# A REPRESENTATION THEOREM FOR MV-ALGEBRAS 

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#### Abstract

An $M V$-pair is a pair $(B, G)$ where $B$ is a Boolean algebra and $G$ is a subgroup of the automorphism group of $B$ satisfying certain conditions. Let $\sim_{G}$ be the equivalence relation on $B$ naturally associated with $G$. We prove that for every MV-pair $(B, G)$, the effect algebra $B / \sim_{G}$ is an MV-effect algebra. Moreover, for every MV-effect algebra $M$ there is an MV-pair $(B, G)$ such that $M$ is isomorphic to $B / \sim_{G}$.


## 1. Introduction

Let $D$ be a bounded distributive lattice. Recall, that a Boolean algebra $B(D)$ is called $R$-generated by $D$ iff $D$ is a 0 , 1-sublattice of $B(D)$ and $D$ generates $B(D)$, as a Boolean algebra. Given $D$, these properties determine $B(D)$ up to isomorphism.

In [12], it was proved that every MV-effect algebra $M$ there is a surjective morphism of effect algebras $\phi_{M}: B(M) \rightarrow M$. Since $\phi_{M}$ is a full morphism of effect algebras, $B / \sim_{\phi_{M}}$ is isomorphic to $M$. A natural question arises: is it possible to express $\phi_{M}$ in terms of $B(M)$, using only the language of Boolean algebras? In this paper, we answer this question in the affirmative. We prove that for every MV-algebra $M$ there exists a subgroup $G(M)$ of the automorphism group of $B(M)$ such that the standard equivalence relation on $B(M)$ associated with $G(M)$ equals $\sim_{\phi_{M}}$. Conversely, we give conditions under which a pair $(B, G)$ gives rise to an MV-effect algebra in aforementioned way; we call such pairs $(B, G) M V$-pairs. Finally, we prove that $(B(M), G(M))$ is an MV-pair.

The origins of the main idea of this paper lie in the paper [5].

## 2. Definitions and basic Relationships

An effect algebra is a partial algebra $(E ; \oplus, 0,1)$ with a binary partial operation $\oplus$ and two nullary operations 0,1 satisfying the following conditions.
(E1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b=b \oplus a$.
(E2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus(b \oplus c)$ are defined and $(a \oplus b) \oplus c=a \oplus(b \oplus c)$.
(E3) For every $a \in E$ there is a unique $a^{\prime} \in E$ such that $a \oplus a^{\prime}=1$.
(E4) If $a \oplus 1$ exists, then $a=0$
Effect algebras were introduced by Foulis and Bennett in their paper [7]. In their papers [14] and [15], Kôpka and Chovanec introduced an essentially equivalent structure called D-poset. Another equivalent structure, called weak orthoalgebras

[^0]was introduced by Giuntini and Greuling in［8］．We refer to the monograph［6］for more information on effect algebras and similar algebraic structures．

For brevity，we denote an effect algebra $(E ; \oplus, 0,1)$ by $E$ ．In an effect algebra $E$ ， we write $a \leq b$ iff there is $c \in E$ such that $a \oplus c=b$ ．It is easy to check that every effect algebra is cancellative，thus $\leq$ is a partial order on $E$ ．In this partial order， 0 is the least and 1 is the greatest element of $E$ ．Moreover，it is possible to introduce a new partial operation $\ominus ; b \ominus a$ is defined iff $a \leq b$ and then $a \oplus(b \ominus a)=b$ ．It can be proved that $a \oplus b$ is defined iff $a \leq b^{\prime}$ iff $b \leq a^{\prime}$ ．Therefore，we denote the domain of $\oplus$ by $\perp$ ．

Let $E_{1}, E_{2}$ be effect algebras．A mapping $\phi: E_{1} \mapsto E_{2}$ is called a morphism of effect algebras iff $\phi(1)=1$ and for all $a, b \in E$ ，the existence of $a \oplus b$ implies the existence of $\phi(a) \oplus \phi(b)$ and $\phi(a \oplus b)=\phi(a) \oplus \phi(b)$ ．A morphism $\phi: E_{1} \rightarrow E_{2}$ is full iff whenever $\phi(a) \perp \phi(b)$ and $\phi(a) \oplus \phi(b) \in \phi\left(E_{1}\right)$ ，then there are $a_{1}, b_{1} \in E_{1}$ such that $\phi(a)=\phi\left(a_{1}\right), \phi(b)=\phi\left(b_{1}\right)$ and $a_{1} \perp b_{1}$ ．A morphism $\phi$ is an isomorphism iff $\phi$ is bijective and full．Note that even if both $E_{1}$ and $E_{2}$ are lattice ordered，a morphism of effect algebras need not to preserve joins and meets．

An $M V$－algebra（c．f．［2］，18）is a $(2,1,0)$－type algebra $(M ; \boxplus, \neg, 0)$ ，such that $\boxplus$ satisfying the identities $(x \boxplus y) \boxplus z=x \boxplus(y \boxplus z), x \boxplus z=y \boxplus x, x \boxplus 0=x, \neg \neg x=x$, $x \boxplus \neg 0=\neg 0$ and

$$
x \boxplus \neg(x \boxplus \neg y)=y \boxplus \neg(y \boxplus \neg x) .
$$

On every MV－algebra，a partial order $\leq$ is defined by the rule

$$
x \leq y \Longleftrightarrow y=x \boxplus \neg(x \boxplus \neg y)
$$

In this partial order，every MV－algebra is a distributive lattice bounded by 0 and $\neg 0$ ．

An $M V$－effect algebra is a lattice ordered effect algebra $M$ in which，for all $a, b \in$ $M,(a \vee b) \ominus a=b \ominus(a \wedge b)$ ．It is proved in 4］that there is a natural，one－to one correspondence between MV－effect algebras and MV－algebras given by the following rules．Let $(M, \oplus, 0,1)$ be an MV－effect algebra．Let $⿴ 囗 十$ be a total operation given by $x \boxplus y=x \oplus\left(x^{\prime} \wedge y\right)$ ．Then $\left(M, \boxplus,,^{\prime}, 0\right)$ is an MV－algebra．Similarly，let $(M, \boxplus, \neg, 0)$ be an MV－algebra．Restrict the operation $\boxplus$ to the pairs $(x, y)$ satisfying $x \leq y^{\prime}$ and call the new partial operation $\oplus$ ．Then $(M, \oplus, 0, \neg 0)$ is an MV－effect algebra．

Among lattice ordered effect algebras，MV－effect algebras can be characterized in a variety of ways．Three of them are given in the following proposition．
Proposition 2．1．1］，4］Let E be a lattice ordered effect algebra．The following are equivalent
（a）$E$ is an $M V$－effect algebra．
（b）For all $a, b \in E, a \wedge b=0$ implies $a \leq b^{\prime}$ ．
（c）For all $a, b \in E, a \ominus(a \wedge b) \leq b^{\prime}$ ．
（d）For all $a, b \in E$ ，there exist $a_{1}, b_{1}, c \in E$ such that $a_{1} \oplus b_{1} \oplus c$ exists， $a_{1} \oplus c=a$ and $b_{1} \oplus c=b$ ．

Notation．In what follows，we will deal with an MV－effect algebra $M$ and a Boolean algebra $B(M)$ such that $M$ is a 0,1 －sublattice of $B(M)$ ．In this particular situation，a small notational problem arises：both $M$ and $B(M)$ are MV－effect algebras，but the $\oplus, \ominus$ and＇operations on $B(M)$ and $M$ differ．To avoid confusion， we denote the partial operation of disjoint join（the $\oplus$ of Boolean algebras）on a Boolean algebra by $\dot{\vee}$ ．The partial difference of comparable elements and the complement in a Boolean algebra are denoted by $\backslash$ and ${ }^{\text {C }}$ ，respectively．

Let $D$ be a bounded distributive lattice. Up to isomorphism, there exists a unique Boolean algebra $B(D)$ such that $D$ is a 0 , 1-sublattice of $B(D)$ and $B$ generates $B(D)$ as a Boolean algebra. This Boolean algebra is called the Boolean algebra Rgenerated by $D$. We refer to [9, section II.4, for an overview of results concerning R-generated Boolean algebras. See also 11] and 17. For every element $x$ of $B(D)$, there exists a finite chain $x_{1} \leq \ldots \leq x_{n}$ in $D$ such that $x=x_{1}+\ldots+x_{n}$. Here, + denotes the symmetric difference, as in Boolean rings. We then say than $\left\{x_{i}\right\}_{i=1}^{n}$ is a $D$-chain representation of $x$. It is easy to see that every element of $B(D)$ has a $D$-chain representation of even length. Note that, for $n=2 k$ we have

$$
x=x_{1}+\cdots+x_{2 k}=\left(x_{2 k} \backslash x_{2 k-1}\right) \oplus \cdots \oplus\left(x_{2} \backslash x_{1}\right)
$$

If $D_{1}, D_{2}$ are bounded distributive lattices and $\psi: D_{1} \rightarrow D_{2}$ is a 0 , 1-lattice homomorphism, then $\psi$ uniquely extends to a homomorphism of Boolean algebras $\psi^{*}: B\left(D_{1}\right) \rightarrow B\left(D_{2}\right)$. Similarly, if $[0, a]_{D}$ is an interval in a bounded distributive lattice $D$, then $B\left([0, a]_{D}\right)$ is naturally isomorphic to the interval $[0, a]_{B(D)}$.

Theorem 2.2. 12] Let $M$ be an $M V$-effect algebra. The mapping $\phi_{M}: B(M) \rightarrow M$ given by

$$
\phi_{M}(x)=\bigoplus_{i=1}^{n}\left(x_{2 i} \ominus x_{2 i-1}\right)
$$

where $\left\{x_{i}\right\}_{i=1}^{2 n}$ is a $M$-chain representation of $x$, is a surjective morphism of effect algebras.

We note that the value of $\phi_{M}(x)$ does not depend on the choice of the $M$-chain representation of $x$. Obviously, for all $x \in M,\{x, 0\}$ is a $M$-chain representation of $x$. Therefore, $\phi_{M}(x)=x \ominus 0=x$, so every $x \in M$ is a fixpoint of $\phi_{M}$.

Example 2.3. Let $M$ be an MV-effect algebra, which is totally ordered. By [9, Corollary II.4.19, $B(M)$ is isomorphic to the Boolean algebra of all subsets of $M$ of the form $\left[a_{1}, b_{1}\right) \dot{\cup} \ldots \dot{\cup}\left[a_{n}, b_{n}\right)$. Here, we denote $[a, b)=\{x \in M: a \leq x<b\}$. The $\phi_{M}: B(M) \rightarrow M$ morphism is then given by

$$
\phi_{M}\left(\left[a_{1}, b_{1}\right) \dot{\cup} \ldots \dot{\cup}\left[a_{n}, b_{n}\right)\right)=\left(b_{1} \ominus a_{1}\right) \oplus \ldots \oplus\left(b_{n} \ominus a_{n}\right) .
$$

Example 2.4. In this example, $[0,1]$ denotes the closed real unit interval. Let $C_{[0,1]}$ be the MV-effect algebra of all real continuous functions $f:[0,1] \rightarrow[0,1]$. Let $B$ be the Boolean algebra

$$
\prod_{x \in[0,1]} B([0,1]),
$$

where $B([0,1])$ is the Boolean algebra generated by semiopen intervals as described in Example 2.3 It is obvious that $C_{[0,1]}$, as a bounded lattice, can be embedded into $B$ by a mapping $\gamma: E \rightarrow B$ given by $\gamma(f)=([0, f(x)))_{x \in[0,1]}$. The image of $E$ under $\gamma$ then generates a Boolean subalgebra of $B$, which we can identify with $B\left(C_{[0,1]}\right)$. The $\phi_{C_{[0,1]}}: B\left(C_{[0,1]}\right) \rightarrow C_{[0,1]}$ mapping can then be constructed as follows.

Let $\left(A_{x}\right)_{x \in[0,1]} \in B\left(C_{[0,1]}\right)$. Fix $x \in[0,1]$ and write $A_{x}=\left[a_{1}, b_{1}\right) \dot{\cup} \ldots \dot{\cup}\left[a_{n}, b_{n}\right)$. The value of the continuous function $\phi_{C_{[0,1]}}\left(\left(A_{x}\right)_{x \in[0,1]}\right)$ at $x$ is then equal to ( $b_{1} \ominus$ $\left.a_{1}\right) \oplus \ldots \oplus\left(b_{n} \ominus a_{n}\right)$.

Let $E$ be an effect algebra. A relation $\sim$ on $E$ is a weak congruence iff the following conditions are satisfied.
(C1) $\sim$ is an equivalence relation.
(C2) If $a_{1} \sim a_{2}, b_{1} \sim b_{2}$ and $a_{1} \oplus b_{1}, a_{2} \oplus b_{2}$ exist, then $a_{1} \oplus b_{1} \sim a_{2} \oplus b_{2}$.
If $E$ is an effect algebra and $\sim$ is a weak congruence on $E$, the quotient $E / \sim$ ( $\oplus$ is defined on $E / \sim$ in an obvious way) need not to be a partial abelian monoid, since the associativity condition may fail (c.f. [10]). This fact motivates the study of sufficient conditions for a weak congruence to preserve associativity. The following condition was considered in 3].
(C5) If $a \sim b \oplus c$, then there are $b_{1}, c_{1}$ such that $b_{1} \sim b, c_{1} \sim c, b_{1} \oplus c_{1}$ exists and $a=b_{1} \oplus c_{1}$.
In [3], it was proved that for a partial abelian monoid $P$ and a weak congruence $\sim$, satisfying (C5), the quotient $P / \sim$ is again a partial abelian monoid. Moreover, it is easy to prove that the eventual positivity of $P$ is preserved for such $\sim$. However, for an effect algebra $E$, the (C5) property of $\sim$ does not guarantee that the ' operation is preserved by $\sim$. If ${ }^{\prime}$ is preserved by $\sim$, that means, if condition
(C6) If $a \sim b$, then $a^{\prime} \sim b^{\prime}$.
is satisfied, then $E / \sim$ is an effect algebra. A relation on an effect algebra satisfying (C1),(C2),(C5),(C6) is called an effect algebra congruence. For every effect algebra congruence $\sim$ on an effect algebra $E$, the mapping $a \rightarrow[a]_{\sim}$ is a full morphism of effect algebras.

We refer the interested reader to [19] and [10] for further details concerning congruences on effect algebras and partial abelian monoids.

The (b) and (c) of the following lemma are just two equivalent $\perp$-to- $\leq$ reformulations of the (C3) property from [10. Thus, the lemma is (implicitly) well known, but we cannot find it in print.

Lemma 2.5. Let $\sim$ be a congruence on an effect algebra $E$. For all $x, y \in E$, the following are equivalent.
(a) $[x]_{\sim} \leq[y]_{\sim}$.
(b) There is $x_{1} \sim x$ such that $x_{1} \leq y$.
(c) There is $y_{1} \sim y$ such that $x \leq y_{1}$.

Proof.
(b) $\Longrightarrow$ (a) and (c) $\Longrightarrow$ (a) are trivial.
(a) $\Longrightarrow(\mathrm{b}):$ As $[x]_{\sim} \leq[y]_{\sim}$, there is $u \in E$ such that $[x]_{\sim} \oplus[u]_{\sim}=[y]_{\sim}$. This implies that there are $x_{0}, u_{0} \in E$ such that $x_{0} \sim x, u_{0} \sim u, x_{0} \oplus u_{0}$ exists, and $x_{0} \oplus u_{0} \sim y$. By the (C5) property, there are $x_{1}, u_{1}$ such that $x_{1} \sim x_{0}, u_{1} \sim u_{0}$, $x_{1} \oplus u_{1}$ exists, and $x_{1} \oplus u_{1}=y$.
(a) $\Longrightarrow$ (c): By the (C6) property, $\left[y^{\prime}\right]_{\sim} \leq\left[x^{\prime}\right]_{\sim}$. As $(\mathrm{a}) \Longrightarrow(\mathrm{b})$, there is $z \sim y^{\prime}$ such that $z \leq x^{\prime}$ and this is equivalent with $x \leq z^{\prime}$. By the (C6) property, $z \sim y^{\prime}$ iff $z^{\prime} \sim y$ and we can put $y_{1}=z^{\prime}$.

Recall that an effect algebra $E$ satisfies the Riesz decomposition property iff for all $u, v_{1}, v_{2} \in E, u \leq v_{1} \oplus v_{2}$ iff there are $u_{1}, u_{2}$ such that $u_{1} \leq v_{1}, u_{2} \leq v_{2}$ and $u=u_{1} \oplus u_{2}$. A lattice ordered effect algebra is an MV-effect algebra iff it satisfies the Riesz decomposition property. There are non-lattice ordered effect algebras satisfying the Riesz decomposition property, for example the effect algebra of all polynomial functions $[0,1]_{\mathbb{R}} \rightarrow[0,1]_{\mathbb{R}}$. By $[20$, every effect algebra satisfying the Riesz decomposition property can be embedded, as an interval in the positive cone,
into a partially ordered abelian group satisfying the Riesz decomposition property. This result is a generalization of the famous result by Mundici from [18.

An effect algebra satisfies the Riesz interpolation property iff for all elements $u_{1}, u_{2}, v_{1}, v_{2}$ such that $u_{i} \leq v_{j}$ for all $i, j \in\{1,2\}$, there is an element $x$ such that $x$ is an upper bound of $u_{1}, u_{2}$ and a lower bound of $v_{1}, v_{2}$. If an effect algebra satisfies the Riesz decomposition property, then it satisfies the Riesz interpolation property. The opposite implication is not true, since every lattice ordered effect algebra satisfies the Riesz interpolation property, but there exist (obviously) some effect algebras that are lattice ordered and non-MV.

## 3. From MV-pairs to MV-effect algebras

Let $B$ be a Boolean algebra. We write $\operatorname{Aut}(B)$ for the group of all automorphisms of $B$. Let $G$ be a subgroup of $\operatorname{Aut}(B)$. For $a, b \in B$, we write $a \sim_{G} b$ iff there exists $f \in G$ such that $b=f(a)$. Obviously, $\sim_{G}$ is an equivalence relation. We write $[a]_{G}$ for the equivalence class of an element $a$ of $B$.

A pair $(B, G)$, where $B$ is a Boolean algebra and $G$ is a subgroup of $\operatorname{Aut}(B)$ is called a BG-pair. BG-pairs are a well-established topic in the theory of Boolean algebras, see for example Chapter 15 of the handbook [16.

Let $(P, \leq)$ be a poset. Let us write,

$$
\max (P)=\{m \in P: x \leq m \Longrightarrow x=m\}
$$

that means, $\max (P)$ is the set of all maximal elements of the poset $P$.
Let $B$ be a Boolean algebra, let $G$ be a subgroup of $\operatorname{Aut}(B)$. For all $a, b \in B$, we write

$$
\begin{aligned}
L(a, b) & =\{a \wedge f(b): f \in G\} \text { and } \\
L^{+}(a, b) & =\{g(a) \wedge f(b): f, g \in G\} .
\end{aligned}
$$

Note that $L(a, b) \subseteq L^{+}(a, b)$ and that $L^{+}(a, b)$ is closed with respect to any $h \in G$; this implies that $L^{+}(a, b)$ is a union of equivalence classes of $\sim_{G}$.
Definition 3.1. Let $B$ be a Boolean algebra, let $G$ be a subgroup of $\operatorname{Aut}(B)$. We say that $(B, G)$ is an $M V$-pair iff the following two conditions are satisfied.
(MVP1) For all $a, b \in B, f \in G$ such that $a \leq b$ and $f(a) \leq b$, there is $h \in G$ such that $h(a)=f(a)$ and $h(b)=b$.
(MVP2) For all $a, b \in B$ and $x \in L(a, b)$, there exists $m \in \max (L(a, b))$ with $m \geq x$.
Example 3.2. For every finite Boolean algebra $B,(B, \operatorname{Aut}(B))$ is an MV-pair.
Example 3.3. Let $B$ be a Boolean algebra with three atoms $a_{1}, a_{2}, a_{3}$. The mapping $f$ given by

| $x$ | 0 | $a_{1}$ | $a_{2}$ | $a_{3}$ | $a_{1}^{\complement}$ | $a_{2}^{\complement}$ | $a_{3}^{\complement}$ | 1 |
| ---: | :--- | :--- | :--- | :--- | :--- | :--- | :--- | ---: |
| $f(x)$ | 0 | $a_{2}$ | $a_{3}$ | $a_{1}$ | $a_{2}^{\complement}$ | $a_{3}^{\complement}$ | $a_{1}^{\complement}$ | 1 |

is an automorphism of $B$ and $G=\left\{i d, f, f^{2}\right\}$ is a subgroup of $\operatorname{Aut}(B)$. However, $(B, G)$ is not an MV-pair. Indeed, we have $a_{1} \leq a_{3}^{\complement}$ and $f\left(a_{1}\right)=a_{2} \leq a_{3}^{\complement}$, but there is no $h \in G$ such that $h\left(a_{1}\right)=f\left(a_{1}\right)$ and $h\left(a_{3}^{\complement}\right)=a_{3}^{\complement}$.

Example 3.4. Let $B$ be the Boolean algebra of all Borel subsets of the real unit interval $[0,1]_{\mathbb{R}}$ that are unions of a finite number of intervals. (as usual, we identify the Borel sets that differ by a set of measure 0.) Let $W$ the subgroup of the
permutation group of $[0,1]_{\mathbb{R}}$ that is generated by the set of all bijections $p_{a, b}$ given by

$$
p_{a, b}(x)= \begin{cases}x & \text { if } x \in[0, a] \\ a+b-x & \text { if } x \in(a, b) \\ x & \text { if } x \in[b, 1]\end{cases}
$$

where $0 \leq a \leq b \leq 1$. For every $p \in W$, let $f_{p}$ be the mapping $f_{p}: B \rightarrow B$ given by $f_{p}(X)=p(X)$ and let $G=\left\{f_{p}: p \in W\right\}$. Obviously, $G$ is a subgroup of $\operatorname{Aut}(B)$. Then $(B, G)$ is an MV-pair; the proof of this fact is a bit longer, but straightforward. Note that every $f_{p} \in G$ preserves measure.

Example 3.5. Let $2^{\mathbb{Z}}$ be the Boolean algebra of all subsets of $\mathbb{Z}$. Then $\left(2^{\mathbb{Z}}, \operatorname{Aut}\left(2^{\mathbb{Z}}\right)\right)$ is not an MV-pair. Indeed, let $f \in \operatorname{Aut}\left(2^{\mathbb{Z}}\right)$ be the automorphism of $2^{\mathbb{Z}}$ associated with the permutation $f(n)=n+1$. Let $A=B=\mathbb{N}$. We see that $f(A)=A \backslash\{0\}$, $A \subseteq B$ and $f(A) \subseteq B$. However, there is no $h \in \operatorname{Aut}\left(2^{\mathbb{Z}}\right)$ such that $h(A)=f(A)$ and $h(B)=B$, simply because $A=B$ implies that $h(A)=h(B)$, but $f(A) \neq B$.

The (MVP1) condition can be reformulated:
Lemma 3.6. Let $B$ be a Boolean algebra, let $G$ be a subgroup of $\operatorname{Aut}(B)$. Then the following conditions are equivalent.
(a) (MVP1)
(b) For all $a, b \in B, f \in G$ such that $a \leq b$ and $a \leq f(b)$, there is $h \in G$ such that $h(b)=f(b)$ and $h(a)=a$.
(c) For all $a, b \in B, f \in G$ such that $a \wedge b=0$ and $a \wedge f(b)=0$, there is $h \in G$ such that $h(b)=f(b)$ and $h(a)=a$.
Proof.
$(\mathrm{a}) \Longrightarrow(\mathrm{b})$ : Replace $a$ with $b^{\complement}$ and $b$ with $a^{\complement}$ and apply the fact that $f$ is an automorphism.
$(\mathrm{b}) \Longrightarrow(\mathrm{c})$ : Replace $b$ with $b^{\text {C }}$.
$(c) \Longrightarrow(a)$ : Replace $b$ with $a$ and $a$ with $b^{\complement}$.
Lemma 3.7. Let $(B, G)$ be an $M V$-pair, let $a, b \in B$ and let $m$ be a maximal element of $L(a, b)$. For all $f \in G, f(m)$ is a maximal element of $L^{+}(a, b)$.
Proof. Suppose that there is some element in $y \in L^{+}(a, b)$ with $y \geq f(m)$ and write $y=g_{1}(a) \wedge f_{1}(b)$, where $g_{1}, f_{1} \in G$. Since $m \in L(a, b), a \geq m$ and since

$$
a \wedge g_{1}^{-1}\left(f_{1}(b)\right)=g_{1}^{-1}\left(g_{1}(a) \wedge f_{1}(b)\right)=g_{1}^{-1}(y) \geq g_{1}^{-1}(f(m))=\left(g_{1}^{-1} \circ f\right)(m)
$$

we see that $a \geq\left(g_{1}^{-1} \circ f\right)(m)$.
By (MVP1), $a \geq\left(g_{1}^{-1} \circ f\right)(m)$ and $a \geq m$ imply that there exists $h \in G$ such that $h(a)=a$ and $h(m)=\left(g_{1}^{-1} \circ f\right)(m)$. We apply $h^{-1}$ to both sides of the inequality

$$
a \wedge g_{1}^{-1}\left(f_{1}(b)\right) \geq\left(g_{1}^{-1} \circ f\right)(m)
$$

to obtain

$$
h^{-1}\left(a \wedge g_{1}^{-1}\left(f_{1}(b)\right)\right)=a \wedge h^{-1}\left(g_{1}^{-1}\left(f_{1}(b)\right)\right) \geq h^{-1}\left(\left(g_{1}^{-1} \circ f\right)(m)\right)=m
$$

Since $m$ is a maximal element of $L(a, b), a \wedge h^{-1}\left(g_{1}^{-1}\left(f_{1}(b)\right)\right) \geq m$ implies that $a \wedge h^{-1}\left(g_{1}^{-1}\left(f_{1}(b)\right)\right)=m$. After we apply the mapping $g_{1} \circ h$ on both sides of the latter equality we obtain $y=g_{1}(a) \wedge f_{1}(b)=f(m)$. Thus, $f(m)$ is maximal in $L^{+}(a, b)$.

Note that Lemma 3.7 implies that $\max (L(a, b)) \subseteq \max \left(L^{+}(a, b)\right)$.
Corollary 3.8. Let $(B, G)$ be an MV-pair. For all $a, b \in B$ and $x \in L^{+}(a, b)$, there exists $m \in \max \left(L^{+}(a, b)\right)$ with $m \geq x$.

Proof. As $x \in L^{+}(a, b)$, we have $x=g_{1}(a) \wedge f_{1}(b)$ for some $f_{1}, g_{1} \in G$. Then

$$
g_{1}^{-1}\left(g_{1}(a) \wedge f_{1}(b)\right)=a \wedge g_{1}^{-1}\left(a_{1}(b)\right) \in L(a, b)
$$

By (MVP2), there is $m \in \max (L(a, b))$ with $m \geq a \wedge g_{1}^{-1}\left(a_{1}(b)\right)$. This implies that $g_{1}(m) \geq g_{1}(a) \wedge f_{1}(b)$. By Lemma 3.7 $g_{1}(m) \in \max \left(L^{+}(a, b)\right)$.

Theorem 3.9. Let $(B, G)$ be an $M V$-pair. Then $\sim_{G}$ is an effect algebra congruence on $B$ and $B / \sim_{G}$ is an $M V$-effect algebra.

Proof. We shall prove that the equivalence $\sim_{G}$ is an effect congruence. It is easy to see that $\sim_{G}$ preserves the ${ }^{\complement}$ operation, so (C6) is satisfied. To prove (C5), let $a_{1}, a_{2} \in B$ be such that $a_{1} \dot{\vee} a_{2}$ exists and $a_{1} \dot{\vee} a_{2} \sim_{G} b$. Then there is $f \in G$ such that $f\left(a_{1} \dot{\vee} a_{2}\right)=b$ and we may put $b_{1}=f\left(a_{1}\right)$ and $b_{2}=f\left(a_{2}\right)$.

Let us prove (C2). Let $a_{1}, a_{2}, b_{1}, b_{2} \in B$ be such that $a_{1} \sim_{G} a_{2}, b_{1} \sim_{G} b_{2}$, and $a_{1} \dot{\vee} b_{1}, a_{2} \dot{\vee} b_{2}$ exist. There are $f_{a}, f_{b} \in G$ such that $f_{a}\left(a_{1}\right)=a_{2}$ and $f_{b}\left(b_{1}\right)=b_{2}$.

We see that $b_{2}^{\complement} \geq a_{2}$ and that implies

$$
b_{1}^{\complement}=f_{b}^{-1}\left(b_{2}^{\complement}\right) \geq f_{b}^{-1}\left(a_{2}\right)=f_{b}^{-1}\left(f_{a}\left(a_{1}\right)\right)=\left(f_{b}^{-1} \circ f_{a}\right)\left(a_{1}\right)
$$

By (MVP1), $a_{1} \leq b_{1}^{\complement}$ and $\left(f_{b}^{-1} \circ f_{a}\right)\left(a_{1}\right) \leq b_{1}^{\complement}$ imply that there is $h \in G$ such that $h\left(a_{1}\right)=\left(f_{b}^{-1} \circ f_{a}\right)\left(a_{1}\right)$ and $h\left(b_{1}^{\complement}\right)=b_{1}^{\complement}$. Therefore,
$f_{b}\left(h\left(a_{1} \dot{\vee} b_{1}\right)\right)=f_{b}\left(h\left(a_{1}\right) \dot{\vee} h\left(b_{1}\right)\right)=f_{b}\left(\left(f_{b}^{-1} \circ f_{a}\right)\left(a_{1}\right) \dot{\vee} b_{1}\right)=f_{a}\left(a_{1}\right) \dot{\vee} f_{b}\left(b_{1}\right)=a_{2} \dot{\vee} b_{2}$, and $a_{1} \dot{\vee} b_{1} \sim_{G} a_{2} \dot{\vee} b_{2}$.

Since $\sim_{G}$ is an effect congruence, $B / \sim_{G}$ is an effect algebra. By Proposition 4.3 of [13], since $B$ satisfies the Riesz decomposition property, $B / \sim_{G}$ satisfies the Riesz decomposition property as well. It remains to prove that $B / \sim_{G}$ is a lattice. Since an effect algebra is a lattice iff it is a (join or meet) semilattice, it suffices to prove that for all $a, b \in B,[a]_{G} \wedge[b]_{G}$ exists in $B / \sim_{G}$.

Let $a, b \in B$. We shall prove that every common lower bound of $[a]_{G},[b]_{G}$ is under a maximal common lower bound of $[a]_{G},[b]_{G}$.

If $[c]_{G} \leq[a]_{G},[b]_{G}$ then, by Lemma 2.5 there is $c_{1} \sim_{G} c$ such that $c_{1} \leq a$ and, again by Lemma [2.5] $b_{1} \sim_{G} b$ such that $c_{1} \leq b$. As $b_{1} \sim_{G} b$, there is $f \in G$ such that $b_{1}=f(b)$. Thus,

$$
c \sim_{G} c_{1} \leq a \wedge f(b) \in L(a, b)
$$

By (MVP2), there is $m \in \max (L(a, b))$ with $a \wedge f(b) \leq m$. Obviously, $m \in L(a, b)$ implies that $[m]_{G} \leq[a]_{G},[b]_{G}$. Therefore, for every common lower bound $[c]_{G}$ of $[a]_{G},[b]_{G}$, there is $m \in \max (L(a, b))$ such that

$$
[c]_{G} \leq[m]_{G} \leq[a]_{G},[b]_{G}
$$

Let us prove that $[m]_{G}$ is a maximal common lower bound of $[a]_{G},[b]_{G}$ in $B / \sim_{G}$. Suppose that

$$
[m]_{G} \leq[x]_{G} \leq[a]_{G},[b]_{G}
$$

By Lemma 2.5 there are $m_{1} \sim_{G} m, x_{1} \sim_{G} x$ and $b_{1} \sim_{G} b$ such that

$$
m_{1} \leq x_{1} \leq a, b_{1}
$$

There is $f \in G$ such that $b_{1}=f(b)$. We see that $x_{1} \leq a \wedge f(b) \in L(a, b) \subseteq L^{+}(a, b)$. There is $g \in G$ such that $m_{1}=g(m)$. By Lemma 3.7 $m_{1}=g(m)$ is maximal element of $L^{+}(a, b)$. Therefore, $m_{1}=a \wedge f(b)$ and hence $x_{1}=m_{1}$. This implies that $[m]_{G}=[x]_{G}$.

Let $\left[m_{1}\right]_{G},\left[m_{2}\right]_{G}$ be maximal common lower bounds of $[a]_{G},[b]_{G}$. Since $B / \sim_{G}$ satisfies the Riesz decomposition property, $B / \sim_{G}$ satisfies the Riesz interpolation property. By the Riesz interpolation property, there is $[m]_{G}$ such that $\left[m_{1}\right]_{G},\left[m_{2}\right]_{G} \leq[m]_{G} \leq[a]_{G},[b]_{G}$. Since $\left[m_{1}\right]_{G},\left[m_{2}\right]_{G}$ are maximal, $\left[m_{1}\right]_{G}=$ $[m]_{G}=\left[m_{2}\right]_{G}$. Since every common lower bound of $[a]_{G},[b]_{G}$ is under a maximal one, and there is a single maximal common lower bound of $[a]_{G},[b]_{G},[a]_{G} \wedge[b]_{G}$ exists.

Note that we have proved that $[a]_{G} \wedge[b]_{G}=L^{+}(a, b)$. In particular, $L^{+}(a, b)$ is a single equivalence class of $\sim_{G}$.

In what follows we shall denote the MV-effect algebra arising from an MV-pair $(B, G)$ in the way indicated above by $\mathcal{A}(B, G)$.

## 4. From MV-effect algebras to MV-pairs

We have proved that for every MV-pair $(B, G)$ there is an MV-effect algebra $\mathcal{A}(B, G)$ arising from it. In this section, we shall prove that for every MV-effect algebra there is a MV-pair $(B, G)$ such that $\mathcal{A}(B, G) \simeq M$.

Let $M$ be an MV-effect algebra. Let $S$ be a subset of $B(M)$. We say that a mapping $f: S \rightarrow B$ is $\phi_{M}$-preserving iff, for all $x \in S, \phi_{M}(x)=\phi_{M}(f(x))$ or, in other words, $\phi_{M}$ restricted to $S$ equals $\phi_{M} \circ f$.

Theorem 4.1. Let $M$ be an $M V$-effect algebra. Let $G(M)$ be the set of all $\phi_{M^{-}}$ preserving automorphisms of $B(M)$. Then $(B(M), G(M))$ is an $M V$-pair and $\mathcal{A}(B(M), G(M))$ is isomorphic to $M$.

We have divided the proof into a sequence of lemmas. In this section, $M$ is an MV-effect algebra and $G(M)$ is the subgroup of $\operatorname{Aut}(B(M))$ described in Theorem 4.1

Lemma 4.2. Let $c, d \in M, d \leq c$. There is a $\phi_{M}$-preserving isomorphism

$$
\psi: B\left([0, c \ominus d]_{M}\right) \rightarrow[0, c \backslash d]_{B(M)}
$$

Proof. Consider the mapping $\psi_{0}:[0, c \ominus d]_{M} \rightarrow[0, c \backslash d]_{B(M)}$, given by $\psi_{0}(x)=$ $(x \oplus d) \backslash d$. We see that $\psi_{0}(0)=0, \psi_{0}(c \ominus d)=c \backslash d$ and, since $\psi_{0}$ is just a composition of a translation in $M$ and a translation in $B(M), \psi_{0}$ preserves joins and meets. Moreover, it is easy to see that $\psi_{0}$ is injective, hence $\psi_{0}$ is a 0 , 1-lattice embedding of $[0, c \ominus d]_{M}$ into $[0, c \backslash d]_{B(M)}$. We shall prove that the range of $\psi_{0}$ R-generates the Boolean algebra $[0, c \backslash d]_{B(M)}$. $\psi_{0}$ then uniquely extends to an isomorphism $\psi: B\left([0, c \ominus d]_{M}\right) \rightarrow[0, c \backslash d]_{B(M)}$.

Let $x \in[0, c \backslash d]_{B(M)}$. Let $\left\{x_{i}\right\}_{i=1}^{2 n}$ be an $M$-chain representation of $x$. For all $1 \leq i \leq n, x_{2 i} \backslash x_{2 i-1} \leq c \backslash d$. By elementary Boolean calculus, this implies that

$$
x_{2 i} \backslash x_{2 i-1}=\left(\left(x_{2 i} \vee d\right) \wedge c\right) \backslash\left(\left(x_{2 i-1} \vee d\right) \wedge c\right)
$$

For all $1 \leq j \leq 2 n,\left(x_{j} \vee d\right) \wedge c \in[d, c]$ Therefore, $x$ has a $M$-chain representation $\left\{y_{j}\right\}_{j=1}^{2 n} \subseteq[d, c]_{M}$. Since, for all $1 \leq i \leq n$,

$$
y_{2 i} \backslash y_{2 i-1}=\left(y_{2 i} \backslash d\right) \backslash\left(y_{2 i-1} \backslash d\right)
$$

$\left\{y_{i} \backslash d\right\}_{i=1}^{2 n}$ is a chain representation of $x$. It remains to observe that, for all $1 \leq i \leq$ $2 n$,

$$
y_{i} \backslash d=\left(\left(y_{i} \ominus d\right) \oplus d\right) \backslash d=\phi_{0}\left(y_{i} \ominus d\right)
$$

and that $y_{i} \ominus d \in[0, c \ominus d]_{M}$. Thus, every element of $[0, c \backslash d]_{B(M)}$ has a $\psi_{0}\left([0, c \ominus d]_{M}\right)$ chain representation.

Let us prove that $\psi$ is a $\phi_{M}$-preserving mapping. Let $z \in B\left([0, c \ominus d]_{M}\right)$, let $\left\{z_{i}\right\}_{i=1}^{2 n}$ be a $[0, c \ominus d]_{M}$-chain representation of $z$. Then

$$
\begin{array}{r}
\phi_{M}(\psi(z))=\phi_{M}\left(\psi\left(\dot{\vee}_{i=1}^{n}\left(z_{2 i} \backslash z_{2 i-1}\right)\right)\right)= \\
=\phi_{M}\left(\dot{\vee}_{i=1}^{n} \psi\left(z_{2 i} \backslash z_{2 i-1}\right)\right)=\bigoplus_{i=1}^{n} \phi_{M}\left(\psi\left(z_{2 i} \backslash z_{2 i-1}\right)\right)
\end{array}
$$

and, for all $1 \leq i \leq n$,

$$
\begin{array}{r}
\phi_{M}\left(\psi\left(z_{2 i} \backslash z_{2 i-1}\right)\right)=\phi_{M}\left(\psi\left(z_{2 i}\right) \backslash \psi\left(z_{2 i-1}\right)\right)= \\
=\phi_{M}\left(\left(\left(z_{2 i} \oplus d\right) \backslash d\right) \backslash\left(\left(z_{2 i} \oplus d\right) \backslash d\right)\right)= \\
=\phi_{M}\left(\left(z_{2 i} \oplus d\right) \backslash\left(z_{2 i} \oplus d\right)\right)=\phi_{M}\left(z_{2 i} \oplus d\right) \ominus \phi_{M}\left(z_{2 i} \oplus d\right)= \\
=\left(z_{2 i} \oplus d\right) \ominus\left(z_{2 i-1} \oplus d\right)=z_{2 i} \ominus z_{2 i-1}=\phi_{M}\left(z_{2 i} \backslash z_{2 i-1}\right)
\end{array}
$$

so we obtain

$$
\phi_{M}(\psi(z))=\bigoplus_{i=1}^{n} \phi_{M}\left(\psi\left(z_{2 i} \backslash z_{2 i-1}\right)\right)=\bigoplus_{i=1}^{n} \phi_{M}\left(z_{2 i} \backslash z_{2 i-1}\right)=\phi_{M}(z)
$$

Corollary 4.3. Let $c_{1}, d_{1}, c_{2}, d_{2} \in M$ be such that $c_{1} \geq d_{1}, c_{2} \geq d_{2}$ and $c_{1} \ominus d_{1}=$ $c_{2} \ominus d_{2}$. There is a $\phi_{M}$-preserving isomorphism $\psi:\left[0, c_{1} \backslash d_{1}\right]_{B(M)} \rightarrow\left[0, c_{2} \backslash d_{2}\right]_{B(M)}$.
Proof. Use Lemma 4.2 twice.
Lemma 4.4. For every $a \in B(M)$, there is a $\phi_{M}$-preserving isomorphism of Boolean algebras $\psi: B\left(\left[0, \phi_{M}(a)\right]_{M}\right) \rightarrow[0, a]_{B(M)}$.

Proof. Let $\left\{a_{i}\right\}_{i=1}^{2 n}$ be an $M$-chain representation of $a$. Then $\left\{a_{2 i} \backslash a_{2 i-1}\right\}_{i=1}^{n}$ is a decomposition of unit in the Boolean algebra $[0, a]_{B(M)}$ and $\phi_{M}(a)=\bigoplus_{i=1}^{n}\left(a_{2 i} \ominus\right.$ $a_{2 i-1}$. For $j \in\{0, \ldots, n\}$, write $b_{j}=\bigoplus_{i=1}^{j}\left(a_{2 i} \ominus a_{2 i-1}\right)$. Then $\left\{b_{j}\right\}_{j=0}^{n}$ is a finite chain in $\left[0, \phi_{M}(a)\right]_{M}$ with $b_{0}=0$ and $b_{n}=\phi_{M}(a)$. Thus, $\left\{b_{j} \backslash b_{j-1}\right\}_{j=1}^{n}$ is a decomposition of unit in the Boolean algebra $B\left(\left[0, \phi_{M}(a)\right]_{M}\right)$. For every $x \in$ $B\left(\left[0, \phi_{M}(a)\right]_{M}\right), x=\dot{\bigvee}_{j=1}^{n} x \wedge\left(b_{j} \backslash b_{j-1}\right)$. Since, for all $j, b_{j} \ominus b_{j-1}=a_{2 j} \ominus a_{2 j-1}$, Corollary 4.3 implies that, for all $1 \leq i \leq n$, there is a a $\phi_{M}$-preserving isomorphism $\psi_{j}:\left[0, b_{j} \backslash b_{j-1}\right]_{B(M)} \rightarrow\left[0, a_{2 j} \backslash a_{2 j-1}\right]_{B(M)}$. Define $\psi(x)=\dot{\bigvee}_{i=1}^{n} \psi_{j}\left(x \wedge\left(b_{j} \backslash b_{j-1}\right)\right)$.

The proof that $\psi$ is a $\phi_{M}$-preserving isomorphism of Boolean algebras is trivial and thus omitted.

Corollary 4.5. Let $a, b \in B(M)$ be such that $\phi_{M}(a)=\phi_{M}(b)$. Then there is $a$ $\phi_{M}$-preserving isomorphism $\psi:[0, a]_{B(M)} \rightarrow[0, b]_{B(M)}$.

Proof. Use Lemma 4.4 twice.

Lemma 4.6. Let $u, v \in B(M), u \wedge v=0, \phi_{M}(u)=\phi_{M}(v)$. Then there is a $\phi_{M}$-preserving automorphism $f$ of $B(M)$ such that $f(u)=v, f(v)=u$ and for all $x \leq(u \dot{\vee} v)^{\complement}, f(x)=x$.
Proof. By Corollary 4.5 there is an isomorphism $\psi:[0, u]_{B(M)} \rightarrow[0, v]_{B(M)}$. Let $f: B(M) \rightarrow B(M)$ be a mapping given by

$$
f(x)=\psi^{-1}(x \wedge v) \dot{\vee} \psi(x \wedge u) \dot{\vee}\left(x \wedge(u \dot{\vee} v)^{\complement}\right)
$$

It is easy to check that, for all $x \in B(M), f(f(x))=x$. Thus, $f$ is a bijection. Moreover, we see that $f(0)=0, f(1)=1$ and, for all $x, y \in B(M)$,

$$
\begin{aligned}
f(x \vee y)= & \psi^{-1}((x \vee y) \wedge v) \dot{\vee} \psi((x \vee y) \wedge u) \dot{\vee}\left((x \vee y) \wedge(u \dot{\vee} v)^{\complement}\right)= \\
= & \psi^{-1}((x \wedge v) \vee(y \wedge v)) \dot{\vee} \psi((x \wedge u) \vee(y \wedge u)) \dot{\vee}\left(\left(x \wedge(u \dot{\vee} v)^{\complement}\right) \vee\right. \\
& \left.\vee\left(y \wedge(u \vee \dot{\vee} v)^{\complement}\right)\right)= \\
= & \left(\psi^{-1}(x \wedge v) \dot{\vee} \psi(x \wedge u) \dot{\vee}\left(x \wedge(u \dot{\vee} v)^{\complement}\right)\right) \vee \\
& \quad \vee\left(\psi^{-1}(x \wedge v) \dot{\vee} \psi(x \wedge u) \dot{\vee}\left(x \wedge(u \dot{\vee} v)^{\complement}\right)\right)= \\
= & f(x) \vee f(y)
\end{aligned}
$$

and

$$
\begin{aligned}
f\left(x^{\complement}\right) & =\psi^{-1}\left(x^{\complement} \wedge v\right) \dot{\vee} \psi\left(x^{\complement} \wedge u\right) \dot{\vee}\left(x^{\complement} \wedge(u \dot{\vee} v)^{\complement}\right)= \\
& =\psi^{-1}(v \backslash(x \wedge v)) \dot{\vee} \psi(u \backslash(x \wedge u)) \dot{\vee}\left(x^{\complement} \wedge(u \dot{\vee} v)^{\complement}\right)= \\
& =\left(u \backslash \psi^{-1}(x \wedge v)\right) \dot{\vee}(v \backslash \psi(x \wedge u)) \dot{\vee}\left(x^{\complement} \wedge(u \dot{\vee} v)^{\complement}\right)= \\
& =\left(\psi^{-1}(x \wedge v) \dot{\vee} \psi(x \wedge u) \dot{\vee}(x \wedge(u \dot{\vee} v))\right)^{\complement}
\end{aligned}
$$

The latter equality follows by elementary Boolean calculus. Since $f$ preserves $0,1, \vee$ and ${ }^{\complement}$, it is a homomorphism of Boolean algebras.

Lemma 4.7. Let $u, v \in B(M), \phi_{M}(u)=\phi_{M}(v)$. Then there is a $\phi_{M}$-preserving automorphism $f$ of $B(M)$ such that $f(u)=v, f(v)=u$ and for all $x \leq(u \dot{\vee} v)^{\complement}$, $f(x)=x$.

Proof. Put $u_{0}=u \backslash u \wedge v$ and $v_{0}=v \backslash u \wedge v$. Since

$$
\phi_{M}\left(u_{0}\right) \oplus \phi_{M}(u \wedge v)=\phi_{M}(u)=\phi_{M}(v)=\phi_{M}\left(v_{0}\right) \oplus \phi_{M}(u \wedge v)
$$

$\phi_{M}\left(u_{0}\right)=\phi_{M}\left(v_{0}\right)$. By Lemma 4.6 there is $f \in G(M)$ such that $f\left(u_{0}\right)=v_{0}$, $f\left(v_{0}\right)=u_{0}$ and for all $x \in B$ such that $x \leq\left(u_{0} \dot{\vee} v_{0}\right)^{\complement}$ we have $f(x)=0$. Since $u \wedge v \leq\left(u_{0} \dot{\vee} v_{0}\right)^{\complement}, f(u \wedge v)=u \wedge v$. Therefore,

$$
f(u)=f\left(u_{0} \dot{\vee}(u \wedge v)\right)=f\left(u_{0}\right) \dot{\vee}(u \wedge v)=v_{0} \dot{\vee}(u \wedge v)=v
$$

and, similarly, $f(v)=u$.
Let $x \leq(u \vee v)^{\complement}$. Since $x \leq\left(u_{0} \dot{\vee} v_{0}\right)^{\complement}, f(x)=x$.
Corollary 4.8. For all $u, v \in B(M), u \sim_{G(M)} v$ iff $\phi_{M}(u)=\phi_{M}(v)$.
Proof. One implication follows by the definition of $G(M)$, the other one follows by Lemma 4.7

Corollary 4.9. For all $u \in B(M), u \sim_{G} \phi_{M}(u)$.
Proof. Put $v=\phi_{M}(u)$ in Corollary 4.8

## Proof of Theorem 4.1.

(MVP1): Let $a, b \in B(M), f \in G$ be such that $a \leq b, a \leq f(b)$. Let $u=$ $b \backslash(b \wedge f(b)), v=f(b) \backslash(b \wedge f(b))$. We have

$$
\begin{aligned}
\phi_{M}(u) & \left.=\phi_{M}(b \backslash(b \wedge f(b)))=\phi_{M}(b) \ominus \phi_{M}(b \wedge f(b))\right)= \\
& \left.=\phi_{M}(f(b)) \ominus \phi_{M}(b \wedge f(b))\right)=\phi_{M}(f(b) \backslash(b \wedge f(b)))=\phi_{M}(v)
\end{aligned}
$$

By Lemma 4.6 there is a $\phi_{M}$-preserving automorphism $h$ of $B(M)$ with $h(u)=v$. Moreover, since $a \wedge u=a \wedge v=0$ and $(b \wedge f(b)) \wedge u=(b \wedge f(b)) \wedge v=0$, we have $h(a)=a$ and $h(b \wedge f(b))=b \wedge f(b)$. This implies that

$$
h(b)=h((b \wedge f(b)) \dot{\vee} u)=h((b \wedge f(b))) \dot{\vee} h(u)=(b \wedge f(b)) \dot{\vee} v=f(b)
$$

Thus, there is $h \in G$ such that $h(a)=a$ and $h(b)=f(b)$. By Lemma 3.6 this implies (MVP1).
(MVP2): Let $a \wedge f(b)$ be an element of $L(a, b)$. By Corollary 4.9 there is $f_{1} \in G$ such that $f_{1}(a)=\phi_{M}(a)$. Since $f_{1}$ is $\phi_{M}$-preserving, $\phi_{M}\left(f_{1}(a \wedge f(b))\right)=\phi_{M}(a \wedge$ $f(b))$. By Corollary 4.9 there is $g \in G$ such that $g\left(f_{1}(a \wedge f(b))\right)=\phi_{M}(a \wedge f(b))$. Since

$$
f_{1}(a \wedge f(b)) \leq f_{1}(a)=\phi_{M}(a)
$$

and

$$
g\left(f_{1}(a \wedge f(b))\right)=\phi_{M}(a \wedge f(b)) \leq \phi_{M}(a)
$$

(MVP1) implies that there is $h \in G$ such that $h\left(f_{1}(a \wedge f(b))\right)=\phi_{M}(a \wedge f(b))$ and $h\left(\phi_{M}(a)\right)=\phi_{M}(a)$.

Put $y=a \wedge f_{1}^{-1}\left(h^{-1}\left(\phi_{M}(f(b))\right)\right)$. We shall prove that $y \geq a \wedge f(b)$ and that $y$ is a maximal element of $L(a, b)$.

Indeed, we have

$$
h\left(f_{1}(a)\right)=h\left(\phi_{M}(a)\right)=\phi_{M}(a)
$$

therefore

$$
\begin{aligned}
h\left(f_{1}(y)\right) & =h\left(f_{1}\left(a \wedge f_{1}^{-1}\left(h^{-1}\left(\phi_{M}(f(b))\right)\right)\right)\right)= \\
& =h\left(f_{1}(a)\right) \wedge h\left(f_{1}\left(f_{1}^{-1}\left(h^{-1}\left(\phi_{M}(f(b))\right)\right)\right)\right)= \\
& =\phi_{M}(a) \wedge \phi_{M}(f(b))=\phi_{M}(a) \wedge \phi_{M}(b)
\end{aligned}
$$

and

$$
h\left(f_{1}(a \wedge f(b))\right)=\phi_{M}(a \wedge(f(b))) \leq \phi_{M}(a) \wedge \phi_{M}(f(b))=h\left(f_{1}(y)\right)
$$

Since both $h$ and $f_{1}$ are automorphisms of $B(M)$, the latter inequality clearly implies that $a \wedge f(b) \leq y$. Moreover, since $h$ and $f_{1}$ are $\phi_{M}$-preserving and $\phi_{M}$ restricted to $M$ is the identity mapping, we obtain

$$
\phi_{M}(y)=\phi_{M}\left(h\left(f_{1}(y)\right)\right)=\phi_{M}\left(\phi_{M}(a) \wedge \phi_{M}(b)\right)=\phi_{M}(a) \wedge \phi_{M}(b)
$$

Let us prove that $y$ is maximal in $L(a, b)$. Suppose that $z \in L(a, b), z \geq y$. Since $z=a \wedge f_{2}(b)$ for some $f_{2} \in G$, we see that

$$
\phi_{M}(z)=\phi_{M}\left(a \wedge f_{2}(b)\right) \leq \phi_{M}(a) \wedge \phi_{M}\left(f_{2}(b)\right)=\phi_{M}(y)
$$

This implies that $\phi_{M}(z)=\phi_{M}(y)$. As $\phi_{M}(z \backslash y)=\phi_{M}(z) \ominus \phi_{M}(y)=0$ and $\phi_{M}$ is faithful, $z \backslash y=0$ and hence $z=y$.

Let us prove that $\mathcal{A}(B(M), G(M))$ is isomorphic to $M$. The isomorphism $\psi$ : $\mathcal{A}(B(M), G(M)) \rightarrow M$ is given by

$$
\psi\left([a]_{G(M)}\right)=\phi_{M}(u)
$$

By Corollary4.8 $\psi$ is well-defined and injective. Since, for all $a \in M, \psi\left([a]_{G(M)}\right)=$ $a, \psi$ is surjective. Obviously, $\psi\left([1]_{G(M)}\right)=1$. Let $[a]_{G(M)},[b]_{G(M)} \in \mathcal{A}(B(M), G(M))$ be such that $[a]_{G(M)},[b]_{G(M)}$. We may always select the elements $a, b \in B(M)$ so that $a \dot{\vee} b$ exists, that means, $a \wedge b=0$. Since $\phi_{M}$ is a morphism of effect algebras, $\phi_{M}(a) \oplus \phi_{M}(b)$ exists in $M$ and we may compute

$$
\begin{aligned}
\psi\left([a]_{G(M)} \oplus[b]_{G(M)}\right. & =\psi\left([a \dot{\vee} b]_{G(M)}\right)=\phi_{M}(a \dot{\vee} b)= \\
& =\phi_{M}(a) \oplus \phi_{M}(b)=\psi\left([a]_{G(M)}\right) \oplus \psi\left([b]_{G(M)}\right)
\end{aligned}
$$

hence $\psi$ is a morphism of effect algebras. It remains to prove that $\psi$ is a full morphism. Suppose that $\psi\left([a]_{G(M)}\right) \oplus \psi\left([b]_{G(M)}\right)$ exists in $M$. Consider the elements $\phi_{M}(a)$ and $\left(\phi_{M}(a) \oplus \phi_{M}(b)\right) \backslash \phi_{M}(a)$ of $B(M)$. We see that

$$
\phi_{M}(a) \wedge\left(\left(\phi_{M}(a) \oplus \phi_{M}(b)\right) \backslash \phi_{M}(a)\right)=0
$$

that means, $\phi_{M}(a) \dot{\vee}\left(\left(\phi_{M}(a) \oplus \phi_{M}(b)\right) \backslash \phi_{M}(a)\right)$ exists in $B(M)$. This implies that $\left.\left[\phi_{M}(a)\right]_{G(M)} \oplus\left[\left(\phi_{M}(a) \oplus \phi_{M}(b)\right) \backslash \phi_{M}(a)\right)\right]_{G(M)}$ exists in $\mathcal{A}(B(M), G(M))$. Finally,

$$
\psi\left(\left[\phi_{M}(a)\right]_{G(M)}\right)=\phi_{M}\left(\phi_{M}(a)\right)=\phi_{M}(a)=\psi\left([a]_{G(M)}\right)
$$

and

$$
\begin{aligned}
& \psi\left(\left[\left(\phi_{M}(a) \oplus \phi_{M}(b)\right) \backslash \phi_{M}(a)\right]_{G(M)}\right)=\phi_{M}\left(\left(\phi_{M}(a) \oplus \phi_{M}(b)\right) \backslash \phi_{M}(a)\right)= \\
& =\phi_{M}\left(\phi_{M}(a) \oplus \phi_{M}(b)\right) \ominus \phi_{M}\left(\phi_{M}(a)\right)=\left(\phi_{M}(a) \oplus \phi_{M}(b)\right) \ominus \phi_{M}(a)= \\
& =\phi_{M}(b)=\psi\left([b]_{G(M)}\right) .
\end{aligned}
$$

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