0$, $P(s)$ from (6.7) satisfies the following bounds:

$$P(s) \leq \frac{p\rho}{1 - e^{-p\rho T_{\max}}} \int_0^{T_{\max}} e^{-p\rho\tau} g(\tau) d\tau + e^{-p\rho(s - T_{\min})}, \quad (6.14)$$

$$P(s) \geq \frac{p\rho}{1 - e^{-p\rho T}} \int_0^{T_{\min}} e^{-p\rho\tau} h(\tau) d\tau - e^{-p\rho(s - T_{\min})}. \quad (6.15)$$

Proof. Using the assumption $0 < \omega_{\min} \leq \omega_{\max} \leq 2\omega_{\min}$, it is easy to verify that

$$\frac{\pi}{2\omega_{\max}} \leq \frac{\pi}{2\omega_{\min}} \leq \frac{3\pi}{\omega_{\min} + \omega_{\max}} \leq \frac{2\pi}{\omega_{\max}}, \quad \frac{\pi}{\omega_{\min} + \omega_{\max}} < \frac{3\pi}{2\omega_{\max}} \leq \frac{3\pi}{2\omega_{\min}} < \frac{2\pi}{\omega_{\min}}.$$

This means that g and h in (6.12) and (6.13) are defined precisely on the intervals $[0, T_{\max}]$ and $[0, T_{\min}]$. Given $s \geq 0$, fix $0 = \tau_0 < \tau_1 < \dots < \tau_n \leq s$ such that

$$\int_0^{\tau_k} \omega_\rho(s - \sigma) d\sigma = 2k\pi \quad \forall k \in \{1, 2, \dots, n\} \quad \text{and} \quad \int_{\tau_n}^s \omega_\rho(s - \sigma) d\sigma < 2\pi.$$

From (6.11) we get that for each $0 \leq k \leq n$ and all $\tau_k \leq \tau \leq s$,

$$\omega_{\min}(\tau - \tau_k) \leq \int_{\tau_k}^{\tau} \omega_\rho(s - \sigma) d\sigma \leq \omega_{\max}(\tau - \tau_k). \quad (6.16)$$

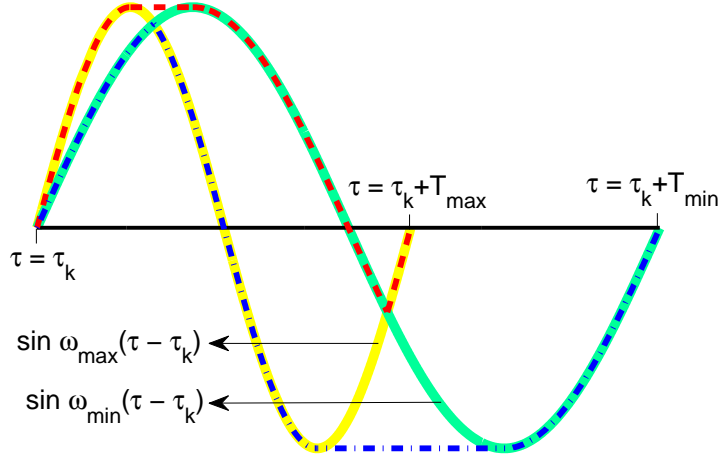


Figure 5. The function $\sin(\omega_{\max}(\tau - \tau_k))$ on the interval $[\tau_k, \tau_k + T_{\max}]$ is plotted in yellow, while $\sin(\omega_{\min}(\tau - \tau_k))$ on the interval $[\tau_k, \tau_k + T_{\min}]$ is plotted in green. Here $T_{\max} = 2\pi/\omega_{\max}$ and $T_{\min} = 2\pi/\omega_{\min}$. The dashed line in red is the function g_k used in the proof of Lemma 6.2 to obtain an upper bound for $P(s)$. The dash-dot line in blue is the function h_k used in the same proof to obtain a lower bound for $P(s)$.

For each $k < n$, by letting $\tau = \tau_{k+1}$ in (6.16) it follows from the second inequality that $\tau_{k+1} - \tau_k \geq T_{\max}$ and from the first inequality that $\tau_{k+1} - \tau_k \leq T_{\min}$. Therefore

$$kT_{\max} \leq \tau_k \leq kT_{\min} \quad \forall k \in \{1, 2, \dots, n\}. \quad (6.17)$$

From (6.7) we have

$$\begin{aligned} P(s) &= p\rho \sum_{k=1}^n \int_{\tau_{k-1}}^{\tau_k} e^{-p\rho\tau} \sin \left(\int_{\tau_{k-1}}^{\tau} \omega_\rho(s - \sigma) d\sigma \right) d\tau, \\ &\quad + p\rho \int_{\tau_n}^s e^{-p\rho\tau} \sin \left(\int_{\tau_n}^{\tau} \omega_\rho(s - \sigma) d\sigma \right) d\tau. \end{aligned} \quad (6.18)$$

For each $k \in \{0, 1, \dots, n\}$, define the function g_k on the interval $[\tau_k, \tau_k + T_{\max}]$ by $g_k(\tau) = g(\tau - \tau_k)$ (see Figure 5). Using $\omega_{\min} \leq \omega_{\max} \leq 2\omega_{\min}$ and (6.16) it can be verified that for each $k \in \{0, 1, \dots, n\}$ and all $\tau_k \leq \tau \leq \min\{\tau_k + T_{\max}, s\}$,

$$\sin \left(\int_{\tau_k}^{\tau} \omega_\rho(s - \sigma) d\sigma \right) d\tau \leq g_k(\tau) \quad (6.19)$$

and when $\min\{\tau_k + T_{\max}, s\} < \tau \leq \min\{\tau_{k+1}, s\}$ (if $k = n$, then let $\tau_{k+1} = s$)

$$\sin \left(\int_{\tau_k}^{\tau} \omega_\rho(s - \sigma) d\sigma \right) d\tau \leq 0. \quad (6.20)$$

Using (6.19) and (6.20), we obtain from (6.18) that

$$\begin{aligned} P(s) &\leq p\rho \sum_{k=1}^n \int_{\tau_{k-1}}^{\tau_{k-1} + T_{\max}} e^{-p\rho\tau} g_{k-1}(\tau) d\tau + p\rho \int_{\tau_n}^{\min\{s, \tau_n + T_{\max}\}} e^{-p\rho\tau} g_n(\tau) d\tau \\ &= p\rho \sum_{k=1}^n e^{-p\rho\tau_{k-1}} \int_0^{T_{\max}} e^{-p\rho\tau} g(\tau) d\tau + p\rho e^{-p\rho\tau_n} \int_0^{\min\{s - \tau_n, T_{\max}\}} e^{-p\rho\tau} g(\tau) d\tau. \end{aligned}$$

By letting $\tau = s$ and $k = n$ in (6.16), we get that $s - T_{\min} < \tau_n$. This, and the inequality $|g(\tau)| \leq 1$ for all $\tau \in [0, T_{\max}]$, means that the second term on the right side of the above expression can be bounded in absolute value by $e^{-p\rho(s - T_{\min})}$. From the above expression, using the easily verifiable fact $\int_0^{T_{\max}} e^{-p\rho\tau} g(\tau) d\tau > 0$ and the inequalities in (6.17), the upper bound in (6.14) follows.

For each $k \in \{0, 1, \dots, n\}$, define the function h_k on the interval $[\tau_k, \tau_k + T_{\min}]$ by $h_k(\tau) = h(\tau - \tau_k)$ (see Figure 5). Using $\omega_{\min} \leq \omega_{\max} \leq 2\omega_{\min}$ and (6.16) it can be verified that for each $k \in \{0, 1, \dots, n\}$ and all $\tau_k \leq \tau \leq \min\{\tau_{k+1}, s\}$ (if $k = n$, then let $\tau_{k+1} = s$),

$$\sin \left(\int_{\tau_k}^{\tau} \omega_\rho(s - \sigma) d\sigma \right) d\tau \geq h_k(\tau) \quad (6.21)$$

and when $\min\{\tau_{k+1}, s\} < \tau \leq \min\{\tau_k + T_{\min}, s\}$, $h_k(\tau) \leq 0$. Using this and (6.21), we obtain from (6.18) that

$$\begin{aligned} P(s) &\geq p\rho \sum_{k=1}^n \int_{\tau_{k-1}}^{\tau_{k-1} + T_{\min}} e^{-p\rho\tau} h_{k-1}(\tau) d\tau + p\rho \int_{\tau_n}^s e^{-p\rho\tau} h_n(\tau) d\tau \\ &= p\rho \sum_{k=1}^n e^{-p\rho\tau_{k-1}} \int_0^{T_{\min}} e^{-p\rho\tau} h(\tau) d\tau + p\rho e^{-p\rho\tau_n} \int_0^{s - \tau_n} e^{-p\rho\tau} h(\tau) d\tau. \end{aligned}$$

Using $s - T_{\min} < \tau_n$ (shown earlier) and the inequality $|h(\tau)| \leq 1$ for all $\tau \in [0, T_{\min}]$, it is easy to see that the second term on the right side of the above expression can be bounded in absolute value by $e^{-p\rho(s-T_{\min})}$. From the above expression, using the inequalities in (6.17), the lower bound in (6.15) follows. \square

The next theorem is the main result of this paper. It presents checkable conditions for the almost global asymptotic stability of the SG model (2.9). Recall the notations V_r and P_∞ from (6.1), Γ from (6.6) and $p = R_s/L_s$. As discussed earlier, the conditions are specified in terms of a nonlinear map $\mathcal{N} : (0, \Gamma] \rightarrow [0, \infty)$. We show that if the graph of this map is below the graph of $F(x) = x$, then the SG model in (2.9) is almost globally asymptotically stable.

Theorem 6.3. *Consider the SG model (2.9). For each $d \in (0, \Gamma]$ let $\omega_{\max}^d = \omega_p + \rho\omega_g$ and $\omega_{\min}^d = \omega_n + \rho\omega_g$, where ω_p and ω_n are obtained from (6.8) using α and β given in (6.2) and ρ is given by (6.1). Then $\omega_{\min}^d < \omega_{\max}^d$ for all d . If $\omega_{\max}^d \leq 2\omega_{\min}^d$, then recall the functions g and h from (6.12) and (6.13), where we take $\omega_{\max} = \omega_{\max}^d$ and $\omega_{\min} = \omega_{\min}^d$ so that $T_{\max} = 2\pi/\omega_{\max}^d$ and $T_{\min} = 2\pi/\omega_{\min}^d$, and define the numbers*

$$P_u^d = \frac{p\rho}{1 - e^{-p\rho T_{\max}}} \int_0^{T_{\max}} e^{-p\rho\tau} g(\tau) d\tau, \quad P_l^d = \frac{p\rho}{1 - e^{-p\rho T}} \int_0^{T_{\min}} e^{-p\rho\tau} h(\tau) d\tau,$$

where $T = T_{\max}$ if $\int_0^{T_{\min}} e^{-p\rho\tau} h(\tau) d\tau < 0$ and $T = T_{\min}$ otherwise. If $\omega_{\max}^d > 2\omega_{\min}^d$, then let $P_u^d = 1$ and $P_l^d = 0$. Define the map $\mathcal{N} : (0, \Gamma] \rightarrow \mathbb{R}$ by

$$\mathcal{N}(d) = V_r \max\{P_u^d - P_\infty, P_\infty - P_l^d\}. \quad (6.22)$$

If $\mathcal{N}(d) < d$ and $\omega_{\max}^d \leq 2\omega_{\min}^d$ for each $d \in (0, \Gamma]$, then every trajectory of the SG model (2.9) converges to an equilibrium point. In addition, if all the equilibrium points of (2.9) are hyperbolic, then (2.9) is almost globally asymptotically stable.

Proof. Throughout this proof we assume that $\mathcal{N}(d) < d$ and $\omega_{\max}^d \leq 2\omega_{\min}^d$ for each $d \in (0, \Gamma]$ and that α and β are as in the theorem. First we claim that for any solution $(i_d, i_q, \omega, \delta)$ of (2.9), if the corresponding solution ψ of (6.3) is such that γ in (6.4) satisfies $\limsup |\gamma(s)| = 0$, then $(i_d, i_q, \omega, \delta)$ converges to an equilibrium point. Indeed, for each small d , $\mathcal{N}(d) < d$ implies that $|P_u^d - P_l^d|$ and hence (using the definitions of g and h and the fact that $\omega_{\min}^d < \omega_{\max}^d \leq (1 + |\beta| + d)/\alpha + \rho\omega_g$) $\omega_{\max}^d - \omega_{\min}^d$ are both small, proportional to d . This, and the observation using (6.8) that if (5.16) does not hold for a d , then $\omega_{\max}^d - \omega_{\min}^d > 2/\alpha$, implies that (5.16) holds for all sufficiently small d . Therefore, given a ψ as above satisfying $\limsup |\gamma(s)| = 0$, we can apply Theorem 5.14 and Remark 5.15 to (6.3) and conclude that for every d sufficiently small there exists a $T_d > 0$ such that

$$|\psi(s_1) - \psi(s_2)| < \psi_1 - \psi_2 + 4d/\alpha^2 \quad \forall s_1, s_2 > T_d. \quad (6.23)$$

Here ψ_1 and ψ_2 are as in (5.2). Since $d > 0$ can be arbitrarily small, (6.23) implies that $\lim_{s \rightarrow \infty} \psi(s) = \psi_l$ for some finite ψ_l . It follows from (6.3) that, since $|\psi'(s)| \leq |\psi'(0)| + (|\beta| + \Gamma + 1)/\alpha + \|f\|_{L^\infty}/(i_v \alpha)$ for all $s \geq 0$ (by the argument in (5.3)), ψ'' is a continuous bounded function of time. We can therefore apply Barb al at's

lemma to ψ' to conclude that $\lim_{s \rightarrow \infty} \psi'(s) = 0$. Since $\eta(\rho s) = \psi(s)$ for all $s \geq 0$ by definition, it follows using (4.6) that $\lim_{t \rightarrow \infty} \delta(t) = \delta_l = \psi_l - \phi - 3\pi/2$ and $\lim_{t \rightarrow \infty} \dot{\delta}(t) = 0$. This, along with (2.8), gives that $\lim_{t \rightarrow \infty} \omega(t) = \omega_g$. Note that both i_d and i_q are bounded functions on $[0, \infty)$. This follows from the discussion at the beginning of Section 3 (if either $|i_d(t)|$, $|i_q(t)|$ or $|\omega(t)|$ is sufficiently large, then $\dot{W}(t) < 0$ which ensures that i_d , i_q and ω are bounded). Hence (4.1) can be rewritten as a second-order exponentially stable linear system driven by an input which is a sum of a constant vector and a vanishing vector as follows:

$$\begin{aligned} \begin{bmatrix} \dot{i}_d \\ \dot{i}_q \end{bmatrix} &= \begin{bmatrix} -p & \omega_g \\ -\omega_g & -p \end{bmatrix} \begin{bmatrix} i_d \\ i_q \end{bmatrix} + \frac{1}{L_s} \begin{bmatrix} V \sin \delta_l \\ V \cos \delta_l - m i_f \omega_g \end{bmatrix} \\ &+ \frac{1}{L_s} \begin{bmatrix} L_s(\omega - \omega_g)i_q + V(\sin \delta - \sin \delta_l) \\ -L_s(\omega - \omega_g)i_d + V(\cos \delta - \cos \delta_l) - m i_f(\omega - \omega_g) \end{bmatrix}. \end{aligned}$$

The second term on the right side of the above equation is a constant, while the third term decays to zero asymptotically (because $\lim_{t \rightarrow \infty} \omega(t) = \omega_g$ and $\lim_{t \rightarrow \infty} \delta(t) = \delta_l$). This means that i_d and i_q converge to some constants $i_{d,l}$ and $i_{q,l}$, i.e.

$$\lim_{t \rightarrow \infty} (i_d(t), i_q(t), \omega(t), \delta(t)) = (i_{d,l}, i_{q,l}, \omega_g, \delta_l).$$

It is now easy to verify using (2.9) that $(\ddot{i}_d, \ddot{i}_q, \ddot{\omega}, \ddot{\delta})$ are bounded continuous functions. Therefore, by applying Barbălat's lemma to $(\dot{i}_d, \dot{i}_q, \dot{\omega}, \dot{\delta})$, we can conclude that $\lim_{t \rightarrow \infty} (\dot{i}_d(t), \dot{i}_q(t), \dot{\omega}(t), \dot{\delta}(t)) = 0$ and so $(i_{d,l}, i_{q,l}, \omega_g, \delta_l)$ is an equilibrium point for (2.9). This completes the proof of our claim.

Next we show that for each solution $(i_d, i_q, \omega, \delta)$ of (2.9), the corresponding solution ψ of (6.3) is such that γ in (6.4) satisfies $\limsup |\gamma(s)| = 0$. This and the claim established above imply that every trajectory of (2.9) converges to an equilibrium point which in turn implies, using Lemma 3.5, that (2.9) is almost globally asymptotically stable whenever all its equilibrium points are hyperbolic. Below, we will use the fact that the nonlinear function \mathcal{N} is right-continuous if $\omega_{\max}^d \leq 2\omega_{\min}^d$ for all $d \in (0, \Gamma]$. This follows from two (easily verifiable) facts: (1) P_u^d and P_l^d , which are determined by ω_{\max}^d and ω_{\min}^d using the functions g and h , depend continuously on ω_{\max}^d and ω_{\min}^d and (2) ω_p and ω_n (defined in (6.8)), and consequently ω_{\max}^d and ω_{\min}^d , are right-continuous functions of d .

Consider a solution $(i_d, i_q, \omega, \delta)$ of (2.9) and the corresponding solution ψ of (6.3). We will show that $d_0 = \limsup |\gamma(s)| = 0$. Suppose that $d_0 \neq 0$. It follows from (6.4) and (6.5) that $d_0 \leq \Gamma$. In fact, $d_0 < \Gamma$ since $\limsup |P(s)| < 1$. The latter inequality is a consequence of two simple facts: (i) the integrand in (6.7) is 0 when $\tau = 0$ and (ii) ψ' is a bounded function on $[0, \infty)$ (as stated below (6.23)) and therefore so is $\omega_\rho = \psi' + \rho\omega_g$. Fix $\varepsilon > 0$ such that $d_0 + \varepsilon < \Gamma$. Let $d = d_0 + \varepsilon$ and define $\omega_{\max} = \omega_{\max}^d$ and $\omega_{\min} = \omega_{\min}^d$. Lemma 6.1 gives that there exists $T_1 > 0$ such that for all $s \geq T_1$, $\omega_{\min} < \omega_\rho(s) < \omega_{\max}$. From (6.7) we get that for all $s \geq T_1$,

$$\begin{aligned} P(s) &= p\rho \int_0^{\tilde{s}} e^{-p\rho\tau} \sin \left(\int_0^\tau \tilde{\omega}_\rho(\tilde{s} - \sigma) d\sigma \right) d\tau \\ &+ p\rho \int_{\tilde{s}}^s e^{-p\rho\tau} \sin \left(\int_0^\tau \omega_\rho(s - \sigma) d\sigma \right) d\tau, \end{aligned} \quad (6.24)$$

where $\tilde{s} = s - T_1$ and $\tilde{\omega}_\rho(\tilde{s} - \sigma) = \omega_\rho(s - \sigma)$ for all $0 \leq \sigma \leq \tilde{s}$. The second term on the right side of (6.24) decays exponentially to zero as $s \rightarrow \infty$. Denote the first term on the right side of (6.24) by $\tilde{P}(\tilde{s})$. It is easy to see that $\omega_{\min} < \tilde{\omega}_\rho(\tau) < \omega_{\max}$ for all $0 \leq \tau \leq \tilde{s}$. Using this and the assumption $\omega_{\max} \leq 2\omega_{\min}$, we can apply the bounds (6.14) and (6.15) derived in Lemma 6.2 for $P(s)$ to $\tilde{P}(\tilde{s})$ to conclude that

$$P_l^d - e^{-p\rho(\tilde{s}-T_{\min})} \leq \tilde{P}(\tilde{s}) \leq P_u^d + e^{-p\rho(\tilde{s}-T_{\min})},$$

where $T_{\min} = 2\pi/\omega_{\min}$. This, together with (6.24), implies that $P_l^d \leq \liminf P(s) \leq \limsup P(s) \leq P_u^d$. It now follows from (6.4) that $d_0 \leq V_r \max\{P_u^d - P_\infty, P_\infty - P_l^d\} = \mathcal{N}(d_0 + \varepsilon)$. Thus we have shown that $d_0 \leq \mathcal{N}(d_0 + \varepsilon)$ for all $\varepsilon > 0$ satisfying $d_0 + \varepsilon < \Gamma$ which, due to the right-continuity of \mathcal{N} , implies that $d_0 \leq \mathcal{N}(d_0)$. If $d_0 \neq 0$, this contradicts our assumption that $\mathcal{N}(d) < d$ for all $d \in (0, \Gamma]$. Hence $d_0 = 0$. \square

In general, the conditions in the above theorem are hard to verify analytically, but it is straightforward to verify them numerically. This will be demonstrated using an example in the next section.

Remark 6.4. In Section 3 we showed that (2.9) has two sequences of equilibrium points if and only if the right side of (3.3), denoted as Λ , satisfies $|\Lambda| < 1$. It is easy to check that $\beta = -\Lambda$ (β is defined in (6.2)). The condition $\mathcal{N}(d) < d$ for all $d \in (0, \Gamma]$ in Theorem 6.3 implies that $|\beta| < 1$. Indeed, if $|\beta| \geq 1$, then irrespective of d , $\omega_p - \omega_n > 2/\alpha$ and so $\omega_{\max}^d - \omega_{\min}^d > 2/\alpha$. This means that even when we let $d \rightarrow 0$, $P_u^d - P_l^d$ will remain bounded away from 0 which, along with (6.22), implies that $\mathcal{N}(d) > d$ for small d .

Remark 6.5. In [23, Theorem 5.1] we presented a simple set of conditions, which can be easily verified analytically, under which (2.9) is aGAS. These conditions were derived using the standard form (6.3)–(6.5) of the ESE. They were stated in [23] under the assumptions that $0 < \beta < 1$, $\|\gamma\|_{L^\infty} < d < \beta$ for some $d > 0$ and $\beta + d < 1$, because in that work the asymptotic bounds for the forced pendulum equation were derived under these assumptions. In Section 5, we have derived the same asymptotic bounds under the less restrictive assumptions $|\beta| < 1$, $\limsup |\gamma(s)| < d$ and $|\beta| + d < 1$ for some $d > 0$. Hence the conclusions of [23, Theorem 5.1] continue to hold under these less restrictive assumptions as well. For the simple conditions of that theorem to hold V_r must be small (much less than 1), but for nominal SG parameters typically $V_r > 1$. Nevertheless, that theorem enabled us to identify a large range of (not necessarily practical) SG parameters for which (2.9) is aGAS. For instance, given a set of SG parameters, if we increase V by a factor of n and decrease J by the same factor, then for all n sufficiently large the simple stability conditions will hold.

For the sufficient stability conditions in Theorem 6.3 to hold, it is necessary that $|\beta| < 1$ (see Remark 6.4) and it is desirable that the damping coefficient α be large. Indeed, for any given $d > 0$ it follows from (6.8) that $|\omega_p|$ and $|\omega_n|$ are inversely proportional to α and it follows from the definitions of g , h , \mathcal{N} in (6.12), (6.13) and (6.22), respectively, that $\mathcal{N}(d)$ is proportional to $\max\{|\omega_p|, |\omega_n|\}$. Hence for

any given β and Γ with $|\beta| < 1$, if α is sufficiently large, then $\mathcal{N}(d) < d$ and $\omega_{\max}^d \leq 2\omega_{\min}^d$ for all $d \in (0, \Gamma]$, i.e. the conditions of Theorem 6.3 will hold. On an intuitive level, the need for large α and $|\beta| < 1$ for global asymptotic stability can be anticipated from the ESE (6.3). It is easy to see from (6.2) that

$$\alpha = \frac{D_p \sqrt{L_s} \sqrt[4]{p^2 + \omega_g^2}}{\sqrt{mi_f V J}}, \quad \beta = \frac{T_a L_s \sqrt{p^2 + \omega_g^2}}{mi_f V} - \frac{mi_f p \omega_g}{V \sqrt{p^2 + \omega_g^2}},$$

where, as usual, $p = R_s/L_s$ and $T_a = T_m - D_p \omega_g$ is the actual mechanical torque. The next two remarks contain suggestions for choosing the parameters of the SG model to increase α , so that the sufficient conditions of Theorem 6.3 are satisfied. These suggestions may be useful for designing a synchronverter.

Remark 6.6. Let P_n denote the nominal power of the SG. Then $T_a = P_n/\omega_g$ and so it is independent of D_p . Clearly, by increasing D_p or decreasing J , α can be increased without changing β . We can also increase α by increasing L_s , but this will result in an increase in β as well. The constraint $|\beta| < 1$ gives an upper limit for L_s . Similarly, increasing p also increases α (to a smaller extent since typically $p < \omega_g$). Again the restriction $|\beta| < 1$ imposes an upper bound on the possible values for p .

Our numerical experiments with the stability conditions of Theorem 6.3 suggest that a smaller value for $|\beta|$ is preferable. Let β_1 and β_2 be the first and second terms, respectively, in the above expression for β . For typical SG parameters $p \ll \omega_g$. Hence increasing p (in a certain range) will scale up β_2 more than it does β_1 . On the other hand, when we increase L_s , β_1 increases while β_2 remains constant. Since $\beta = \beta_1 - \beta_2$, it is possible to increase p and L_s simultaneously such that β remains small. Increasing p and L_s increases α , which is desirable.

Remark 6.7. Modifying any SG parameter other than D_p to increase α , while keeping β small, will change at least one of V_r , p and ρ . Also, when L_s is increased we must increase mi_f , for instance according to (7.1). Thus when we increase α , we cause an unintentional change in the value of the function \mathcal{N} corresponding to the variations that we induce in the values of V_r , p , ρ and mi_f . Our numerical studies indicate that the desired changes in \mathcal{N} caused by increasing α are often far more significant than these unintentional changes. Thus increasing α according to Remark 6.6 typically helps to satisfy the stability conditions of Theorem 6.3. This is demonstrated using an example in the next section.

7. Application and examples

In this section, on the basis of the results of Section 6, we propose a modification to the design of synchronverters to enhance their global stability properties (by artificially enlarging the filter inductors). Using this modification, we choose the main parameters of a 500kW synchronverter so that they satisfy the stability conditions

of Theorem 6.3, for a suitably chosen constant field current. Thus for this set of parameters (2.9) is almost globally asymptotically stable.

To select a set of nominal parameters for a synchronverter (without using our modification), we follow the empirical guidelines used in the design of SGs and commercial inverters. As usual, let ω_g be the grid frequency and let V be the line voltage so that the rms voltage on each phase is $V_{\text{rms}} = V/\sqrt{3}$. Let P_n denote the nominal active power supplied by a SG. Then the nominal active mechanical torque generated by its prime mover is $T_a = P_n/\omega_g$. Following empirical guidelines, the moment of inertia J of the SG rotor is chosen so that $(J\omega_g^2/2)/P_n$ lies between 2 and 12 seconds. The frequency droop constant D_p is selected such that if the SG rotor frequency drops below the nominal grid frequency by $d_p\%$ of ω_g , then the active power should increase by the amount P_n . Thus, $D_p = 100P_n/(d_p\omega_g^2)$ (typically $d_p \approx 3$). By definition $T_m = T_a + D_p\omega_g$. To compute L_s and R_s we assume that in steady state the stator current i and the grid voltage v are in phase. Then $P_n = 3V_{\text{rms}}I_{\text{rms}}$, where I_{rms} is the nominal rms value of the current on each phase. In commercial inverters the inductance L_s of the filter inductor is chosen so that the voltage drop across L_s , given by $L_s\omega_g I_{\text{rms}}$, is 3–5% of V_{rms} . The resistance R_s of the filter inductor is normally such that the rms voltage drop across R_s is below 0.5% of V_{rms} . The expression $mi_f = \sqrt{\frac{3}{2}}M_f i_f$ is determined by (2.4) (with $\omega = \omega_g$) and the condition

$$e_{\text{rms}} \approx \sqrt{V_{\text{rms}}^2 + \omega_g^2 L_s^2 I_{\text{rms}}^2}, \quad (7.1)$$

where e_{rms} is the rms of the electromotive force e in each phase at steady state.

We have briefly introduced synchronverters in Section 1, but so far we have not described their structure. Without going into too much detail, the synchronverter is based on an inverter having three legs built from electronic switches which operate at a high switching frequency, see [32, 34] for details. It has a DC side which is normally connected to a DC energy source (or a storage device), three AC output terminals corresponding to the three phases of the power grid and a neutral line (which serves as reference for all the voltages). We denote the vector of voltages on the AC terminals, averaged over one switching period, by $g = [g_a \ g_b \ g_c]^\top$. These AC terminals are connected to passive low-pass filters, each of which may be an inductor, or two inductors and a capacitor (the so-called LCL filter) or they may have a more complicated structure. The purpose of these filters is to transfer the power from the inverter to the grid while eliminating the voltage and current ripples at the switching frequency and its higher harmonics. If there is an LCL filter, then for the purpose of modeling, we neglect the capacitor and approximate the filter with a single inductor whose inductance L_s is the sum of the two inductances in the circuit. (The same goes for the series resistances of these inductors.) This is justified because, up to the grid frequency, the impedance of the capacitor is much larger (in absolute value) than the impedances of the inductors.

For a synchronverter designed as in [34], the voltages g_a , g_b and g_c represent the synchronous internal voltages in the stator windings of the virtual synchronous

generator, while an LCL filter represents the inductance L_s and the series resistance R_s of the stator windings (by ignoring the capacitor). As discussed in Section 1, the rotor dynamics is implemented in software. According to the design in [32, 34], $g = e$, where e is computed by the synchronverter algorithm using the measured stator currents and the equations (2.4) and (2.7). This e is then provided to the stator coil, as depicted in Figure 6 with $n = 1$. We now think that choosing $g = e$ is not the best approach, because the inductance L_s is far too small. Indeed, for reasons of size and cost, the inductance of the filter inductor of a typical commercial inverter is usually much smaller than the stator inductance of a SG of the same power rating (about 50 times smaller), see for instance [19, Example 3.1]. This fact alone justifies increasing L_s artificially, in order to make the synchronverter more similar to a SG. There is an additional reason for increasing L_s artificially, and this is to improve the stability of the system, as explained below.

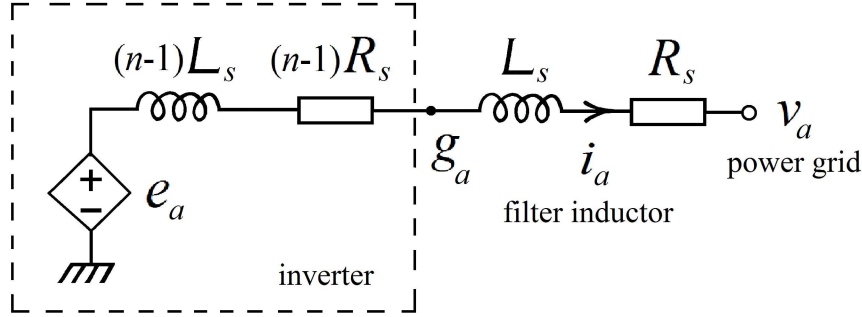


Figure 6. An inverter operated as a synchronverter, with filter inductor L_s and its series resistance R_s , connected to the utility grid. Only phase a is shown. The synchronverter algorithm provides the synchronous internal voltage e_a according to (2.4). The inductor and resistor multiplied with $(n - 1)$ are virtual. For $n = 1$ we get a usual synchronverter as in [34]. The actual (short time average) voltage generated by the inverter is g_a .

The parameters of synchronverters selected according to the empirical guidelines described earlier typically do not satisfy the stability conditions of Theorem 6.3. But as discussed in Section 6 (above Remark 6.6), if we increase the damping factor α to a sufficiently large value, then the stability conditions will hold. We see from Remark 6.6 that α can be increased primarily by increasing D_p , decreasing J or increasing L_s . In a synchronverter (in normal operation), the parameters D_p and J are chosen based on grid requirements (standard droop behavior and inertia) as discussed earlier and we cannot change them to increase α . Thus, the only way to increase α is via increasing L_s . Replacing the existing inductor with a much larger one (designed for the same nominal current, of course) would be very expensive and the larger inductor would be very bulky. We propose a method to virtually increase the inductance L_s of the inductor (and also its series resistance R_s) by a factor n (for instance, $n = 30$), by only changing the synchronverter control algorithm. The idea is to create a virtual inductor of value $(n - 1)L_s$ and with series resistance $(n - 1)R_s$

in series with the real inductor, as shown in Figure 6 (which shows only one phase out of three, phase a). We see from the figure (and a trivial computation) that

$$g_a = \frac{(n-1)v_a + e_a}{n}. \quad (7.2)$$

Here e_a is the synchronous internal voltage given by (2.4) while g_a is the average voltage (over one switching period) at the output of the switches in the inverter. Thus, by enforcing g computed as in (7.2) we create the effect of providing the synchronous internal voltage $e = [e_a \ e_b \ e_c]^\top$ to stator coils with inductance nL_s and resistance nR_s . By this method we increase the effective inductance and resistance of the stator coils by a factor of n and α by a factor of \sqrt{n} . For $n = 1$ we recover the structure of the synchronverter in [34].

In the sequel, we present a choice for the main parameters of a 500kW synchronverter which (after the modification in Figure 6) satisfy the conditions of Theorem 6.3. Thus for this set of parameters the grid-connected SG model (2.9) is almost globally asymptotically stable (aGAS). We also consider other values for the SG parameters to illustrate the different types of global dynamic behavior that the system (2.9) can exhibit.

Example 7.1. Consider a synchronverter designed for the grid frequency $\omega_g = 100\pi$ rad/sec and line voltage $V = 6000\sqrt{3}$ Volts. The synchronverter supplies a nominal active power $P_n = 500$ kW and operates with a 3% frequency droop coefficient, i.e. $d_p = 3$. Following the empirical guidelines discussed earlier, we choose $D_p = 168.87$ N·m/(rad/sec), $T_m = 54.64$ kN·m, $J = 20.26$ Kg·m²/rad, $L_s = 27.5$ mH, $R_s = 1.08 \Omega$ and $mi_f = 33.11$ Volt·sec. For this set of parameters the sufficient stability conditions in Theorem 6.3 do not hold. Although the stability conditions can be satisfied by increasing D_p or decreasing J , this is not desirable from an operational standpoint. So following the discussion earlier in this section we increase the effective inductance and resistance by a factor of 30, i.e. $n = 30$ in Figure 6. Then (the effective) $L_s = 825.06$ mH, $R_s = 32.4 \Omega$ and $mi_f = 51.67$ Volt·sec.

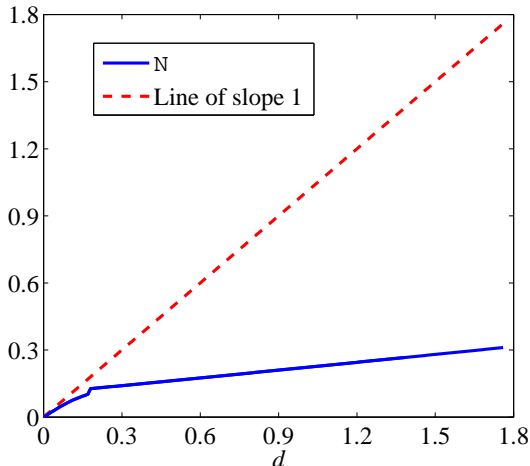


Figure 7. Plot of \mathcal{N} for the SG parameters in Example 7.1.

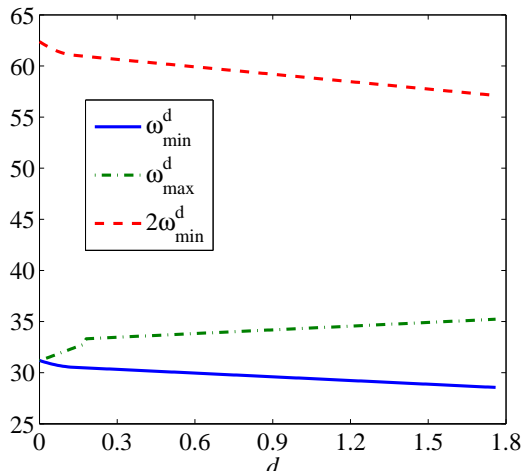


Figure 8. Plots of ω_{\max}^d , ω_{\min}^d and $2\omega_{\min}^d$ for the SG parameters in Example 7.1.

Consider the SG model (2.9) and the corresponding pendulum system (6.3) with the above parameter values and $n = 30$. This yields $p = 39.27$ rad/sec, $i_v = 39.78$ A, $\alpha = 0.83$ sec/rad, $\beta = 0.58$, $\rho = 0.1$ s/ $\sqrt{\text{rad}}$, $V_r = 1.57$ and $P_\infty = 0.12$. For the chosen parameter values, all the equilibrium points of (2.9) are hyperbolic. Figure 7 is the plot of the function \mathcal{N} defined in (6.22) on the interval $(0, (1 + P_\infty)V_r]$. Figure 8 is the plot of ω_{\max}^d , ω_{\min}^d and $2\omega_{\min}^d$ (as defined in Theorem 6.3) on the same interval. It is clear from these figures that the conditions in Theorem 6.3 are satisfied. Hence the SG model (2.9) with the virtual inductor is aGAS. \square

Numerical simulations suggest that the SG model (2.9) is aGAS for the parameter values in the above example even when we take $n = 1$ (no virtual inductance). But we cannot prove this since the conditions of Theorem 6.3 do not hold. In fact it may be hard to prove this analytically because if we make small changes in the value of R_s (while keeping all other parameter values same), then the SG model loses the aGAS property. Indeed, if we increase R_s so that the voltage drop across the resistor is 1% (instead of 0.5%) of V_{rms} , then the SG model is not aGAS. In this case, the SG model has a sequence of stable and unstable equilibrium points. It also has a sequence of periodic solutions (two periodic solutions in this sequence differ only in their value of δ and the difference is a multiple of 2π). Along each periodic solution, $\omega < \omega_g$ and ω , i_d and i_q oscillate with a time period of about 0.16 seconds while δ decreases monotonically (δ is periodic modulo 2π).

Suppose that we choose $D_p = 15$ Nm/(rad/sec) and let all the other SG parameters be as in Example 7.1 with $n = 1$. Then the SG model (2.9) has two sequences of equilibrium points, both of them unstable (this cannot happen in the case of a pendulum equation with constant forcing). The SG model also has a sequence of periodic solutions.

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