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Sampled-data Output Regulation of Unstable Well-posed Infinite-dimensional Systems with Constant Reference and Disturbance Signals

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Abstract We study the sample-data control problem of output tracking and disturbance rejection for unstable well-posed linear infinite-dimensional systems with constant reference and disturbance signals. We obtain a sufficient condition for the existence of finite-dimensional sampled-data controllers that are solutions of this control problem. To this end, we study the problem of output tracking and disturbance rejection for infinite-dimensional discrete-time systems and propose a design method of finite-dimensional controllers by using a solution of the Nevanlinna-Pick interpolation problem with both interior and boundary conditions. We apply our results to systems with state and output delays.

Keywords Frequency-domain methods · Output regulation · Sampled-data control · State-space methods · Well-posed infinite-dimensional systems

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1 Introduction

Due to the development of computer technology, digital controllers are commonly implemented for continuous-time plants. We call such closed-loop systems *sampled-data systems*. In addition to their practical motivation, sampled-data systems yield theoretically interesting problems related to the combination of both continuous-time and discrete-time dynamics, and various techniques such as the lifting approach [5, 52, 54] and the frequency response operator approach [1, 15] have been also developed for the analysis and synthesis of sampled-data finite-dimensional systems. Sampled-data control theory for infinite-dimensional systems has been developed, e.g., in [18–20, 25–28, 37–39, 43]. Several specifically relevant studies will be cited below again. In this paper, we study the problem of sampled-data output regulation for unstable well-posed systems. The main objective in our control problem is to find finite-dimensional digital controllers achieving the output tracking of given constant reference signals in the presence of external constant disturbances. A theory for well-posed systems has been extensively developed; see, e.g., the survey [45, 50] and the book [44]. Well-posed systems allow unbounded control and observation operators and provide a framework to formulate control problems for systems governed by partial differential equations with point control and observation and by functional differential equations with delays in the state, input, and output variables.

Our output regulation method is based on the internal model principle, which was originally developed for finite-dimensional systems in [12] and was later generalized for infinite-dimensional systems with finite-dimensional and infinite-dimensional exosystems in [33–36, 53] and references therein. In particular, output regulation of nonsmooth periodic signals has been extensively studied as *repetitive control* [17]. For regular systems, which is a subclass of well-posed systems, the authors of [7, 31, 33, 51] have provided design methods of continuous-time controllers for robust output regulation. For stable well-posed systems with finite-dimensional exosystems, low-gain controllers suggested by the internal model principle have been constructed for the continuous-time setup in [29, 40] and for the sampled-data setup in [18–20, 28]. The difficulty of the problem we consider arises from the instability of well-posed systems. If the system is unstable, then low-gain controllers cannot achieve closed-loop stability. Ukai and Iwazumi [46] have developed a state-space-based design method of finite-dimensional controllers for output regulation of unstable continuous-time infinite-dimensional systems, by using residue mode filters proposed in [42]. On the other hand, we employ a frequency-domain technique based on coprime factorizations as in [16, 22, 23, 28, 29]. In particular, we extend a design method of stabilizing sampled-data controllers in [25] to output regulation.

Let (A, B, C) and $\mathbf{G}$ be generating operators and a transfer function of a well-posed system Σ, respectively. The operator A is the generator of a strongly continuous semigroup $\mathbf{T}$, which governs the dynamics of the system without control. The operators B and C are control and observation operators, respectively. We consider only infinite-dimensional systems that has finite-dimensional input and output spaces with the same dimension. In other words, the transfer function $\mathbf{G}$ is a square-matrix-valued function. The well-posed system is connected with a discrete-time linear time-invariant controller Σ_{d} through a zero-order hold $\mathcal{H}_\tau$ and a generalized sampler $\mathcal{S}_\tau$, where $\tau > 0$ is a sampling period. Let u, y be the input and output of the well-posed system Σ and $u_{\mathrm{d}}, y_{\mathrm{d}}$ be the input and output of the digital controller Σ_{d}, respectively. The generalized sampler $\mathcal{S}_\tau$ is written as

$$(\mathcal{S}_\tau y)(k) = \int_0^\tau w(t) y(k\tau + t) dt \qquad \forall k \in \mathbb{Z}_+,$$

where the scalar weighting function w belongs to $L^2(0,\tau)$ and satisfies $\int_0^\tau w(t)dt = 1$. In well-posed systems, the output y is in L^2_{loc}, and hence the ideal sampling, i.e., point evaluation does not make sense. The weighting function w should be chosen so that the sampled-data system is detectable.

Using the zero-order hold $\mathcal{H}_\tau$ and the generalized sampler $\mathcal{S}_\tau$, we consider the sampled-data feedback of the form

$$u = \mathcal{H}_\tau y_{\mathrm{d}} + v\mathbb{1}_{\mathbb{R}_+} \qquad u_{\mathrm{d}} = y_{\mathrm{ref}}\mathbb{1}_{\mathbb{Z}_+} - \mathcal{S}_\tau y,$$

where $v\mathbb{1}_{\mathbb{R}_+}$ and $y_{\mathrm{ref}}\mathbb{1}_{\mathbb{Z}_+}$ are constant external reference and disturbance signals, respectively. Fig. 1 illustrates the sampled-data system we study. Since the output y of well-posed systems belongs to L^2_{loc}, the output y is not guaranteed to converge to y_{ref} as $t \to \infty$. In this paper, we therefore consider the convergence of the output in the "energy" sense, i.e., there exist constants $\Gamma_{\mathrm{ref}} > 0$ and $\alpha < 0$ such that

$$\|y - y_{\mathrm{ref}}\mathbb{1}_{\mathbb{R}_+}\|_{L^2_\alpha} \leq \Gamma_{\mathrm{ref}} \left(\left\| \begin{bmatrix} x(0) \\ x_{\mathrm{d}}(0) \end{bmatrix} \right\|_{X \times X_{\mathrm{d}}} + \|v\|_{\mathbb{C}^p} + \|y_{\mathrm{ref}}\|_{\mathbb{C}^p} \right)$$

for all initial states $x(0) \in X$ of Σ and $x_{\mathrm{d}}(0) \in X_{\mathrm{d}}$ of Σ_{d} and all $v, y_{\mathrm{ref}} \in \mathbb{C}^p$, where L^2_α is the L^2-space weighted by the exponential function $e^{-\alpha t}$. The above condition means that as $t \to \infty$, the "energy" of the restricted tracking error $(y - y_{\mathrm{ref}})|_{[t,\infty)}$ exponentially converges to zero. If we embed a smoothing precompensator between the plant and the zero-order hold as proposed in [25], then the output y exponentially converges to y_{ref} in the usual sense under a certain regularity assumption on initial states.

Before studying the sampled-data output regulation problem, we investigate an output regulation problem for infinite-dimensional discrete-time systems. In the discrete-time setup, we propose a design method of finite-dimensional controllers that achieve output regulation. Although in the sampled-data setup, we consider only constant reference and disturbance signals, the proposed method in the discrete-time setup allows reference and disturbance signals that are finite superpositions of sinusoids. The construction of regulating controllers consists of two steps: First we design stabilizing controllers with a free parameter in H^∞, using the techniques developed in [24, 25]. Next we choose

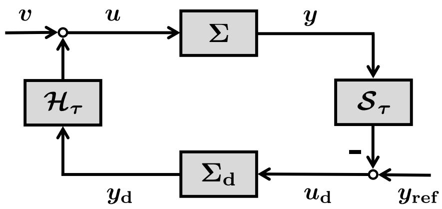

Fig. 1 Sampled-data system.

the free parameter so that the controller incorporates an internal model for output regulation. The design problem of regulating controllers is reduced to the Nevanlinna-Pick interpolation problem with both interior and boundary conditions. In the reduced interpolation problem, interior conditions are required for stabilization, whereas boundary conditions arise from output tracking and disturbance rejection.

Our main result, Theorem 3.10, states that there exists a finite-dimensional digital controller that achieves output regulation for constant reference and disturbance signals if the following conditions are satisfied:

(i) The resolvent set of A contains 0.
(ii) $\det \mathbf{G}(0) \neq 0$.
(iii) The unstable part of the spectrum of A consists of finitely many eigenvalues with finite multiplicities.
(iv) The semigroup generated by the stable part of A is exponentially stable.
(v) The unstable part of (A, B, C) is controllable and observable.
(vi) For every nonzero integer ℓ, $2\ell\pi i/\tau$ does not belong to the spectrum of A.
(vii) For every unstable eigenvalues λ of A, $\int_0^\tau w(t)e^{\lambda t}dt \neq 0$.
(viii) For every unstable eigenvalues λ, μ of A and nonzero integer ℓ, $\tau(\lambda - \mu) \neq 2\ell\pi i$.
(ix) The multiplicities of all unstable eigenvalues of A are one.

The assumptions (iii)– (viii) are used for sampled-data stabilization in [25]. In fact, (iii)–(vii) are sufficient for the existence of sampled-data stabilizing controllers, and further, (iii)–(viii) are necessary and sufficient in the single-input and single-output case. In particular, (v)–(viii) is used to guarantee that the unstable part of the sampled-data system is controllable and observable. We place the assumptions (i) and (ii) for output regulation. The remaining assumption (ix) is used to reduce the design problem of stabilizing controllers to an interpolation problem of functions in the H^∞-space. In the multi-input and multi-output case, the assumption (ix) makes it easy to obtain the associated interpolation conditions. We can remove (ix) in the single-input and single-output case.

The paper is structured as follows. In Section 2, we study an output regulation problem for infinite-dimensional discrete-time systems. In Section 3, we obtain a sufficient condition for the existence of finite-dimensional sampled-data regulating controllers for unstable well-posed systems with constant reference and disturbance signals. In Section 4, we illustrate our results by applying them to systems with state and output delays.

Notation and terminology

We denote by $\mathbb{Z}_+$ and $\mathbb{R}_+$ the set of nonnegative integers and the set of nonnegative real numbers, respectively. For $\alpha \in \mathbb{R}$, we define $\mathbb{C}_\alpha := \{s \in \mathbb{C} : \operatorname{Re} s > \alpha\}$, and for $\eta > 0$, $\mathbb{E}_\eta := \{z \in \mathbb{C} : |z| > \eta\}$. We also define $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. For a set $\Omega \subset \mathbb{C}$, its closure is denoted by $\operatorname{cl}(\Omega)$. For an arbitrary set Ω_0, the indicator function of $\Omega \subset \Omega_0$ is denoted by $\mathbb{1}_\Omega$. For a matrix $M \in \mathbb{C}^{p\times m}$, let us denote by M^*, $\bar{M}$, and M^{adj} the conjugate transpose, the matrix with complex conjugate entries, and the adjugate matrix of M, respectively.

Let X and Y be Banach spaces. Let $\mathcal{L}(X,Y)$ denote the space of all bounded linear operators from X to Y. We set $\mathcal{L}(X) := \mathcal{L}(X,X)$. An operator $T \in \mathcal{L}(X)$ is said to be *power stable* if there exist $\Gamma \geq 1$ and $\rho \in (0,1)$ such that $\|T^k\|_{\mathcal{L}(X)} \leq \Gamma \rho^k$ for every $k \in \mathbb{Z}_+$. Let $\mathbf{T} = (\mathbf{T}_t)_{t\geq 0}$ be a strongly continuous semigroup on X. The exponential growth bound of $\mathbf{T}$ is denoted by $\omega(\mathbf{T})$, that is, $\omega(\mathbf{T}) := \lim_{t\to\infty} \ln \|\mathbf{T}_t\|/t$. We say that the strongly continuous semigroup $\mathbf{T}$ is *exponentially stable* if $\omega(\mathbf{T}) < 0$. For a linear operator A from X to Y, let $\operatorname{dom}(A)$ denote the domain of A. The spectrum and resolvent set of a linear operator $A : \operatorname{dom}(A) \subset X \to X$ are denoted by $\sigma(A)$ and $\varrho(A)$, respectively.

For $\alpha \in \mathbb{R}$, we define the weighted L^2-space $L^2_\alpha(\mathbb{R}_+, X)$ by $L^2_\alpha(\mathbb{R}_+, X) := \{f : \mathbb{R}_+ \to X : e_{-\alpha} f \in L^2(\mathbb{R}_+, X)\}$, where $e_{-\alpha}(t) := e^{-\alpha t}$ for $t \in \mathbb{R}_+$, with the norm $\|f\|_{L^2_\alpha} := \|e_{-\alpha} f\|_{L^2}$. The space of all functions from $\mathbb{Z}_+$ to X is denoted by $F(\mathbb{Z}_+, X)$. Set $f^\triangledown(k) := f(k+1)$ for every $k \in \mathbb{Z}_+$ and every $f \in F(\mathbb{Z}_+, X)$. Let $\Omega = \mathbb{C}_\alpha$, $\Omega = \mathbb{E}_\eta$, or $\Omega = \mathbb{D}$. Let $H^\infty(\Omega, \mathbb{C}^{p\times m})$ denote the space of all bounded holomorphic functions from Ω to $\mathbb{C}^{p\times m}$. The norm of $H^\infty(\Omega, \mathbb{C}^{p\times m})$ is given by $\|\Phi\|_\infty := \sup_{s\in\Omega} \|\Phi(s)\|$. We write $H^\infty(\Omega)$ for $H^\infty(\Omega, \mathbb{C})$.

2 Discrete-time output regulation

In this section, we construct finite-dimensional controllers for the robust output regulation of infinite-dimensional discrete-time systems. Before proceeding to technical details, we describe the overview of this section. A fundamental assumption throughout this paper is that an infinite-dimensional plant can be decomposed into a finite-dimensional unstable part and an infinite-dimensional stable part. To avoid spill-over effects [3], we cannot ignore the infinite-dimensional stable part completely in the design of stabilizing controllers. However, it has been shown in [24, 25] that if the infinite-dimensional stable part is appropriately approximated by a finite-dimensional stable system, then we can design finite-dimensional stabilizing controllers. Now one may ask the following question for the problem of output regulation:

By such an approximation-based method, can we always construct stabilizing controllers that incorporate an internal model?

To stabilize the plant, the approximation error should be small. However, it is possible that if the approximation error is smaller than a certain threshold, then we cannot design stabilizing controllers with internal models by using the finite-dimensional approximating system. We will show that such a situation does not occur under certain assumptions on the plant. The proof is based on two key facts: First, controllers incorporate internal models if and only if their free parameters in $H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ satisfy certain interpolation conditions on the boundary $\mathbb{T}$. Second, the boundary Nevanlinna-Pick interpolation problem (see Problem A.7 in the appendix for details) is always solvable.

In Section 2.1, we formulate the problem of robust output regulation and recall the concept of p-copy internal models. In Section 2.2, we introduce assumptions of the plant and provide the main result of this section, Theorem 2.5. We provide the proof of this theorem in Sections 2.3 and 2.4. In particular, Sections 2.3 is devoted to preliminary lemmas for the multi-input and the multi-output case. Section 2.3 may be skipped by the readers interested only in the single-input and single-output case.

2.1 Control objective

In this section, we consider the following infinite-dimensional discrete-time system:

$$x^\triangledown(k) = Ax(k) + Bu(k), \quad x(0) = x^0 \in X \tag{2.1a}$$
$$y(k) = Cx(k) + Du(k), \tag{2.1b}$$

where the state space X is a separable complex Hilbert space, $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(\mathbb{C}^p, X)$, $C \in \mathcal{L}(X, \mathbb{C}^p)$, and $D \in \mathbb{C}^{p\times p}$. We use a strictly causal controller

$$x_{\mathrm{d}}^\triangledown(k) = Px_{\mathrm{d}}(k) + Qu_{\mathrm{d}}(k), \quad x_{\mathrm{d}}(0) = x_{\mathrm{d}}^0 \in X_{\mathrm{d}} \tag{2.2a}$$
$$y_{\mathrm{d}}(k) = Rx_{\mathrm{d}}(k), \tag{2.2b}$$

where the state space X_{d} is a complex Hilbert space, $P \in \mathcal{L}(X_{\mathrm{d}})$, $Q \in \mathcal{L}(\mathbb{C}^p, X_{\mathrm{d}})$, and $R \in \mathcal{L}(X_{\mathrm{d}}, \mathbb{C}^p)$. The control objective is that the output y tracks a given reference signal y_{ref} in the presence of an external disturbance signal v. The reference and disturbance signals y_{ref} and v are assumed to be generated by an exosystem of the form

$$\xi^\triangledown(k) = S\xi(k), \qquad \xi(0) = \xi^0 \in \mathbb{C}^n \tag{2.3a}$$
$$v(k) = E\xi(k) \tag{2.3b}$$
$$y_{\mathrm{ref}}(k) = F\xi(k), \tag{2.3c}$$

where $E \in \mathbb{C}^{p\times n}$, $F \in \mathbb{C}^{p\times n}$, and

$$S := \mathrm{diag}\big(e^{i\theta_1}, \ldots, e^{i\theta_n}\big) \text{ with } \theta_1, \ldots, \theta_n \in [0, 2\pi) \text{ distinct.}$$

The input u of the plant and the input $u_{\rm d}$ of the controller are given by

$$u(k) = y_{\rm d}(k) + v(k), \qquad u_{\rm d}(k) = y_{\rm ref}(k) - y(k) =: e(k).$$

We can write the dynamics of the closed-loop system as

$$x_{\rm e}^{\triangledown}(k) = A_{\rm e}x_{\rm e}(k) + B_{\rm e}\xi(k), \quad x_{\rm e}(0) = x_{\rm e}^0 \tag{2.4a}$$
$$e(k) = C_{\rm e}x_{\rm e}(k) + D_{\rm e}\xi(k), \tag{2.4b}$$

where $x_{\rm e}(k) = \begin{bmatrix} x(k) \\ x_{\rm d}(k) \end{bmatrix}$, $x_{\rm e}^0 = \begin{bmatrix} x^0 \\ x_{\rm d}^0 \end{bmatrix}$, and

$$A_{\rm e} := \begin{bmatrix} A & BR \\ -QC & P - QDR \end{bmatrix}, \quad B_{\rm e} := \begin{bmatrix} BE \\ Q(F - DE) \end{bmatrix} \tag{2.5a}$$
$$C_{\rm e} := -\begin{bmatrix} C & DR \end{bmatrix}, \quad D_{\rm e} := F - DE. \tag{2.5b}$$

For the controller in (2.2) represented by the operators (P, Q, R), we consider a set of perturbed plants and exosystems $\mathcal{O}(P, Q, R)$ defined as follows.

Definition 2.1 (Set of perturbed plants and exosystems) *For given operators $P \in \mathcal{L}(X_{\rm d})$, $Q \in \mathcal{L}(\mathbb{C}^p, X_{\rm d})$, and $R \in \mathcal{L}(X_{\rm d}, \mathbb{C}^p)$, $\mathcal{O}(P, Q, R)$ is the set of all $(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}, \widetilde{E}, \widetilde{F})$ satisfying the following two conditions:*

1. *$\widetilde{A} \in \mathcal{L}(X)$, $\widetilde{B} \in \mathcal{L}(\mathbb{C}^p, X)$, $\widetilde{C} \in \mathcal{L}(X, \mathbb{C}^p)$, $\widetilde{D} \in \mathbb{C}^{p\times p}$, $\widetilde{E} \in \mathbb{C}^{p\times n}$, and $\widetilde{F} \in \mathbb{C}^{p\times n}$.*
2. *The perturbed operator $\widetilde{A}_{\rm e}$ defined by*

$$\widetilde{A}_{\rm e} := \begin{bmatrix} \widetilde{A} & \widetilde{B}R \\ -Q\widetilde{C} & P - Q\widetilde{D}R \end{bmatrix}$$

is power stable.

If $A_{\rm e}$ is power stable, the conditions above are satisfied for any bounded perturbations of sufficiently small norms.

In this section, we study a robust output regulation problem.

Problem 2.2 (Robust output regulation for discrete-time systems) *Given the plant (2.1) and the exosystem (2.3), find a controller (2.2) satisfying the following properties:*

Stability: The operator $A_{\rm e}$ is power stable.

Tracking: There exist $M_e > 0$ and $\rho_e \in (0, 1)$ such that for every initial state $x^0 \in X$, $x_{\rm d}^0 \in X_{\rm d}$, and $\xi^0 \in \mathbb{C}^n$, the tracking error e satisfies

$$\|e(k)\|_{\mathbb{C}^p} \leq M_e\rho_e^k \left(\left\| \begin{bmatrix} x^0 \\ x_{\rm d}^0 \end{bmatrix} \right\|_{X\times X_{\rm d}} + \|\xi^0\|_{\mathbb{C}^n} \right) \qquad \forall k \in \mathbb{Z}_+.$$

Robustness: If the operators (A, B, C, D, E, F) are changed to $(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}, \widetilde{E}, \widetilde{F}) \in \mathcal{O}(P, Q, R)$, then the above tracking condition still holds.

Before proceeding to the construction of finite-dimensional regulating controllers, we recall the internal model principle. In [35], a p-copy internal model has been introduced for continuous-time systems. The discrete-time counterpart has appeared in Section IV.B of [34].

Definition 2.3 (Definition 6.1 in [35]) *A controller* (2.2) *is said to incorporate a p-copy internal model of the exosystem* (2.3) *if*

$$\dim \ker(e^{i\theta_\ell} I - P) \geq p \qquad \forall \ell \in \{1, \dots, n\}. \tag{2.6}$$

Theorem 2.4 (Theorem IV.5 in [34]) *Suppose that A_{e} is power stable. The controller* (2.2) *incorporates a p-copy internal model of the exosystem* (2.3) *if and only if it is a solution of Problem 2.2.*

2.2 Output regulation by a finite-dimensional controller

Throughout this section, we impose the following assumptions:

⟨a1⟩ $e^{i\theta_\ell} \in \varrho(A)$ for every $\ell \in \{1, \dots, n\}$.
⟨a2⟩ $\det \mathbf{G}(e^{i\theta_\ell}) \neq 0$ for every $\ell \in \{1, \dots, n\}$.
⟨a3⟩ There exist subspaces X^+ and X^- of X such that $\dim X^+ < \infty$ and $X = X^+ \oplus X^-$.
⟨a4⟩ $AX^+ \subset X^+$ and $AX^- \subset X^-$.

Let us denote the projection operator from X to X^+ by Π, and define

$$\begin{aligned} A^+ &:= A|_{X^+}, \quad B^+ := \Pi B, \quad C^+ := C|_{X^+} \\ A^- &:= A|_{X^-}, \quad B^- := (I - \Pi)B, \quad C^- := C|_{X^-}. \end{aligned}$$

We place the remaining assumptions.

⟨a5⟩ $\sigma(A) \cap \mathrm{cl}(\mathbb{E}_1)$ consists of finitely many eigenvalues with finite algebraic multiplicities, $\sigma(A^+) = \sigma(A) \cap \mathrm{cl}(\mathbb{E}_1)$, and there exists $\eta_0 \in (0,1)$ such that $\sigma(A^-) = \sigma(A) \cap \big(\mathbb{C} \setminus \mathrm{cl}(\mathbb{E}_{\eta_0})\big)$.
⟨a6⟩ (A^+, B^+, C^+) is controllable and observable.
⟨a7⟩ The zeros of $\det(zI - A^+)$ are simple.

We place the assumptions ⟨a1⟩ and ⟨a2⟩ for robust output regulation. The assumptions ⟨a3⟩–⟨a6⟩ are used for stabilization of infinite-dimensional discrete-time systems; see, e.g., [24]. We will show in Lemma 2.7 below that the assumption ⟨a7⟩ guarantees that $\dim \ker(\lambda I - A^+) = 1$ for every $\lambda \in \mathbb{C}$ satisfying $\det(\lambda I - A^+) = 0$. This allows us to reduce the design problem of stabilizing controllers to the problem of finding functions in $H^\infty(\mathbb{E}_1, \mathbb{C}^{p \times p})$ that satisfy elementary interpolation conditions, which will be shown in Lemma 2.10. In the single-input and single-output case $p = 1$, we can remove ⟨a7⟩ as mentioned at the end of this section. This is because it is much easier to translate stabilization into interpolation in the scalar-valued case than in the matrix-valued case.

Under ⟨a4⟩ and ⟨a5⟩, we fix $\eta \in (\eta_0, 1)$ and define the transfer function $\mathbf{G}$ of the plant (2.1) by

$$\mathbf{G}(z) := C(zI - A)^{-1}B + D \qquad \forall z \in \mathbb{E}_\eta \cap \varrho(A).$$

We can decompose $\mathbf{G}$ into

$$\mathbf{G}(z) = \mathbf{G}^+(z) + \mathbf{G}^-(z) \qquad \forall z \in \mathbb{E}_\eta \cap \varrho(A),$$

where

$$\mathbf{G}^+(z) := C^+(zI - A^+)^{-1}B^+, \quad \mathbf{G}^-(z) := C^-(zI - A^-)^{-1}B^- + D \tag{2.7}$$

and $\mathbf{G}^- \in H^\infty(\mathbb{E}_\eta, \mathbb{C}^{p\times p})$. By ⟨a6⟩, the unstable part $\mathbf{G}^+$ of the plant has no unstable pole-zero cancellations. There exist $\mathbf{N}_+$, $\mathbf{D}_+$ with rational entries in $H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ such that

$$\mathbf{G}^+ = \mathbf{D}_+^{-1}\mathbf{N}_+$$

and $\mathbf{N}_+, \mathbf{D}_+$ are left coprime over the sets of rational functions in $H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$. Choose such $\mathbf{N}_+, \mathbf{D}_+$ arbitrarily, and let $\chi_1, \dots, \chi_\Upsilon$ be the zeros of $\det \mathbf{D}_+$ in $\mathrm{cl}(\mathbb{E}_1)$. Together with ⟨a6⟩ and ⟨a7⟩, Lemma A.7.39 of [10] shows that these zeros are equal to the eigenvalues of A^+ and that the orders of the zeros are one.

The objective of this section is to prove the following theorem constructively:

Theorem 2.5 *Assume that ⟨a1⟩–⟨a7⟩ hold. There exists a finite-dimensional controller (2.2) that is a solution of the robust output regulation problem, Problem 2.2.*

2.3 Preliminary lemmas

Before proceeding to the proof of Theorem 2.5, we show three preliminary results, all of which are used for the multi-input and multi-output case $p > 1$. Hence the readers who are interested only in the single-input and single-output case $p = 1$ can skip this subsection.

The first lemma provides an upper bound on the norm of inverse matrices.

Lemma 2.6 *Let $V, W \in \mathbb{C}^{p\times p}$. If V is invertible and if*

$$\|V^{-1}\|_{\mathbb{C}^{p\times p}} \cdot \|V - W\|_{\mathbb{C}^{p\times p}} < 1,$$

then W is also invertible and

$$\|W^{-1}\|_{\mathbb{C}^{p\times p}} \leq \frac{\|V^{-1}\|_{\mathbb{C}^{p\times p}}}{1 - \|V^{-1}\|_{\mathbb{C}^{p\times p}} \cdot \|V - W\|_{\mathbb{C}^{p\times p}}}. \tag{2.8}$$

Proof Since

$$\|I - V^{-1}W\|_{\mathbb{C}^{p\times p}} \le \|V^{-1}\|_{\mathbb{C}^{p\times p}} \cdot \|V - W\|_{\mathbb{C}^{p\times p}} < 1,$$

it follows that $V^{-1}W$ and hence W are invertible.

Using the identity

$$V^{-1} - W^{-1} = V^{-1}(W - V)W^{-1},$$

we obtain

$$\|V^{-1} - W^{-1}\|_{\mathbb{C}^{p\times p}} \le \|V^{-1}\|_{\mathbb{C}^{p\times p}} \cdot \|V - W\|_{\mathbb{C}^{p\times p}} \cdot \|W^{-1}\|_{\mathbb{C}^{p\times p}}.$$

This yields

$$\begin{aligned} \|W^{-1}\|_{\mathbb{C}^{p\times p}} &\le \|V^{-1}\|_{\mathbb{C}^{p\times p}} + \|V^{-1} - W^{-1}\|_{\mathbb{C}^{p\times p}} \\ &\le \|V^{-1}\|_{\mathbb{C}^{p\times p}} + \|V^{-1}\|_{\mathbb{C}^{p\times p}} \cdot \|V - W\|_{\mathbb{C}^{p\times p}} \cdot \|W^{-1}\|_{\mathbb{C}^{p\times p}}. \end{aligned}$$

Thus, we obtain the desired inequality (2.8). □

The second preliminary result characterizes adjugate matrices.

Lemma 2.7 *For a region $\Omega \subset \mathbb{C}$, consider a holomorphic function $\Delta : \Omega \to \mathbb{C}^{p\times p}$. Suppose that $z_0 \in \Omega$ is a simple zero of* $\det \Delta$. *Then* $\dim \ker \Delta(z_0) = 1$. *Furthermore, if a nonzero vector $\psi \in \mathbb{C}^p$ satisfies* $\ker \Delta(z_0)^* = \{\alpha\psi : \alpha \in \mathbb{C}\}$, *then there exist $\alpha_1, \dots, \alpha_p \in \mathbb{C}$ such that $\alpha_\ell \neq 0$ for some $\ell \in \{1, \dots, p\}$ and $\Delta^{\mathrm{adj}}(z_0)$ can be written as*

$$\Delta^{\mathrm{adj}}(z_0) = \begin{bmatrix} \alpha_1 \psi^* \\ \vdots \\ \alpha_p \psi^* \end{bmatrix}. \tag{2.9}$$

Proof Suppose, to get a contradiction, that $\dim \ker \Delta(z_0) \ge 2$. There exist nonzero vectors $\psi_1, \psi_2 \in \mathbb{C}^p$ such that ψ_1, ψ_2 are linearly independent and $\Delta(z_0)\psi_1 = 0$, $\Delta(z_0)\psi_2 = 0$. Let $\mathrm{e}_1, \dots, \mathrm{e}_p$ be the standard basis of the p-dimensional Euclidean space. There exists an invertible matrix $U \in \mathbb{C}^{p\times p}$ such that $\psi_1 = U\mathrm{e}_1$ and $\psi_2 = U\mathrm{e}_2$. Let us denote by Δ_ℓ the ℓth column vector of the product ΔU. Then

$$\Delta_\ell(z_0) = \Delta(z_0)U\mathrm{e}_\ell = \Delta(z_0)\psi_\ell = 0 \qquad \forall \ell \in \{1, 2\}.$$

Since each element of ΔU is holomorphic, there exist vector-valued functions $\widehat{\Delta}_1$ and $\widehat{\Delta}_2$ with each entry holomorphic such that $\Delta_1(z) = (z - z_0)\widehat{\Delta}_1(z)$ and $\Delta_2(z) = (z - z_0)\widehat{\Delta}_2(z)$. Thus,

$$\begin{aligned} \det \Delta(z) &= \det(\Delta(z)U) \det U^{-1} \\ &= (z - z_0)^2 \det \begin{bmatrix} \widehat{\Delta}_1(z) & \widehat{\Delta}_2(z) & \Delta_3(z) & \cdots & \Delta_p(s) \end{bmatrix} \det U^{-1}, \end{aligned}$$

which contradicts that z_0 is a simple zero.

To prove the second assertion, we employ Cramer's rule

$$\Delta\Delta^{\mathrm{adj}} = \Delta^{\mathrm{adj}}\Delta = \det \Delta \cdot I. \tag{2.10}$$

We obtain

$$\Delta^{\mathrm{adj}}(z_0)\Delta(z_0) = \det \Delta(z_0) I = 0.$$

Since $\ker \Delta(z_0)^* = \{\alpha\psi : \alpha \in \mathbb{C}\}$, it follows that all the row vectors of $\Delta^{\mathrm{adj}}(z_0)$ can be written as $\alpha\psi^*$ for some $\alpha \in \mathbb{C}$. Thus (2.9) holds.

Finally, let us show the existence of a nonzero coefficient α_ℓ. By contradiction, assume that $\alpha_\ell = 0$ in (2.9) for every $\ell \in \{1, \dots, p\}$. Then $\Delta^{\mathrm{adj}}(z_0) = 0$. Since Δ^{adj} and $\det \Delta$ are holomorphic, then there exist holomorphic functions F and f such that

$$\Delta^{\mathrm{adj}}(s) = (s - z_0)F, \qquad \det \Delta(s) = (s - z_0)f. \tag{2.11}$$

Since z_0 is a simple zero of $\det \Delta$, it follows that $f(z_0) \neq 0$. Substituting (2.11) to Cramer's rule (2.10), we obtain

$$\Delta F = fI.$$

It follows that

$$0 = \psi^*\Delta(z_0)F(z_0) = f(z_0)\psi^*,$$

which contradicts $f(z_0) \neq 0$ and $\psi \neq 0$. □

The third preliminary lemma provides a stabilizable and detectable realization of the series interconnection of two finite-dimensional systems.

Lemma 2.8 *For $\ell \in \{1, 2\}$, consider the matrix pair $(P_\ell, Q_\ell, R_\ell, S_\ell)$ with appropriate dimensions and define the transfer function*

$$\mathbf{K}_\ell(z) := R_\ell(zI - P_\ell)^{-1}Q_\ell + S_\ell.$$

Assume that $\sigma(P_1) \cap \sigma(P_2) \cap \mathrm{cl}(\mathbb{E}_1) = \emptyset$. Assume also that $\mathbf{K}_1(\lambda)$ is full column rank for every $\lambda \in \sigma(P_2) \cap \mathrm{cl}(\mathbb{E}_1)$ and that $\mathbf{K}_2(\lambda)$ is full row rank for every $\lambda \in \sigma(P_1) \cap \mathrm{cl}(\mathbb{E}_1)$. If $(P_\ell, Q_\ell, R_\ell, S_\ell)$ is stabilizable and detectable for $\ell \in \{1, 2\}$, then the realization of $\mathbf{K}_1\mathbf{K}_2$ given by

$$\left(\begin{bmatrix} P_1 & Q_1R_2 \\ 0 & P_2 \end{bmatrix}, \begin{bmatrix} Q_1S_2 \\ Q_2 \end{bmatrix}, \begin{bmatrix} R_1 & S_1R_2 \end{bmatrix}, S_1S_2 \right) \tag{2.12}$$

is stabilizable and detectable.

Proof It is well known that (2.12) is a realization of $\mathbf{K}_1\mathbf{K}_2$; see, e.g., Section 3.6 of [56]. It suffices to show that the realization (2.12) is stabilizable and detectable.

Assume, to reach a contradiction, that the realization (2.12) is not stabilizable. Then there exist an eigenvalue $\lambda \in \sigma(P_1) \cup \sigma(P_2)$ with $|\lambda| \geq 1$ and vectors ψ_1, ψ_2 such that

$$\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \neq 0, \quad \begin{bmatrix} \psi_1^* & \psi_2^* \end{bmatrix} \begin{bmatrix} \lambda I - P_1 & -Q_1R_2 \\ 0 & \lambda I - P_2 \end{bmatrix} = 0, \quad \begin{bmatrix} \psi_1^* & \psi_2^* \end{bmatrix} \begin{bmatrix} Q_1S_2 \\ Q_2 \end{bmatrix} = 0.$$

For the case $\lambda \in \sigma(P_2)$, we obtain $\lambda \in \varrho(P_1)$ by the assumption $\sigma(P_1) \cap \sigma(P_2) \cap \mathrm{cl}(\mathbb{E}_1) = \emptyset$, and hence $\psi_1 = 0$ from $\psi_1^*(\lambda I - P_1) = 0$. Therefore,

$$\psi_2^*(\lambda I - P_2) = 0, \quad \psi_2^* Q_2 = 0.$$

Using the stabilizability of (P_2, Q_2), we find $\psi_2 = 0$. This is a contradiction.

Suppose next that $\lambda \in \sigma(P_1)$. Then $\lambda I - P_2$ is invertible by the assumption $\sigma(P_1) \cap \sigma(P_2) \cap \mathrm{cl}(\mathbb{E}_1) = \emptyset$. Therefore,

$$\psi_2^* = \psi_1^* Q_1 R_2 (\lambda I - P_2)^{-1}. \tag{2.13}$$

We obtain

$$\psi_1^* Q_1 \mathbf{K}_2(\lambda) = \psi_1^* Q_1 (R_2(\lambda I - P_2)^{-1} Q_2 + S_2) = \begin{bmatrix} \psi_1^* & \psi_2^* \end{bmatrix} \begin{bmatrix} Q_1 S_2 \\ Q_2 \end{bmatrix} = 0.$$

Since $\mathbf{K}_2(\lambda)$ is full row rank, it follows that $\psi_1^* Q_1 = 0$. Together with $\psi_1^*(\lambda I - P_1) = 0$, this implies $\psi_1 = 0$ by the stabilizability of (P_1, Q_1). Hence $\psi_2 = 0$ by (2.13). This is a contradiction. Thus, the realization (2.12) is stabilizable. The detectability of the realization (2.12) can be obtained in a similar way. □

2.4 Proof of Theorem 2.5

Let us start to prove Theorem 2.5, by using Lemmas 2.6–2.8. To construct finite-dimensional regulating controllers, we approximate the infinite-dimensional stable part $\mathbf{G}^-$ in (2.7) by a rational function. In the next result, the approximation error is used to characterize the norm of a certain matrix, which will appear in interpolation conditions on the boundary $\mathbb{T}$.

Lemma 2.9 *Assume that* $\langle$a1$\rangle$–$\langle$a7$\rangle$ *hold. Define*

$$\delta^* := \max\left\{ \left\| (\mathbf{D}_+ \mathbf{G})^{-1}(e^{i\theta_\ell}) \right\|_{\mathbb{C}^{p\times p}} : \ell \in \{1, \ldots, n\} \right\}. \tag{2.14}$$

For every rational function $\mathbf{R} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ *satisfying*

$$\|\mathbf{G}^- - \mathbf{R}\|_{H^\infty(\mathbb{E}_1)} < \frac{1}{2\delta^* \|\mathbf{D}_+\|_{H^\infty(\mathbb{E}_1)}}, \tag{2.15}$$

we obtain

$$\left\| (\mathbf{N}_+ + \mathbf{D}_+ \mathbf{R})^{-1}(e^{i\theta_\ell}) \right\|_{\mathbb{C}^{p\times p}} < 2\delta^* \qquad \forall \ell \in \{1, \ldots, n\}. \tag{2.16}$$

Proof The assumption $\langle$a1$\rangle$ yields $\det \mathbf{D}_+(e^{i\theta_\ell}) \neq 0$ for every $\ell \in \{1, \ldots, n\}$, which together with $\langle$a2$\rangle$ implies that δ^* is well defined. Since

$$\mathbf{G} = \mathbf{G}^+ + \mathbf{G}^- = \mathbf{D}_+^{-1} \mathbf{N}_+ + \mathbf{G}^-,$$

we have from (2.14) that

$$\left\| (\mathbf{N}_+ + \mathbf{D}_+ \mathbf{G}^-)^{-1}(e^{i\theta_\ell}) \right\|_{\mathbb{C}^{p\times p}} = \left\| (\mathbf{D}_+ \mathbf{G})^{-1}(e^{i\theta_\ell}) \right\|_{\mathbb{C}^{p\times p}} \leq \delta^*$$

for every $\ell \in \{1,\dots,n\}$. Moreover, for every $\ell \in \{1,\dots,n\}$,

$$\begin{aligned} \left\|\mathbf{D}_+(e^{i\theta_\ell})\right\|_{\mathbb{C}^{p\times p}} &\leq \|\mathbf{D}_+\|_{H^\infty(\mathbb{E}_1)} \\ \left\|\mathbf{G}^-(e^{i\theta_\ell}) - \mathbf{R}(e^{i\theta_\ell})\right\|_{\mathbb{C}^{p\times p}} &\leq \|\mathbf{G}^- - \mathbf{R}\|_{H^\infty(\mathbb{E}_1)}. \end{aligned}$$

Thus we conclude from Lemma 2.6 and (2.15) that for every $\ell \in \{1,\dots,n\}$, the matrix $(\mathbf{N}_+ + \mathbf{D}_+\mathbf{R})(e^{i\theta_\ell})$ is invertible and satisfies

$$\begin{aligned} &\left\|(\mathbf{N}_+ + \mathbf{D}_+\mathbf{R})^{-1}(e^{i\theta_\ell})\right\|_{\mathbb{C}^{p\times p}} \\ &\quad \leq \frac{\left\|(\mathbf{N}_+ + \mathbf{D}_+\mathbf{G}^-)^{-1}(e^{i\theta_\ell})\right\|_{\mathbb{C}^{p\times p}}}{1 - \|(\mathbf{N}_+ + \mathbf{D}_+\mathbf{G}^-)^{-1}(e^{i\theta_\ell})\|_{\mathbb{C}^{p\times p}} \cdot \|\mathbf{D}_+\|_{H^\infty(\mathbb{E}_1)} \cdot \|\mathbf{G}^- - \mathbf{R}\|_{H^\infty(\mathbb{E}_1)}} \\ &\quad < 2\delta^*, \end{aligned}$$

which is the desired inequality. □

For the rational functions $\mathbf{N}_+, \mathbf{D}_+$, which are left coprime over the sets of rational functions in $H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$, there exists a strictly proper rational function $\mathbf{Y}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ and a rational function $\mathbf{Z}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ such that the Bézout identity

$$\mathbf{N}_+\mathbf{Y}_+ + \mathbf{D}_+\mathbf{Z}_+ = I \tag{2.17}$$

holds; see, e.g., Lemma 5.2.9 of [47] and its proof. We provide interpolation conditions that such a rational function $\mathbf{Y}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ satisfies, as in Theorem IV.3 of [49]. To that purpose, we see from Lemma 2.7 and $\langle$a7$\rangle$ that, for every $r \in \{1,\dots,\Upsilon\}$, there exists a nonzero vector $\psi_r \in \mathbb{C}^p$ such that $\ker \mathbf{D}_+(\chi_r)^* = \{\alpha\psi_r : \alpha \in \mathbb{C}\}$.

Lemma 2.10 *Suppose that* $\langle$a1$\rangle$–$\langle$a7$\rangle$ *are satisfied. A rational function* $\mathbf{Y}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ *is strictly proper and satisfies the Bézout identity* (2.17) *for some rational function* $\mathbf{Z}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ *if and only if the interpolation conditions*

$$\mathbf{Y}_+(\infty) = 0, \qquad \psi_r^*\mathbf{N}_+(\chi_r)\mathbf{Y}_+(\chi_r) = \psi_r^* \quad \forall r \in \{1,\dots,\Upsilon\} \tag{2.18}$$

hold. Moreover, if the latter part of the interpolation conditions (2.18) *holds, then a rational function*

$$\mathbf{Z}_+ := \frac{\mathbf{D}_+^{\mathrm{adj}} - \mathbf{D}_+^{\mathrm{adj}}\mathbf{N}_+\mathbf{Y}_+}{\det \mathbf{D}_+} \tag{2.19}$$

satisfies $\mathbf{Z}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ *and the Bézout idendity* (2.17).

Proof It is clear that the strict properness of $\mathbf{Y}_+$ is equivalent to $\mathbf{Y}_+(\infty) = 0$. Suppose that rational functions $\mathbf{Y}_+, \mathbf{Z}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ satisfy the Bézout identity (2.17). Using Cramer's rule for $\mathbf{D}_+$, we obtain

$$\mathbf{D}_+^{\mathrm{adj}} = \mathbf{D}_+^{\mathrm{adj}}(\mathbf{N}_+\mathbf{Y}_+ + \mathbf{D}_+\mathbf{Z}_+) = \mathbf{D}_+^{\mathrm{adj}}\mathbf{N}_+\mathbf{Y}_+ + \det \mathbf{D}_+ \cdot \mathbf{Z}_+. \tag{2.20}$$

For every $r \in \{1,\dots,\Upsilon\}$, we obtain $\det \mathbf{D}_+(\chi_r) = 0$ and hence

$$\mathbf{D}_+^{\mathrm{adj}}(\chi_r) = (\mathbf{D}_+^{\mathrm{adj}}\mathbf{N}_+\mathbf{Y}_+)(\chi_r).$$

The second statement of Lemma 2.7 shows that $\psi_r^*\mathbf{N}_+(\chi_r)\mathbf{Y}_+(\chi_r) = \psi_r^*$ for every $r \in \{1,\dots,\Upsilon\}$.

Conversely, suppose that a rational function $\mathbf{Y}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ satisfies the interpolation conditions (2.18). To show that the Bézout identity (2.17) holds for some rational function $\mathbf{Z}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$, it suffices to prove that $\mathbf{Z}_+$ defined by (2.19) satisfies the Bézout identity (2.17) and $\mathbf{Z}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$.

Using Cramer's rule for $\mathbf{D}_+$, we find that $\mathbf{Z}_+$ satisfies the Bézout identity (2.17). By way of contradiction, assume that $\mathbf{Z}_+ \not\in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$. Let the (j,ℓ)th entry $\mathbf{Z}_+^{j,\ell}$ of $\mathbf{Z}_+$ satisfy $\mathbf{Z}_+^{j,\ell} \not\in H^\infty(\mathbb{E}_1)$. By definition, $\mathbf{Z}_+^{j,\ell}$ is rational. Using again Cramer's rule for $\mathbf{D}_+$, we derive

$$\det \mathbf{D}_+ \cdot \mathbf{Z}_+ = \mathbf{D}_+^{\mathrm{adj}}(I - \mathbf{N}_+\mathbf{Y}_+) \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p}).$$

Since a rational function $\det \mathbf{D}_+$ is not strictly proper by Theorem 4.3.12 of [47], it follows that $\mathbf{Z}_+^{j,\ell}$ is proper. Therefore, there exists a pole of the rational function $\mathbf{Z}_+^{j,\ell}$ in $\mathrm{cl}(\mathbb{E}_1)$ that is equal to a zero χ_{r_0} of $\det \mathbf{D}_+$. Since χ_{r_0} is a simple zero, it follows that

$$\big(\det \mathbf{D}_+ \cdot \mathbf{Z}_+^{j,\ell}\big)(\chi_{r_0}) \neq 0. \tag{2.21}$$

However, by the latter part of the interpolation conditions (2.18) and Lemma 2.7, we obtain

$$(\det \mathbf{D}_+ \cdot \mathbf{Z}_+)(\chi_{r_0}) = \mathbf{D}_+^{\mathrm{adj}}(\chi_{r_0})\big(I - \big(\mathbf{N}_+\mathbf{Y}_+)(\chi_{r_0})\big) = 0.$$

This contradicts (2.21). □

Set $M > 0$ as in

$$M > \inf\big\{\|Y\|_{H^\infty(\mathbb{E}_1)} : \mathbf{Y}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p}) \text{ is rational and satisfies (2.18)}\big\}. \tag{2.22}$$

Since there always exists a rational function $\mathbf{Y}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ satisfying the interpolation conditions (2.18), the right side of (2.22) belongs to $\mathbb{R}_+$.

The boundary interpolation conditions in Lemma 2.11 below is used for the incorporation of a p-copy internal model.

Lemma 2.11 *Assume that* $\langle$a1$\rangle$–$\langle$a7$\rangle$ *hold, and define* $\delta^* > 0$ *by* (2.14). *For every rational function* $\mathbf{R} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ *satisfying* (2.16), *there exist a strictly proper rational function* $\mathbf{Y}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ *and a rational function* $\mathbf{Z}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ *such that* $\mathbf{Y}_+$ *satisfies the interpolation conditions*

$$\mathbf{Y}_+(e^{i\theta_\ell}) = (\mathbf{N}_+ + \mathbf{D}_+\mathbf{R})^{-1}(e^{i\theta_\ell}) \qquad \forall \ell = \{1,\dots,n\} \tag{2.23a}$$

$$\begin{aligned}\mathbf{Y}_+'(e^{i\theta_\ell}) = -(\mathbf{N}_+ + \mathbf{D}_+\mathbf{R})^{-1}(e^{i\theta_\ell})\big(&\mathbf{D}_+(e^{i\theta_\ell}) \\ &+ (\mathbf{N}_+ + \mathbf{D}_+\mathbf{R})'(e^{i\theta_\ell})(\mathbf{N}_+ + \mathbf{D}_+\mathbf{R})^{-1}(e^{i\theta_\ell})\big) \qquad \forall \ell = \{1,\dots,n\},\end{aligned} \tag{2.23b}$$

the norm condition

$$\|\mathbf{Y}_+\|_{H^\infty(\mathbb{E}_1)} < \max\{2\delta^*, M\}, \tag{2.24}$$

and the Bézout identity (2.17) *hold.*

Proof Lemma 2.10 shows that a rational function $\mathbf{Y}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ is strictly proper and satisfies the Bézout identity (2.17) for some rational function $\mathbf{Z}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ if and only if the interpolation conditions (2.18) hold. Hence the problem of finding the desired $\mathbf{Y}_+, \mathbf{Z}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ is equivalent to that of finding a rational function $\mathbf{Y}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ satisfying the interior interpolation conditions (2.18), the boundary interpolation conditions (2.23), and the norm condition (2.24), which is called the *Nevanlinna-Pick interpolation problem with both interior and boundary conditions*; see Appendix A for details. This interpolation problem is solvable if R satisfies (2.16). Once we obtain a solution $\mathbf{Y}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ of the interpolation problem, $\mathbf{Z}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ defined by (2.19) satisfies the Bézout identity (2.17). □

Lemma 2.12 *Assume that* ⟨a1⟩–⟨a7⟩ *hold. Suppose that a rational function* $\mathbf{R} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ *satisfies* (2.16). *Let a strictly proper rational function* $\mathbf{Y}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ *and a proper rational function* $\mathbf{Z}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ *satisfy the interpolation conditions* (2.23) *and the Bézout identity* (2.17). *Then the following results hold:*

(a) $\mathbf{Y}_+$ *and* $\mathbf{Z}_+ - \mathbf{R}\mathbf{Y}_+$ *are right coprime over the set of rational functions in* $H^\infty(\mathbb{E}_1)$.
(b) *The rational function defined by*

$$\mathbf{K} := \mathbf{Y}_+(\mathbf{Z}_+ - \mathbf{R}\mathbf{Y}_+)^{-1} \tag{2.25}$$

is strictly proper and satisfies

$$\mathbf{K} = \mathbf{Y}_+\big(I - (\mathbf{N}_+ + \mathbf{D}_+\mathbf{R})\mathbf{Y}_+\big)^{-1}\mathbf{D}_+. \tag{2.26}$$

(c) *There exists a rational function* $\widehat{\mathbf{Z}}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ *such that*

$$(\mathbf{Z}_+ - \mathbf{R}\mathbf{Y}_+)(z) = \prod_{\ell=1}^{n}(z - e^{i\theta_\ell}) \cdot \widehat{\mathbf{Z}}_+(z) \tag{2.27a}$$

$$\det \widehat{\mathbf{Z}}_+(e^{i\theta_\ell}) \neq 0. \tag{2.27b}$$

Proof (a) By the Bézout identity (2.17),

$$(\mathbf{N}_+ + \mathbf{D}_+\mathbf{R})\mathbf{Y}_+ + \mathbf{D}_+(\mathbf{Z}_+ - \mathbf{R}\mathbf{Y}_+) = I.$$

Hence $\mathbf{Y}_+$ and $\mathbf{Z}_+ - \mathbf{R}\mathbf{Y}_+$ are right coprime over the sets of rational functions in $H^\infty(\mathbb{E}_1)$.

(b) Since $\mathbf{Y}_+(\infty) = 0$, it follows from the Bézout identity (2.17) that $\mathbf{Z}_+(\infty)$ is invertible. Therefore, $\mathbf{K}(\infty) = 0$ and $\mathbf{K}$ is strictly proper. The Bézout identity (2.17) also yields

$$\mathbf{Z}_+ - \mathbf{R}\mathbf{Y}_+ = \mathbf{D}_+^{-1}(I - \mathbf{N}_+\mathbf{Y}_+) - \mathbf{R}\mathbf{Y}_+ = \mathbf{D}_+^{-1}(I - (\mathbf{N}_+ + \mathbf{D}_+\mathbf{R})\mathbf{Y}_+). \tag{2.28}$$

Therefore, we obtain (2.26).

(c) To show the existence of a rational function $\widehat{\mathbf{Z}}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ satisfying (2.27), it suffices to prove

$$(\mathbf{Z}_+ - \mathbf{R}\mathbf{Y}_+)(e^{i\theta_\ell}) = 0 \qquad \forall \ell \in \{1, \dots, n\} \tag{2.29}$$

and $(\mathbf{Z}_+ - \mathbf{R}\mathbf{Y}_+)'(e^{i\theta_\ell})$ is invertible for all $\ell \in \{1, \dots, n\}$. We immediately obtain (2.29) from (2.23a) and (2.28). We also have

$$(\mathbf{Z}_+ - \mathbf{R}\mathbf{Y}_+)'(e^{i\theta_\ell}) = \big(\mathbf{D}_+^{-1}\big)(e^{i\theta_\ell})\big(I - (\mathbf{N}_+ + \mathbf{D}_+\mathbf{R})\mathbf{Y}_+\big)'(e^{i\theta_\ell})$$

for every $\ell \in \{1, \dots, n\}$. The interpolation condition (2.23b) yields

$$\big(I - (\mathbf{N}_+ + \mathbf{D}_+\mathbf{R})\mathbf{Y}_+\big)'(e^{i\theta_\ell}) = \mathbf{D}_+(e^{i\theta_\ell}) \qquad \forall \ell \in \{1, \dots, n\}.$$

Thus, $(\mathbf{Z}_+ - \mathbf{R}\mathbf{Y}_+)'(e^{i\theta_\ell}) = I$ for all $\ell \in \{1, \dots, n\}$. This completes the proof. □

For δ^* in (2.14) and M in (2.22), define

$$M_1 := \max\{2\delta^*, M\}. \tag{2.30}$$

The following lemma provides a sufficient condition for the robust output regulation problem to be solvable.

Lemma 2.13 *Assume that* ⟨a1⟩–⟨a7⟩ *hold. Choose a rational function* $\mathbf{R} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ *so that*

$$\|\mathbf{G}^- - \mathbf{R}\|_{H^\infty(\mathbb{E}_1)} < \frac{1}{M_1\|\mathbf{D}_+\|_{H^\infty(\mathbb{E}_1)}}. \tag{2.31}$$

Let a strictly proper rational function $\mathbf{Y}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ *and a proper rational function* $\mathbf{Z}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ *satisfy the interpolation conditions* (2.23), *the norm condition* (2.24), *and the Bézout identity* (2.17). *Then there exists a realization* (P, Q, R) *of the rational function* $\mathbf{K}$ *defined by* (2.25) *such that the controller* (2.2) *with this realization* (P, Q, R) *is a solution of Problem 2.2.*

Proof Let a rational function $\mathbf{R} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ satisfy (2.31). Since

$$\frac{1}{M_1\|\mathbf{D}_+\|_{H^\infty(\mathbb{E}_1)}} \le \frac{1}{2\delta^*\|\mathbf{D}_+\|_{H^\infty(\mathbb{E}_1)}},$$

Lemma 2.9 shows that $\mathbf{R}$ satisfies (2.16).

Due to Theorem 2.4, it is enough to prove that there exists a realization (P, Q, R) of the rational function $\mathbf{K}$ defined by (2.25) such that A_e defined by (2.5) is power stable and (2.6) holds.

Let us first find a stabilizable and detectable realization (P, Q, R) of $\mathbf{K}$ satisfying (2.6). In the single-input and single-output case $p = 1$, Lemma A.7.39 of [10] directly shows that a minimal realization (P, Q, R) of $\mathbf{K}$ satisfies (2.6). For the multi-input and multi-output case $p > 1$, we decompose $\mathbf{K}$ and then use Lemma 2.8. Fix $a \in (-1, 1)$, and let a strictly proper rational function $\mathbf{Y}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ and a proper rational function $\mathbf{Z}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ satisfy the interpolation conditions (2.23), the norm condition (2.24), and the Bézout identity (2.17). Choose a rational function $\widehat{\mathbf{Z}}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ satisfying (2.27). Define

$$\mathbf{K}_1(z) := \prod_{\ell=1}^{n} \frac{z-a}{z-e^{i\theta_\ell}} I, \quad \mathbf{K}_2(z) := \mathbf{Y}_+(z)\big((z-a)^n \widehat{\mathbf{Z}}_+(z)\big)^{-1}. \tag{2.32}$$

Then $\mathbf{K} = \mathbf{K}_1\mathbf{K}_2$.

For every $\ell \in \{1, \dots, n\}$, let $c_\ell \in \mathbb{C}$ be the residue of $\prod_{j=1}^{n} \frac{z-a}{z-e^{i\theta_j}}$ at $z = e^{i\theta_\ell}$. Using the identity matrix I with dimension p, we define

$$P_1 := \operatorname{diag}\big(e^{i\theta_1} I, \dots, e^{i\theta_n} I\big),\ Q_1 := \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix},\ R_1 := \begin{bmatrix} c_1 I \cdots c_n I \end{bmatrix},\ S_1 := I. \tag{2.33}$$

Then (P_1, Q_1, R_1, S_1) is a minimal realization of $\mathbf{K}_1$, and

$$\dim\ker(e^{i\theta_\ell} I - P_1) \geq p \qquad \forall \ell \in \{1, \dots, n\}. \tag{2.34}$$

Let (P_2, Q_2, R_2, S_2) be a minimal realization of $\mathbf{K}_2$. Since $\mathbf{K}_2$ is strictly proper, it follows that $S_2 = 0$. In addition, the realizations (P_1, Q_1, R_1, S_1) and (P_2, Q_2, R_2, S_2) satisfy the conditions in Lemma 2.8. By (a) of Lemma 2.12, $\mathbf{Y}_+$ and $(z-a)^n\widehat{\mathbf{Z}}_+$ are right coprime. Lemma A.7.39 of [10] shows that every $\ell \in \{1, \dots, n\}$ satisfies $e^{i\theta_\ell} \notin \sigma(P_2)$ by (2.27b). Hence $\sigma(P_1) \cap \sigma(P_2) \cap \operatorname{cl}(\mathbb{E}_1) = \emptyset$. By definition, $\det \mathbf{K}_1(\lambda) \neq 0$ for every $\lambda \in \sigma(P_2) \cap \operatorname{cl}(\mathbb{E}_1)$. Since the interpolation condition (2.23a) implies that $\mathbf{Y}_+(e^{i\theta_\ell})$ is invertible for every $\ell \in \{1, \dots, n\}$, it follows that $\det \mathbf{K}_2(\lambda) \neq 0$ for every $\lambda \in \sigma(P_1) \cap \operatorname{cl}(\mathbb{E}_1) = \{e^{i\theta_1}, \dots, e^{i\theta_n}\}$. Therefore, Lemma 2.8 shows that the realization (P, Q, R) of $\mathbf{K} = \mathbf{K}_1\mathbf{K}_2$ in the form (2.12) is stabilizable and detectable. By (2.34), (2.6) is satisfied.

We can see the power stability of A_e from the same argument as in the proofs of Theorem 7 in [24] and Theorem 9 in [25]. Using (2.31) and $\|\mathbf{Y}_+\|_{H^\infty(\mathbb{E}_1)} < M_1$, we derive

$$\begin{aligned} \|\mathbf{D}_+(\mathbf{G}^- - \mathbf{R})\mathbf{Y}_+\|_{H^\infty(\mathbb{E}_1)} &\leq \|\mathbf{D}_+\|_{H^\infty(\mathbb{E}_1)} \cdot \|\mathbf{G}^- - \mathbf{R}\|_{H^\infty(\mathbb{E}_1)} \cdot \|\mathbf{Y}_+\|_{H^\infty(\mathbb{E}_1)} \\ &< 1. \end{aligned}$$

Therefore $\mathbf{U} := (\mathbf{D}_+\mathbf{G}^- + \mathbf{N}_+)\mathbf{Y}_+ + \mathbf{D}_+(\mathbf{Z}_+ - \mathbf{R}\mathbf{Y}_+)$ satisfies

$$\|\mathbf{U} - I\|_{H^\infty(\mathbb{E}_1)} = \|\mathbf{D}_+(\mathbf{G}^- - \mathbf{R})\mathbf{Y}_+\|_{H^\infty(\mathbb{E}_1)} < 1,$$

which yields $\mathbf{U}, \mathbf{U}^{-1} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$. Since

$$(I + \mathbf{G}\mathbf{K})^{-1} = (\mathbf{Z}_+ - \mathbf{R}\mathbf{Y}_+)\mathbf{U}^{-1}\mathbf{D}_+, \quad \mathbf{G} = \mathbf{D}_+^{-1}(\mathbf{N}_+ + \mathbf{D}_+\mathbf{G}^-),$$

it follows that

$$\begin{aligned}\begin{bmatrix} I & -\mathbf{K} \\ \mathbf{G} & I \end{bmatrix}^{-1} &= \begin{bmatrix} I - \mathbf{K}(I+\mathbf{G}\mathbf{K})^{-1}\mathbf{G} & \mathbf{K}(I+\mathbf{G}\mathbf{K})^{-1} \\ -(I+\mathbf{G}\mathbf{K})^{-1}\mathbf{G} & (I+\mathbf{G}\mathbf{K})^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I - \mathbf{Y}_+\mathbf{U}^{-1}(\mathbf{N}_+ + \mathbf{D}_+\mathbf{G}^-) & \mathbf{Y}_+\mathbf{U}^{-1}\mathbf{D}_+ \\ -(\mathbf{Z}_+ - \mathbf{R}\mathbf{Y}_+)\mathbf{U}^{-1}(\mathbf{N}_+ + \mathbf{D}_+\mathbf{G}^-) & (\mathbf{Z}_+ - \mathbf{R}\mathbf{Y}_+)\mathbf{U}^{-1}\mathbf{D}_+ \end{bmatrix} \\ &\in H^\infty(\mathbb{E}_1, \mathbb{C}^{2p\times 2p}).\end{aligned}$$

A routine calculation similar to that for the finite-dimensional case in Lemma 5.3 of [56] shows that for the transfer functions $\mathbf{G}$ of the plant (2.1) and $\mathbf{K}$ of the controller (2.2),

$$\begin{bmatrix} I & 0 \\ D & I \end{bmatrix} \begin{bmatrix} I & -\mathbf{K} \\ \mathbf{G} & I \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ D & I \end{bmatrix}$$

is the transfer function of the system

$$\left(A_{\mathrm{e}}, \begin{bmatrix} B & 0 \\ 0 & Q \end{bmatrix}, \begin{bmatrix} 0 & R \\ -C & 0 \end{bmatrix}, \begin{bmatrix} I & 0 \\ D & I \end{bmatrix}\right).$$

Hence Theorem 2 of [24] shows that A_{e} is power stable if

$$\left(A_{\mathrm{e}}, \begin{bmatrix} B & 0 \\ 0 & Q \end{bmatrix}\right), \quad \left(\begin{bmatrix} 0 & R \\ -C & 0 \end{bmatrix}, A_{\mathrm{e}}\right) \tag{2.35}$$

is stabilizable and detectable, respectively, which is equivalent to the stabilizablity and detectablility of (A, B, C) and (P, Q, R). These properties of (A, B, C) follow from $\langle$a6$\rangle$, and we have already proved that (P, Q, R) is stabilizable and detectable. This completes the proof. $\square$

We are now in a position to prove Theorem 2.5.

Proof (of Theorem 2.5) Due to Lemma 2.13, it remains to show the existence of a rational function $\mathbf{R} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ satisfying (2.31).

Since $\mathbf{G}^- \in H^\infty(\mathbb{E}_\eta, \mathbb{C}^{p\times p})$ for some $\eta \in (0, 1)$, it follows that the Taylor expansion of $\mathbf{G}^-$ at ∞,

$$\mathbf{G}^-(z) = \sum_{j=0}^{\infty} G_j z^{-j},$$

converges uniformly in $\mathbb{E}_1$, i.e.,

$$\lim_{N\to\infty} \sup_{z\in\mathbb{E}_1} \left\| \mathbf{G}^-(z) - \sum_{j=0}^{N} G_j z^{-j} \right\|_{\mathbb{C}^{p\times p}} = 0.$$

Thus (2.31) holds with

$$\mathbf{R}^-(z) := \sum_{j=0}^{N} G_j z^{-j}$$

for all sufficiently large $N \in \mathbb{N}$. $\square$

We summarize the proposed method for the construction of finite-dimensional regulating controllers. The problem of finding rational functions in the steps 2 and 5 of the procedure below is called the Nevanlinna-Pick interpolation problem; see Appendix A for details.

Design procedure of controllers

1. Obtain a left-coprime factorization $\mathbf{D}_+^{-1}\mathbf{N}_+$ of a rational function $\mathbf{G}^+$ over the set of rational functions in $H^\infty(\mathbb{E}_1)$.
2. Find $M > 0$ satisfying (2.22).
3. Set $M_1 > 0$ as in (2.30).
4. Find a rational function $\mathbf{R} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ satisfying the norm condition (2.31).
5. Find a rational function $\mathbf{Y}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ satisfying the interpolation conditions (2.18), (2.23) and the norm condition $\|\mathbf{Y}_+\|_{H^\infty(\mathbb{E}_1)} < M_1$.
6. Define a rational function $\mathbf{Z}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ by (2.19).
7. Calculate a rational function $\widehat{\mathbf{Z}}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ satisfying (2.27).
8. Define the minimal realization (P_1, Q_1, R_1, S_1) as in (2.33) and compute a minimal realization (P_2, Q_2, R_2) of $\mathbf{K}_2$ defined by (2.32).
9. Calculate a realization (2.12), which is a realization of a regulating controller.

In the single-input and single-output case $p = 1$, we can remove the assumption $\langle$a7$\rangle$ and the redundant steps 6–8 in the above design procedure. To see this, let the multiplicity of the zeros $\chi_1, \dots, \chi_\Upsilon$ in $\mathrm{cl}(\mathbb{E}_1)$ of $\det(sI - A^+)$ be $J_r \in \mathbb{N}$ for $r \in \{1, \dots, \Upsilon\}$. If $M_1 > 0$ is sufficiently large, then there exists a rational function $\mathbf{Y}_+ \in H^\infty(\mathbb{E}_1)$ satisfying the interpolation conditions

$$\mathbf{Y}_+(\infty) = 0, \quad \mathbf{Y}_+(\chi_r) = \frac{1}{\mathbf{N}_+(\chi_r)} \tag{2.18$'$a}$$

$$\mathbf{Y}_+^{(j)}(\chi_r) = \frac{-1}{\mathbf{N}_+(\chi_r)} \sum_{\ell=0}^{j-1} \frac{j!}{\ell!(j-\ell)!} \mathbf{N}^{(j-\ell)}(\chi_r)\mathbf{Y}_+^{(\ell)}(\chi_r) \tag{2.18$'$b}$$

for all $r \in \{1, \dots, \Upsilon\}$, $j \in \{1, \dots, J_r\}$ and

$$\mathbf{Y}_+(e^{i\theta_\ell}) = \frac{1}{\mathbf{N}_+(e^{i\theta_\ell}) + \mathbf{D}_+(e^{i\theta_\ell})\mathbf{R}(e^{i\theta_\ell})} \tag{2.23$'$}$$

for all $\ell \in \{1, \dots, n\}$ and the norm condition $\|\mathbf{Y}_+\|_{H^\infty(\mathbb{E}_1)} < M_1$. See, e.g., [30] for an algorithm to compute a rational function $\mathbf{Y}_+ \in H^\infty(\mathbb{E}_1)$ satisfying these interpolation and norm conditions. For a rational function $\mathbf{R} \in H^\infty(\mathbb{E}_1)$ satisfying (2.31),

$$\mathbf{K} := \frac{\mathbf{Y}_+\mathbf{D}_+}{1 - (\mathbf{N}_+ + \mathbf{D}_+\mathbf{R})\mathbf{Y}_+}$$

is strictly proper, has a pole at $z = e^{i\theta_\ell}$ for all $\ell \in \{1, \dots, n\}$, and satisfies

$$\begin{bmatrix} 1 & -\mathbf{K} \\ \mathbf{G} & 1 \end{bmatrix}^{-1} \in H^\infty(\mathbb{E}_1, \mathbb{C}^{2\times 2}).$$

As commented in the proof of Lemma 2.13, we see from Lemma A.7.39 of [10] that a minimal realization (P, Q, R) of $\mathbf{K}$ satisfies (2.6). Thus, (P, Q, R) is a realization of a regulating controller. Since this result can be obtained by a slight modification of the argument for the multi-input multi-output case $p > 1$, we omit the details for the sake of brevity.

3 Sampled-data output regulation for constant reference and disturbance signals

In this section, we investigate sampled-data robust output regulation for unstable well-posed systems with constant reference and disturbance signals. To this end, we employ the results for discrete-time systems developed in Section 2. However, there remains two issues to be solved:

– *What conditions are required for the original continuous-time system in order to guarantee the conditions* $\langle$a1$\rangle$–$\langle$a7$\rangle$ *of the discretized system?*
– *Does output regulation at sampling instants imply continuous-time output regulation?*

The main difficulty of the first problem is to obtain the relationship between the transfer function $\mathbf{G}(s)$ of the original continuous-time system and the transfer function $\mathbf{G}_\tau(z)$ of the discretized system with sampling period $\tau > 0$. We here show that $\mathbf{G}_\tau(1) = \mathbf{G}(0)$. This equality allows us to check the assumption $\langle$a2$\rangle$, $\det \mathbf{G}_\tau(1) \neq 0$, by using only $\mathbf{G}(s)$. For exponentially stable well-posed systems, $\mathbf{G}_\tau(1) = \mathbf{G}(0)$ has been proved in Proposition 4.3 of [28] and Proposition 3.1 of [20]. We extend these results to systems whose unstable part is finite-dimensional. The point of the proof is to decompose $\mathbf{G}(s)$ into the unstable part $\mathbf{G}^+(s)$ and the stable part $\mathbf{G}^-(s)$.

For the second issue, we first prove that the output has the limit as $t \to \infty$ in the "energy" sense. Next, we show that this limit coincides the value of the constant reference signal if output regulation at sampling instants is achieved. We further prove that if a smoothing precompensator is embedded between the zero-order hold and the plant, then the output exponentially converges to the constant reference signal in the usual sense under a certain regularity condition on the initial states.

In Section 3.1, we recall briefly some facts on well-posed continuous-time systems. In Section 3.2, we introduce sampled-data systems and formulate the problem of sampled-data robust output regulation for constant reference and disturbance signals. We place assumptions on the original continuous-time systems in Section 3.3 and reduce them to the assumptions $\langle$a1$\rangle$–$\langle$a7$\rangle$ on the discretized system in Section 3.4. Finally, Section 3.5 is devoted to solving the sampled-data output regulation problem.

3.1 Preliminaries on well-posed systems

We provide brief preliminaries on well-posed linear systems and refer the readers to the surveys [45, 50] and the book [44] for more details. As a plant, we consider a well-posed system Σ with state space X, input space $\mathbb{C}^p$, and output space $\mathbb{C}^p$, generating operators (A, B, C), transfer function $\mathbf{G}$, and input-output operator G. Here X is a separable complex Hilbert space with norm $\|\cdot\|$ and A is the generator of a strongly continuous semigroup $\mathbf{T} = (\mathbf{T}_t)_{t\geq 0}$ on X. The spaces X_1 and X_{-1} are the interpolation and extrapolation spaces associated with $\mathbf{T}$, respectively. For $\lambda \in \varrho(A)$, the space X_1 is defined as $\operatorname{dom}(A)$ endowed with the norm $\|\zeta\|_1 := \|(\lambda I - A)\zeta\|$, and X_{-1} is the completion of X with respect to the norm $\|\zeta\|_{-1} := \|(\lambda I - A)^{-1}\zeta\|$. Different choices of λ lead to equivalent norms on X_1 and X_{-1}. The semigroup $\mathbf{T}$ restricts to a strongly continuous semigroup on X_1, and the generator of the restricted semigroup is the part of A in X_1. Similarly, $\mathbf{T}$ can be uniquely extended to a strongly continuous semigroup on X_{-1}, and the generator of the extended semigroup is an extension of A with domain X. The restriction and extension of $\mathbf{T}$ have the same exponential growth bound as the original semigroup $\mathbf{T}$. We denote the restrictions and extensions of $\mathbf{T}$ and A by the same symbols. We refer the reader to Section II.5 of [11] and Section 2.10 of [45] for more details on the interpolation and extrapolation spaces.

We place the following conditions for the system node $(A, B, C, \mathbf{G})$ to be well posed:

- The operator B satisfies $B \in \mathcal{L}(\mathbb{C}^p, X_{-1})$ and is an admissible control operator for $\mathbf{T}$, that is, for every $t \geq 0$, there exists $b_t \geq 0$ such that

$$\left\| \int_0^t \mathbf{T}_{t-s} B u(s) ds \right\| \leq b_t \|u\|_{L^2(0,t)} \qquad \forall u \in L^2([0,t], \mathbb{C}^p).$$

- The operator C satisfies $C \in \mathcal{L}(X_1, \mathbb{C}^p)$ and is an admissible observation operator for $\mathbf{T}$, that is, for every $t \geq 0$, there exists $c_t \geq 0$ such that

$$\left(\int_0^t \|C\mathbf{T}_s \zeta\|^2_{\mathbb{C}^p} ds \right)^{1/2} \leq c_t \|\zeta\| \qquad \forall \zeta \in X_1.$$

- The transfer function $\mathbf{G} : \mathbb{C}_{\omega(\mathbf{T})} \to \mathbb{C}^{p \times p}$ satisfies

$$\mathbf{G}(s) - \mathbf{G}(\lambda) = -(s - \lambda) C (sI - A)^{-1} (\lambda I - A)^{-1} B \quad \forall s, \lambda \in \mathbb{C}_{\omega(\mathbf{T})} \quad (3.1)$$

  and $\mathbf{G} \in H^\infty(\mathbb{C}_\alpha, \mathbb{C}^{p\times p})$ for every $\alpha > \omega(\mathbf{T})$.

The transfer function $\mathbf{G}$ may have an analytic extension to a half plane $\mathbb{C}_\alpha$ with $\alpha < \omega(\mathbf{T})$. If it exists, we say that $\mathbf{G}$ is holomorphic (meromorphic) on $\mathbb{C}_\alpha$ and use the same symbol $\mathbf{G}$ for an analytic extension to a larger right half plane. For every $\alpha > \omega(\mathbf{T})$, the input-output operator $G : L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{C}^p) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{C}^p)$ satisfies $G \in \mathcal{L}\big(L^2_\alpha(\mathbb{R}_+, \mathbb{C}^p), L^2_\alpha(\mathbb{R}_+, \mathbb{C}^p)\big)$ and

$$\big(\mathfrak{L}(Gu)(s)\big) = \mathbf{G}(s)\big(\mathfrak{L}(u)\big)(s) \qquad \forall s \in \mathbb{C}_\alpha,\ \forall u \in L^2_\alpha(\mathbb{R}_+, \mathbb{C}^p),$$

where $\mathfrak{L}$ denotes the Laplace transform.

The Λ-extension C_Λ of C is defined by

$$C_\Lambda \zeta := \lim_{s\to\infty,\, s\in\mathbb{R}} Cs(sI-A)^{-1}\zeta$$

with domain $\mathrm{dom}(C_\Lambda)$ consisting of those $\zeta \in X$ for which the limit exists. For every $\zeta \in X$, we obtain $\mathbf{T}_t \zeta \in \mathrm{dom}(C_\Lambda)$ for a.e. $t \geq 0$. By the admissibility of C, for every $t \geq 0$, there exists $c_t \geq 0$ such that

$$\left(\int_0^t \|C_\Lambda \mathbf{T}_s \zeta\|^2_{\mathbb{C}^p} ds\right)^{1/2} \leq c_t \|\zeta\| \qquad \forall \zeta \in X.$$

If we define the operator $\Psi : X \to L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{C}^p)$ by

$$(\Psi\zeta)(t) := C_\Lambda \mathbf{T}_t \zeta \qquad \forall \zeta \in X,\ \text{a.e. } t \geq 0,$$

then Ψ satisfies $\Psi \in \mathcal{L}\big(X, L^2_\alpha(\mathbb{R}_+, \mathbb{C}^p)\big)$ for every $\alpha > \omega(\boldsymbol{T})$. The Laplace transform of $\Phi\zeta$ is given by $C(sI-A)^{-1}\zeta$ for every $\zeta \in X$ and every $s \in \mathbb{C}_{\omega(\mathbf{T})}$.

Fix $\lambda \in \mathbb{C}_{\omega(\mathbf{T})}$ arbitrarily. Let x and y denote, respectively, the state and output functions of the well-posed system Σ with the initial condition $x(0) = x^0 \in X$ and the input function $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{C}^p)$. The state x and the output y satisfy

$$x(t) = \mathbf{T}_t x^0 + \int_0^t \mathbf{T}_{t-s} Bu(s) ds \qquad \forall t \geq 0, \tag{3.2}$$

$x(t) - (\lambda I - A)^{-1}Bu(t) \in \mathrm{dom}(C_\Lambda)$ for a.e. $t \geq 0$, and

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x^0 \in X \quad \text{a.e. } t \geq 0 \tag{3.3a}$$

$$y(t) = C_\Lambda\big(x(t) - (\lambda I - A)^{-1}Bu(t)\big) + \mathbf{G}(\lambda)u(t) \quad \text{a.e. } t \geq 0, \tag{3.3b}$$

where the differential equation (3.3a) is interpreted on X_{-1}. We have from (3.2) and (3.3b) that for every $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{C}^p)$ and a.e. $t \geq 0$, the input-output operator G satisfies

$$(Gu)(t) = C_\Lambda \left(\int_0^t \mathbf{T}_{t-s} Bu(s) ds - (\lambda I - A)^{-1} Bu(t)\right) + \mathbf{G}(\lambda)u(t). \tag{3.4}$$

3.2 Closed-loop system and control objective

Let $\tau > 0$ denote the sampling period. The zero-order hold operator $\mathcal{H}_\tau : F(\mathbb{Z}_+, \mathbb{C}^p) \to L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{C}^p)$ is defined by

$$(\mathcal{H}_\tau f)(k\tau + t) := f(k) \qquad \forall t \in [0,\tau),\ \forall k \in \mathbb{Z}_+.$$

The generalized sampling operator $\mathcal{S}_\tau : L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{C}^p) \to F(\mathbb{Z}_+, \mathbb{C}^p)$ is defined by

$$(\mathcal{S}_\tau g)(k) := \int_0^\tau w(t) g(k\tau + t) dt \qquad \forall k \in \mathbb{Z}_+,$$

where the scalar weighting function w satisfies $w \in L^2(0,\tau)$ and

$$\int_0^\tau w(t)dt = 1.$$

The outputs of well-posed systems are in L^2_{loc}, and hence the above type of generalized sampling is reasonable. Note that controllers connected to the sampler above need to be strictly causal, i.e., have no feedforward term.

We connect the continuous-time system (3.3) and the discrete-time controller (2.2) via the following sampled-data feedback law:

$$u = \mathcal{H}_\tau y_{\mathrm{d}} + v\mathbb{1}_{\mathbb{R}_+}, \qquad u_{\mathrm{d}} = y_{\mathrm{ref}}\mathbb{1}_{\mathbb{Z}_+} - \mathcal{S}_\tau y,$$

where $y_{\mathrm{ref}}\mathbb{1}_{\mathbb{Z}_+}$ and $v\mathbb{1}_{\mathbb{R}_+}$ with $y_{\mathrm{ref}} \in \mathbb{C}^p$ and $v \in \mathbb{C}^p$ are the constant reference and disturbance signals, respectively. These signals are constant, but their values y_{ref} and v are unknown when we design controllers. The dynamics of the sampled-data system is given by

$$\dot{x} = Ax + B(\mathcal{H}_\tau y_{\mathrm{d}} + v\mathbb{1}_{\mathbb{R}_+}), \qquad x(0) = x^0 \in X \tag{3.5a}$$
$$y = C_\Lambda\big(x - (\lambda I - A)^{-1}B(\mathcal{H}_\tau y_{\mathrm{d}} + v\mathbb{1}_{\mathbb{R}_+})\big) + \mathbf{G}(\lambda)(\mathcal{H}_\tau y_{\mathrm{d}} + v\mathbb{1}_{\mathbb{R}_+}) \tag{3.5b}$$
$$x_{\mathrm{d}}^{\triangledown} = Px_{\mathrm{d}} + Q(y_{\mathrm{ref}}\mathbb{1}_{\mathbb{Z}_+} - \mathcal{S}_\tau y), \quad x_{\mathrm{d}}(0) = x_{\mathrm{d}}^0 \in X_{\mathrm{d}} \tag{3.5c}$$
$$y_{\mathrm{d}} = Rx_{\mathrm{d}}. \tag{3.5d}$$

We define the exponential stability of this sampled-data system.

Definition 3.1 (Exponential stability) *The sampled-data system (3.5) is exponentially stable if there exist $\Gamma \geq 1$ and $\gamma > 0$ such that*

$$\left\| \begin{bmatrix} x(k\tau + t) \\ x_{\mathrm{d}}(k) \end{bmatrix} \right\|_{X\times X_{\mathrm{d}}} \leq \Gamma \left(e^{-\gamma(k\tau+t)} \left\| \begin{bmatrix} x^0 \\ x_{\mathrm{d}}^0 \end{bmatrix} \right\|_{X\times X_{\mathrm{d}}} + \|y_{\mathrm{ref}}\|_{\mathbb{C}^p} + \|v\|_{\mathbb{C}^p} \right) \tag{3.6}$$
$$\forall k \in \mathbb{Z}_+,\ \forall t \in [0,\tau),\ \forall x^0 \in X,\ \forall x_{\mathrm{d}}^0 \in X_{\mathrm{d}},\ \forall y_{\mathrm{ref}}, v \in \mathbb{C}^p.$$

We consider a set of perturbed plants $\mathcal{O}_{\mathrm{s}}(P,Q,R)$ defined as follows.

Definition 3.2 (Set of perturbed plants) *For given operators $P \in \mathcal{L}(X_{\mathrm{d}})$, $Q \in \mathcal{L}(\mathbb{C}^p, X_{\mathrm{d}})$, and $R \in \mathcal{L}(X_{\mathrm{d}}, \mathbb{C}^p)$, $\mathcal{O}_{\mathrm{s}}(P,Q,R)$ is the set of system nodes $(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{\mathbf{G}})$ satisfying the following two conditions:*

1. *The operators $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ and the transfer function $\widetilde{\mathbf{G}}$ generate a well-posed system with state space X, input space $\mathbb{C}^p$, and output space $\mathbb{C}^p$.*
2. *The perturbed sampled-data system, in which the system node $(A, B, C, \mathbf{G})$ is changed to $(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{\mathbf{G}})$, is exponentially stable.*

In this section, we study the following sampled-data robust output regulation problem.

Problem 3.3 (Robust output regulation for sampled-data systems) *Find a controller (2.2) such that the following three properties hold for the sampled-data system (3.5):*

Stability: The sampled-data system (3.5) *is exponentially stable.*
Tracking: There exist $\Gamma_{\mathrm{ref}} > 0$ *and* $\alpha < 0$ *such that*

$$\|y - y_{\mathrm{ref}}\mathbb{1}_{\mathbb{R}_+}\|_{L^2_\alpha} \leq \Gamma_{\mathrm{ref}}\left(\left\|\begin{bmatrix} x^0 \\ x^0_{\mathrm{d}} \end{bmatrix}\right\|_{X\times X_{\mathrm{d}}} + \|y_{\mathrm{ref}}\|_{\mathbb{C}^p} + \|v\|_{\mathbb{C}^p}\right) \quad (3.7)$$
$$\forall x^0 \in X,\ \forall x^0_d \in X_d,\ \forall y_{\mathrm{ref}}, v \in \mathbb{C}^p.$$

Robustness: If the system node $(A, B, C, \mathbf{G})$ *is changed to* $(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{\mathbf{G}}) \in \mathcal{O}_{\mathrm{s}}(P, Q, R)$*, then the above tracking property still holds.*

3.3 Assumptions on well-posed systems

In what follows, we impose several assumptions on the well-posed system (3.3).

⟨b1⟩ $0 \in \varrho(A)$.
⟨b2⟩ $\det \mathbf{G}(0) \neq 0$.
⟨b3⟩ There exists $\varepsilon > 0$ such that $\sigma(A) \cap \mathrm{cl}(\mathbb{C}_{-\varepsilon})$ consists of finitely many isolated eigenvalues of A with finite algebraic multiplicities.

Under the assumption ⟨b3⟩, we obtain the following spectral decomposition of X for A; see, e.g., Lemma 2.5.7 of [10] or Proposition IV.1.16 of [11]. There exists a rectifiable, closed, simple curve Φ in $\mathbb{C}$ enclosing an open set that contains $\sigma(A) \cap \mathrm{cl}(\mathbb{C}_0)$ in its interior and $\sigma(A) \cap \big(\mathbb{C} \setminus \mathrm{cl}(\mathbb{C}_0)\big)$ in its exterior. The operator

$$\Pi := \frac{1}{2\pi i}\int_{\Phi} (sI - A)^{-1} ds \quad (3.8)$$

is a projection on X. Define $X^+ := \Pi X$ and $X^- := (I - \Pi)X$. Then $X = X^+ \oplus X^-$, $\dim X^+ < \infty$, and $X^+ \subset X_1$. The subspaces X^+ and X^- are $\mathbf{T}_t$-invariant for all $t \geq 0$.

Define

$$A^+ := A|_{X^+}, \quad \mathbf{T}^+_t := \mathbf{T}_t|_{X^+}, \quad A^- := A|_{X_1 \cap X^-}, \quad \mathbf{T}^-_t := \mathbf{T}_t|_{X^-}.$$

Then

$$\sigma(A^+) = \sigma(A) \cap \mathrm{cl}(\mathbb{C}_0), \quad \sigma(A^-) = \sigma(A) \cap \big(\mathbb{C} \setminus \mathrm{cl}(\mathbb{C}_0)\big),$$

and $\mathbf{T}^+ := (\mathbf{T}^+_t)_{t\geq 0}$ and $\mathbf{T}^- := (\mathbf{T}^-_t)_{t\geq 0}$ are strongly continuous semigroups on X^+ and X^- with generators A^+ and A^-, respectively. The projection operator Π on X can be extended to a projection Π_{-1} on X_{-1}, and $\Pi_{-1}X_{-1} = \Pi X = X^+$. We define

$$B^+ := \Pi_{-1}B, \quad C^+ := C|_{X^+}, \quad B^- := (I - \Pi_{-1})B, \quad C^- := C|_{X_1 \cap X^-}.$$

We can uniquely extend the semigroup $\mathbf{T}^-_t$ to a strongly continuous semigroup on $(X^-)_{-1}$, and the generator of the extended semigroup is an extension of A^-. The same symbols $\mathbf{T}^-_t$ and A^- will be used to denote the extensions. Note that we can identity $(X^-)_{-1}$ and $(X_{-1})^- := (I - \Pi_{-1})X_{-1}$ as mentioned in the footnote 2 on p. 1357 of [27].

We are now in a position to formulate the remaining assumptions.

⟨b4⟩ The strongly continuous semigroup $\mathbf{T}^- = (\mathbf{T}^-_t)_{t\geq 0}$ is exponentially stable.
⟨b5⟩ (A^+, B^+, C^+) is controllable and observable.
⟨b6⟩ $2\ell\pi i/\tau \notin \sigma(A^+)$ for every $\ell \in \mathbb{Z} \setminus \{0\}$.
⟨b7⟩ $\int_0^\tau w(t)e^{\lambda t}dt \neq 0$ for every $\lambda \in \sigma(A^+)$.
⟨b8⟩ $\tau(\lambda - \mu) \neq 2\ell\pi i$ for every $\lambda, \mu \in \sigma(A^+)$ and for every $\ell \in \mathbb{Z} \setminus \{0\}$.
⟨b9⟩ The zeros of $\det(sI - A^+)$ are simple.

As in the discrete-time case, we assume ⟨b1⟩ and ⟨b2⟩ for output regulation. For the design of regulating controllers, we place the assumption ⟨b9⟩ but can remove it in the single-input and single-output case $p = 1$, as commented in Section 2. Proposition 5 and Theorem 9 of [25] show that for the existence of stabilizing controllers, the conditions ⟨b3⟩–⟨b7⟩ are sufficient, and the conditions ⟨b3⟩–⟨b8⟩ are necessary and sufficient in the case $p = 1$.

Define the input-output operator G^+ of the finite-dimensional system (A^+, B^+, C^+) by

$$(G^+u)(t) := \int_0^t C^+e^{A^+(t-s)}B^+u(s)ds \qquad \forall t \geq 0,\ \forall u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{C}^p).$$

and define $G^- := G - G^+$. We use the following result on the decomposition of the output:

Lemma 3.4 (Lemma 4.2 in [27]) *Assume that* ⟨b3⟩ *holds. There exists a well-posed system* Σ^- *with generating operator* (A^-, B^-, C^-) *and input-output operator* G^-*. For every* $x^0 \in X$ *and every* $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{C}^p)$*, the output* y *of the well-posed system* (3.3) *can be written in the form*

$$y(t) = C^+\Pi x(t) + (C^-)_\Lambda \mathbf{T}^-_t (I - \Pi)x^0 + (G^-u)(t) \qquad \text{a.e. } t \geq 0. \tag{3.9}$$

The Λ*-extension of* C^- *satisfies*

$$(C^-)_\Lambda \zeta = C_\Lambda \zeta \qquad \forall \zeta \in \mathrm{dom}\left((C^-)_\Lambda\right) = \mathrm{dom}(C_\Lambda) \cap X^-. \tag{3.10}$$

3.4 Properties of discretized systems

To employ the discrete-time result developed in Section 2, we here convert the sampled-data system to a discretized system and then obtain the properties of the discretized system.

First, we recall the discrete-time dynamics of the plant combined with the zero-order hold and the sampler. Define

$$A_\tau := \mathbf{T}_\tau \in \mathcal{L}(X).$$

By the admissibility of B, the operator $B_\tau : L^2([0, \tau], \mathbb{C}^p) \to X$ defined by

$$B_\tau g := \int_0^\tau \mathbf{T}_t B g(\tau - t)dt \qquad \forall g \in L^2([0, \tau], \mathbb{C}^p)$$

satisfies $B_\tau \in \mathcal{L}(L^2([0,\tau],\mathbb{C}^p), X)$. Similarly, by the admissibility of C, the operator $C_\tau : X \to \mathbb{C}^p$ defined by

$$C_\tau \zeta := \int_0^\tau w(t) C_\Lambda \mathbf{T}_t \zeta dt \qquad \forall \zeta \in X$$

satisfies $C_\tau \in \mathcal{L}(X, \mathbb{C}^p)$. We define the operator $D_\tau : L^2([0,\tau],\mathbb{C}^p) \to \mathbb{C}^p$ by

$$D_\tau g := \int_0^\tau w(t)(Gg)(t) dt \qquad \forall g \in L^2([0,\tau],\mathbb{C}^p),$$

which satisfies $D_\tau \in \mathcal{L}\big(L^2([0,\tau],\mathbb{C}^p),\mathbb{C}^p\big)$. For simplicity of notation , we set

$$B_\tau \psi := B_\tau\big(\psi \mathbb{1}_{[0,\tau]}\big), \quad D_\tau \psi := D_\tau\big(\psi \mathbb{1}_{[0,\tau]}\big) \qquad \forall \psi \in \mathbb{C}^p.$$

Lemma 3.5 (Lemma 2 of [25]) *Let $u = \mathcal{H}_\tau f + g$, where $f \in F(\mathbb{Z}_+, \mathbb{C}^p)$ and $g \in L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{C}^p)$, and let $x^0 \in X$. Set $x(t)$ as in (3.2). Then*

$$\begin{aligned} x\big((k+1)\tau\big) &= A_\tau x(k\tau) + B_\tau f(k) + B_\tau \mathbf{L}_{k\tau} g \\ (\mathcal{S}_\tau y)(k) &= C_\tau x(k\tau) + D_\tau f(k) + D_\tau \mathbf{L}_{k\tau} g, \end{aligned}$$

where $\mathbf{L}_{k\tau} g \in L^2([0,\tau],\mathbb{C}^p)$ is defined by $(\mathbf{L}_{k\tau} g)(t) = g(k\tau + t)$ for all $t \in [0,\tau]$.

Remark 3.6 Throughout this section, we exploit the discretized system in Lemma 3.5. Another approach for the analysis and synthesis of sampled-data systems is to lift the plant and then apply a discrete-time technique for the lifted discrete-time plant. This lifting approach is well established for finite-dimensional systems and has the advantage that one can treat the intersample behavior of sampled-data systems in a unified, time-invariant fashion; see, e.g., [5,52,54]. There are two major reasons why we do not use the lifting approach in this study. First, our problem, output regulation for constant reference and disturbance signals, is so simple that we do not need to analyze intersample behaviors of sampled-data systems by the lifting approach. Second, the transfer function of the lifted system is an operator-valued function, and hence the discrete-time results developed in Section 2 is not applicable. This is because, to apply the Nevanlinna-Pick interpolation problem, we consider in Section 2 discrete-time systems whose transfer function is matrix-valued.

We provide two lemmas on the discretized system. These lemmas will be used to guarantee that the assumptions $\langle$a1$\rangle$–$\langle$a7$\rangle$ introduced in Section 2 are satisfied for the discretized system.

Define

$$\begin{aligned} A_\tau^+ &:= \mathbf{T}_\tau^+ = A_\tau|_{X^+}, \quad B_\tau^+ := \Pi B_\tau, \quad C_\tau^+ := C_\tau|_{X^+} \\ A_\tau^- &:= \mathbf{T}_\tau^- = A_\tau|_{X^-}, \quad B_\tau^- := (I - \Pi) B_\tau, \quad C_\tau^- := C_\tau|_{X^-} \end{aligned}$$

and $D_\tau^+ : L^2([0,\tau],\mathbb{C}^p) \to \mathbb{C}^p$ by

$$D_\tau^+ g := \int_0^\tau w(t)(G^+ g)(t) dt \qquad \forall g \in L^2([0,\tau],\mathbb{C}^p).$$

For $\psi \in \mathbb{C}^p$, we also set

$$B_\tau^+\psi := B_\tau^+(\psi \mathbb{1}_{[0,\tau]}), \quad B_\tau^-\psi := B_\tau^-(\psi \mathbb{1}_{[0,\tau]}), \quad D_\tau^+\psi := D_\tau^+(\psi \mathbb{1}_{[0,\tau]}).$$

Let $\eta \in \big(e^{\tau\omega(T^-)}, 1\big)$. On $\mathbb{E}_\eta \cap \varrho(A_\tau)$, we define the transfer function $\mathbf{G}_\tau$ of the discretized system by

$$\mathbf{G}_\tau(z) := C_\tau(zI - A_\tau)^{-1}B_\tau + D_\tau. \tag{3.11}$$

The first lemma provides a property of the resolvent set of A_τ.

Lemma 3.7 *If* ⟨b1⟩, ⟨b3⟩, ⟨b4⟩, *and* ⟨b6⟩ *hold, then* $1 \in \varrho(A_\tau)$.

Proof Since X^+ and X^- are A_τ-invariant, it is enough to show that $1 \in \varrho(A_\tau^+) \cap \varrho(A_\tau^-)$. By ⟨b1⟩, we obtain $0 \in \varrho(A^+)$. Together with ⟨b6⟩, this yields $2\ell\pi i/\tau \notin \sigma(A^+)$ for every $\ell \in \mathbb{Z}$. By the spectral mapping theorem,

$$\sigma\big(e^{\tau A^+}\big) = e^{\tau\sigma(A^+)}. \tag{3.12}$$

Therefore, $1 \notin \sigma\big(e^{\tau A^+}\big) = \sigma(A_\tau^+)$. On the other hand, ⟨b4⟩ leads to the power stability of A_τ^-, and hence $1 \in \varrho(A_\tau^-)$. This completes the proof □

The second lemma gives a relationship between the transfer functions of the original continuous-time system and the discretized system. This result will be used to verify the assumption ⟨a2⟩ on the discretized system as well as to obtain δ^* in (2.14).

Lemma 3.8 *If* ⟨b1⟩, ⟨b3⟩, *and* ⟨b4⟩ *hold, then* $\mathbf{G}_\tau(1) = \mathbf{G}(0)$.

Proof Define

$$\mathbf{G}^+(s) := C^+(sI - A^+)^{-1}B^+, \qquad \mathbf{G}^-(s) := \mathbf{G}(s) - \mathbf{G}_+(s).$$

Clearly, $\mathbf{G}^+$ is the transfer function of a finite-dimensional system with generating matrices (A^+, B^+, C^+) and input-output operator G^+. By Lemma 3.4, $\mathbf{G}^-$ is the transfer function of the exponentially stable well-posed system with generating operators (A^-, B^-, C^-) and input-output operator G^-.

We first show that

$$\mathbf{G}^+(0)\psi = -C_\tau^+(A^+)^{-1}B^+\psi + D_\tau^+\psi \qquad \forall \psi \in \mathbb{C}^p, \tag{3.13}$$

where A^+ is invertible by ⟨b1⟩. Since if $g(t) \equiv \psi \in \mathbb{C}^p$, then

$$(G^+g)(t) = C^+(e^{A^+t} - I)(A^+)^{-1}B^+\psi,$$

it follows from $\int_0^\tau w(t)dt = 1$ that

$$D_\tau^+\psi = C_\tau^+(A^+)^{-1}B^+\psi - C^+(A^+)^{-1}B^+\psi \qquad \forall \psi \in \mathbb{C}^p.$$

Thus, (3.13) holds.

Since $\mathbf{T}^-$ is exponentially stable by $\langle$b4$\rangle$, A^- is boundedly invertible. Next we shall prove that

$$\mathbf{G}^-(0)\psi = -C_\tau^-(A^-)^{-1}B^-\psi + D_\tau\psi - D_\tau^+\psi \qquad \forall \psi \in \mathbb{C}^p. \tag{3.14}$$

By definition,

$$D_\tau g - D_\tau^+ g = \int_0^\tau w(t)(G^-g)(t)dt \qquad \forall g \in L^2([0,\tau],\mathbb{C}^p).$$

Similarly to (3.4), we obtain

$$(G^-g)(t) = (C^-)_\Lambda \left(\int_0^t \mathbf{T}_s^- B^- g(t-s)ds + (A^-)^{-1}B^-g(t)\right) + \mathbf{G}^-(0)g(t)$$
$$\forall g \in L^2_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{C}^p),\ \text{a.e. } t \geq 0,$$

Using

$$\int_0^t \mathbf{T}_s^- B^-\psi ds = \mathbf{T}_t^-(A^-)^{-1}B^-\psi - (A^-)^{-1}B^-\psi \qquad \forall \psi \in \mathbb{C}^p,$$

and $\int_0^\tau w(t)dt = 1$, we obtain

$$\int_0^\tau w(t)\big(G^-(\psi \mathbb{1}_{[0,\tau]})\big)(t)dt = \int_0^\tau w(t)(C^-)_\Lambda \mathbf{T}_t^-(A^-)^{-1}B^-\psi dt + \mathbf{G}^-(0)\psi$$

for every $\psi \in \mathbb{C}^p$. By (3.10),

$$\begin{aligned} D_\tau\psi - D_\tau^+\psi &= \int_0^\tau w(t) C_\Lambda \mathbf{T}_t(A^-)^{-1}B^-\psi dt + \mathbf{G}^-(0)\psi \\ &= C_\tau^-(A^-)^{-1}B^-\psi + \mathbf{G}^-(0)\psi \qquad \forall \psi \in \mathbb{C}^p, \end{aligned}$$

and (3.14) holds.

By definition

$$\mathbf{G}_\tau(z)\psi = C_\tau^+(zI - A_\tau^+)^{-1}B_\tau^+\psi + C_\tau^-(zI - A_\tau^-)^{-1}B_\tau^-\psi + D_\tau\psi$$

for every $\psi \in \mathbb{C}^p$ and every $z \in \mathbb{E}_\eta \cap \varrho(A_\tau)$ with $\eta \in \big(e^{\tau\omega(T^-)}, 1\big)$. Combining (3.13), (3.14), and

$$B_\tau^+\psi = (A_\tau^+ - I)(A^+)^{-1}B^+\psi, \quad B_\tau^-\psi = (A_\tau^- - I)(A^-)^{-1}B^-\psi \qquad \forall \psi \in \mathbb{C}^p,$$

we obtain

$$\begin{aligned} \mathbf{G}_\tau(1)\psi &= C_\tau^+(I - A_\tau^+)^{-1}B_\tau^+\psi + C_\tau^-(I - A_\tau^-)^{-1}B_\tau^-\psi + D_\tau\psi \\ &= -C_\tau^+(A^+)^{-1}B^+\psi - C_\tau^-(A^-)^{-1}B^-\psi + D_\tau\psi \\ &= (\mathbf{G}^+(0) - D_\tau^+)\psi + (\mathbf{G}^-(0) - D_\tau + D_\tau^+)\psi + D_\tau\psi \\ &= \mathbf{G}^+(0)\psi + \mathbf{G}^-(0)\psi = \mathbf{G}(0)\psi \qquad \forall \psi \in \mathbb{C}^p. \end{aligned}$$

Thus we obtain $\mathbf{G}_\tau(1) = \mathbf{G}(0)$. $\square$

3.5 Output regulation by a finite-dimensional digital controller

Using Theorem 2.5, here we present two results on sampled-data output regulation for constant reference and disturbance signals. First, we show that the output converges to the constant reference signal in the "energy" sense. Next, we consider sampled-data systems with smoothing precompensators. The output of such a sampled-data system is continuous under a certain regularity condition on the initial states. Hence we can prove that the output exponentially converges to the constant reference signal in the usual sense.

The following lemma, which is a part of Proposition 3 in [25], connects the power stability of the discretized system and the exponential stability of the sampled-data system.

Lemma 3.9 (Proposition 3 in [25]) *The sampled-data system* (3.5) *is exponentially stable if and only if the operator* A_{e} *defined by*

$$A_{\mathrm{e}} := \begin{bmatrix} A_\tau & B_\tau R \\ -QC_\tau & P - QD_\tau R \end{bmatrix} \tag{3.15}$$

is power stable.

Theorem 3.10 *Assume that* ⟨b1⟩–⟨b9⟩ *hold. There exists a finite-dimensional controller* (2.2) *that is a solution of Problem 3.3.*

Proof One can say that the constant reference and disturbance signals $y_{\mathrm{ref}}, v \in \mathbb{C}^p$ are generated from the exosystem (2.3) with $S = 1$:

$$\xi^{\nabla}(k) = \xi(k), \qquad \xi(0) = \xi^0 \in \mathbb{C} \tag{3.16a}$$
$$v(k) = E\xi(k) \tag{3.16b}$$
$$y_{\mathrm{ref}}(k) = F\xi(k) \tag{3.16c}$$

for some unknown constant matrices $E \in \mathbb{C}^{p\times 1}$ and $F \in \mathbb{C}^{p\times 1}$. Since

$$u = \mathcal{H}_\tau y_{\mathrm{d}} + v\mathbb{1}_{\mathbb{R}_+}$$

Lemma 3.5 yields the following closed-loop dynamics at sampling instants:

$$x_{\mathrm{e}}^{\nabla}(k) = A_{\mathrm{e}} x_{\mathrm{e}}(k) + B_{\mathrm{e}}\xi^0, \quad x_{\mathrm{e}}(0) = x_{\mathrm{e}}^0 \tag{3.17a}$$
$$e(k) = C_{\mathrm{e}} x_{\mathrm{e}}(k) + D_{\mathrm{e}}\xi^0, \tag{3.17b}$$

where $e(k) := y_{\mathrm{ref}} - (\mathcal{S}_\tau y)(k)$, $x_{\mathrm{e}}(k) := \begin{bmatrix} x(k\tau) \\ x_{\mathrm{d}}(k) \end{bmatrix}$, $x_{\mathrm{e}}^0 := \begin{bmatrix} x^0 \\ x_{\mathrm{d}}^0 \end{bmatrix}$, A_{e} is defined by (3.15), and

$$B_{\mathrm{e}} := \begin{bmatrix} B_\tau E \\ Q(F - D_\tau E) \end{bmatrix}, \quad C_{\mathrm{e}} := -\begin{bmatrix} C_\tau & D_\tau R \end{bmatrix}, \quad D_{\mathrm{e}} := F - D_\tau E. \tag{3.18}$$

To employ the discrete-time result, Theorem 2.5, we first show that the assumptions in Theorem 2.5 are satisfied for the discrete-time plant $(A_\tau, B_\tau, C_\tau, D_\tau)$. By Lemmas 3.7 and 3.8, we find that

⟨a1′⟩ $1 \in \varrho(A_\tau)$;
⟨a2′⟩ $\det \mathbf{G}_\tau(1) \neq 0$.

The assumption ⟨b3⟩ implies that

⟨a3′⟩ There exist subspaces X^+ and X^- with $\dim X^+ < \infty$ such that $X = X^+ \oplus X^-$.
⟨a4′⟩ $A_\tau X^+ \subset X^+$ and $A_\tau X^- \subset X^-$

By ⟨b3⟩–⟨b8⟩, the following conditions hold:

⟨a5′⟩ $\sigma(A_\tau) \cap \mathrm{cl}(\mathbb{E}_1)$ consists of finitely many eigenvalues with finite algebraic multiplicities, $\sigma(A_\tau^+) = \sigma(A_\tau) \cap \mathrm{cl}(\mathbb{E}_1)$, and there exists $\eta \in (0,1)$ such that $\sigma(A_\tau^-) = \sigma(A_\tau) \cap \big(\mathbb{C} \setminus \mathrm{cl}(\mathbb{E}_\eta)\big)$.
⟨a6′⟩ $(A_\tau^+, B_\tau^+, C_\tau^+)$ is controllable and observable.

Here we used Proposition 5 and Theorem 9 in [25] to see that ⟨A6′⟩ holds. Finally we find from ⟨b8⟩, ⟨b9⟩, and the spectral mapping theorem (3.12) that

⟨a7′⟩ The zeros of $\det(zI - A_\tau^+)$ are simple.

Thus, Theorem 2.5 shows the existence of a finite-dimensional controller that is a solution of the robust output regulation problem, Problem 2.2, for the discrete-time plant $(A_\tau, B_\tau, C_\tau, D_\tau)$ and the exosystem (3.16). The power stability of A_{e} is equivalent to the exponential stability (3.6) by Lemma 3.9.

We next show that the tracking property holds. Let $x^0 \in X$, $x_{\mathrm{d}}^0 \in X_{\mathrm{d}}$, and $y_{\mathrm{ref}}, v \in \mathbb{C}^p$ be given. Since A_{e} is power stable, it follows that $(I - A_{\mathrm{e}})$ is invertible. By (3.17a),

$$\begin{aligned} x_{\mathrm{e}}^{\nabla}(k) - (I - A_{\mathrm{e}})^{-1} B_{\mathrm{e}} \xi^0 &= A_{\mathrm{e}} x_{\mathrm{e}}(k) + (I - (I - A_{\mathrm{e}})^{-1}) B_{\mathrm{e}} \xi^0 \\ &= A_{\mathrm{e}} (x_{\mathrm{e}}(k) - (I - A_{\mathrm{e}})^{-1} B_{\mathrm{e}} \xi^0) \qquad \forall k \in \mathbb{Z}_+. \end{aligned}$$

Using again the power stability of A_{e}, we find that there exist $\Gamma_1 > 0$ and $\rho \in (0,1)$ such that

$$\|x_{\mathrm{e}}(k) - (I - A_{\mathrm{e}})^{-1} B_{\mathrm{e}} \xi^0\|_{X \times X_{\mathrm{d}}} \leq \Gamma_1 \rho^k \big(\|x_{\mathrm{e}}^0\|_{X \times X_{\mathrm{d}}} + \|y_{\mathrm{ref}}\|_{\mathbb{C}^p} + \|v\|_{\mathbb{C}^p}\big). \tag{3.19}$$

Define

$$\begin{bmatrix} x^\infty \\ x_{\mathrm{d}}^\infty \end{bmatrix} := (I - A_{\mathrm{e}})^{-1} B_{\mathrm{e}} \xi^0, \quad u^\infty := R x_{\mathrm{d}}^\infty + v.$$

As shown in the proof of Theorem 10 in [25], we have from the assumptions ⟨b3⟩, ⟨b4⟩, and ⟨b6⟩ that

$$A x^\infty + B u^\infty = 0 \tag{3.20}$$

and

$$x^\infty = \mathbf{T}_t x^\infty + \int_0^t \mathbf{T}_s B u^\infty ds \qquad \forall t \in [0, \tau].$$

Since

$$x(k\tau + t) = \mathbf{T}_t x(k\tau) + \int_0^t \mathbf{T}_s B (R x_{\mathrm{d}}(k) + v) ds \qquad \forall t \in [0, \tau],\ \forall k \in \mathbb{Z}_+,$$

together with the admissibility of B (or Lemma 2.2 of [26]), (3.19) implies that there exists $\Gamma_2 > 0$ such that

$$\begin{aligned}\big\|x(k\tau + t) - x^\infty\big\| &\leq \|\mathbf{T}_t\| \cdot \big\|x(k\tau) - x^\infty\big\| + \left\|\int_0^t \mathbf{T}_s BR(x_{\mathrm{d}}(k) - x_{\mathrm{d}}^\infty)ds\right\| \\ &\leq \Gamma_2 \rho^k \big(\|x_{\mathrm{e}}^0\|_{X\times X_{\mathrm{d}}} + \|y_{\mathrm{ref}}\|_{\mathbb{C}^p} + \|v\|_{\mathbb{C}^p}\big)\end{aligned}$$

for all $t \in [0,\tau]$ and all $k \in \mathbb{Z}_+$. Using (3.19) again, we have that for $\Gamma_3 := \|R\|\Gamma_1$,

$$\big\|u(k\tau + t) - u^\infty\big\|_{\mathbb{C}^p} \leq \Gamma_3 \rho^k \big(\|x_{\mathrm{e}}^0\|_{X\times X_{\mathrm{d}}} + \|y_{\mathrm{ref}}\|_{\mathbb{C}^p} + \|v\|_{\mathbb{C}^p}\big)$$

for all $t \in [0,\tau]$ and all $k \in \mathbb{Z}_+$. Therefore, there exist $\Gamma_4 > 0$ and $\alpha_1 < 0$ such that

$$\|x - x^\infty \mathbb{1}_{\mathbb{R}_+}\|_{L^2_{\alpha_1}} + \|u - u^\infty \mathbb{1}_{\mathbb{R}_+}\|_{L^2_{\alpha_1}} \leq \Gamma_4\big(\|x_{\mathrm{e}}^0\|_{X\times X_{\mathrm{d}}} + \|y_{\mathrm{ref}}\|_{\mathbb{C}^p} + \|v\|_{\mathbb{C}^p}\big). \tag{3.21}$$

Define

$$x_-^\infty := (I - \Pi)x^\infty, \quad x_-^0 := (I-\Pi)x^0, \quad y^\infty := \mathbf{G}^-(0)u^\infty + C^+\Pi x^\infty.$$

Recall that the output y can be written in the form (3.9). Then we obtain

$$y(t) - y^\infty \mathbb{1}_{\mathbb{R}_+} = y_1(t) + y_2(t) + y_3(t) \qquad \text{a.e. } t \geq 0, \tag{3.22}$$

where

$$\begin{aligned} y_1 &:= (C^-)_\Lambda \mathbf{T}^- x_-^\infty + G^-(u^\infty \mathbb{1}_{\mathbb{R}_+}) - \mathbf{G}^-(0)u^\infty \mathbb{1}_{\mathbb{R}_+} \\ y_2 &:= (C^-)_\Lambda \mathbf{T}^-(x_-^0 - x_-^\infty) + G^-(u - u^\infty \mathbb{1}_{\mathbb{R}_+}) \\ y_3 &:= C^+\Pi(x - x^\infty \mathbb{1}_{\mathbb{R}_+}).\end{aligned}$$

By (3.20),

$$A^- x_-^\infty + B^- u^\infty = (I - \Pi_{-1})(Ax^\infty + Bu^\infty) = 0.$$

Since (3.1) yields

$$\begin{aligned}\mathcal{L}(G^-(u^\infty \mathbb{1}_{\mathbb{R}_+}) - \mathbf{G}^-(0)u^\infty \mathbb{1}_{\mathbb{R}_+})(s) &= \frac{\mathbf{G}^-(s) - \mathbf{G}^-(0)}{s}u^\infty \\ &= C^-(sI - A^-)^{-1}(A^-)^{-1}B^-u^\infty\end{aligned}$$

for every $s \in \mathbb{C}_0$, the Laplace transform of y_1 satisfies

$$\mathfrak{L}(y_1)(s) = C^-(sI - A^-)^{-1}(A^-)^{-1}(A^-x_-^\infty + B^-u^\infty) = 0 \qquad \forall s \in \mathbb{C}_0.$$

The uniqueness of the Laplace transform (see, e.g., Theorem 1.7.3 in [2]) yields

$$y_1(t) = 0 \qquad \text{a.e. } t \geq 0. \tag{3.23}$$

By the exponential stability of $\mathbf{T}_t^-$ and the admissibility of C^-, there exists $\Gamma_5 > 0$ and $\alpha_2 < 0$ such that

$$\|y_2\|_{L^2_{\alpha_2}} \leq \Gamma_5\big(\|x^0 - x^\infty\| + \|u - u^\infty \mathbb{1}_{\mathbb{R}_+}\|_{L^2_{\alpha_2}}\big). \tag{3.24}$$

By definition, there exists $\Gamma_6 > 0$ such that

$$\|x^\infty\| \leq \Gamma_6\big(\|y_{\mathrm{ref}}\|_{\mathbb{C}^p} + \|v\|_{\mathbb{C}^p}\big). \tag{3.25}$$

In terms of y_3, we obtain

$$\|y_3\|_{L^2_{\alpha_1}} \leq \|C^+ \Pi\|_{\mathcal{L}(X,\mathbb{C}^p)} \cdot \|x - x^\infty \mathbb{1}_{\mathbb{R}_+}\|_{L^2_{\alpha_1}}. \tag{3.26}$$

Combining (3.23)–(3.26) with (3.22), we have that there exists $\Gamma_7 > 0$ and $\alpha_3 := \max\{\alpha_1, \alpha_2\} < 0$ such that

$$\|y - y^\infty \mathbb{1}_{\mathbb{R}_+}\|_{L^2_{\alpha_3}} \leq \Gamma_7\big(\|x_{\mathrm{e}}^0\|_{X \times X_{\mathrm{d}}} + \|v\|_{\mathbb{C}^p} + \|y_{\mathrm{ref}}\|_{\mathbb{C}^p}\big), \tag{3.27}$$

which yields

$$\int_0^\tau \|y(k\tau + t) - y^\infty\|^2_{\mathbb{C}^p}\, dt \to 0 \qquad (k \to \infty).$$

Since $\int_0^\tau w(t)dt = 1$, it follows that

$$\begin{aligned}\|(\mathcal{S}_\tau y)(k) - y^\infty\|_{\mathbb{C}^p} &\leq \int_0^\tau \|w(t)(y(k\tau + t) - y^\infty)\|_{\mathbb{C}^p} dt \\ &\leq \sqrt{\int_0^\tau |w(t)|^2 dt} \cdot \sqrt{\int_0^\tau \|y(k\tau + t) - y^\infty\|^2_{\mathbb{C}^p} dt} \to 0\end{aligned}$$

as $k \to \infty$. Therefore, the sampled output $\mathcal{S}_\tau y$ converges to y^∞.

On the other hand, the tracking property and the robustness property with respect to exosystems of the discretized system implies that for every $y_{\mathrm{ref}}, v \in \mathbb{C}^p$, $(S_\tau y)(k) \to y_{\mathrm{ref}}$ as $k \to \infty$. This means that $y^\infty = y_{\mathrm{ref}}$. Thus, the tracking property is obtained from (3.27).

Finally, we prove the robustness property. Let (P, Q, R) be the realization of the controller (2.2) and $(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{\mathbf{G}})$ be the perturbed system node in $\mathcal{O}_{\mathrm{s}}(P, Q, R)$. Define the operator $\widetilde{A}_{\mathrm{e}}$ as in (3.15) by using $(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{\mathbf{G}})$. By assumption, the perturbed sampled-data system is exponentially stable. Using Lemma 3.9, we find that $\widetilde{A}_{\mathrm{e}}$ is power stable. Hence Theorem 2.4 shows that for every $y_{\mathrm{ref}}, v \in \mathbb{C}^p$, the sampled output $S_\tau y$ of the perturbed plant satisfies $\lim_{k\to\infty}(S_\tau y)(k) = y_{\mathrm{ref}}$. In the argument to obtain (3.27), we used only the well-posedness of the system node $(A, B, C, \mathbf{G})$, the power stability of A_{e}, the assumptions $\langle$b3$\rangle$, $\langle$b4$\rangle$, and $\langle$b6$\rangle$. The perturbed system node $(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{\mathbf{G}})$ is well posed by assumption. The power stability of $\widetilde{A}_{\mathrm{e}}$ has been already proved. By Proposition 3 and Theorem 9 in [25], the assumptions $\langle$b3$\rangle$, $\langle$b4$\rangle$, and $\langle$b6$\rangle$ hold for $(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{\mathbf{G}})$. Hence, repeating the argument as above, we obtain the tracking property of the perturbed sampled-data system. □

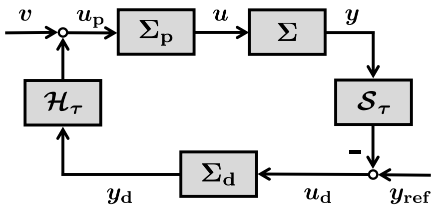

Fig. 2 Sampled-data system with precompensator.

Remark 3.11 As seen in the proof of Theorem 3.10, the states $x(t)$ and $x_{\rm d}(k)$ exponentially converge to x^∞ and $x_{\rm d}^\infty$, respectively, where

$$\begin{bmatrix} x^\infty \\ x_{\rm d}^\infty \end{bmatrix} = \begin{bmatrix} I - A_\tau & B_\tau R \\ -QC_\tau & I - P + QD_\tau R \end{bmatrix}^{-1} \begin{bmatrix} B_\tau v \\ Q(y_{\rm ref} - D_\tau v) \end{bmatrix}.$$

Since the output y may not be continuous, Theorem 3.10 does not guarantee that $y(t) \to y_{\rm ref}$ as $t \to \infty$. To address this issue, we use a smoothing stable precompensator $\Sigma_{\rm p}$ of the form

$$\dot{x}_{\rm p} = -ax_{\rm p} + u_{\rm p}, \quad x_{\rm p}(0) = x_{\rm p}^0 \in \mathbb{C}^p, \tag{3.28}$$

where $a > 0$. Consider the sampled-data system consisting of the digital controller (2.2), the well-posed plant (3.3), the precompensator (3.28), and the feedback law

$$u = x_{\rm p}, \qquad u_{\rm p} = \mathcal{H}_\tau y_{\rm d} + v\mathbb{1}_{\mathbb{R}_+}, \qquad u_{\rm d} = y_{\rm ref}\mathbb{1}_{\mathbb{Z}_+} - \mathcal{S}_\tau y.$$

Fig. 2 illustrates the sampled-data system with a precompensator.

The new plant $\widehat{\Sigma}$, which is the interconnection of the plant Σ and the precompensator $\Sigma_{\rm p}$, is a well-posed system with state space $\widehat{X} := X \times \mathbb{C}^p$, input space $\mathbb{C}^p$, and output space $\mathbb{C}^p$. The generating operators $(\widehat{A}, \widehat{B}, \widehat{C})$ of $\widehat{\Sigma}$ are given by

$$\widehat{A} := \begin{bmatrix} A & B \\ 0 & -aI \end{bmatrix} \text{ with } \operatorname{dom}(\widehat{A}) := \left\{ \begin{bmatrix} x \\ x_{\rm p} \end{bmatrix} \in X \times \mathbb{C}^p : Ax + Bx_{\rm p} \in X \right\}$$

$$\widehat{B} := \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \widehat{C} \begin{bmatrix} x \\ x_{\rm p} \end{bmatrix} := C(x - (\lambda I - A)^{-1} B x_{\rm p}) + \mathbf{G}(\lambda) x_{\rm p} \quad \forall \begin{bmatrix} x \\ x_{\rm p} \end{bmatrix} \in \operatorname{dom}(\widehat{A}),$$

where $\lambda \in \varrho(A)$. The transfer function $\widehat{\mathbf{G}}$ of $\widehat{\Sigma}$ is $\widehat{\mathbf{G}}(s) := \mathbf{G}(s)/(s + a)$.

Theorem 3.12 *If the assumptions $\langle$b1$\rangle$–$\langle$b9$\rangle$ hold, then there exists a finite-dimensional controller (2.2) that is a solution of Problem 3.3 in the context of the interconnected plant $\widehat{\Sigma}$. Furthermore, if a controller in the form (2.2) satisfies the stability property and the tracking property in Problem 3.3 for the interconnected plant $\widehat{\Sigma}$, then the following convergence property holds: Let $x_{\rm d}^0 \in X_{\rm d}$ and $v, y_{\rm ref} \in \mathbb{C}^p$ be arbitrary and let the initial states $x^0 \in X$ and $x_{\rm p}^0 \in \mathbb{C}^p$ be such that $\mathbf{T}_{t_0}(Ax^0 + Bx_{\rm p}^0) \in X$ for some $t_0 \geq 0$. Then there exist a function $y_{\rm c} : \mathbb{R}_+ \to \mathbb{C}^p$ and a constant $\alpha < 0$ such that*

1. *y_{c} coincides with the output y of $\widehat{\Sigma}$ for a.e. $t \geq 0$, is continuous on $[t_0, \infty)$, and satisfies*
$$\lim_{t\to\infty} (y_{\mathrm{c}}(t) - y_{\mathrm{ref}})e^{-\alpha t} = 0;$$
2. *α is independent of $x^0 \in X, x^0_{\mathrm{p}} \in \mathbb{C}^p$, $x^0_{\mathrm{d}} \in X_{\mathrm{d}}$, and $y_{\mathrm{ref}}, v \in \mathbb{C}^p$.*

Proof Due to Theorem 3.10, the first assertion follows if the assumptions ⟨b1⟩–⟨b9⟩ are satisfied in the context of the interconnected plant $\widehat{\Sigma}$. Among these assumptions, ⟨b3⟩–⟨b7⟩ hold in the context of $\widehat{\Sigma}$ by Proposition 5 and the proof of Theorem 11 in [25]. By the definition of $\widehat{A}$ and $\widehat{\mathbf{G}}$, the remaining assumptions ⟨b1⟩, ⟨b2⟩, ⟨b8⟩, and ⟨b9⟩ hold in the context of $\widehat{\Sigma}$.

We prove the second assertion. Define the operator $\widehat{A}_{\mathrm{e}}$ as in (3.15) by using the interconnected plant $\widehat{\Sigma}$. By Lemma 3.9, the stability property implies the power stability of $\widehat{A}_{\mathrm{e}}$. By Proposition 3 and Theorems 9 and 10 in [25], the assumptions ⟨b3⟩, ⟨b4⟩, and ⟨b6⟩ hold in the context of both Σ and $\widehat{\Sigma}$.

Let $x^0_{\mathrm{d}} \in X_{\mathrm{d}}$ and $v, y_{\mathrm{ref}} \in \mathbb{C}^p$ be given, and let $t_0 \geq 0$, $x^0 \in X$, and $x^0_{\mathrm{p}} \in \mathbb{C}^p$ be such that $\mathbf{T}_{t_0}(Ax^0 + Bx^0_{\mathrm{p}}) \in X$. It can be shown as in the proof of Theorem 3.10 that there exist $x^\infty \in X$, $x^\infty_{\mathrm{p}} \in \mathbb{C}^p$, $x^\infty_{\mathrm{d}} \in X_{\mathrm{d}}$, $\widehat{\Gamma} > 0$, and $\widehat{\rho} \in (0, 1)$ such that

$$\begin{aligned}&\|x(k\tau + t) - x^\infty\| + \|x_{\mathrm{p}}(k\tau + t) - x^\infty_{\mathrm{p}}\|_{\mathbb{C}^p} + \|x_{\mathrm{d}}(k) - x^\infty_{\mathrm{d}}\|_{X_{\mathrm{d}}} \\ &\quad\leq \widehat{\Gamma}\widehat{\rho}^k\big(\|x^0\| + \|x^0_{\mathrm{p}}\|_{\mathbb{C}^p} + \|x^0_{\mathrm{d}}\|_{X_{\mathrm{d}}} + \|y_{\mathrm{ref}}\|_{\mathbb{C}^p} + \|v\|_{\mathbb{C}^p}\big) \quad \forall t \in [0, \tau),\ \forall k \in \mathbb{Z}_+.\end{aligned} \tag{3.29}$$

Similarly to (3.20), we obtain

$$\widehat{A}\begin{bmatrix} x^\infty \\ x^\infty_{\mathrm{p}}\end{bmatrix} + \widehat{B}(Rx^\infty_{\mathrm{d}} + v) = 0. \tag{3.30}$$

Using the projection Π on X given in (3.8), we define

$$x^\infty_- := (I - \Pi)x^\infty, \quad x^0_- := (I - \Pi)x^0, \quad y^\infty := \mathbf{G}^-(0)x^\infty_{\mathrm{p}} + C^+\Pi x^\infty.$$

Lemma 3.4 yields

$$y(t) - y^\infty \mathbb{1}_{\mathbb{R}_+} = y_1(t) + y_2(t) + y_3(t) \qquad \text{a.e. } t \geq 0,$$

where

$$\begin{aligned}y_1 &:= (C^-)_\Lambda \mathbf{T}^- x^\infty_- + G^-(x^\infty_{\mathrm{p}}\mathbb{1}_{\mathbb{R}_+}) - \mathbf{G}^-(0)x^\infty_{\mathrm{p}}\mathbb{1}_{\mathbb{R}_+} \\ y_2 &:= (C^-)_\Lambda \mathbf{T}^-(x^0_- - x^\infty_-) + G^-(x_{\mathrm{p}} - x^\infty_{\mathrm{p}}\mathbb{1}_{\mathbb{R}_+}) \\ y_3 &:= C^+\Pi(x - x^\infty\mathbb{1}_{\mathbb{R}_+}).\end{aligned}$$

We can show that $y_1(t) = 0$ for a.e. $t \geq 0$ in the same way as in the proof of Theorem 3.10. In fact, using (3.30), we obtain

$$A^- x^\infty_- + B^- x^\infty_{\mathrm{p}} = (I - \Pi_{-1})(Ax^\infty + Bx^\infty_{\mathrm{p}}) = 0. \tag{3.31}$$

By (3.1),

$$\begin{aligned}\mathfrak{L}(G^-(x_\mathrm{p}^\infty \mathbb{1}_{\mathbb{R}_+}) - \mathbf{G}^-(0)x_\mathrm{p}^\infty \mathbb{1}_{\mathbb{R}_+})(s) &= \frac{\mathbf{G}^-(s) - \mathbf{G}^-(0)}{s} x_\mathrm{p}^\infty \\ &= C^-(sI - A^-)^{-1}(A^-)^{-1}B^- x_\mathrm{p}^\infty\end{aligned}$$

for every $s \in \mathbb{C}_0$. Hence the Laplace transform of y_1 is given by

$$\mathfrak{L}(y_1)(s) = C^-(sI - A^-)^{-1}(A^-)^{-1}(A^- x_-^\infty + B^- x_\mathrm{p}^\infty) = 0 \qquad \forall s \in \mathbb{C}_0.$$

Thus we obtain $y_1(t) = 0$ for a.e. $t \geq 0$.

We next investigate continuity and convergence of y_2. By Proposition 2.1 of [27], if

$$\mathbf{T}_{t_0}^-\big(A^-(x_-^0 - x_-^\infty) + B^-(x_\mathrm{p}^0 - x_\mathrm{p}^\infty)\big) \in X^- \tag{3.32}$$

and if $x_\mathrm{p} - x_\mathrm{p}^\infty \mathbb{1}_{\mathbb{R}_+} \in L^2_{\beta_2}(\mathbb{R}_+, \mathbb{C}^p)$ with $\dot{x}_\mathrm{p} \in L^2_{\beta_2}(\mathbb{R}_+, \mathbb{C}^p)$ for some $\beta_2 \in (\omega(\mathbf{T}^-), 0)$, then there exists a function $y_{2,\mathrm{c}} : \mathbb{R}_+ \to \mathbb{C}^p$ such that $y_{2,\mathrm{c}}$ coincides with y_2 for a.e. $t \geq 0$, is continuous on $[t_0, \infty)$, and satisfies $\lim_{t\to\infty} y_{2,\mathrm{c}}(t)e^{-\beta_2 t} = 0$.

Since $\mathbf{T}_{t_0}(Ax^0 + Bx_\mathrm{p}^0) \in X$ by assumption, it follows that

$$\mathbf{T}_{t_0}^-(A^- x_-^0 + B^- x_\mathrm{p}^0) = \mathbf{T}_{t_0}^-(I - \mathit{\Pi}_{-1})(Ax^0 + Bx_\mathrm{p}^0) \in X^-.$$

This together with (3.31) yields (3.32).

Let us show that $x_\mathrm{p} - x_\mathrm{p}^\infty \mathbb{1}_{\mathbb{R}_+} \in L^2_{\beta_2}(\mathbb{R}_+, \mathbb{C}^p)$ and $\dot{x}_\mathrm{p} \in L^2_{\beta_2}(\mathbb{R}_+, \mathbb{C}^p)$ for some $\beta_2 \in (\omega(\mathbf{T}^-), 0)$. Recall that

$$\dot{x}_\mathrm{p} = -a x_\mathrm{p} + \mathcal{H}_\tau R x_\mathrm{d} + v\mathbb{1}_{\mathbb{R}_+}.$$

Since (3.30) yields

$$-a x_\mathrm{p}^\infty + R x_\mathrm{d}^\infty + v = 0,$$

it follows that

$$\dot{x}_\mathrm{p} = -a(x_\mathrm{p} - x_\mathrm{p}^\infty \mathbb{1}_{\mathbb{R}_+}) + \mathcal{H}_\tau R(x_\mathrm{d} - x_\mathrm{d}^\infty \mathbb{1}_{\mathbb{Z}_+}).$$

By (3.29), there exists $\beta_2 \in (\omega(\mathbf{T}^-), 0)$ such that $x_\mathrm{p} - x_\mathrm{p}^\infty \mathbb{1}_{\mathbb{R}_+} \in L^2_{\beta_2}(\mathbb{R}_+, \mathbb{C}^p)$ and $\dot{x}_\mathrm{p} \in L^2_{\beta_2}(\mathbb{R}_+, \mathbb{C}^p)$.

Since x is continuous, it follows that y_3 is also continuous. Invoking (3.29), we have that $\lim_{t\to\infty} y_3(t)e^{-\beta_3 t} = 0$ for some $\beta_3 < 0$. Thus $y_\mathrm{c} := y_{2,\mathrm{c}} + y_3 + y^\infty \mathbb{1}_{\mathbb{R}_+}$ coincides y almost everywhere in $\mathbb{R}_+$, is continuous on $[t_0, \infty)$, and $\lim_{t\to\infty}(y_\mathrm{c}(t) - y^\infty)e^{-\alpha t} = 0$ for $\alpha := \max\{\beta_2, \beta_3\} < 0$. By construction, α is independent of $x^0 \in X, x_\mathrm{p}^0 \in \mathbb{C}^p$, $x_\mathrm{d}^0 \in X_\mathrm{d}$, and $y_\mathrm{ref}, v \in \mathbb{C}^p$.

Finally, we prove that $y_\mathrm{ref} = y^\infty$. Since $\int_0^\tau w(t)dt = 1$, it follows that, for every $k \in \mathbb{Z}_+$ with $k\tau > t_0$,

$$\begin{aligned}\big\|(\mathcal{S}_\tau y)(k) - y^\infty\big\|_{\mathbb{C}^p} &\leq \int_0^\tau \|w(t)(y_\mathrm{c}(k\tau + t) - y^\infty)\|_{\mathbb{C}^p} dt \\ &\leq \sqrt{\tau}\|w\|_{L^2(0,\tau)} \max_{0\leq t\leq \tau} \|y_\mathrm{c}(k\tau + t) - y^\infty\|_{\mathbb{C}^p}.\end{aligned}$$

Therefore, $\lim_{k\to\infty}(\mathcal{S}_\tau y)(k) = y^\infty$. On the other hand, from the tracking property, it follows that $\|y - y_{\mathrm{ref}}\mathbb{1}_{\mathbb{R}_+}\|_{L^2} < \infty$. Hence

$$\begin{aligned}\|(\mathcal{S}_\tau y)(k) - y_{\mathrm{ref}}\|_{\mathbb{C}^p} &\leq \int_0^\tau \|w(t)(y(k\tau+t) - y_{\mathrm{ref}})\|_{\mathbb{C}^p} dt \\ &\leq \sqrt{\int_0^\tau |w(t)|^2 dt} \cdot \sqrt{\int_0^\tau \|y(k\tau+t) - y_{\mathrm{ref}}\|^2_{\mathbb{C}^p} dt} \to 0\end{aligned}$$

as $k\to\infty$. Thus, $y_{\mathrm{ref}} = y^\infty$. This completes the proof. □

4 Application to delay systems

In this section, we study sampled-data output regulation for systems with state and output delays. This illustrates Theorem 3.10 and the design procedure of finite-dimensional regulating controllers in Section 2. For delay systems, the problem of output regulation has been investigated in [8, 13, 55] and the reference therein. Recently, the solvability of the output regulation problem for delay systems with infinite-dimensional state spaces has been characterized by the associated regulator equations in [32]. In the studies above, continuous-time output regulation is considered, whereas we here study sampled-data output regulation for delay systems, focusing on constant reference and disturbance signals.

First, the delay system we consider and its state-space representation are introduced. Next, in Section 4.1, we decompose delay systems into a finite-dimensional unstable part and an infinite-dimensional stable part, and then approximate the infinite-dimensional stable part by a finite-dimensional system for the design of regulating controllers. In Section 4.2, we finally present a numerical example to illustrate the proposed design method. Throughout this section, we use the same notation as in Section 3.

For $q, \widehat{q} \in \mathbb{N}$, let $h_q > h_{q-1} > \cdots > h_1 > 0$ and $h_q \geq \widehat{h}_{\widehat{q}} > \widehat{h}_{\widehat{q}-1} > \cdots > \widehat{h}_1 \geq 0$. Consider the following delay system:

$$\dot{z}(t) = A_0 z(t) + \sum_{j=1}^{q} A_j z(t-h_j) + bu(t), \quad t \geq 0 \tag{4.1a}$$

$$y(t) = \sum_{\ell=1}^{\widehat{q}} c_\ell z(t-\widehat{h}_\ell), \quad t\geq 0 \tag{4.1b}$$

$$z(0) = z^0, \qquad z(\theta) = \varpi(\theta), \quad \theta\in[-h_q, 0], \tag{4.1c}$$

where $z(t) \in \mathbb{C}^n, u(t), y(t) \in \mathbb{C}$ are the state, the input, and the output of the system, respectively, $A_j \in \mathbb{C}^{n\times n}$, $b\in\mathbb{C}^{n\times 1}$, $c_\ell \in \mathbb{C}^{1\times n}$ for every $j \in \{0,\dots,q\}$ and for every $\ell \in \{1,\dots,\widehat{q}\}$, $z^0\in\mathbb{C}^n$, and $\varpi \in L^2\big([-h_q,0],\mathbb{C}^n\big)$. In (4.1), $h_1,\dots,h_q$ and $\widehat{h}_1,\dots,\widehat{h}_{\widehat{q}}$ represent the state delay and the output delay, respectively. We assume that the input u satisfies $u \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$.

The state space of the delay system (4.1) is given by $X = \mathbb{C}^n \oplus L^2\big([-h_q, 0], \mathbb{C}^n\big)$ with the standard inner product:

$$\left(\begin{bmatrix} \zeta_1 \\ \varpi_1 \end{bmatrix}, \begin{bmatrix} \zeta_2 \\ \varpi_2 \end{bmatrix} \right) := (\zeta_1, \zeta_2)_{\mathbb{C}^n} + (\varpi_1, \varpi_2)_{L^2(-h_q,0)}.$$

The generating operators (A, B, C) of the delay system (4.1) are given by

$$A \begin{bmatrix} \zeta \\ \varpi \end{bmatrix} = \begin{bmatrix} A_0\zeta + \sum_{j=1}^{q} A_j \varpi(-h_j) \\ \frac{d\varpi}{d\theta} \end{bmatrix}$$

with domain

$$\mathrm{dom}(A) = \left\{ \begin{bmatrix} \zeta \\ \varpi \end{bmatrix} \in \mathbb{C}^n \oplus W^{1,2}\big([-h_q, 0], \mathbb{C}^n\big) : \varpi(0) = \zeta \right\}$$

and

$$Bs = \begin{bmatrix} bs \\ 0 \end{bmatrix} \qquad \forall s \in \mathbb{C}$$

$$C \begin{bmatrix} \zeta \\ \varpi \end{bmatrix} = \sum_{\ell=1}^{\widehat{q}} c_\ell \varpi\big(-\widehat{h}_\ell\big) \qquad \forall \begin{bmatrix} \zeta \\ \varpi \end{bmatrix} \in X_1.$$

The transfer function of the delay system (4.1) is given by

$$\mathbf{G}(s) = \sum_{\ell=1}^{\widehat{q}} e^{-\widehat{h}_\ell s} c_\ell \Delta(s)^{-1} b, \quad \text{where } \Delta(s) := sI - A_0 - \sum_{j=1}^{q} A_j e^{-h_j s}.$$

The derivation of the generating operators and the transfer function of delay systems can be found, e.g., in Chapters 2–4 of [10] (for the case without output delays). One can see from Lemma 2.4.3 in [10] that C is admissible. Hence, Theorem 5.1 in [9] implies that the delay system (4.1) defines a well-posed system. See, e.g, [6,14] for the well-posedness of more general delay systems.

Let $\mathbf{T}$ be the strongly continuous semigroup generating A, and define

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} := x(t) := \mathbf{T}(t)x^0 + \int_0^t \mathbf{T}(t-s)Bu(s)ds, \quad x^0 := \begin{bmatrix} z^0 \\ \varpi \end{bmatrix}. \tag{4.2}$$

It is shown in Example 3.1.9 of [10] that $x_1(t) = z(t)$ and $x_2(t) = z(t+\cdot)$ hold for all $t \geq 0$, where z is the solution of (4.1). Furthermore, z is absolutely continuous on $[0, \infty)$; see, e.g., Theorem 2.4.1 in [10]. Hence $x(t) \in X_1$ for every $t \geq h_q$, and y is (absolutely) continuous on $[\widehat{h}_{\widehat{q}}, \infty)$. For completeness, we show in Appendix B that $y(t) = C_\Lambda x(t)$ for a.e. $t \geq 0$.

The output y of this delay system exponentially converges to a constant reference signal without a precompensator. In fact, once we construct a controller that is a solution of Problem 3.3, $z(t)$ exponentially converges to some $z^\infty \in \mathbb{C}^n$; see, e.g, Remark 3.11. Since y is continuous on $[\widehat{h}_{\widehat{q}}, \infty)$, we have from the argument in the last paragraph of the proof of Theorem 3.12 that $y(t)$ also exponentially converges to $y_{\mathrm{ref}} = \sum_{\ell=1}^{\widehat{q}} c_\ell z^\infty$.

4.1 Decomposition of delay systems into stable and unstable parts

By Theorem 2.4.6 of [10], all elements of $\sigma(A)$ are the eigenvalues of A with finite multiplicities, and

$$\sigma(A) = \{s \in \mathbb{C} : \det \Delta(s) = 0\}.$$

For every $\varepsilon \in \mathbb{R}$, $\sigma(A) \cap \mathrm{cl}(\mathbb{C}_{-\varepsilon})$ consists of finitely many isolated eigenvalues of A. Hence the assumption $\langle$b3$\rangle$ in Section 3.3 holds. We place the following assumption on the eigenvalues of A in $\mathrm{cl}(\mathbb{C}_0)$.

Assumption 4.1 *The zeros, $\gamma_1, \dots, \gamma_N$, of $\det \Delta$ in $\mathrm{cl}(\mathbb{C}_0)$ are simple.*

Using Lemma 2.7, we find that $\dim \ker \Delta(\gamma_m) = 1$ for every $m \in \{1, \dots, N\}$ under Assumption 4.1. By Theorem 2.4.6 and Corollary 2.4.7 of [10], the order and the multiplicity of the eigenvalues $\gamma_1, \dots, \gamma_N$ of A are both one. For $m \in \{1, \dots, N\}$, let nonzero vectors $\varsigma_m, \nu_m \in \mathbb{C}^n$ satisfy $\Delta(\gamma_m)\varsigma_m = 0$ and $\Delta(\overline{\gamma}_m)^* \nu_m = 0$, respectively. By Theorem 2.4.6 and Lemma 2.4.9 of [10], the eigenvector ϕ_m of A corresponding to the eigenvalue γ_m and the eigenvector ψ_m of A^* corresponding to the eigenvalue $\overline{\gamma}_m$ are given by

$$\phi_m := \begin{bmatrix} \varsigma_m \\ \varpi_m \varsigma_m \end{bmatrix}, \qquad \psi_m := \frac{1}{\overline{d}_m} \begin{bmatrix} \nu_m \\ \sum_{j=1}^q e^{-\overline{\gamma}_m h_j} \frac{\mathbb{1}_{[-h_j,0]}}{\varpi_m^*} A_j^* \nu_m \end{bmatrix}, \tag{4.3}$$

where $\varpi_m(\theta) := e^{\gamma_m \theta}$, $\varpi_m^*(\theta) := e^{\overline{\gamma}_m \theta}$ for every $\theta \in [-h_q, 0]$ and

$$d_m := (\varsigma_m, \nu_m)_{\mathbb{C}^n} + \sum_{j=1}^q h_j e^{-\gamma_m h_j} (A_j \varsigma_m, \nu_m)_{\mathbb{C}^n}.$$

By definition, ϕ_m and ψ_m satisfy $(\phi_m, \psi_m) = 1$ for every $m \in \{1, \dots, N\}$. In addition, since

$$\gamma_m(\phi_m, \psi_j) = (A\phi_m, \psi_j) = (\phi_m, A^*\psi_j) = \gamma_j(\phi_m, \psi_j) \qquad \forall m, j \in \{1, \dots, N\},$$

it follows that $(\phi_m, \psi_j) = 0$ if $m \neq j$.

Let Φ be a rectifiable, closed, simple curve Φ in $\mathbb{C}$ enclosing an open set that contains $\sigma(A) \cap \mathrm{cl}(\mathbb{C}_0)$ in its interior and $\sigma(A) \cap \big(\mathbb{C} \setminus \mathrm{cl}(\mathbb{C}_0)\big)$ in its exterior. The spectral projection Π corresponding to $\sigma(A) \cap \big(\mathbb{C} \setminus \mathrm{cl}(\mathbb{C}_0)\big)$ is defined by

$$\Pi x := \frac{1}{2\pi i} \int_\Phi (sI - A)^{-1} x ds,$$

which, by Lemma 2.5.7 of [10], satisfies

$$\Pi x = \sum_{m=1}^N (x, \psi_m)\phi_m \qquad \forall x \in X. \tag{4.4}$$

Hence

$$X^+ := \Pi X = \left\{ \sum_{m=1}^{N} s_m \phi_m : s_m \in \mathbb{C} \quad \forall m \in \{1, \dots, N\} \right\}$$

and for $s, s_1, \dots, s_N \in \mathbb{C}$, the operators A^+, B^+, C^+, and $\mathbf{T}^+$ defined as in Section 3.3 satisfy

$$\begin{aligned} A^+ \left(\sum_{m=1}^{N} s_m \phi_m \right) &= \sum_{m=1}^{N} s_m \gamma_m \phi_m \\ B^+ s &= s \sum_{m=1}^{N} \frac{(b, \nu_m)_{\mathbb{C}^n}}{d_m} \phi_m \\ C^+ \left(\sum_{m=1}^{N} s_m \phi_m \right) &= \sum_{m=1}^{N} s_m \sum_{\ell=1}^{\widehat{q}} e^{-\gamma_m \widehat{h}_\ell} c_\ell \varsigma_m \\ \mathbf{T}_t^+ \left(\sum_{m=1}^{N} s_m \phi_m \right) &= \sum_{m=1}^{N} s_m e^{\gamma_m t} \phi_m \qquad \forall t \geq 0. \end{aligned}$$

We obtain

$$\mathbf{G}^+(s) := C^+(sI - A^+)^{-1} B^+ = \sum_{m=1}^{N} \frac{\kappa_m}{s - \gamma_m},$$

where

$$\kappa_m := \frac{(b, \nu_m)_{\mathbb{C}^n}}{d_m} \sum_{\ell=1}^{\widehat{q}} e^{-\gamma_m \widehat{h}_\ell} c_\ell \varsigma_m.$$

Furthermore, as shown in b. of the proof of Theorem 5.2.12 of [10], $\mathbf{T}^-$ is exponentially stable. Therefore the assumption $\langle$b4$\rangle$ in Section 3.3 is satisfied.

Since for every $s, s_1, \dots, s_N \in \mathbb{C}$, the operators $(A_\tau^+, B_\tau^+, C_\tau^+)$ defined as in Section 3.4 satisfy

$$\begin{aligned} A_\tau^+ \left(\sum_{m=1}^{N} s_m \phi_m \right) &= \sum_{m=1}^{N} s_m e^{\gamma_m \tau} \phi_m \\ B_\tau^+ s &= s \sum_{m=1}^{N} \frac{(b, \nu_m)_{\mathbb{C}^n}}{d_m} \frac{e^{\gamma_m \tau} - 1}{\gamma_m} \phi_m \\ C_\tau^+ \left(\sum_{m=1}^{N} s_m \phi_m \right) &= \sum_{m=1}^{N} s_m \int_0^\tau w(t) e^{\gamma_m t} dt \sum_{\ell=1}^{\widehat{q}} e^{-\gamma_m \widehat{h}_\ell} c_\ell \varsigma_m, \end{aligned}$$

it follows that

$$\mathbf{G}_\tau^+(z) := C_\tau^+(zI - A_\tau^+)^{-1} B_\tau^+ = \sum_{m=1}^{N} \frac{\alpha_m}{z - e^{\gamma_m \tau}},$$

where

$$\alpha_m := \kappa_m \frac{e^{\gamma_m \tau} - 1}{\gamma_m} \int_0^\tau w(t) e^{\gamma_m t} dt.$$

As in Example on pp. 1221–1223 of [25], one can construct the approximation $\mathbf{R}$ of $\mathbf{G}_\tau^- = \mathbf{G}_\tau - \mathbf{G}_\tau^+$ as follows. Define the input-output map $G^+ : L^2_{\mathrm{loc}}(\mathbb{R}_+) \to L^2_{\mathrm{loc}}(\mathbb{R}_+)$ by

$$(G^+ u)(t) := \sum_{m=1}^N \kappa_m \int_0^t e^{\gamma_m (t-s)} u(s) ds,$$

whose transfer function is given by $\mathbf{G}^+$. Similarly, we denote by $G_\tau^+ : F(\mathbb{Z}_+) \to F(\mathbb{Z}_+)$ the discrete-time input-output operator associated with the transfer function $\mathbf{G}_\tau^+$:

$$(G_\tau^+ f)(k) := \sum_{m=1}^N \alpha_m \sum_{\ell=0}^{k-1} e^{(k-\ell-1)\gamma_m \tau} f(\ell).$$

A routine calculation shows that

$$\mathcal{S}_\tau G^+ \mathcal{H}_\tau = G_\tau^+ + \sum_{m=1}^N \frac{\kappa_m(\beta_m - 1)}{\gamma_m} I, \quad \text{where } \beta_m := \int_0^\tau w(t) e^{\gamma_m t} dt.$$

This yields

$$\mathcal{S}_\tau G \mathcal{H}_\tau = G_\tau^+ + \sum_{m=1}^N \frac{\kappa_m(\beta_m - 1)}{\gamma_m} I + \mathcal{S}_\tau G^- \mathcal{H}_\tau.$$

Note that $\mathcal{S}_\tau G \mathcal{H}_\tau$ is the discrete-time input-output operator associated with the transfer function $\mathbf{G}_\tau$. Then we obtain

$$\mathbf{G}_\tau^- = \mathbf{G}_\tau - \mathbf{G}_\tau^+ = \sum_{m=1}^N \frac{\kappa_m(\beta_m - 1)}{\gamma_m} + \mathbf{H}_\tau,$$

where $\mathbf{H}_\tau$ is the transfer function of the discrete-time input-output operator $\mathcal{S}_\tau G^- \mathcal{H}_\tau$. Choose a rational function $\mathbf{R} \in H^\infty(\mathbb{E}_1)$ as a constant function

$$\mathbf{R}(z) \equiv \sum_{m=1}^N \frac{\kappa_m(\beta_m - 1)}{\gamma_m}.$$

A simple calculation gives

$$\|\mathcal{H}_\tau\|_{\mathcal{L}(l^2(\mathbb{Z}_+), L^2(\mathbb{R}_+))} = \sqrt{\tau}, \qquad \|\mathcal{S}_\tau\|_{\mathcal{L}(L^2(\mathbb{R}_+), l^2(\mathbb{Z}_+))} = \|w\|_{L^2(0,\tau)}.$$

Noting that the transfer function $\mathbf{G}^- = \mathbf{G} - \mathbf{G}^+$ of an exponentially stable well-posed system satisfies $\mathbf{G}^- \in H^\infty(\mathbb{C}_0)$, we obtain

$$\|\mathbf{G}_\tau^- - \mathbf{R}\|_{H^\infty(\mathbb{E}_1)} = \|\mathbf{H}_\tau\|_{H^\infty(\mathbb{E}_1)} \le \sqrt{\tau}\|w\|_{L^2(0,\tau)} \cdot \|\mathbf{G}^-\|_{H^\infty(\mathbb{C}_0)}.$$

Thus, if

$$\|\mathbf{G}^-\|_{H^\infty(\mathbb{C}_0)} = \|\mathbf{G} - \mathbf{G}^+\|_{H^\infty(\mathbb{C}_0)} < \frac{1}{\sqrt{\tau} M_1 \|w\|_{L^2(0,\tau)} \cdot \|\mathbf{D}_+\|_{H^\infty(\mathbb{E}_1)}}, \quad (4.5)$$

then we can design a regulating controller, where $M_1 > 0$ is defined as in (2.30) and $\mathbf{N}_+/\mathbf{D}_+$ is a coprime factorization of $\mathbf{G}_\tau^+$ over the set of rational functions in $\in H^\infty(\mathbb{E}_1)$.

4.2 Numerical simulation

In what follows, we consider the case $q = \widehat{q} = 1$ $A_0 = A_1 = 0.2$, $b = 1$, $c_1 = 1$, $h_1 = 1$, $\widehat{h}_1 = 0.1$, $\tau = 2$, $w(t) \equiv 1/2$. We first show that

$$g(s) := s - A_0 - A_1 e^{-h_1 s} = s - 0.2 - 0.2e^{-s}$$

has only one zero in $\mathrm{cl}(\mathbb{C}_0)$ in a way similar to Example 5.2.13 of [10]. Define $g_1(s) := s - 1$ and $g_2(s) := 0.8 - 0.2e^{-s}$. For every $s \in \mathbb{C}_0$ satisfying $|s| > 2$, we obtain $|g_1(s)| \geq |s| - 1 > 1$ and $|g_2(s)| \leq 0.8 + 0.2|e^{-s}| \leq 1$. Therefore, $|g_1(s)| > |g_2(s)|$ for every $s \in \mathbb{C}_0$ with $|s| > 2$. On the other hand, for every $\omega \in \mathbb{R}$, $|g_1(i\omega)|^2 = 1 + \omega^2$ and $|g_2(i\omega)|^2 = 0.68 - 0.32\cos\omega$. Hence

$$|g_1(i\omega)| > |g_2(i\omega)| \qquad \forall \omega \in \mathbb{R}. \quad (4.6)$$

Rouche's theorem shows that g_1 and $g = g_1 + g_2$ have the same number of zeros in $\mathbb{C}_0$, where each zero is counted as many times as its multiplicity. Thus, g has only one simple zero in $\mathbb{C}_0$. Moreover, (4.6) yields

$$|g(i\omega)| \geq |g_1(i\omega)| - |g_2(i\omega)| > 0 \qquad \omega \in \mathbb{R},$$

and hence g has no zeros on the imaginary axis. Since $g(s)$ is negative at $s = 0$ and positive at $s = +\infty$, it follows that the zero of g in $\mathrm{cl}(\mathbb{C}_0)$ is real. Thus, the generator A has only an eigenvalue at $s = \gamma \approx 0.3421$ in $\mathrm{cl}(\mathbb{C}_0)$, and the assumption $\langle$b1$\rangle$ in Section 3.3 is satisfied.

The transfer function of the delay system (4.1) is given by

$$\mathbf{G}(s) = \frac{e^{-\widehat{h}_1 s}}{s - A_0 - A_1 e^{-h_1 s}}.$$

Since

$$\mathbf{G}(0) = \frac{-1}{A_0 + A_1} = -\frac{5}{2} \neq 0,$$

it follows that the assumption $\langle$b2$\rangle$ in Section 3.3 holds.

By (4.3), the eigenvectors ϕ of A and ψ of A^* corresponding to the eigenvalue γ are given by

$$\phi = \begin{bmatrix} 1 \\ \varpi_\gamma \end{bmatrix}, \qquad \psi = \frac{1}{d} \begin{bmatrix} 1 \\ A_1 e^{-\gamma h_1}/\varpi_\gamma \end{bmatrix},$$

where $\varpi_\gamma(\theta) := e^{\gamma\theta}$ for every $\theta \in [-h_1, 0]$ and $d := 1 + A_1 h_1 e^{-\gamma h_1}$. Then ϕ and ψ satisfy $(\phi, \psi) = 1$. It follows from (4.4) that the projection Π is given by

$$\Pi x = (x, \psi)\phi \qquad \forall x \in X.$$

Hence, $X^+ = \Pi X = \{s\phi : s \in \mathbb{C}\}$. For $s \in \mathbb{C}$,

$$A^+(s\phi) = s\gamma\phi, \quad B^+ s = \frac{s}{d}\phi, \quad C^+(s\phi) = se^{-\gamma\widehat{h}_1}$$

and

$$\mathbf{T}_t^+(s\phi) = se^{\gamma t}\phi \qquad \forall t \geq 0.$$

In the previous subsection, we have showed that the assumptions $\langle$b3$\rangle$ and $\langle$b4$\rangle$ in Section 3.3 hold. The assumptions $\langle$b5$\rangle$–$\langle$b9$\rangle$ are clearly satisfied. The transfer function of the unstable part of $\mathbf{G}$ is given by

$$\mathbf{G}^+(s) = \frac{\kappa}{s-\gamma} \qquad \text{where} \quad \kappa := \frac{e^{-\gamma\widehat{h}_1}}{d}.$$

Similarly, the transfer function of the unstable part of $\mathbf{G}_\tau$ is

$$\mathbf{G}_\tau^+(z) = \frac{\alpha}{z - e^{\gamma\tau}}, \qquad \text{where} \quad \alpha := \kappa\frac{e^{\gamma\tau}-1}{\gamma}\int_0^\tau w(t)e^{\gamma t}dt.$$

Define

$$\mathbf{N}_+(z) := \frac{\alpha}{z-a}, \quad \mathbf{D}_+(z) := \frac{z - e^{\gamma\tau}}{z-a}, \qquad \text{where } a := 0.9.$$

Then $\mathbf{N}_+/\mathbf{D}_+$ is a coprime factorization of $\mathbf{G}_\tau^+$ over the set of rational functions in $H^\infty(\mathbb{E}_1)$. Using Lemma 3.8, we obtain

$$\delta^* := \left|\frac{1}{\mathbf{D}_+(1)\mathbf{G}_\tau(1)}\right| = \left|\frac{1}{\mathbf{D}_+(1)\mathbf{G}(0)}\right| = \frac{(1-a)(A_0+A_1)}{e^{\gamma\tau}-1}.$$

There exists a rational function $\mathbf{Y}_+ \in H^\infty(\mathbb{E}_1)$ satisfying interpolation conditions (2.18) and the norm condition $\|\mathbf{Y}_+\|_{H^\infty(\mathbb{E}_1)} < 1 =: M$.

Choose a rational function $\mathbf{R} \in H^\infty(\mathbb{E}_1)$ as a constant function

$$\mathbf{R}(z) \equiv \frac{\kappa(\beta-1)}{\gamma}, \quad \text{where } \beta := \int_0^\tau w(t)e^{\gamma t}dt,$$

and let us next show that (4.5) is satisfied. A numerical computation shows that

$$\|\mathbf{G} - \mathbf{G}^+\|_{H^\infty(\mathbb{C}_0)} = \sup_{\omega\in\mathbb{R}} |\mathbf{G}(i\omega) - \mathbf{G}^+(i\omega)| < 0.1.$$

Define

$$M_1 := \max\{2\delta^*, M\} = M = 1.$$

Since $\|w\|_{L^2(0,\tau)} = 1/\sqrt{\tau}$ for the case $w(t) \equiv 1/\tau$, it follows that

$$\frac{1}{\sqrt{\tau}M_1\|w\|_{L^2(0,\tau)}\|\mathbf{D}_+\|_{H^\infty(\mathbb{E}_1)}} = \frac{1}{M_1\|\mathbf{D}_+\|_{H^\infty(\mathbb{E}_1)}} \approx 0.1018,$$

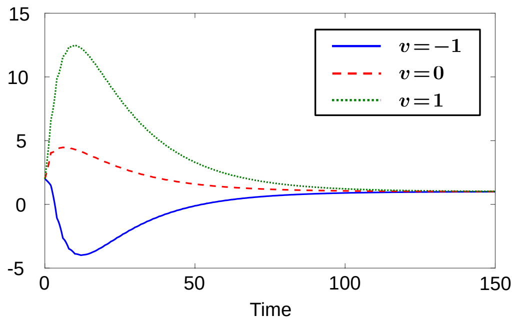

Fig. 3 Time response of y with $y_{\mathrm{ref}} = 1$.

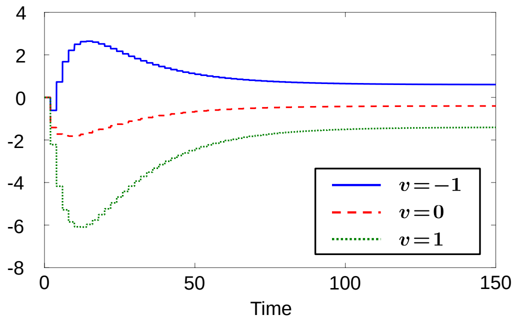

Fig. 4 Time response of u with $y_{\mathrm{ref}} = 1$.

and hence (4.5) is satisfied.

Define a rational function $\mathbf{Y}_+ \in H^\infty(\mathbb{E}_1)$ by

$$\mathbf{Y}_+(z) := \frac{0.7712z - 0.7602}{z^2 - 0.7328z},$$

which satisfies the interpolation conditions (2.18), (2.23) and the norm condition $\|\mathbf{Y}_+\|_{H^\infty(\mathbb{E}_1)} < M_1$. By the construction used in the proof of Theorem 2.5, a minimal realization of the digital controller

$$\mathbf{K}(z) := \frac{(\mathbf{Y}_+\mathbf{D}_+)(z)}{1 - (\mathbf{N}_+ + \mathbf{D}_+\mathbf{R})(z)\mathbf{Y}_+(z)} = \frac{0.7712z - 0.7602}{z^2 - 0.4814z - 0.5186}$$

is a solution of Problem 3.3.

Figs. 3 and 4 illustrate the time responses of the output y and the input u, respectively. The initial states of the plant and the controller are chosen as $z^0 = 2$, $\varpi(\theta) \equiv 2$, and $x_{\mathrm{d}}^0 = \begin{bmatrix} 0 & 0 \end{bmatrix}^*$, respectively. The reference and disturbance signals are given by $y_{\mathrm{ref}} = 1$ and $v \in \{-1, 0, 1\}$, respectively.

5 Conclusion

We have studied the sampled-data output regulation problem for infinite-dimensional systems with constant reference and disturbance signals. Our main contribution is to obtain a sufficient condition for this control problem to be solvable with a finite-dimensional controller. To this end, we have proposed a design method of finite-dimensional controllers for the robust output regulation of infinite-dimensional discrete-time systems. In the controller design, the discrete-time output regulation problem has been reduced to the Nevanlinna-Pick interpolation problem. We have also applied the obtained results to systems with state and output delays. In future work on sampled-data output regulation, we are planning to design generalized hold functions for infinite-dimensional systems with general reference and disturbance signals.

A Nevanlinna-Pick interpolation problem

In this section, we obtain a necessary and sufficient condition for the solvability of the interpolation problem to which we reduce the design problem of regulating controllers. In the process, we also show how to construct a solution of the interpolation problem. Although we consider $H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times q})$ in Section 2, the standard theory of the Nevanlinna-Pick interpolation problem uses $H^\infty(\mathbb{D}, \mathbb{C}^{p\times q})$. Hence, it is convenient to map $\mathbb{E}_1$ to $\mathbb{D}$ via the bilinear transformation $\varphi : \mathbb{E}_1 \to \mathbb{D} : z \mapsto 1/z$.

In Section A.1, we recall basic facts on the Nevanlinna-Pick interpolation problem only with conditions on the interior $\mathbb{D}$. Section A.2 is devoted to solving the Nevanlinna-Pick interpolation problem with conditions on both the interior $\mathbb{D}$ and the boundary $\mathbb{T}$. As in [30, 48], we transform this problem into the Nevanlinna-Pick interpolation problem only with conditions on the boundary $\mathbb{T}$, which is always solvable.

A.1 Interpolation problem only with interior conditions

First we consider interpolation problems only with interior interpolation conditions.

Problem A.1 (Chapter 18 in [4], Section II in [21]) *Suppose that $\alpha_1, \dots, \alpha_n \in \mathbb{D}$ are distinct. Let vector pairs $(\xi_\ell, \eta_\ell) \in \mathbb{C}^p \times \mathbb{C}^q$ satisfy*

$$\|\xi_\ell\|_{\mathbb{C}^p} > \|\eta_\ell\|_{\mathbb{C}^q} \qquad \forall \ell \in \{1, \dots, n\}. \tag{A.1}$$

Find $\Phi \in H^\infty(\mathbb{D}, \mathbb{C}^{p\times q})$ such that $\|\Phi\|_{H^\infty(\mathbb{D})} < 1$ and

$$\xi_\ell^* \Phi(\alpha_\ell) = \eta_\ell^* \qquad \forall \ell \in \{1, \dots, n\}.$$

We call this problem the *Nevanlinna-Pick interpolation problem with n interpolation data $(\alpha_\ell, \xi_\ell, \eta_\ell)_{\ell=1}^n$*. The solvability of Problem A.1 can be characterized by the so-called Pick matrix.

Theorem A.2 (Theorem 18.2.3 in [4], Theorem 2 in [21]) *Consider Problem A.1. Define the Pick matrix P by*

$$P := \begin{bmatrix} P_{1,1} & \cdots & P_{1,n} \\ \vdots & & \vdots \\ P_{n,1} & \cdots & P_{n,n} \end{bmatrix}, \quad \text{where } P_{j,\ell} := \frac{\xi_j^* \xi_\ell - \eta_j^* \eta_\ell}{1 - \alpha_j \overline{\alpha}_\ell} \quad \forall j, \ell \in \{1, \dots, n\}.$$

Problem A.1 is solvable if and only if P is positive definite.

Let us next introduce an algorithm to construct a solution of Problem A.1. To this end, define

$$\mathcal{B} := \{E \in \mathbb{C}^{p \times q} : \|E\|_{\mathbb{C}^{p\times q}} < 1\}.$$

Let I_p and I_q be the identity matrix with dimension p and q, respectively. For a matrix $E \in \mathcal{B}$, define

$$A(E) := (I_p - EE^*)^{-1/2}, \quad B(E) := -(I_p - EE^*)^{-1/2} E \tag{A.2a}$$
$$C(E) := -(I_q - E^*E)^{-1/2} E^*, \quad D(E) := (I_q - E^*E)^{-1/2}, \tag{A.2b}$$

where $M^{-1/2}$ denotes the inverse of the Hermitian square root of a positive definite matrix M. Define the maps U_E and V_E by

$$\begin{aligned} U_E &: \mathbb{C}^p \times \mathbb{C}^q \to \mathbb{C}^p : (\xi, \eta) \mapsto A(E)\xi + B(E)\eta \\ V_E &: \mathbb{C}^p \times \mathbb{C}^q \to \mathbb{C}^q : (\xi, \eta) \mapsto C(E)\xi + D(E)\eta. \end{aligned}$$

The mapping T_E in the lemma below is useful for solving Problem A.1.

Lemma A.3 (Lemma 6.5.10 in [47]) *For a matrix $E \in \mathcal{B}$, define the matrices $A(E)$, $B(E)$, $C(E)$, and $D(E)$ by (A.2). The mapping*

$$T_E : \mathcal{B} \to \mathcal{B} : X \mapsto \big(A(E)X + B(E)\big)\big(C(E)X + D(E)\big)^{-1} \tag{A.3}$$

is well-defined and bijective.

A routine calculation shows that the inverse of T_E is given by

$$\begin{aligned} T_E^{-1}(Y) &= \big(A(E) - YC(E)\big)^{-1}\big(YD(E) - B(E)\big) \\ &= \big(A(E)Y - B(E)\big)\big(-C(E)Y + D(E)\big)^{-1}. \end{aligned} \tag{A.4}$$

Lemma A.4 (Lemma 1 in [21]) *Consider Problem A.1 with n interpolation data $(\alpha_\ell, \xi_\ell, \eta_\ell)_{\ell=1}^n$. Set $E := \xi_1 \eta_1^* / \|\xi_1\|^2_{\mathbb{C}^p}$ and define $A(E)$, $B(E)$, $C(E)$, and $D(E)$ as in (A.2). Define also $\nu := U_E(\xi_1, \eta_1)$ and*

$$\kappa(z) := \begin{cases} \frac{|\alpha_1|}{\alpha_1} \frac{z - \alpha_1}{1 - \overline{\alpha}_1 z} & \text{if } \alpha_1 \neq 0 \\ z & \text{if } \alpha_1 = 0 \end{cases}, \quad X := I_p + (\kappa - 1)\frac{\nu\nu^*}{\|\nu\|^2_{\mathbb{C}^p}}. \tag{A.5}$$

Problem A.1 with n interpolation data $(\alpha_\ell, \xi_\ell, \eta_\ell)_{\ell=1}^n$ is solvable if and only if Problem A.1 with $n-1$ interpolation data

$$\big(\alpha_\ell, X(\alpha_\ell)^* U_E(\xi_\ell, \eta_\ell), V_E(\xi_\ell, \eta_\ell)\big)_{\ell=2}^n \tag{A.6}$$

is solvable. Moreover, if Φ_{n-1} is a solution of the problem with $n-1$ interpolation data given in (A.6)*, then*

$$\Phi_n := T_{-E}\left(X\Phi_{n-1}\right) = \big(A(E)X\Phi_{n-1} - B(E)\big)\big(-C(E)X\Phi_{n-1} + D(E)\big)^{-1} \tag{A.7}$$

is a solution Φ_n of the original problem with n interpolation data $(\alpha_\ell, \xi_\ell, \eta_\ell)_{\ell=1}^n$.

The iterative algorithm derived from Lemma A.4 is called the *Schur-Nevanlinna algorithm*. Lemma A.4 also shows that if the problem is solvable, then there exist always solutions whose elements are rational functions.

Note that ν given in Lemma A.4 is nonzero. In fact, since $\|\xi_1\|_{\mathbb{C}^p} > \|\eta_1\|_{\mathbb{C}^q}$, it follows that

$$A(E)^{-1}\nu = \xi_1 - E\eta_1 = \xi_1 - \frac{\|\eta_1\|^2_{\mathbb{C}^q}}{\|\xi_1\|^2_{\mathbb{C}^p}}\xi_1 \neq 0,$$

and hence $\nu \neq 0$. Furthermore, the matrix X defined by (A.5) satisfies $X(\lambda)^{-1} = X(\lambda)^*$ for all $\lambda \in \mathbb{T}$ and $\|X(z)\|_{\mathbb{C}^{p\times p}} < 1$ for all $z \in \mathbb{D}$.

A.2 Interpolation problem with both interior and boundary conditions

We next study interpolation problems with both interior and boundary conditions.

Problem A.5 *Suppose that $\alpha_1, \dots, \alpha_n \in \mathbb{D}$ and $\lambda_1, \dots, \lambda_m \in \mathbb{T}$ are distinct. Consider vector pairs $(\xi_\ell, \eta_\ell) \in \mathbb{C}^p \times \mathbb{C}^q$ for $\ell \in \{1, \dots, n\}$ and matrices $F_j, G_j \in \mathbb{C}^{p\times q}$ for $j \in \{1, \dots, m\}$, and suppose that*

$$\|\xi_\ell\|_{\mathbb{C}^p} > \|\eta_\ell\|_{\mathbb{C}^q} \qquad \forall \ell \in \{1, \dots, n\} \tag{A.8a}$$

$$\|F_j\|_{\mathbb{C}^{p\times q}} < 1 \qquad \forall j \in \{1, \dots, m\}. \tag{A.8b}$$

Find a rational function $\Phi \in H^\infty(\mathbb{D}, \mathbb{C}^{p\times q})$ such that $\|\Phi\|_{H^\infty(\mathbb{D})} < 1$ and

$$\xi_\ell^* \Phi(\alpha_\ell) = \eta_\ell^* \qquad \forall \ell \in \{1, \dots, n\} \tag{A.9a}$$

$$\Phi(\lambda_j) = F_j, \quad \Phi'(\lambda_j) = G_j \qquad \forall j \in \{1, \dots, m\}. \tag{A.9b}$$

Problem A.5 is called the *Nevanlinna-Pick interpolation problem with interior interpolation data $(\alpha_\ell, \xi_\ell, \eta_\ell)_{\ell=1}^n$ and boundary interpolation data $(\lambda_j, F_j, G_j)_{j=1}^m$.* The scalar-valued case $p = q = 1$ with more general interpolation conditions has been studied in [30].

The following theorem implies that the solvability of Problem A.5 depends only on its interior interpolation data.

Theorem A.6 *Problem A.5 with interior interpolation data $(\alpha_\ell, \xi_\ell, \eta_\ell)_{\ell=1}^n$ and boundary interpolation data $(\lambda_j, F_j, G_j)_{j=1}^m$ is solvable if and only if Problem A.1 with interpolation data $(\alpha_\ell, \xi_\ell, \eta_\ell)_{\ell=1}^n$ is solvable.*

To solve Problem A.5, we transform it to the following problem with boundary conditions only:

Problem A.7 *Suppose that $\lambda_1, \dots, \lambda_m \in \mathbb{T}$ are distinct. Consider matrices $F_j, G_j \in \mathbb{C}^{p\times q}$ for $j \in \{1, \dots, m\}$, and suppose that*

$$\|F_j\|_{\mathbb{C}^{p\times q}} < 1 \qquad \forall j \in \{1, \dots, m\}. \tag{A.10}$$

Find a rational function $\Phi \in H^\infty(\mathbb{D}, \mathbb{C}^{p\times q})$ such that $\|\Phi\|_{H^\infty(\mathbb{D})} < 1$ and

$$\Phi(\lambda_j) = F_j, \quad \Phi'(\lambda_j) = G_j \qquad \forall j \in \{1, \dots, m\}.$$

This problem is referred to as the *boundary Nevanlinna-Pick interpolation problem with interpolation data* $(\lambda_j, F_j, G_j)_{j=1}^m$. The condition (A.10) is necessary for the solvability for Problem A.7, and the lemma below shows that the condition (A.10) is also sufficient. We can prove the sufficiency by extending the Schur-Nevanlinna algorithm in Lemma A.4.

Lemma A.8 *Problem A.7 is always solvable.*

Proof Consider Problem A.7 with interpolation data $(\lambda_j, F_j, G_j)_{j=1}^m$. We first find $m-1$ interpolation data such that if Problem A.7 with these $m-1$ data is solvable, then the original problem with m interpolation data $(\lambda_j, F_j, G_j)_{j=1}^m$ is also solvable. To that purpose, we extend the technique developed in [30] for the scalar-valued case.

Define $A := A(F_1)$, $B := B(F_1)$, $C := C(F_1)$, and $D := D(F_1)$ as in (A.2). For $\epsilon > 0$, set

$$\begin{aligned}\kappa_\epsilon(z) &:= \frac{1}{\lambda_1}\frac{z-\lambda_1}{(1+\epsilon)-\overline{\lambda}_1 z}\\ \widehat{F}_1 &:= \epsilon\lambda_1(I_p - F_1F_1^*)^{-1/2}G_1(I_q - F_1^*F_1)^{-1/2}\end{aligned}$$

and

$$\begin{aligned}\widehat{F}_j &:= \frac{1}{\kappa_\epsilon(\lambda_j)}T_{F_1}(F_j)\\ \widehat{G}_j &:= \frac{1}{\kappa_\epsilon(\lambda_j)}(A - \kappa_\epsilon(\lambda_j)\widehat{F}_jC)G_j(CF_j + D)^{-1} - \frac{\kappa_\epsilon'(\lambda_j)}{\kappa_\epsilon(\lambda_j)}\widehat{F}_j\end{aligned}$$

for $j \in \{2, \dots, m\}$. Let us show that there exists $\epsilon > 0$ such that

$$\|\widehat{F}_j\|_{\mathbb{C}^{p\times q}} < 1 \qquad \forall j \in \{1, \dots, m\}. \tag{A.11}$$

By definition,

$$\|\widehat{F}_1\|_{\mathbb{C}^{p\times q}} \leq \epsilon\|G_1\|_{\mathbb{C}^{p\times q}} \cdot \left\|(I_p - F_1F_1^*)^{-1/2}\right\|_{\mathbb{C}^{p\times p}} \cdot \left\|(I_q - F_1^*F_1)^{-1/2}\right\|_{\mathbb{C}^{q\times q}},$$

and hence if

$$\epsilon < \frac{1}{\|G_1\|_{\mathbb{C}^{p\times q}} \cdot \left\|(I_p - F_1F_1^*)^{-1/2}\right\|_{\mathbb{C}^{p\times p}} \cdot \left\|(I_q - F_1^*F_1)^{-1/2}\right\|_{\mathbb{C}^{q\times q}}}, \tag{A.12}$$

then $\|\widehat{F}_1\|_{\mathbb{C}^{p\times q}} < 1$. Let $j \in \{2,\dots,m\}$ be given. We obtain

$$\|\widehat{F}_j\|_{\mathbb{C}^{p\times q}} \leq \left(1 + \frac{\epsilon}{|\lambda_j - \lambda_1|}\right) \|T_{F_1}(F_j)\|_{\mathbb{C}^{p\times q}}. \tag{A.13}$$

Since $F_j \in \mathcal{B}$, it follows that $\|T_{F_1}(F_j)\|_{\mathbb{C}^{p\times q}} < 1$ by Lemma A.3. If we choose $\epsilon > 0$ so that

$$\epsilon < \min_{j=2,\dots,m} \left(|\lambda_j - \lambda_1| \left(\frac{1}{\|T_{F_1}(F_j)\|_{\mathbb{C}^{p\times q}}} - 1 \right)\right), \tag{A.14}$$

then $\|\widehat{F}_j\|_{\mathbb{C}^{p\times q}} < 1$ for every $j \in \{2,\dots,m\}$. Thus, we obtain the desired inequality (A.11) for $\epsilon > 0$ satisfying (A.12) and (A.14).

Assume that there exists a rational solution $\Psi_{m-1} \in H^\infty(\mathbb{D}, \mathbb{C}^{p\times q})$ such that

$$\|\Phi_{m-1}\|_{H^\infty(\mathbb{D})} < 1 \tag{A.15a}$$

$$\Psi_{m-1}(\lambda_j) = \widehat{F}_j \qquad \forall j \in \{1,\dots,m\} \tag{A.15b}$$

$$\Psi'_{m-1}(\lambda_j) = \widehat{G}_j \qquad \forall j \in \{2,\dots,m\} \tag{A.15c}$$

We shall show that $\Psi_m := T_{F_1}^{-1}(\kappa_\epsilon \Psi_{m-1})$ is a solution of the original problem with m interpolation data $(\lambda_j, F_j, G_j)_{j=1}^m$. By definition, Ψ_m is rational. Since $\|\kappa_\epsilon\|_{H^\infty(\mathbb{D})} < 1$ and $\|\Psi_{m-1}\|_{H^\infty(\mathbb{D})} < 1$, it follows that

$$\kappa_\epsilon(z)\Psi_{m-1}(z) \in \mathcal{B} \qquad \forall z \in \mathrm{cl}(\mathbb{D}).$$

Together with this, Lemma A.3 yields $\Psi_m \in H^\infty(\mathbb{D}, \mathbb{C}^{p\times q})$ and $\|\Psi_m\|_{H^\infty(\mathbb{D})} < 1$.

We now prove that Ψ_m satisfies the interpolation conditions $\Psi_m(\lambda_j) = F_j$ and $\Psi'_m(\lambda_j) = G_j$ for every $j \in \{1,\dots,m\}$. For the case $j = 1$, $\kappa_\epsilon(\lambda_1) = 0$ yields

$$\Psi_m(\lambda_1) = T_{F_1}^{-1}\big(\kappa_\epsilon(\lambda_1)\Psi_{m-1}(\lambda_1)\big) = F_1.$$

By (A.4), we obtain

$$(A - \kappa_\epsilon \Psi_{m-1} C)\Psi_m = \kappa_\epsilon \Psi_{m-1} D - B,$$

which implies

$$(\kappa_\epsilon \Psi'_{m-1} + \kappa'_\epsilon \Psi_{m-1})(C\Psi_m + D) = (A - \kappa_\epsilon \Psi_{m-1} C)\Psi'_m. \tag{A.16}$$

Therefore,

$$\Psi'_m(\lambda_1) = \kappa'_\epsilon(\lambda_1) A^{-1} \widehat{F}_1 (CF_1 + D).$$

Since

$$\kappa'_\epsilon(z) = \frac{1}{\lambda_1} \frac{\epsilon}{\big((1+\epsilon) - \overline{\lambda}_1 z\big)^2},$$

it follows that $\kappa_\epsilon'(\lambda_1) = 1/(\epsilon\lambda_1)$. Using

$$A^{-1} = (I_p - F_1F_1^*)^{1/2}, \quad CF_1 + D = (I_q - F_1^*F_1)^{1/2},$$

we derive $\Psi_m'(\lambda_1) = G_1$.

For $j \in \{2, \dots, m\}$, we have by the definition of $\widehat{F}_j$ that,

$$\Psi_m(\lambda_j) = T_{F_1}^{-1}(\kappa_\epsilon(\lambda_j)\widehat{F}_j) = T_{F_1}^{-1}\big(T_{F_1}(F_j)\big) = F_j.$$

Using (A.16) again, we obtain

$$\kappa_\epsilon(\lambda_j)\widehat{G}_j + \kappa_\epsilon'(\lambda_j)\widehat{F}_j = (A - \kappa_\epsilon(\lambda_j)\widehat{F}_jC)\Psi_m'(\lambda_j)(CF_j + D)^{-1}.$$

By the definition of $\widehat{G}_j$, we find that

$$\Psi_m'(\lambda_j) = G_j \qquad \forall j \in \{2, \dots, m\}.$$

Thus Φ_m is a solution of the original problem with m interpolation conditions.

If we apply this procedure again to the resulting interpolation problem, i.e., the problem of finding a rational solution $\Psi_{m-1} \in H^\infty(\mathbb{D}, \mathbb{C}^{p\times q})$ such that the conditions given in (A.15) hold, then the interpolation condition at $z = \lambda_1$ is removed. Therefore, Problem A.7 with m interpolation data can be reduced to Problem A.7 with $m-1$ interpolation data. Continuing in this way, we finally obtain Problem A.7 with no interpolation conditions, which always admits a solution. Thus Problem A.7 is always solvable. $\square$

By Lemmas A.4 and A.8, we obtain a proof of Theorem A.6.

Proof (of Theorem A.6) The necessity is straightforward. We prove the sufficiency. To this end, it is enough to show that the following problem always has a solution:

Problem A.9 *Assume that Problem A.1 with n interior interpolation data $(\alpha_\ell, \xi_\ell, \eta_\ell)_{\ell=1}^n$ is solvable and that $\|F_j\|_{\mathbb{C}^{p\times q}} < 1$ for every $j \in \{1, \dots, m\}$. Find a solution of Problem A.5 with n interior interpolation data $(\alpha_\ell, \xi_\ell, \eta_\ell)_{\ell=1}^n$ and m boundary interpolation data $(\lambda_j, F_j, G_j)_{j=1}^m$.*

Suppose that Problem A.1 with n interior interpolation data $(\alpha_\ell, \xi_\ell, \eta_\ell)_{\ell=1}^n$ is solvable. Define the matrix E and the function X as in Lemma A.4. Then this lemma shows that Problem A.1 with $n-1$ interior interpolation data

$$\big(\alpha_\ell, X(\alpha_\ell)^*U_E(\xi_\ell, \eta_\ell), V_E(\xi_\ell, \eta_\ell)\big)_{\ell=2}^n \tag{A.17}$$

is solvable. Set $A := A(E)$, $B := B(E)$, $C := C(E)$, and $D := D(E)$ as in (A.2). For $j \in \{1, \dots, m\}$, define also

$$\begin{aligned}
\widehat{F}_j &:= X(\lambda_j)^{-1}T_{-E}^{-1}(F_j) \\
\widehat{G}_j &:= X(\lambda_j)^{-1}(A + F_jC)^{-1}G_j(-CX(\lambda_j)\widehat{F}_j + D) - X(\lambda_j)^{-1}X'(\lambda_j)\widehat{F}_j.
\end{aligned}$$

Since $X(\lambda_j)^{-1} = X(\lambda_j)^*$ for every $j \in \{1,\dots,m\}$, we obtain $\|X(\lambda_j)^{-1}\|_{\mathbb{C}^{p\times p}} = 1$ and hence $\|\widehat{F}_j\|_{\mathbb{C}^{p\times p}} < 1$ for every $j \in \{1,\dots,m\}$. Suppose that Φ_{n-1} is a solution of Problem A.5 with $n-1$ interior interpolation data given in (A.17) and m boundary interpolation data $(\lambda_j, \widehat{F}_j, \widehat{G}_j)_{j=1}^m$. Then $\Phi_n := T_{-E}(X\Phi_{n-1})$ is a solution of Problem A.5 with n interior interpolation data $(\alpha_\ell, \xi_\ell, \eta_\ell)_{\ell=1}^n$ and m boundary interpolation data $(\lambda_j, F_j, G_j)_{j=1}^m$. In fact, Lemma A.4 shows that Φ_n satisfies $\|\Phi_n\|_{H^\infty(\mathbb{D})} < 1$ and $\xi_\ell^*\Phi_n(\alpha_\ell) = \eta_\ell^*$ for every $\ell \in \{1,\dots,n\}$. It remains to show that the boundary conditions hold. We obtain

$$\Phi_n(\lambda_j) = T_{-E}\big(X(\lambda_j)\widehat{F}_j\big) = T_{-E}\big(T_{-E}^{-1}(F_j)\big) = F_j \qquad \forall j \in \{1,\dots,m\}.$$

By the definition of T_{-E}, we obtain

$$\Phi_n(-CX\Phi_{n-1} + D) = (AX\Phi_{n-1} - B),$$

and hence

$$\Phi_n'(-CX\Phi_{n-1} + D) = (A + \Phi_n C)(X\Phi_{n-1})'.$$

This yields

$$\Phi_n'(\lambda_j) = (A + F_jC)(X(\lambda_j)\widehat{G}_j + X'(\lambda_j)\widehat{F}_j)(-CX(\lambda_j)\widehat{F}_j + D)^{-1} = G_j.$$

Thus, we can reduce Problem A.9 with n interior data to that with $n-1$ interior data. Continuing in this way, we reduce Problem A.5 to Problem A.7, which is always solvable by Lemma A.8. This completes the proof. □

In the construction of regulating controllers in Section 2, a rational function $\mathbf{Y}_+ \in H^\infty(\mathbb{E}_1, \mathbb{C}^{p\times p})$ needs to satisfy the interpolation condition $\mathbf{Y}_+(\infty) = 0$. Its counterpart in $H^\infty(\mathbb{D}, \mathbb{C}^{p\times p})$ under the transformation $\varphi : \mathbb{E}_1 \to \mathbb{D} : z \mapsto 1/z$ is given by the interpolation condition $(\mathbf{Y}_+\circ\varphi^{-1})(0) = 0$. Such a condition is excluded in Problem A.5, but we can easily incorporate it into the problem.

Corollary A.10 *Suppose that $\alpha_1,\dots,\alpha_n \in \mathbb{D}\setminus\{0\}$ and $\lambda_1,\dots,\lambda_m \in \mathbb{T}$ are distinct. Consider vector pairs $(\xi_\ell, \eta_\ell) \in \mathbb{C}^p \times \mathbb{C}^q$ for $\ell \in \{1,\dots,n\}$ and matrices $F_j, G_j \in \mathbb{C}^{p\times q}$ for $j \in \{1,\dots,m\}$, and suppose that the norm conditions (A.8) are satisfied. Then the following three statements are equivalent:*

1) *There exists a rational function $\Phi \in H^\infty(\mathbb{D}, \mathbb{C}^{p\times q})$ such that $\|\Phi\|_{H^\infty(\mathbb{D})} < 1$, $\Phi(0) = 0$, and the interpolation conditions (A.9a) and (A.9b) hold.*
2) *There exists a rational function $\Phi \in H^\infty(\mathbb{D}, \mathbb{C}^{p\times q})$ such that $\|\Phi\|_{H^\infty(\mathbb{D})} < 1$, $\Phi(0) = 0$, and the interpolation conditions (A.9a) hold.*
3) *The Pick matrix P defined by*

$$P := \begin{bmatrix} P_{1,1} & \cdots & P_{1,n} \\ \vdots & & \vdots \\ P_{n,1} & \cdots & P_{n,n} \end{bmatrix}, \quad \text{where } P_{j,k} := \frac{\alpha_j\overline{\alpha}_\ell\xi_j^*\xi_\ell - \eta_j^*\eta_\ell}{1 - \alpha_j\overline{\alpha}_\ell} \quad \forall j,\ell \in \{1,\dots,n\}$$

is positive definite.

Proof By a straightforward calculation, we have the following fact: A rational function $\Phi \in H^\infty(\mathbb{D}, \mathbb{C}^{p\times q})$ satisfies the conditions of 1) if and only if $\widehat{\Phi}(z) := \Phi(z)/z$ is a solution of Problem A.5 with the interior interpolation data $(\alpha_\ell, \overline{\alpha}_\ell\xi_\ell, \eta_\ell)_{\ell=1}^n$ and the boundary interpolation data $(\lambda_j, F_j/\lambda_j, G_j/\lambda_j - F_j/\lambda_j^2)_{j=1}^m$. This fact together with Theorem A.6 shows that 1) is true if and only if Problem A.1 with the interpolation data $(\alpha_\ell, \overline{\alpha}_\ell\xi_\ell, \eta_\ell)_{\ell=1}^n$ is solvable. Hence, we obtain 1) $\Leftrightarrow$ 3) by Theorem A.2. Using the fact mentioned above again, we obtain 1) $\Leftrightarrow$ 2). This completes the proof. □

Remark A.11 Suppose that the interpolation data have conjugate symmetry in Problem A.5. In other words, suppose that both (α, ξ, η) and $(\overline{\alpha}, \overline{\xi}, \overline{\eta})$ are in its interior interpolation data and that (λ, F, G) and $(\overline{\lambda}, \overline{F}, \overline{G})$ are in its boundary interpolation data. If the interpolation problem is solvable, then there exists a solution that is a rational function with real coefficients. In fact, for every rational function Φ, there uniquely exist rational functions Φ_R and Φ_I with real coefficients such that $\Phi = \Phi_R + i\Phi_I$. If a rational function Φ is a solution of the interpolation problem, then one can easily prove that its real part Φ_R is also a solution.

Remark A.12 Let $\lambda \in \mathbb{T}$. For a vector pair $(\xi, \eta) \in \mathbb{C}^p \times \mathbb{C}^q$, define a matrix $F := \xi\eta^*/\|\xi\|_{\mathbb{C}^p}^2$. If $\|\xi\|_{\mathbb{C}^p} > \|\eta\|_{\mathbb{C}^q}$, then $\|F\|_{\mathbb{C}^{p\times q}} < 1$. Further, if a rational function $\Psi \in H^\infty(\mathbb{D}, \mathbb{C}^{p\times q})$ satisfies $\Psi(\lambda) = F$, then $\xi^*\Psi(\lambda) = \eta^*$. In this way, we can transform the tangential interpolation condition $\xi^*\Psi(\lambda) = \eta^*$ to the matrix-valued interpolation condition $\Psi(\lambda) = F$. This transformation is used in the design procedure of regulating controllers in Section 2 if unstable eigenvalues of A lie on the boundary $\mathbb{T}$. Moreover, the above observation and Theorem A.6 indicate that for $\lambda \in \mathbb{T}$ and $(\xi, \eta) \in \mathbb{C}^p \times \mathbb{C}^q$ with $\|\xi\|_{\mathbb{C}^p} > \|\eta\|_{\mathbb{C}^q}$, boundary interpolation conditions of the form $\xi^*\Psi(\lambda) = \eta^*$ can be also ignored when we determine the solvability of the Nevanlinna-Pick interpolation problem.

B Λ-extension of output operator of delay systems

Consider the delay system (4.1), and define x as in (4.2). The objective of this section is to show for a.e. $t \geq 0$,

$$\sum_{\ell=1}^{\widehat{q}} c_\ell z(t - \widehat{h}_\ell) = C_\Lambda x(t). \tag{B.1}$$

Since $x(t) \in X_1$ for every $t \geq h_q$ and since $C_\Lambda \zeta = C\zeta$ for every $\zeta \in X_1$, it suffices to show (B.1) a.e. on $[0, h_q)$. For simplicity of notation, we consider the case $\widehat{q} = 1$ and define $\widehat{h} := \widehat{h}_1$ and $c := c_1$.

By Lemma 2.4.5 of [10], there exists $s_0 > 0$ such that

$$(sI - A)^{-1}x(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} \qquad \forall s > s_0,\ \forall t \in [0, h_q),$$

where

$$g_1(t) := \Delta(s)^{-1}\left(z(t) + \sum_{j=1}^{q}\int_{-h_j}^{0} e^{-s(\theta+h_j)}A_j z(t+\theta)d\theta\right)$$

$$\big(g_2(t)\big)(\theta) := e^{s\theta}g_1(t) - \int_0^{\theta} e^{s(\theta-\nu)}z(t+\nu)d\nu \qquad \forall \theta \in [-h_q, 0].$$

Hence for every $s > s_0$ and every $t \in [0, h_q)$, we obtain

$$\begin{aligned} Cs(sI-A)^{-1}x(t) &= sc\big(g_2(t)\big)(-\widehat{h}) \\ &= sc\left(e^{-s\widehat{h}}g_1(t) + \int_0^{\widehat{h}} e^{-s(\widehat{h}-\nu)}z(t-\nu)d\nu\right). \end{aligned}$$

Since

$$\lim_{s\to\infty,\, s\in\mathbb{R}} s\Delta(s)^{-1} = I \quad \text{and} \quad z \in L^1\big((-h_q, h_q), \mathbb{C}^n\big),$$

Lebesgue's dominated convergence theorem implies that in the case $\widehat{h} = 0$,

$$\begin{aligned} &\lim_{s\to\infty,\, s\in\mathbb{R}} sc\left(e^{-s\widehat{h}}g_1(t) + \int_0^{\widehat{h}} e^{-s(\widehat{h}-\nu)}z(t-\nu)d\nu\right) \\ &\quad = \lim_{s\to\infty,\, s\in\mathbb{R}} sc\Delta(s)^{-1}\left(z(t) + \sum_{j=1}^{q}\int_{-h_j}^{0} e^{-s(\theta+h_j)}A_j z(t+\theta)d\theta\right) \\ &\quad = cz(t) \qquad \forall t \in [0, h_q). \end{aligned}$$

Thus, we obtain $cz(t-\widehat{h}) = C_\Lambda x(t)$ for every $t \in [0, h_q)$ if $\widehat{h} = 0$.

In the case $\widehat{h} \in (0, h_q)$, we obtain

$$\lim_{s\to\infty,\, s\in\mathbb{R}} se^{-s\widehat{h}}g_1(t) = 0 \qquad \forall t \in [0, h_q).$$

Since $B \in \mathcal{L}(U, X)$, it follows that $x(t) \in \operatorname{dom}(C_\Lambda)$ for a.e. $t \geq 0$ and

$$\begin{aligned} C_\Lambda x(t) &= \lim_{s\to\infty,\, s\in\mathbb{R}} Cs(sI-A)^{-1}x(t) \\ &= \lim_{s\to\infty,\, s\in\mathbb{R}} s\int_0^{\widehat{h}} e^{-s(\widehat{h}-\nu)}\zeta(t-\nu)d\nu \qquad \text{a.e. } t \geq 0, \end{aligned} \tag{B.2}$$

where $\zeta := cz$. For each $n \in \mathbb{N}$, define

$$f_n(t) := n\int_0^{\widehat{h}} e^{-n(\widehat{h}-\nu)}\zeta(t-\nu)d\nu \qquad \forall t \in [0, h_q).$$

We will show that there exists a subsequence $\{f_{n_\ell} : \ell \in \mathbb{N}\}$ such that $\lim_{\ell\to\infty} f_{n_\ell}(t) = \zeta(t-\widehat{h})$ for a.e. $t \in [0, h_q)$. Together with (B.2), this yields $\zeta(t-\widehat{h}) = C_\Lambda x(t)$ for a.e. $t \in [0, h_q)$ in the case $\widehat{h} \in (0, h_q)$.

Let $s > s_0$. Define

$$\varphi(s) := s\int_0^{\widehat{h}} e^{-s(\widehat{h}-\nu)}d\nu = 1 - e^{-\widehat{h}s}.$$

Since $\zeta \in L^1(-h_q, h_q)$, it follows from Fubini's theorem that

$$\begin{aligned}
&\int_0^{h_q}\left|\zeta(t-\widehat{h}) - s\int_0^{\widehat{h}} e^{-s(\widehat{h}-\nu)}\zeta(t-\nu)d\nu\right| dt \\
&\leq \int_0^{h_q}\left|\big(1-\varphi(s)\big)\zeta(t-\widehat{h})\right| dt + s\int_0^{h_q}\int_0^{\widehat{h}} e^{-s(\widehat{h}-\nu)}\big|\zeta(t-\widehat{h})-\zeta(t-\nu)\big|d\nu dt \\
&\leq e^{-\widehat{h}s}\|\zeta\|_{L^1(-h_q,h_q)} + s\int_0^{\widehat{h}} e^{-s(\widehat{h}-\nu)}\int_0^{h_q}\big|\zeta(t-\widehat{h})-\zeta(t-\nu)\big|dtd\nu.
\end{aligned}$$

Choose $\varepsilon > 0$ arbitrarily. By the strong continuity of the left translation semigroup on $L^1(-h_q, h_q)$ (see, e.g., Example I.5.4 in [11]), there exists $\delta_0 \in (0, \widehat{h})$ such that

$$\int_0^{h_q} |\zeta(t-\widehat{h}) - \zeta(t-\widehat{h}+\delta)|dt < \varepsilon \qquad \forall \delta \in [0, \delta_0).$$

Therefore,

$$s\int_{\widehat{h}-\delta_0}^{\widehat{h}} e^{-s(\widehat{h}-\nu)}\int_0^{h_q}\big|\zeta(t-\widehat{h})-\zeta(t-\nu)\big|dtd\nu < \varepsilon(1-e^{-\delta_0 s}) < \varepsilon.$$

Since

$$\begin{aligned}
s\int_0^{\widehat{h}-\delta_0} e^{-s(\widehat{h}-\nu)}\int_0^{h_q}\big|\zeta(t-\widehat{h})-\zeta(t-\nu)\big|dtd\nu& \\
\leq 2\|\zeta\|_{L^1(-h_q,h_q)}(e^{-\delta_0 s} - e^{-\widehat{h}s})&,
\end{aligned}$$

it follows that there exists $s_1 > s_0$ such that for every $s > s_1$,

$$e^{-\widehat{h}s}\|\zeta\|_{L^1(-h_q,h_q)} < \varepsilon, \quad s\int_0^{\widehat{h}-\delta_0} e^{-s(\widehat{h}-\nu)}\int_0^{h_q}\big|\zeta(t-\widehat{h})-\zeta(t-\nu)\big|dtd\nu < \varepsilon.$$

Hence we obtain

$$\int_0^{h_q}\left|\zeta(t-\widehat{h}) - s\int_0^{\widehat{h}} e^{-s(\widehat{h}-\nu)}\zeta(t-\nu)d\nu\right| dt < 3\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have that $\lim_{n\to\infty}\|\zeta(\cdot - \widehat{h}) - f_n\|_{L^1(0,h_q)} = 0$. Then there exists a subsequence $\{f_{n_\ell} : \ell \in \mathbb{N}\}$ such that $\lim_{\ell\to\infty} f_{n_\ell}(t) = \zeta(t-\widehat{h})$ for a.e. $t \in [0, h_q)$; see, e.g., Theorem 3.12 in [41]. This completes the proof. □

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