# More about sharp and meager elements in Archimedean atomic lattice effect algebras 

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#### Abstract

The aim of our paper is twofold. First, we thoroughly study the set of meager elements $\mathrm{M}(E)$, the center $\mathrm{C}(E)$ and the compatibility center $\mathrm{B}(E)$ in the setting of atomic Archimedean lattice effect algebras $E$. The main result is that in this case the center $\mathrm{C}(E)$ is bifull (atomic) iff the compatibility center $\mathrm{B}(E)$ is bifull (atomic) whenever $E$ is sharply dominating. As a by-product, we give a new descriciption of the smallest sharp element over $x \in E$ via the basic decomposition of $x$. Second, we prove the Triple Representation Theorem for sharply dominating atomic Archimedean lattice effect algebras.


Keywords: lattice effect algebra, center, atom, MacNeille completion, sharp element, meager element

## Introduction

The history of quantum structures started at the beginning of the 20th century. Observable events constitute a Boolean algebra in a classical physical system. Because event structures in quantum mechanics cannot be described by Boolean algebras, Birkhoff and von Neumann introduced orthomodular lattices which were considered as the standard quantum logic. Later on, orthoalgebras were introduced as the generalizations of orthomodular posets, which were considered as "sharp" quantum logic.

In the nineties of the twentieth century, two equivalent quantum structures, D-posets and effect algebras were extensively studied, which were considered as "unsharp" generalizations of the structures which arise in quantum mechanics, in particular, of orthomodular lattices and MV-algebras.

In 16] Paseka and Riečanová published as open problem whether the center $\mathrm{C}(E)$ is a bifull sublattice of an Archimedean atomic lattice effect algebra $E$. This question was answered by M. Kalina in [10] who proved that $\mathrm{C}(E)$ need not be a bifull sublattice of $E$ even if $C(E)$ is atomic.

[^0]The aim of our paper is twofold. First, we thoroughly study the set of meager elements $\mathrm{M}(E)$, the center $\mathrm{C}(E)$ and the compatibility center $\mathrm{B}(E)$ in the setting of atomic Archimedean lattice effect algebras $E$. The main result of Section 2 is that in this case the center $\mathrm{C}(E)$ is bifull (atomic) iff the compatibility center $\mathrm{B}(E)$ is bifull (atomic) whenever $E$ is sharply dominating. As a by-product, we give a new descriciption of the smallest sharp element over $x \in E$ via the basic decomposition of $x$. Second, in Section 3 we prove the Triple Representation Theorem established by G. Jenča in [8] in the setting of complete lattice effect algebras for sharply dominating atomic Archimedean lattice effect algebras.

## 1. Preliminaries and basic facts

Effect algebras were introduced by D.J. Foulis and M.K. Bennett (see [4]) for modelling unsharp measurements in a Hilbert space. In this case the set $E(H)$ of effects is the set of all self-adjoint operators $A$ on a Hilbert space $H$ between the null operator 0 and the identity operator 1 and endowed with the partial operation + defined iff $A+B$ is in $E(H)$, where + is the usual operator sum.

In general form, an effect algebra is in fact a partial algebra with one partial binary operation and two unary operations satisfying the following axioms due to D.J. Foulis and M.K. Bennett.

Definition 1.1. 22] A partial algebra $(E ; \oplus, 0,1)$ is called an effect algebra if 0,1 are two distinct elements and $\oplus$ is a partially defined binary operation on $E$ which satisfy the following conditions for any $x, y, z \in E$ :
(Ei) $\quad x \oplus y=y \oplus x$ if $x \oplus y$ is defined,
(Eii) $(x \oplus y) \oplus z=x \oplus(y \oplus z)$ if one side is defined,
(Eiii) for every $x \in E$ there exists a unique $y \in E$ such that $x \oplus y=1$ (we put $x^{\prime}=y$ ),
(Eiv) if $1 \oplus x$ is defined then $x=0$.
We often denote the effect algebra $(E ; \oplus, 0,1)$ briefly by $E$. On every effect algebra $E$ a partial order $\leq$ and a partial binary operation $\ominus$ can be introduced as follows:

$$
x \leq y \text { and } y \ominus x=z \text { iff } x \oplus z \text { is defined and } x \oplus z=y
$$

If $E$ with the defined partial order is a lattice (a complete lattice) then ( $E ; \oplus, 0,1$ ) is called a lattice effect algebra (a complete lattice effect algebra).

Definition 1.2. Let $E$ be an effect algebra. Then $Q \subseteq E$ is called a sub-effect algebra of $E$ if
(i) $1 \in Q$
(ii) if out of elements $x, y, z \in E$ with $x \oplus y=z$ two are in $Q$, then $x, y, z \in Q$.

If $E$ is a lattice effect algebra and $Q$ is a sub-lattice and a sub-effect algebra of $E$, then $Q$ is called a sub-lattice effect algebra of $E$.

Note that a sub-effect algebra $Q$ (sub-lattice effect algebra $Q$ ) of an effect algebra $E$ (of a lattice effect algebra $E$ ) with inherited operation $\oplus$ is an effect algebra (lattice effect algebra) in its own right.

For an element $x$ of an effect algebra $E$ we write ord $(x)=\infty$ if $n x=x \oplus x \oplus$ $\cdots \oplus x$ ( $n$-times) exists for every positive integer $n$ and we write $\operatorname{ord}(x)=n_{x}$ if $n_{x}$ is the greatest positive integer such that $n_{x} x$ exists in $E$. An effect algebra $E$ is Archimedean if ord $(x)<\infty$ for all $x \in E$.

A minimal nonzero element of an effect algebra $E$ is called an atom and $E$ is called atomic if below every nonzero element of $E$ there is an atom.

For a poset $P$ and its subposet $Q \subseteq P$ we denote, for all $X \subseteq Q$, by $\bigvee_{Q} X$ the join of the subset $X$ in the poset $Q$ whenever it exists. Recall also $Q \subseteq P$ is densely embedded in $P$ if for every element $x \in P$ there exist $S, T \subseteq Q$ such that $x=\bigvee_{P} S=\bigwedge_{P} T$.

We say that a finite system $F=\left(x_{k}\right)_{k=1}^{n}$ of not necessarily different elements of an effect algebra $(E ; \oplus, 0,1)$ is orthogonal if $x_{1} \oplus x_{2} \oplus \cdots \oplus x_{n}$ (written $\bigoplus_{k=1}^{n} x_{k}$ or $\oplus F)$ exists in $E$. Here we define $x_{1} \oplus x_{2} \oplus \cdots \oplus x_{n}=\left(x_{1} \oplus x_{2} \oplus \cdots \oplus x_{n-1}^{k=1}\right) \oplus x_{n}$ supposing that $\bigoplus_{k=1}^{n-1} x_{k}$ is defined and $\bigoplus_{k=1}^{n-1} x_{k} \leq x_{n}^{\prime}$. We also define $\bigoplus \emptyset=0$. An arbitrary system $G=\left(x_{\kappa}\right)_{\kappa \in H}$ of not necessarily different elements of $E$ is called orthogonal if $\bigoplus K$ exists for every finite $K \subseteq G$. We say that for a orthogonal system $G=\left(x_{\kappa}\right)_{\kappa \in H}$ the element $\bigoplus G$ exists iff $\bigvee\{\bigoplus K \mid K \subseteq G$ is finite $\}$ exists in $E$ and then we put $\bigoplus G=\bigvee\{\bigoplus K \mid K \subseteq G$ is finite $\}$. We say that $\bigoplus G$ is the orthogonal sum of $G$. (Here we write $G_{1} \subseteq G$ iff there is $H_{1} \subseteq H$ such that $\left.G_{1}=\left(x_{\kappa}\right)_{\kappa \in H_{1}}\right)$.

An element $u \in E$ is called finite if either $u=0$ or there is a finite sequence $\left\{a_{1}, a_{2}, \ldots, a_{n}\right\}$ of not necessarily different atoms of $E$ such that $u=a_{1} \oplus a_{2} \oplus$ $\cdots \oplus a_{n}$. Note that any atom of $E$ is evidently finite. An element $v \in E$ is called cofinite if $v^{\prime} \in E$ is finite.

Elements $x$ and $y$ of a lattice effect algebra $E$ are called compatible $(x \leftrightarrow y$ for short) if $x \vee y=x \oplus(y \ominus(x \wedge y))$ (see [13, 20]).

Remarkable sub-lattice effect algebras of a lattice effect algebra $E$ are
(1) A block $M$ of $E$, which is any maximal subset of pairwise compatible elements of $E$ (in fact $M$ is a maximal sub- $M V$-algebra of $E$, see 20 ).
(2) The set $\mathrm{S}(E)=\left\{x \in E \mid x \wedge x^{\prime}=0\right\}$ of sharp elements of $E$ (see [6], 7]), which is an orthomodular lattice (see [9]).
(3) The compatibility center $\mathrm{B}(E)$ of $E, \mathrm{~B}(E)=\bigcap\{M \subseteq E \mid M$ is a block of $E\}=\{x \in E \mid x \leftrightarrow y$ for every $y \in E\}$ which is in fact an $M V$-algebra ( $M V$-effect algebra).
(4) The center $\mathrm{C}(E)=\left\{x \in E \perp y=(y \wedge x) \vee\left(y \wedge x^{\prime}\right)\right.$ for all $\left.y \in E\right\}$ of $E$ is a Boolean algebra (see [5]). In every lattice effect algebra it holds $\mathrm{C}(E)=\mathrm{B}(E) \cap \mathrm{S}(E)=\mathrm{S}(\mathrm{B}(E))$ (see 18] and [19]).

All these sub-lattice effect algebras of a lattice effect algebra $E$ are in fact full sub-lattice effect algebras of $E$. This means that they are closed with respect to all suprema and infima existing in $E$ of their subsets [9, 21].

The $M V$-effect algebras $E$ are precisely lattice effect algebras with a unique block (i.e., $E=\mathrm{B}(E)$ ).

The following statements are well known.
Statement 1.3. Let $E$ be a lattice effect algebra. Then
(i) [9, Theorem 2.1] Assume $b \in E, A \subseteq E$ are such that $\bigvee A$ exists in $E$ and $b \leftrightarrow a$ for all $a \in A$. Then
(a) $b \leftrightarrow \bigvee A$.
(b) $\bigvee\{b \wedge a: a \in A\}$ exists in $E$ and equals $b \wedge(\bigvee A)$.
(ii) 9, Theorem 3.7], 21, Theorem 2.8] $\mathrm{S}(E), \mathrm{B}(E)$ and $\mathrm{C}(E)$ are full sublattice effect algebras of $E$.
(iii) 15, Lemma 3.3] Let $x, y \in E$. Then $x \wedge y=0$ and $x \leq y^{\prime}$ iff $k x \wedge l y=0$ and $k x \leq(l y)^{\prime}$, whenever $k x$ and ly exist in $E$.
(iv) 17, Proposition 1] Let $\left\{b_{\alpha} \mid \alpha \in \Lambda\right\}$ be a family of elements in $E$ and let $a \in E$ with $a \leq b_{\alpha}$ for all $\alpha \in \Lambda$. Then

$$
\left(\bigvee\left\{b_{\alpha} \mid \alpha \in \Lambda\right\}\right) \ominus a=\bigvee\left\{b_{\alpha} \ominus a \mid \alpha \in \Lambda\right\}
$$

if one side is defined.
(v) 25, Theorem 3.5] For every atom $a \in E$ with ord $(a)<\infty, n_{a} a$ is the smallest sharp element over $a$.
(vi) 19, Corollary 4.3] Let $x, y \in E$. Then $x \oplus y=(x \vee y) \oplus(x \wedge y)$ whenever $x \oplus y$ exists.
(vii) 33, Proposition 1.8.7] Let $b \in E, A \subseteq E$ are such that $\bigvee A$ exists in $E$ and $b \oplus a$ exists for all $a \in A$. Then $\bigvee\{b \oplus a: a \in A\}=b \oplus \bigvee A$.
(viii) 24, Lemma 4.1] Assume that $z \in \mathrm{C}(E)$. Then, for all $x, y \in E$ with $x \leq y^{\prime},(x \oplus y) \wedge z=(x \wedge z) \oplus(y \wedge z)$.
Statement 1.4. 23, Theorem 3.3] Let $E$ be an Archimedean atomic lattice effect algebra. Then to every nonzero element $x \in E$ there are mutually distinct atoms $a_{\alpha} \in E$ and positive integers $k_{\alpha}, \alpha \in \mathcal{E}$ such that

$$
x=\bigoplus\left\{k_{\alpha} a_{\alpha} \mid \alpha \in \mathcal{E}\right\}=\bigvee\left\{k_{\alpha} a_{\alpha} \mid \alpha \in \mathcal{E}\right\}
$$

and $x \in S(E)$ iff $k_{\alpha}=n_{a_{\alpha}}=\operatorname{ord}\left(a_{\alpha}\right)$ for all $\alpha \in \mathcal{E}$.
Statement 1.5. 14, Theorem 8] Let $E$ be an atomic Archimedean lattice effect algebra and let $\mathcal{M}=\left\{M_{\kappa} \mid \kappa \in H\right\}$ be a family of all atomic blocks of $E$. For each $\kappa \in H$ let $A_{\kappa}$ be the set of all atoms of $M_{\kappa}$. Then:
(i) For each $\kappa \in H, A_{\kappa}$ is a maximal pairwise compatible set of atoms of $E$.
(ii) For $x \in E$ and $\kappa \in H$ it holds $x \in M_{\kappa}$ iff $x \leftrightarrow A_{\kappa}$.
(iii) $M \in \mathcal{M}$ iff there exists a maximal pairwise compatible set $A$ of atoms of $E$ such that $A \subseteq M$ and if $M_{1}$ is a block of $E$ with $A \subseteq M_{1}$ then $M=M_{1}$.
(iv) $E=\bigcup\left\{M_{\kappa} \mid \kappa \in H\right\}$.
(v) $B(E)=\bigcap\left\{M_{\kappa} \mid \kappa \in H\right\}$.
(vi) $C(E)=\bigcap\left\{C\left(M_{\kappa}\right) \mid \kappa \in H\right\}=\bigcap\left\{S\left(M_{\kappa}\right) \mid \kappa \in H\right\}$.
(vii) $S(E)=\bigcup\left\{C\left(M_{\kappa}\right) \mid \kappa \in H\right\}=\bigcup\left\{S\left(M_{\kappa}\right) \mid \kappa \in H\right\}$.

Lemma 1.6. Let $E$ be a lattice effect algebra and let $b \in E, A \subseteq E$ are such that $\bigvee A$ exists in $E$ and $b \oplus a$ exists for all $a \in A$. Then $b \oplus \bigvee A$ exists in $E$ and $b \oplus \bigvee A=(b \vee \bigvee A) \oplus \bigvee\{b \wedge a: a \in A\}$.
Proof. Clearly $b \leftrightarrow a$ for all $a \in A$. By Statement 1.3, (i) we have that $b \leftrightarrow \bigvee A$ and $\bigvee\{b \wedge a: a \in A\}=b \wedge(\bigvee A)$. Furthermore, $b \leq a^{\prime}$ for all $a \in A$ and hence $b \leq \bigwedge\left\{a^{\prime} \mid a \in A\right\}=(\bigvee A)^{\prime}$. Therefore $b \oplus \bigvee A$ exists. In view of Statement 1.3. (vi)

$$
b \oplus \bigvee A=(b \vee \bigvee A) \oplus(b \wedge \bigvee A)=(b \vee \bigvee A) \oplus \bigvee\{b \wedge a: a \in A\}
$$

## 2. Bifull sub-lattice effect algebras of lattice effect algebras

Definition 2.1. For a poset $L$ and a subset $D \subseteq L$ we say that $D$ is a $\bigvee$-bifull sub-poset of $L$ iff, for any $X \subseteq D, \bigvee_{L} X$ exists iff $\bigvee_{D} X$ exists, in which case $\bigvee_{L} X=\bigvee_{D} X$. Dually, the notion of $\bigwedge$-bifull sub-poset of $L$ is defined. We call a subset $D \subseteq L$ to be a bifull sub-poset of $L$ if it is both $\bigvee$-bifull and $\bigwedge$-bifull.

Remark 2.2. Clearly, if $L$ is a complete lattice then $D \subseteq L$ is a complete sub-lattice of $L$ (i.e., $D$ inherits all suprema and infima of its subsets existing in $L$ ) iff $D$ is a bifull sub-poset of $L$. Moreover, if $E$ is a lattice effect algebra then a sub-lattice effect algebra $D$ of $E$ is a bifull sub-lattice effect algebra of $E$ iff it is $\bigvee$-bifull.

An important class of effect algebras was introduced by S. Gudder in [6] and [7]. Fundamental example is the standard Hilbert spaces effect algebra $\mathcal{E}(\mathcal{H})$.

For an element $x$ of an effect algebra $E$ we denote

$$
\begin{array}{ll}
\widetilde{x}=\bigvee_{E}\{s \in \mathrm{~S}(E) \mid s \leq x\} & \\
\widehat{x}=\bigwedge_{E}\{s \in \mathrm{~S}(E) \mid s \geq x\} & \\
\text { if it exists and belongs to } \mathrm{S}(E) \\
\widehat{x}=x \text { and belongs to } \mathrm{S}(E) .
\end{array}
$$

Definition 2.3. ([6], [7].) An effect algebra $(E, \oplus, 0,1)$ is called sharply dominating if for every $x \in E$ there exists $\widehat{x}$, the smallest sharp element such that $x \leq \widehat{x}$. That is $\widehat{x} \in \mathrm{~S}(E)$ and if $y \in \mathrm{~S}(E)$ satisfies $x \leq y$ then $\widehat{x} \leq y$.

Recall that evidently an effect algebra $E$ is sharply dominating iff for every $x \in E$ there exists $\widetilde{x} \in \mathrm{~S}(E)$ such that $\widetilde{x} \leq x$ and if $u \in \mathrm{~S}(E)$ satisfies $u \leq x$ then $u \leq \widetilde{x}$ iff for every $x \in E$ there exist a smallest sharp element $\widehat{x}$ over $x$ and a greatest sharp element $\widetilde{x}$ below $x$.

In what follows set (see 88, 25])

$$
\mathrm{M}(E)=\{x \in E \mid \text { if } v \in \mathrm{~S}(E) \text { satisfies } v \leq x \text { then } v=0\}
$$

An element $x \in \mathrm{M}(E)$ is called meager. Moreover, $x \in \mathrm{M}(E)$ iff $\widetilde{x}=0$. Recall that $x \in \mathrm{M}(E), y \in E, y \leq x$ implies $y \in \mathrm{M}(E)$ and $x \ominus y \in \mathrm{M}(E)$.

Lemma 2.4. Let $E$ be an effect algebra in which $\mathrm{S}(E)$ is a sub-effect algebra of $E$ and let $x \in \mathrm{M}(E)$ such that $\widehat{x}$ exists. Then
(i) $\widehat{x} \ominus x \in \mathrm{M}(E)$.
(ii) If $y \in \mathrm{M}(E)$ such that $x \oplus y$ exists and $x \oplus y=z \in \mathrm{~S}(E)$ then $\widehat{x}=z$.

Moreover, if $E$ is a lattice effect algebra then $\widehat{y}$ exists and $\widehat{y}=\widehat{\widehat{x} \ominus x}=z$.
Proof. (i): Let $u \in \mathrm{~S}(E)$ such that $u \leq \widehat{x} \ominus x$. Then $x \leq \widehat{x} \ominus u \in \mathrm{~S}(E)$ which yields that $\widehat{x} \leq \widehat{x} \ominus u$. Hence $u=0$, i.e., $\widehat{x} \ominus x \in \mathrm{M}(E)$.
(ii): Since $x \leq z$ and hence $\widehat{x} \leq z$ we have $x \oplus y=z=\widehat{x} \oplus(z \ominus \widehat{x})$ and $\widehat{x}=x \oplus(\widehat{x} \ominus x)$. This yields $x \oplus y=x \oplus(\widehat{x} \ominus x) \oplus(z \ominus \widehat{x})$. By the cancellation law we get $y=(\widehat{x} \ominus x) \oplus(z \ominus \widehat{x})$. Hence $z \ominus \widehat{x}=0$, i.e., $z=\widehat{x}$.

$$
\underbrace{}_{\in S(E)}
$$

Now, assume that $E$ is a lattice effect algebra. Let $u \in S(E), u \geq y$. Then also $u \wedge z \geq y, u \wedge z \in \mathrm{~S}(E)$ and $z \ominus(u \wedge z) \in \mathrm{S}(E)$. Then $x \oplus y=z=$ $y \oplus((u \wedge z) \ominus y) \oplus(z \ominus(u \wedge z))$. Therefore $x=((u \wedge z) \ominus y) \oplus(z \ominus(u \wedge z))$. Since $z \ominus(u \wedge z) \in \mathrm{S}(E)$ this yields $z \ominus(u \wedge z)=0$, i.e., $z=u \wedge z \leq u$.

Lemma 2.5. Let $E$ be an effect algebra in which $\mathrm{S}(E)$ is a sub-effect algebra of $E$ and let $x \in E$ such that $\widetilde{x}$ exists. Then
(i) $x \ominus \widetilde{x} \in \mathrm{M}(E)$ and $x=\widetilde{x} \oplus(x \ominus \widetilde{x})$ is the unique decomposition $x=$ $x_{S} \oplus x_{M}$, where $x_{S} \in \mathrm{~S}(E)$ and $x_{M} \in \mathrm{M}(E)$. Moreover, $x_{S} \wedge x_{M}=0$ and if $E$ is a lattice effect algebra then $x=x_{S} \vee x_{M}$.
(ii) If $E$ is a lattice effect algebra such that $\widehat{x}$ exists then $\widehat{x \ominus \widetilde{x}}$ and $\widehat{\widehat{x \ominus x}}$ exist, $\widehat{x} \ominus \widetilde{x}=\widehat{x \ominus \widetilde{x}}=\widehat{\widehat{x} \ominus x}, \widehat{x}=\widetilde{x} \oplus \widehat{x \ominus \widetilde{x}}=\widetilde{x} \vee \widehat{x \ominus \widetilde{x}}$ and $\widetilde{x} \wedge \widehat{x \ominus \widetilde{x}}=0$. Moreover, $\widehat{x} \ominus \widetilde{x}=(\widehat{x} \ominus \widetilde{x}) \ominus(\widehat{x} \ominus x)$.
Proof. (i): Let $v \in \mathrm{~S}(E), v \leq x \ominus \widetilde{x}$. Then $v \oplus \widetilde{x} \leq x$ and $v \oplus \widetilde{x} \in \mathrm{~S}(E)$. Hence $v \oplus \widetilde{x} \leq \widetilde{x}$, i.e., $v=0, x_{M} \in \mathrm{M}(E)$ and $x=\widetilde{x} \oplus(x \ominus \widetilde{x})$.

Assume that there is a decomposition $x=x_{S} \oplus x_{M}$ such that $x_{S} \in \mathrm{~S}(E)$ and $x_{M} \in \mathrm{M}(E)$. Then $x_{S} \leq \widetilde{x}$ and $x_{M}=x \ominus x_{S}=(x \ominus \widetilde{x}) \oplus\left(\widetilde{x} \ominus x_{S}\right) \geq \widetilde{x} \ominus x_{S} \in \mathrm{~S}(E)$. It follows that $\widetilde{x} \ominus x_{S}=0$ because $x_{M} \in \mathrm{M}(E)$. Therefore, $x_{M}=x \ominus \widetilde{x}$ and $x_{S}=\widetilde{x}$.

We have $x_{S}^{\prime}=1 \ominus x_{S} \geq x \ominus x_{S}=x_{M}$. Hence $x_{S} \wedge x_{M} \leq x_{S} \wedge x_{S}^{\prime}=0$.
Let $E$ be a lattice effect algebra. By Statement 1.3, (vi) we have that $x=x_{S} \oplus x_{M}=\left(x_{S} \vee x_{M}\right) \oplus\left(x_{S} \wedge x_{M}\right)=x_{S} \vee x_{M}$.
(ii): We have that $\widehat{x} \ominus \widetilde{x} \geq x \ominus \widetilde{x}, \widehat{x} \ominus \widetilde{x} \in \mathrm{~S}(E)$. Let $z \in \mathrm{~S}(E)$ such that $z \geq x \ominus \widetilde{x}$. Let us put $w=z \wedge(\widehat{x} \ominus \widetilde{x})$. Then $w \leq \widehat{x} \ominus \widetilde{x}$ hence $w \oplus \widetilde{x} \in \mathrm{~S}(E)$ exists and $w \oplus \widetilde{x} \geq x$. This yields that $w \oplus \tilde{x} \geq \widehat{x}$, i.e., $z \geq w \geq \widehat{x} \ominus \widetilde{x}$. Therefore, $\widehat{x} \ominus \widetilde{x}=\widehat{x \ominus \widetilde{x}}$.

Since $\widehat{x} \ominus \widetilde{x} \leq 1 \ominus \widetilde{x}=(\widetilde{x})^{\prime}$ we obtain that $\widetilde{x} \wedge \widehat{x \ominus \widetilde{x}} \leq \widetilde{x} \wedge(\widetilde{x})^{\prime}=0$.
We proceed similarly to prove that $\widehat{x} \ominus \widetilde{x}=\widehat{\widehat{x} \ominus x}$. Evidently, $\widehat{x} \ominus \widetilde{x} \geq \widehat{x} \ominus x$. Let $z \in \mathrm{~S}(E)$ such that $z \geq \widehat{x} \ominus x$. We put $w=z \wedge(\widehat{x} \ominus \widetilde{x})$. Then $\widehat{x} \ominus x \leq$ $w \leq \widehat{x} \ominus \widetilde{x} \leq \widehat{x}$. It follows that $\widehat{x} \ominus w \leq x$ and $\widehat{x} \ominus w \in \mathrm{~S}(E)$ which yields that $\widehat{x} \ominus w \leq \widetilde{x}$. Hence $\widehat{x} \ominus \widetilde{x} \leq w \leq z$.

Moreover, $(\widehat{x} \ominus \widehat{x}) \ominus(\widehat{x} \ominus x)=\widehat{x \ominus \widetilde{x}}=\widehat{x} \ominus \widetilde{x}$.

As proved in [1] $S(E)$ is always a sub-effect algebra in a sharply dominating effect algebra $E$.
Corollary 2.6. [8, Proposition 15] Let $E$ be a sharply dominating effect algebra. Then every $x \in E$ has a unique decomposition $x=x_{S} \oplus x_{M}$, where $x_{S} \in \mathrm{~S}(E)$ and $x_{M} \in \mathrm{M}(E)$, namely $x=\widetilde{x} \oplus(x \ominus \widetilde{x})$.

Moreover, the following statement holds.
Statement 2.7. Let $E$ be a lattice effect algebra. Then
(i) 12, Corollary 1] If $E$ is a sharply dominating then $S(E)$ is bifull in $E$.
(ii) 16, Lemma 2.7] If $E$ is Archimedean and atomic then $S(E)$ is bifull in $E$.

First, we shall need an extension of Statement 1.3, (iii).
Lemma 2.8. Let $E$ be a lattice effect algebra, $x_{1}, \ldots, x_{n} \in E, k_{1}, \ldots, k_{n} \in \mathbb{N}$, $n \geq 2$ such that $k_{i} x_{i}$ exist in $E$ for all $1 \leq i \leq n$. Then

$$
x_{i} \wedge x_{j}=0 \text { and } x_{i} \leq x_{j}^{\prime} \text { for all } 1 \leq i<j \leq n
$$

iff

$$
\begin{aligned}
& \bigoplus_{j=1}^{n} k_{j} x_{j} \text { exists and } \bigoplus_{j=1}^{n} k_{j} x_{j}=\bigvee_{j=1}^{n} k_{j} x_{j} \\
& \bigoplus_{i \in I} k_{i} x_{i} \wedge \bigoplus_{j \in J} k_{j} x_{j}=0 \text { and } \bigoplus_{j \in J} k_{j} x_{j} \leq\left(\bigoplus_{i \in I} k_{i} x_{i}\right)^{\prime}
\end{aligned}
$$

$$
\text { for all } \emptyset \neq I \subset\{1, \ldots, n\}, J=\{1, \ldots, n\} \backslash I
$$

Proof. Assume that $x_{i} \wedge x_{j}=0$ and $x_{i} \leq x_{j}^{\prime}$ for all $1 \leq i<j \leq n$. Let $k_{i} x_{i}$ exist in $E$ for all $1 \leq i \leq n$. If $n=2$ then from Statement 1.3, (iii) we know that $k_{1} x_{1} \wedge k_{2} x_{2}=0, k_{1} x_{1} \leq\left(k_{2} x_{2}\right)^{\prime}$ and $k_{2} x_{2} \leq\left(k_{1} x_{1}\right)^{\prime}$. Since $k_{1} x_{1} \leftrightarrow k_{2} x_{2}$ we have that $k_{1} x_{1} \vee k_{2} x_{2}=k_{1} x_{1} \oplus\left(k_{2} x_{2} \ominus\left(k_{1} x_{1} \wedge k_{2} x_{2}\right)\right)=k_{1} x_{1} \oplus k_{2} x_{2}$. We shall proceed by induction. Let $n \in \mathbb{N}$ be arbitrary, $n \geq 3$ and assume that the statement holds for every $m<n$. Let us take $\emptyset \neq I \subset\{1, \ldots, n\}$ arbitrarily and put $J=\{1, \ldots, n\} \backslash I$. Hence $|I|<n$ and $|J|<n$. Then we have (again by Statement 1.3, (iii)) that $k_{i} x_{i} \wedge k_{j} x_{j}=0$ and $k_{j} x_{j} \leq\left(k_{i} x_{i}\right)^{\prime}$ for all $i \in I$ and $j \in J$. This and the induction assumption yield that $\bigoplus_{j \in J} k_{j} x_{j}=\bigvee_{j \in J} k_{j} x_{j} \leq$ $\left(k_{i} x_{i}\right)^{\prime}$ for all $i \in I$. This is equivalent to $k_{i} x_{i} \leq\left(\bigoplus_{j \in J} k_{j} x_{j}\right)^{\prime}$ for all $i \in I$ i.e., $\bigoplus_{i \in I} k_{i} x_{i}=\bigvee_{i \in I} k_{i} x_{i} \leq\left(\bigoplus_{j \in J} k_{j} x_{j}\right)^{\prime}$. Furthermore, $k_{j} x_{j} \leftrightarrow k_{i} x_{i}$ for all $i \in I$ and $j \in J$ implies by Statement 1.3, (i) that

$$
\bigoplus_{i \in I} k_{i} x_{i} \wedge \bigoplus_{j \in J} k_{j} x_{j}=\bigvee_{i \in I} k_{i} x_{i} \wedge \bigvee_{j \in J} k_{j} x_{j}=\bigvee_{i \in I} \bigvee_{j \in J}(\underbrace{k_{i} x_{i} \wedge k_{j} x_{j}}_{=0})=0
$$

Similarly by the induction assumption and Statement 1.3, (iii) and (vii),

$$
\begin{aligned}
\bigoplus_{j=1}^{n} k_{j} x_{j} & =\left(\bigoplus_{j=1}^{n-1} k_{j} x_{j}\right) \oplus k_{n} x_{n}=\left(\bigvee_{j=1}^{n-1} k_{j} x_{j}\right) \oplus k_{n} x_{n} \\
& =\bigvee_{j=1}^{n-1}\left(k_{j} x_{j} \oplus k_{n} x_{n}\right)=\bigvee_{j=1}^{n-1}\left(k_{j} x_{j} \vee k_{n} x_{n}\right)=\bigvee_{j=1}^{n} k_{j} x_{j}
\end{aligned}
$$

The converse implication is evident.

Corollary 2.9. Let $E$ be an Archimedean lattice effect algebra and $a_{1}, \ldots, a_{n}$ mutually compatible different atoms from $E, 1 \leq k_{i} \leq n_{a_{i}}$ for all $1 \leq i \leq n$. Then $k_{1} a_{1} \oplus \cdots \oplus k_{n} a_{n}$ exists and $k_{1} a_{1} \oplus \cdots \oplus k_{n} a_{n}=k_{1} a_{1} \vee \cdots \vee k_{n} a_{n}$. Moreover, $n_{a_{1}} a_{1} \oplus \cdots \oplus n_{a_{n}} a_{n}=n_{a_{1}} a_{1} \vee \cdots \vee n_{a_{n}} a_{n}$ is the smallest sharp element over $k_{1} a_{1} \oplus \cdots \oplus k_{n} a_{n}$.

Theorem 2.10. Let $E$ be an atomic Archimedean lattice effect algebra and let $x \in \mathrm{M}(E)$. Let us denote $A_{x}=\{a \mid a$ an atom of $E$, $a \leq x\}$ and, for any $a \in A_{x}$, we shall put $k_{a}^{x}=\max \{k \in \mathbb{N} \mid k a \leq x\}$. Then
(i) For any $a \in A_{x}$ we have $k_{a}^{x}<n_{a}$.
(ii) The set $F_{x}=\left\{k_{a}^{x} a \mid a \in A_{x}\right\}$ is orthogonal and

$$
x=\bigoplus\left\{k_{a}^{x} a \mid a \text { an atom of } E, a \leq x\right\}=\bigoplus F_{x}=\bigvee F_{x}
$$

Moreover, for all $B \subseteq A_{x}$ and all natural numbers $l_{b}<n_{b}, b \in B$ such that $x=\bigoplus\left\{l_{b} b \mid b \in B\right\}$ we have that $B=A_{x}$ and $l_{a}=k_{a}^{x}$ for all $a \in A_{x}$ i.e., $F_{x}$ is the unique set of multiples of atoms from $A_{x}$ such that its orthogonal sum is $x$.
(iii) For every atomic block $M$ of $E, x \in M$ implies that $[0, x] \subseteq M$.
(iv) $x \in \mathrm{~B}(E)$ implies that $[0, x] \subseteq \mathrm{B}(E)$.
(v) If $\widehat{x}$ exists then

$$
\widehat{x}=\widehat{\widehat{x} \ominus x}=\bigoplus\left\{n_{a} a \mid a \text { an atom of } E, a \leq x\right\}=\bigvee\left\{n_{a} a \mid a \in A_{x}\right\}
$$

and

$$
\widehat{x} \ominus x=\bigoplus\left\{\left(n_{a}-k_{a}^{x}\right) a \mid a \in A_{x}\right\}=\bigvee\left\{\left(n_{a}-k_{a}^{x}\right) a \mid a \in A_{x}\right\}
$$

(vi) If $x$ is finite then $[0, x]$ is a finite lattice, $x=\bigoplus_{i=1}^{n} k_{i} a_{i}=\bigvee_{i=1}^{n} k_{i} a_{i}$ for a suitable finite set $A_{x}=\left\{a_{1}, \ldots, a_{n}\right\}$ of atoms of $E$ and $[0, x] \cong$ $\prod_{i=1}^{n}\left[0, k_{i} a_{i}\right]$.

Proof. (i): Let $a \in A_{x}$. Since $E$ is Archimedean we have $k_{a}^{x} \leq n_{a}$. Assume that $k_{a}^{x}=n_{a}$. Then $0<n_{a} a \leq x$ and $n_{a} a \in \mathrm{~S}(E)$ by Statement 1.4 i.e., $x \notin \mathrm{M}(E)$, a contradiction.
(ii): From Statement 1.4, (i) we know that there is a subset $B \subseteq A_{x}$ and natural numbers $l_{b}<n_{b}, b \in B$ such that

$$
x=\bigoplus\left\{l_{b} b \mid b \in B\right\}=\bigvee\left\{l_{b} b \mid b \in B\right\}
$$

Let us show that $F_{x}=\left\{l_{b} b \mid b \in B\right\}$. Evidently, $l_{b} \leq k_{b}^{x}<n_{b}$ and $l_{b} b \leq x$ for all $b \in B$. Hence, for any finite subset $D \subseteq B$ and for any $c \in B$, we have by Corollary 2.9 that $c \oplus \bigoplus\left\{l_{b} b \mid b \in D\right\}$ exists. This yields that $\bigoplus\left\{l_{b} b \mid b \in D\right\} \leq$ $c^{\prime}$ and therefore $x \leq c^{\prime}$ for all $c \in B$. Now, let $a \in A_{x}$. Then $a \leq x \leq c^{\prime}$ for all $c \in B$ i.e., $a \leftrightarrow c$.

We then have

$$
\begin{aligned}
0 \neq k_{a}^{x} a & =k_{a}^{x} a \wedge x=k_{a}^{x} a \wedge \bigvee\left\{l_{b} b \mid b \in B\right\} \\
& =\bigvee\left\{k_{a}^{x} a \wedge l_{b} b \mid b \in B\right\}=k_{a}^{x} a \wedge l_{a} a
\end{aligned}
$$

The third equation follows from Statement 1.3, (i) and the last equation follows from the fact that $a \neq b, a \leftrightarrow b$ implies $k_{a}^{x} a \wedge l_{b} b=0$. Hence $k_{a}^{x} \leq l_{a} \leq k_{a}^{x}$ i.e., $a \in B$ and $l_{a}^{x}=k_{a}$. Therefore $A_{x}=B$ and $F_{x}=\left\{l_{b} b \mid b \in B\right\}$. The remaining part of the statement is evident.
(iii): Let $y \leq x$. Then $y \in \mathrm{M}(E), A_{y} \subseteq A_{x}$ and $k_{a}^{y} \leq k_{a}^{x}$ for all $a \in A_{y}$. Recall that by 16, Lemma 2.7 (i)] we know that $M$ is a bifull sub-lattice effect algebra of $E$. Since $M$ is atomic we have that $x=\bigoplus_{M}\left\{l_{b} b \mid b \in A_{x}^{M}\right\}=\bigoplus_{E}\left\{l_{b} b \mid b \in\right.$ $\left.A_{x}^{M}\right\}$; here $A_{x}^{M}=\{a \mid a$ an atom of $M, a \leq x\}$. This immediately implies by (ii) that the sets $A_{x}^{M}$ and $A_{x}$ coincide. Therefore, $A_{x} \subseteq M$. Note also that $M$ is closed under arbitrary joins existing in $E$. Hence $y=\bigvee\left\{k_{a}^{y} a \mid a \leq y\right\} \in M$. (iv): It follows immediately from (iii) and by $B(E)=\bigcap\{M \subseteq E \mid M$ is an atomic block of $E\}$ (see 1.5).
(v): We have that $x=\bigoplus\left\{k_{a}^{x} a \mid a\right.$ an atom of $\left.E, a \leq x\right\}$. Let $a \in A_{x}$. Then $a \leq x \leq \widehat{x} \in \mathrm{~S}(E)$. Therefore $n_{a} a \leq \widehat{x}$. Assume that $z \in \mathrm{~S}(E), n_{a} a \leq z$ for all $a \in A_{x}$. Then $k_{a}^{x} a \leq z$ for all $a \in A_{x}$, i.e., $x \leq z$. This yields that $\widehat{x} \leq z$, i.e. $\widehat{x}=\bigvee_{\mathrm{S}(E)}\left\{n_{a} a \mid a\right.$ an atom of $\left.E, a \leq x\right\}$. By Statement 2.7, (ii) we obtain that $\widehat{x}=\bigvee_{E}\left\{n_{a} a \mid a\right.$ an atom of $\left.E, a \leq x\right\}$. Let $G \subseteq A_{x}, G$ finite. Then $\bigoplus\left\{n_{a} a \mid\right.$ $a \in G\}=\bigvee\left\{n_{a} a \mid a \in G\right\} \leq \widehat{x}$. Hence $\widehat{x}=\bigoplus\left\{n_{a} a \mid a\right.$ an atom of $\left.E, a \leq x\right\}$.

Further, we have

$$
\begin{aligned}
\widehat{x} \ominus x & =\left(\bigoplus\left\{n_{a} a \mid a \in A_{x}\right\}\right) \ominus\left(\bigoplus\left\{k_{a}^{x} a \mid a \in A_{x}\right\}\right) \\
& \geq\left(\bigoplus\left\{n_{a} a \mid a \in A_{x}, a \neq b\right\} \oplus n_{b} b\right) \ominus\left(\bigoplus\left\{n_{a} a \mid a \in A_{x}, a \neq b\right\} \oplus k_{b}^{x} b\right) \\
& =\left(n_{b}-k_{b}^{x}\right) b
\end{aligned}
$$

for all $b \in A_{x}$. Now, let $z \in E$ such that $z \geq\left(n_{b}-k_{b}^{x}\right) b$ for all $b \in A_{x}$. Then also $z \wedge(\widehat{x} \ominus x) \geq\left(n_{b}-k_{b}^{x}\right) b$. Hence $(z \wedge(\widehat{x} \ominus x)) \oplus k_{b}^{x} b \geq n_{b} b$ i.e., $(z \wedge(\widehat{x} \ominus x)) \oplus \bigoplus\left\{k_{a}^{x} a \mid a \in A_{x}\right\} \geq \bigvee\left\{n_{a} a \mid a \in A_{x}\right\}$. This yields

$$
z \geq z \wedge(\widehat{x} \ominus x) \geq\left(\bigoplus\left\{n_{a} a \mid a \in A_{x}\right\}\right) \ominus\left(\bigoplus\left\{k_{a}^{x} a \mid a \in A_{x}\right\}\right)=\widehat{x} \ominus x
$$

Therefore $\widehat{x} \ominus x=\bigoplus\left\{\left(n_{a}-k_{a}^{x}\right) a \mid a \in A_{x}\right\}=\bigvee\left\{\left(n_{a}-k_{a}^{x}\right) a \mid a \in A_{x}\right\}$.
The equality $\widehat{x}=\widehat{\widehat{x} \ominus x}$ follows from Lemma 2.5. (ii). (vi): Let $x=\bigoplus_{i=1}^{n} k_{i} a_{i}$. By (ii) we have that the only atoms below $x$ are $a_{1}, \ldots, a_{n}$. Hence $x=\bigoplus_{i=1}^{n} k_{i} a_{i}=\bigvee_{i=1}^{n} k_{i} a_{i}$. From the proof of (iii) we know that any element of $[0, x]$ is of the form $\bigvee_{i=1}^{n} l_{i} a_{i}$ for uniquely determined natural numbers $0 \leq l_{i}<n_{a_{i}}, 1 \leq i \leq n$ and conversely, for any system of natural numbers $0 \leq l_{i}<n_{a_{i}}, 1 \leq i \leq n, \bigvee_{i=1}^{n} l_{i} a_{i} \in[0, x]$. This yields the required isomorphism between $[0, x]$ and $\prod_{i=1}^{n}\left[0, k_{i} a_{i}\right]$.

Note that Theorem 2.10 (ii), (iv) immediately yields that the set of meager (finite meager) elements of an atomic Archimedean lattice effect algebra is a dual of a weak implication algebra introduced in [2].

Motivated by [8, Proposition 15] we have the following proposition.

Proposition 2.11. Let $E$ be an atomic Archimedean MV-effect algebra. Then:
(i) Let $x \in \mathrm{M}(E)$ and $y \in E$ such that $x \wedge y=0$ and $\widehat{x}$ exists. Then $\widehat{x} \wedge y=0$.
(ii) $\mathrm{M}(E)$ is a $\bigvee$-bifull sub-poset of $E$.
(iii) $\mathrm{M}(E)$ is a lattice ideal of $E$.

Proof. (i): As in Theorem 2.10 let us put $A_{x}=\{a \mid a$ an atom of $E, a \leq x\}$. Evidently, $a \wedge y=0$ and $y \leq a^{\prime}$ for all $a \in A_{x}$. Therefore by Statement 1.3. (iii) $n_{a} a \wedge y=0$ for all $a \in A_{x}$. Then Theorem 2.10 (v) yields that $\widehat{x} \wedge y=\bigvee\left\{n_{a} a \mid a \in A_{x}\right\} \wedge y=\bigvee\left\{n_{a} a \wedge y \mid a \in A_{x}\right\}=0$.
(ii): Let $X \subseteq \mathrm{M}(E)$. Assume that $z=\bigvee_{\mathrm{M}(E)} X$ exists. Let $u \in E$ be an upper bound of $X$. Hence also $u \wedge z$ is an upper bound of $X$ and clearly $u \wedge z$ is meager. Therefore $z=u \wedge z \leq u$, i.e., $z=\bigvee_{E} X$.

Now, assume that $z=\bigvee_{E} X$ exists. It is enough to check that $z \in \mathrm{M}(E)$. Let $t \in \mathrm{~S}(E), t \leq z, t \neq 0$. Then there exists an atom $b \in E$ such that $b \leq t$. Let us put $k_{b}^{x}=\max \{k \mid k b \leq x\}<n_{b}$ (since any $x \in X$ is meager) and $k_{b}=\max \left\{k_{b}^{x} \mid x \in X\right\}<n_{b}$. Hence also $n_{b} b \leq t \leq z$ and $n_{b} b=n_{b} b \wedge \bigvee_{E} X=$ $\bigvee_{E}\left\{n_{b} b \wedge x \mid x \in X\right\}=\bigvee_{E}\left\{k_{b}^{x} b \mid x \in X\right\}=k_{b} b<n_{b} b$, a contradiction. Hence $\widetilde{z}=0$ and $z \in \mathrm{M}(E)$.
(iii): It follows immediately from (ii) because $\mathrm{M}(E)$ is a downset in $E$ and $E$ is a lattice.

Moreover we have
Proposition 2.12. Let $E$ be an atomic Archimedean lattice effect algebra. Then
(i) For all $X \subseteq \mathrm{~B}(E) \cap \mathrm{M}(E), \bigvee_{E} X$ exists iff $\bigvee_{\mathrm{B}(E)} X$ exists, in which case $\bigvee_{E} X=\bigvee_{\mathrm{B}(E)} X \in \mathrm{M}(E)$.
(ii) $\mathrm{B}(E) \cap \mathrm{M}(E)$ is a $\bigvee$-bifull sub-poset of $E$.

Proof. (i): Let $X \subseteq \mathrm{~B}(E) \cap \mathrm{M}(E)$. Assume first that $z=\bigvee_{\mathrm{B}(E)} X$ exists. Any $x \in X$ is by Theorem 2.10 of the form $x=\bigvee_{E}\left\{k_{a}^{x} a \mid a \in A_{x}\right\}=\bigvee_{\mathrm{B}(E)}\left\{k_{a}^{x} a \mid\right.$ $\left.a \in A_{x}\right\}, A_{x} \subseteq \mathrm{~B}(E) \cap \mathrm{M}(E)$. Hence $z=\bigvee_{\mathrm{B}(E)}\left\{\bigvee_{\mathrm{B}(E)}\left\{k_{a}^{x} a \mid a \in A_{x}\right\} \mid x \in X\right\}$. Let us put $k_{a}=\max \left\{k_{a}^{x} \mid x \in X\right\}<n_{a}$. Then $z=\bigvee_{\mathrm{B}(E)}\left\{k_{a} a \mid a \in A_{x}, x \in X\right\}$.

First, we shall show that $z \in \mathrm{M}(E)$. Assume that there is $y \neq 0, y \leq z$, $y \in \mathrm{~S}(E)$. Then there is an atom $c \in E$ such that $c \leq y$ i.e., also $n_{c} c \leq y \leq z$. Either $c \in A_{x}$ for some $x \in X$ or $c \wedge a=0$ for all $a \in A_{x}, x \in X$. Let $c \in A_{x}$ for some $x \in X$. Then $n_{c} c \in B(E)$. Therefore

$$
\begin{aligned}
n_{c} c & =n_{c} c \wedge z=n_{c} c \wedge \bigvee_{\mathrm{B}(E)}\left\{k_{a} a \mid a \in A_{x}, x \in X\right\} \\
& =\bigvee_{\mathrm{B}(E)}\left\{n_{c} c \wedge k_{a} a \mid a \in A_{x}, x \in X\right\}=k_{c} c<n_{c} c
\end{aligned}
$$

a contradiction. Now, let $c \wedge a=0$ for all $a \in A_{x}, x \in X$. Then $c \leftrightarrow a$ yields that $k_{a} a \leq\left(n_{c} c\right)^{\prime} \in C(E)$. Hence $z \leq\left(n_{c} c\right)^{\prime}$. But $n_{c} c=n_{c} c \wedge z \leq n_{c} c \wedge\left(n_{c} c\right)^{\prime}=0$ and we have a contradiction again. Hence $z \in \mathrm{M}(E)$.

Now, let $u \in E$ be an upper bound of $X$. Then also $u \wedge z$ is an upper bound of $X, u \wedge z \leq z \in \mathrm{~B}(E) \cap \mathrm{M}(E)$. From Theorem 2.10 we have that $u \wedge z \in \mathrm{~B}(E) \cap \mathrm{M}(E)$. Hence $z \leq u \wedge z \leq u$ i.e., $z=\bigvee_{E} X$.

Now, assume that $\bigvee_{E} X$ exists. Then $\bigvee_{E} X=\bigvee_{B(E)} X$ by Statement 1.3 (i). Hence $\bigvee_{E} X \in \mathrm{M}(E)$ by the above argument.
(ii): It follows immediately from (i) because $\mathrm{B}(E) \cap \mathrm{M}(E)$ is a downset in $\mathrm{B}(E)$.

The following statement is well known.
Statement 2.13. Let $E$ be a sharply dominating Archimedean atomic lattice effect algebra. Then
(i) 25, Theorem 3.4] For every $x \in E, x \neq 0$ there exists the unique $w_{x} \in$ $\mathrm{S}(E)$, unique set of atoms $\left\{a_{\alpha} \mid \alpha \in \Lambda\right\}$ and unique positive integers $k_{\alpha} \neq$ $\operatorname{ord}\left(a_{\alpha}\right)$ such that

$$
x=w_{x} \oplus\left(\bigoplus\left\{k_{\alpha} a_{\alpha} \mid \alpha \in \Lambda\right\}\right)
$$

We call such a decomposition the basic decomposition (BDE for short) of $x$.
(ii) 16, Theorem 3.2] $B(E)$ is sharply dominating and for every $x \in B(E), x \neq$ 0 there exists the unique $w_{x} \in C(E)$, unique set $\left\{a_{\alpha} \mid \alpha \in \Lambda\right\} \subseteq B(E)$ of atoms of $E$ and unique positive integers $k_{\alpha} \neq \operatorname{ord}\left(a_{\alpha}\right)$ such that

$$
x=w_{x} \oplus\left(\bigoplus\left\{k_{\alpha} a_{\alpha} \mid \alpha \in \Lambda\right\}\right)
$$

(iii) [16, Theorem 3.1] Let $M \subseteq E$ be an atomic block of $E$. Then $M$ is sharply dominating and, for every $x \in M$, there exists $B D E$ of $x$ in $M$ and it coincides with $B D E$ of $x$ in $E$.

Proposition 2.14. Let E be a sharply dominating atomic Archimedean lattice effect algebra and let $B \subseteq E$ be an atomic block of $E$. Then $\mathrm{M}(B) \subseteq \mathrm{M}(E)$.

Proof. Let $x \in \mathrm{M}(B)$. Then by Theorem 2.10, (ii) $x=0 \oplus\left(\bigoplus_{B}\left\{k_{\alpha} a_{\alpha} \mid \alpha \in \Lambda\right\}\right)$ for a set of atoms $\left\{a_{\alpha} \mid \alpha \in \Lambda\right\}$ of $B$ and positive integers $k_{\alpha} \neq \operatorname{ord}\left(a_{\alpha}\right)$. Since $B$ is a bifull sub-lattice effect algebra of $E$ (see [16, Lemma 2.7 (i)]) we obtain that $x=0 \oplus\left(\bigoplus_{E}\left\{k_{\alpha} a_{\alpha} \mid \alpha \in \Lambda\right\}\right)$. As $E$ is sharply dominating we have from Statement 2.13, (i) that $\widetilde{x}=0$ and hence $x \in \mathrm{M}(E)$.

Let us recall the following statement
Statement 2.15. 11, Lemma 2] Let $(E ; \oplus, 0,1)$ be an Archimedean atomic lattice effect algebra, $x, y \in E, x \leftrightarrow y$. Then there is an atomic block $B$ of $E$ such that $x, y \in B$.

Similarly to [8, Proposition 23] for complete lattice effect algebras we have now the following proposition.

Proposition 2.16. Let $E$ be a sharply dominating atomic Archimedean lattice effect algebra and let $x, y \in \mathrm{M}(E)$. Then
(i) $x \leftrightarrow y$ if and only if $x \vee y \in \mathrm{M}(E)$,
(ii) If $x \oplus y$ exists and $x \oplus y=z \in \mathrm{~S}(E)$ then $z=\widehat{x}=\widehat{y}$.

Proof. (i): Assume first that $x \leftrightarrow y$. Then by Statement 2.15 there is an atomic block $B$ of $E$ such that $x, y \in B$. Since $B$ is an atomic Archimedean MV-effect algebra and $E$ is sharply dominating we have from Propositions 2.11 and 2.14 that $x \vee y \in \mathrm{M}(B) \subseteq \mathrm{M}(E)$.

Now, assume that $x \vee y \in \mathrm{M}(E)$. Then from Theorem 2.10, (iii) we obtain that $[0, x \vee y]$ is an MV-effect algebra. This yields that $x \leftrightarrow y$.
(ii): It follows immediately from Lemma 2.4.

Theorem 2.17. Let $E$ be a sharply dominating atomic Archimedean lattice effect algebra. Then for every $x \in E, x \neq 0$ there exists unique set of atoms $\left\{a_{\alpha} \mid \alpha \in \Lambda\right\}$ (namely $\{a \in E \mid a$ an atom of $E, a \leq x \ominus \widetilde{x}\}$ ) and unique positive integers $k_{\alpha} \neq n_{a_{\alpha}}\left(\right.$ namely $\left.k_{\alpha}=\max \left\{k \in \mathbb{N} \mid k a_{\alpha} \leq x\right\}\right)$ such that

$$
x=\widetilde{x} \oplus\left(\bigoplus\left\{k_{\alpha} a_{\alpha} \mid \alpha \in \Lambda\right\}\right)
$$

Moreover,

$$
\begin{aligned}
x & =\widetilde{x} \oplus\left(\bigvee\left\{k_{\alpha} a_{\alpha} \mid \alpha \in \Lambda\right\}\right)=\widetilde{x} \vee\left(\bigvee\left\{k_{\alpha} a_{\alpha} \mid \alpha \in \Lambda\right\}\right), \\
0 & =\widetilde{x} \wedge\left(\bigvee\left\{k_{\alpha} a_{\alpha} \mid \alpha \in \Lambda\right\}\right)=\widetilde{x} \wedge\left(\bigoplus\left\{k_{\alpha} a_{\alpha} \mid \alpha \in \Lambda\right\}\right), \\
\widehat{x} & =\bigvee\left\{\widetilde{x} \oplus n_{a_{\alpha}} a_{\alpha} \mid \alpha \in \Lambda\right\}=\widetilde{x} \oplus\left(\bigvee\left\{n_{a_{\alpha}} a_{\alpha} \mid \alpha \in \Lambda\right\}\right) \\
& =\widetilde{x} \oplus\left(\bigoplus\left\{n_{a_{\alpha}} a_{\alpha} \mid \alpha \in \Lambda\right\}\right)=\widetilde{x} \vee\left(\bigvee\left\{n_{a_{\alpha}} a_{\alpha} \mid \alpha \in \Lambda\right\}\right), \\
0 & =\widetilde{x} \wedge\left(\bigvee\left\{n_{a_{\alpha}} a_{\alpha} \mid \alpha \in \Lambda\right\}\right)=\widetilde{x} \wedge\left(\bigoplus\left\{n_{a_{\alpha}} a_{\alpha} \mid \alpha \in \Lambda\right\}\right), \\
\widehat{x} & =x \oplus \bigoplus\left\{\left(n_{a_{\alpha}}-k_{\alpha}\right) a_{\alpha} \mid \alpha \in \Lambda\right\} \\
& =x \oplus\left(\bigvee\left\{\left(n_{a_{\alpha}}-k_{\alpha}\right) a_{\alpha} \mid \alpha \in \Lambda\right\}\right) .
\end{aligned}
$$

Proof. The first part of the statement follows immediately from Statement 2.13 (i) and Theorem 2.10 Let us show the second, third and fourth parts.

We have by Theorem 2.10 that $x \ominus \widetilde{x}=\bigoplus\left\{k_{\alpha} a_{\alpha} \mid \alpha \in \Lambda\right\}=\bigvee\left\{k_{\alpha} a_{\alpha} \mid \alpha \in\right.$ $\Lambda\}$. Hence by Lemma 2.5, (i) $(x \ominus \widetilde{x}) \wedge \widetilde{x}=0$ and $\widetilde{x} \vee(x \ominus \widetilde{x})=\widetilde{x} \oplus(x \ominus \widetilde{x})=x$. Since $\widehat{x \ominus \widetilde{x}}$ exists we have from Theorem[2.10, (v) that $\widehat{x \ominus \widetilde{x}}=\bigvee\left\{n_{a_{\alpha}} a_{\alpha} \mid \alpha \in\right.$ $\Lambda$ \}. Therefore by by Lemma 2.5, (ii) we have that $\widehat{x}=\widetilde{x} \oplus \widehat{x \ominus \widetilde{x}}=\widetilde{x} \vee x \ominus \widetilde{x}$ and $\widetilde{x} \wedge \widehat{x \ominus \widetilde{x}}=0$. Moreover, by Statement 1.3, (vii) $\widetilde{x} \oplus\left(\bigvee\left\{n_{a_{\alpha}} a_{\alpha} \mid \alpha \in \Lambda\right\}\right)=$ $\bigvee\left\{\widetilde{x} \oplus n_{a_{\alpha}} a_{\alpha} \mid \alpha \in \Lambda\right\}$.

The fourth part follows immediately from the precedings parts. Namely, by Theorem 2.10. (v)

$$
\begin{aligned}
\widehat{x} \ominus x & =(\widehat{x} \ominus \widetilde{x}) \ominus(x \ominus \widetilde{x})=\bigoplus\left\{\left(n_{a_{\alpha}}-k_{\alpha}\right) a_{\alpha} \mid \alpha \in \Lambda\right\} \\
& =\bigvee\left\{\left(n_{a_{\alpha}}-k_{\alpha}\right) a_{\alpha} \mid \alpha \in \Lambda\right\} .
\end{aligned}
$$

Theorem 2.18. Let $E$ be a sharply dominating atomic Archimedean lattice effect algebra. Then the following conditions are equivalent:
(i) $\mathrm{B}(E)$ is bifull in $E$.
(ii) $\mathrm{C}(E)$ is bifull in $E$.

Proof. (i) $\Longrightarrow$ (ii): Note that from Statement 2.13, (ii) we know that $\mathrm{B}(E)$ is sharply dominating. Hence by Statement 2.7 we obtain that $\mathrm{C}(E)=\mathrm{S}(\mathrm{B}(E))$ is bifull in $\mathrm{B}(E)$. Since $\mathrm{B}(E)$ is bifull in $E$ we have that $\mathrm{C}(E)$ is bifull in $E$.
(ii) $\Longrightarrow$ (i): Let $S \subseteq \mathrm{~B}(E)$ and $x=\bigvee_{\mathrm{B}(E)} S$ exists. Assume first that $x \in$ $\mathrm{C}(E)$. Then
$x=\bigvee_{\mathrm{B}(E)}\{s \mid s \in S\}=\bigvee_{\mathrm{B}(E)}\{\widehat{s} \mid s \in S\}=\bigvee_{\mathrm{C}(E)}\{\widehat{s} \mid s \in S\}=\bigvee_{E}\{\widehat{s} \mid s \in S\}$
since $\mathrm{C}(E)$ is bifull in $E$ and $\widehat{s} \in \mathrm{C}(E)$.
Now, let $z \in E, z \geq s$ for all $s \in S$. Assume for a moment that $z \nsupseteq x$. Then $z \wedge x<x$ which yields that there is an atom $c \in E$ such that $c \leq x \ominus(x \wedge z)$. Then $c=c \wedge x=c \wedge \bigvee_{E}\{\widehat{s} \mid s \in S\}=\bigvee_{E}\{c \wedge \widehat{s} \mid s \in S\}$ since $c \leftrightarrow s$ and hence $x \leftrightarrow \widehat{s}$ for all $a \in S$. Therefore, there is an element $s \in S$ such that $c \wedge \widehat{s}=c$ i.e. $c \leq \widehat{s}$.

By Theorem 2.17 and from Statement 2.13 (ii) we have that there exists the unique set $\left\{a_{\alpha} \mid \alpha \in \Lambda\right\} \subseteq \mathrm{B}(E)$ of atoms of $E$ and unique positive integers $k_{\alpha} \neq n_{a_{\alpha}}$ such that

$$
\begin{aligned}
& s=\widetilde{s} \oplus\left(\bigoplus_{E}\left\{k_{\alpha} a_{\alpha} \mid \alpha \in \Lambda\right\}\right)=\widetilde{s} \oplus\left(\bigvee_{E}\left\{k_{\alpha} a_{\alpha} \mid \alpha \in \Lambda\right\}\right), \\
& \widehat{s}=\widetilde{s} \oplus\left(\bigvee_{E}\left\{n_{a_{\alpha}} a_{\alpha} \mid \alpha \in \Lambda\right\}\right)=\widetilde{s} \vee\left(\bigvee_{E}\left\{n_{a_{\alpha}} a_{\alpha} \mid \alpha \in \Lambda\right\}\right)
\end{aligned}
$$

This gives rise to

$$
c=c \wedge \widehat{s}=(c \wedge \widetilde{s}) \vee\left(c \wedge\left(\bigvee_{E}\left\{n_{a_{\alpha}} a_{\alpha} \mid \alpha \in \Lambda\right\}\right)\right)
$$

since $c \leftrightarrow \widetilde{s} \in \mathrm{~B}(E)$ and $c \leftrightarrow a_{\alpha} \in \mathrm{B}(E)$ for all $\alpha \in \Lambda$.
Assume first that $c=c \wedge \widetilde{s}$. Then from Statement 1.3, (v) we have $n_{c} c \leq$ $\widetilde{s} \leq z \wedge x$, a contradiction with $c \leq x \ominus(x \wedge z)$. So we obtain that

$$
c=c \wedge\left(\bigvee_{E}\left\{n_{a_{\alpha}} a_{\alpha} \mid \alpha \in \Lambda\right\}\right)=\bigvee_{E}\left\{c \wedge n_{a_{\alpha}} a_{\alpha} \mid \alpha \in \Lambda\right\}
$$

Hence there is an atom $a_{\alpha}$ of $E, a_{\alpha} \in \mathrm{B}(E), \alpha \in \Lambda$ such that $c=c \wedge n_{a_{\alpha}} a_{\alpha}$. Assume for a moment that $c \wedge a_{\alpha}=0$. We have that $c \leftrightarrow a_{\alpha}$ i.e., $c \oplus a_{\alpha}$ exists. By Statement 1.3, (iii) we have that $c \wedge n_{a_{\alpha}} a_{\alpha}=0$, a contradiction. So we have shown that $c=a_{\alpha} \in \mathrm{B}(E)$. But $x \wedge c \leq x \ominus c<x, x \ominus c \in \mathrm{~B}(E)$ and $x \ominus c$ is an upper bound of $S$, a contradiction with $x=\bigvee_{\mathrm{B}(E)}\{s \mid s \in S\}$. Therefore $z \geq x$ and hence $\bigvee_{\mathrm{B}(E)} S=\bigvee_{E} S$.

Now, let us assume that $\bigvee_{\mathrm{B}(E)} S=x \in \mathrm{~B}(E)$. Then we have that

$$
\widehat{x}=x \oplus(\widehat{x} \ominus x)=\bigvee_{\mathrm{B}(E)} S \oplus(\widehat{x} \ominus x)=\bigvee_{\mathrm{B}(E)}\{s \oplus(\widehat{x} \ominus x) \mid s \in S\} \in \mathrm{C}(E)
$$

Therefore by above considerations also $\widehat{x}=\bigvee_{E}\{s \oplus(\widehat{x} \ominus x) \mid s \in S\}$. This and Statement 1.3, (iv) yield

$$
x=\widehat{x} \ominus(\widehat{x} \ominus x)=\left(\bigvee_{E}\{s \oplus(\widehat{x} \ominus x) \mid s \in S\}\right) \ominus(\widehat{x} \ominus x)=\bigvee_{E} S
$$

Conversely, let $S \subseteq \mathrm{~B}(E)$ and $x=\bigvee_{E} S$ exists. Then by Statement 1.3, (ii) we get that $x=\bigvee_{\mathrm{B}(E)} S$.

Theorem 2.19. Let $E$ be a sharply dominating atomic Archimedean lattice effect algebra. Then the following conditions are equivalent:
(i) $\mathrm{B}(E)$ is atomic.
(ii) $\mathrm{C}(E)$ is atomic.

Proof. (i) $\Longrightarrow$ (ii): Let $c \in \mathrm{C}(E) \subseteq \mathrm{B}(E), c \neq 0$. Then there is an atom $a \in \mathrm{~B}(E)$ such that $a \leq c$. Therefore by Statement 1.3, (v) $n_{a} a \leq c$ and $n_{a} a \in \mathrm{C}(E)=\mathrm{B}(E) \cap \mathrm{S}(E)$ since $\mathrm{B}(E)$ is a sub-lattice effect algebra of $E$. It follows that $\left[0, n_{a} a\right]=\left\{0, a, \ldots, n_{a} a\right\} \subseteq B(E)$, as for every atom $b$ of $E$, $b \neq a$ we have $b \leftrightarrow a$, which gives that $b \wedge n_{a} a=0$, by Statement 1.3, (iii) and this yields by Statement 1.4, (i) that any element below $n_{a} a$ is of the form $k a$, $0 \leq k \leq n_{a} a$. Hence $\left\{0, a, 2 a, \ldots, n_{a} a\right\} \cap \mathrm{C}(E)=\left\{0, n_{a} a\right\}$. This yields that $n_{a} a$ is an atom of $\mathrm{C}(E)$ below $c$.
(ii) $\Longrightarrow$ (i): Let $x \in \mathrm{~B}(E), x \neq 0$. If $x \notin \mathrm{~S}(E)$ then by Statement 2.13, (ii) there is an atom $a \in \mathrm{~B}(E)$ such that $a \leq x \ominus \widetilde{x} \leq x$. So let us assume that $x \in \mathrm{~S}(E) \cap \mathrm{B}(E)=\mathrm{C}(E)$. Then there is by (ii) an atom $c$ from $\mathrm{C}(E), c \leq x$. Assume that there is an element $y \in \mathrm{~B}(E)$ such that $y<c$. Then we have the following possibilities:
(i) $y \notin \mathrm{~S}(E)$ and by the above argument there is an atom $a \in \mathrm{~B}(E)$ such that $a \leq y<c \leq x$. Otherwise we have
(ii) $y \in \mathrm{~S}(E) \cap \mathrm{B}(E)=\mathrm{C}(E)$ which implies that $y=0$.

Hence we obtain that $\mathrm{B}(E)$ is atomic.
Corollary 2.20. Let $E$ be a sharply dominating atomic Archimedean lattice effect algebra with a finite center $\mathrm{C}(E)$. Then $\mathrm{B}(E)$ is atomic and bifull in $E$.

## 3. Triple Representation Theorem for sharply dominating atomic Archimedean lattice effect algebras

In what follows $E$ will be always a sharply dominating atomic Archimedean lattice effect algebra. Then $\mathrm{S}(E)$ is a sub-lattice effect algebra of $E$ and $\mathrm{M}(E)$ equipped with a partial operation $\oplus_{\mathrm{M}(E)}$ which is defined, for all $x, y \in \mathrm{M}(E)$, by $x \oplus_{\mathrm{M}(E)} y$ exists if and only if $x \oplus_{E} y$ exists and $x \oplus_{E} y \in \mathrm{M}(E)$ in which case $x \oplus_{\mathrm{M}(E)} y=x \oplus_{E} y$ is a generalized effect algebra. Recall only that, for any meager atom $a \in E$, we have that $\operatorname{ord}_{\mathrm{M}(E)}(a)=\operatorname{ord}_{E}(a)-1$. We are therefore able to reconstruct the isotropic index in $E$ of any atom from $\mathrm{M}(E)$. Moreover, we have a map $h: \mathrm{S}(E) \rightarrow 2^{\mathrm{M}(E)}$ that is given by $h(s)=\{x \in \mathrm{M}(E) \mid x \leq s\}$. As in [8] for complete lattice effect algebras we will prove the following theorem.

Triple Representation Theorem The triple $(\mathrm{S}(E), \mathrm{M}(E)$, h) characterizes $E$ up to isomorphism.

We have to construct an isomorphic copy of the original effect algebra $E$ from the triple $(\mathrm{S}(E), \mathrm{M}(E), h)$. To do this we will first construct the following mappings in terms of the triple.
(M1) The mapping ${ }^{\wedge}: \mathrm{M}(E) \rightarrow \mathrm{S}(E)$.
(M2) For every $s \in \mathrm{~S}(E)$, a mapping $\pi_{s}: \mathrm{M}(E) \rightarrow h(s)$, which is given by $\pi_{s}(x)=x \wedge_{E} s$.
(M3) The mapping $R: \mathrm{M}(E) \rightarrow \mathrm{M}(E)$ given by $R(x)=\widehat{x} \ominus_{E} x$.
(M4) The partial mapping $S: \mathrm{M}(E) \times \mathrm{M}(E) \rightarrow \mathrm{S}(E)$ given by $S(x, y)$ is defined if and only if the set $\mathcal{S}(x, y)=\left\{z \in \mathrm{~S}(E) \mid z=(z \wedge x) \oplus_{E}(z \wedge y)\right\}$ has a top element $z_{0} \in \mathcal{S}(x, y)$ in which case $S(x, y)=z_{0}$.

Since $E$ is sharply dominating and $S(E)$ is bifull in $E$ we have that, for all $x \in \mathrm{M}(E)$,

$$
\widehat{x}=\bigwedge_{E}\{s \in \mathrm{~S}(E) \mid x \in h(s)\}=\bigwedge_{\mathrm{S}(E)}\{s \in \mathrm{~S}(E) \mid x \in h(s)\} .
$$

Similarly, for all $s \in \mathrm{~S}(E)$ and for all $x \in \mathrm{M}(E), x \wedge_{E} s \in \mathrm{M}(E)$. Hence

$$
\begin{aligned}
\pi_{s}(x) & =x \wedge_{E} s=\bigvee_{E}\{y \in E \mid y \leq x, y \leq s\} \\
& =\bigvee_{E}\{y \in \mathrm{M}(E) \mid y \leq x, y \in h(s)\}=\bigvee_{M(E)}\{y \in \mathrm{M}(E) \mid y \leq x, y \in h(s)\}
\end{aligned}
$$

Now, let us construct the mapping $R$. Let $x \in \mathrm{M}(E)$. If $x=0$ we put $R(x)=0$. Let $x \neq 0$. As before let us denote by $A_{x}=\{a \mid a$ an atom of $E$, $a \leq x\}=\{a \mid a$ an atom of $\mathrm{M}(E), a \leq x\} \neq \emptyset$ and, for any $a \in A_{x}$, we shall put $k_{a}^{x}=\max \{k \in \mathbb{N} \mid k a \leq x\}$ and $n_{a}=\operatorname{ord}_{M(E)}(a)+1$. Hence $1 \leq k_{a}^{x} \leq$ $\operatorname{ord}_{\mathrm{M}(E)}(a)$. Therefore $\left\{k_{a}^{x} a \mid a \in A_{x}\right\} \cup\left\{\left(n_{a}-k_{a}^{x}\right) a \mid a \in A_{x}\right\} \subseteq \mathrm{M}(E)$. We know from Theorem 2.10, (ii) and (v) that $x=\bigvee_{E}\left\{k_{a}^{x} a \mid a \in A_{x}\right\}=$ $\bigvee_{\mathrm{M}(E)}\left\{k_{a}^{x} a \mid a \in A_{x}\right\}, \widehat{x} \ominus x=\bigvee_{E}\left\{\left(n_{a}-k_{a}^{x}\right) a \mid a \in A_{x}\right\}=\bigvee_{\mathrm{M}(E)}\left\{\left(n_{a}-k_{a}^{x}\right) a \mid\right.$ $\left.a \in A_{x}\right\} \neq 0, \widehat{x} \ominus x \in \mathrm{M}(E)$.

What remains is the partial mapping $S$. Let $x, y \in \mathrm{M}(E)$. By Lemma 2.16. (ii) $\mathcal{S}(x, y)=\left\{z \in \mathrm{~S}(E) \mid z=(z \wedge x) \oplus_{E}(z \wedge y)\right\}=\{z \in \mathrm{~S}(E) \mid z=$ $\widehat{\pi_{z}(x)}$ and $\left.R\left(\pi_{z}(x)\right)=\pi_{z}(y)\right\}$. Hence whether $S(x, y)$ is defined or not we are able to decide in terms of the triple. Since the eventual top element $z_{0}$ of $\mathcal{S}(x, y)$ is in $\mathrm{S}(E)$ our definition of $S(x, y)$ is correct.

Lemma 3.1. Let $E$ be a sharply dominating atomic Archimedean lattice effect algebra, $x, y \in \mathrm{M}(E)$. Then $x \oplus_{E} y$ exists in $E$ iff $S(x, y)$ is defined in terms of the triple $(\mathrm{S}(E), \mathrm{M}(E), h)$ and $\left(x \ominus_{\mathrm{M}(E)}(S(x, y) \wedge x)\right) \oplus_{\mathrm{M}(E)}\left(y \ominus_{\mathrm{M}(E)}(S(x, y) \wedge y)\right)$
exists in $\mathrm{M}(E)$ such that $\left(x \ominus_{\mathrm{M}(E)}(S(x, y) \wedge x)\right) \oplus_{\mathrm{M}(E)}\left(y \ominus_{\mathrm{M}(E)}(S(x, y) \wedge y)\right) \in$ $h\left(S(x, y)^{\prime}\right)$. Moreover, in that case

$$
x \oplus_{E} y=\underbrace{S(x, y)}_{\in \mathrm{S}(E)} \oplus_{E}(\underbrace{\left(x \ominus_{\mathrm{M}(E)}(S(x, y) \wedge x)\right) \oplus_{\mathrm{M}(E)}\left(y \ominus_{\mathrm{M}(E)}(S(x, y) \wedge y)\right)}_{\in \mathrm{M}(E)}) .
$$

Proof. Assume first that $x \oplus_{E} y$ exists in $E$ and let us put $z=x \oplus_{E} y$. Then $z=z_{S} \oplus_{E} z_{M}$ such that $z_{S} \in \mathrm{~S}(E)$ and $z_{M} \in \mathrm{M}(E)$ is BDE of $z$ in $E$. Since $x \leftrightarrow y$ by Statement 2.15 there is an atomic block $B$ of $E$ such that $x, y, z \in B$. We know from Statement 2.13, (iii) that $B$ is sharply dominating and BDE of $z \in B$ in $B$ and BDE of $z$ in $E$ coincide. This yields that $z_{S}, z_{M} \in B$. Therefore $z_{S} \in \mathrm{C}(B)$ and by Statement 1.3, (viii) we have that $z_{S}=z_{S} \wedge\left(x \oplus_{E} y\right)=$ $z_{S} \wedge\left(x \oplus_{B} y\right)=\left(z_{S} \wedge x\right) \oplus_{B}\left(z_{S} \wedge y\right)=\left(z_{S} \wedge x\right) \oplus_{E}\left(z_{S} \wedge y\right)$. Hence $z_{S} \in \mathcal{S}(x, y)$. Now, assume that $u \in \mathcal{S}(x, y)$. Then $u=(u \wedge x) \oplus_{E}(u \wedge y) \leq x \oplus_{E} y$. Since $u \in \mathrm{~S}(E)$ we have that $u \leq z_{S}$, i.e., $z_{S}$ is the top element of $\mathcal{S}(x, y)$. Moreover, we have

$$
\begin{aligned}
z_{S} \oplus_{E} z_{M}= & x \oplus_{E} y \\
= & \left((S(x, y) \wedge x) \oplus_{E}\left(x \ominus_{E}(S(x, y) \wedge x)\right)\right) \oplus_{E} \\
& \left((S(x, y) \wedge y) \oplus_{E}\left(y \ominus_{E}(S(x, y) \wedge y)\right)\right) \\
= & S(x, y) \oplus_{E}\left(\left(x \ominus_{\mathrm{M}(E)}(S(x, y) \wedge x)\right) \oplus_{E}\left(y \ominus_{\mathrm{M}(E)}(S(x, y) \wedge y)\right)\right) .
\end{aligned}
$$

Because $z_{S}=S(x, y)$ it follows that $z_{M}=\left(x \ominus_{\mathrm{M}(E)}(S(x, y) \wedge x)\right) \oplus_{E}\left(y \ominus_{\mathrm{M}(E)}\right.$ $(S(x, y) \wedge y))$, i.e., $z_{M}=\left(x \ominus_{\mathrm{M}(E)}(S(x, y) \wedge x)\right) \oplus_{\mathrm{M}(E)}\left(y \ominus_{\mathrm{M}(E)}(S(x, y) \wedge y)\right)$ and evidently $z_{M} \in h\left(z_{S}^{\prime}\right)$.

Conversely, let us assume that $S(x, y)$ is defined in terms of $(\mathrm{S}(E), \mathrm{M}(E), h)$, $\left(x \ominus_{\mathrm{M}(E)}(S(x, y) \wedge x)\right) \oplus_{\mathrm{M}(E)}\left(y \ominus_{\mathrm{M}(E)}(S(x, y) \wedge y)\right)$ exists in $\mathrm{M}(E)$ and $\left(x \ominus_{\mathrm{M}(E)}\right.$ $(S(x, y) \wedge x)) \oplus_{\mathrm{M}(E)}\left(y \ominus_{\mathrm{M}(E)}(S(x, y) \wedge y)\right) \in h\left(S(x, y)^{\prime}\right)$. Then $\left(x \ominus_{\mathrm{M}(E)}(S(x, y) \wedge\right.$ $x)) \oplus_{\mathrm{M}(E)}\left(y \ominus_{\mathrm{M}(E)}(S(x, y) \wedge y)\right) \leq S(x, y)^{\prime}$, i.e.,

$$
\begin{aligned}
& S(x, y) \oplus_{E}\left(\left(x \ominus_{\mathrm{M}(E)}(S(x, y) \wedge x)\right) \oplus_{\mathrm{M}(E)}\left(y \ominus_{\mathrm{M}(E)}(S(x, y) \wedge y)\right)\right) \\
= & \left((S(x, y) \wedge x) \oplus_{E}(S(x, y) \wedge y)\right) \oplus_{E} \\
& \left(\left(x \ominus_{E}(S(x, y) \wedge x)\right) \oplus_{E}\left(y \ominus_{E}(S(x, y) \wedge y)\right)\right)=_{E} \oplus_{E} y
\end{aligned}
$$

is defined.
Theorem 3.2. Let $E$ be a sharply dominating atomic Archimedean lattice effect algebra. Let $\mathrm{T}(E)$ be a subset of $\mathrm{S}(E) \times \mathrm{M}(E)$ given by

$$
\mathrm{T}(E)=\left\{\left(z_{S}, z_{M}\right) \in \mathrm{S}(E) \times \mathrm{M}(E) \mid z_{M} \in h\left(z_{S}^{\prime}\right)\right\}
$$

Equip $\mathrm{T}(E)$ with a partial binary operation $\oplus_{\mathrm{T}(E)}$ with $\left(x_{S}, x_{M}\right) \oplus_{\mathrm{T}(E)}\left(y_{S}, y_{M}\right)$ is defined if and only if
(i) $S\left(x_{M}, y_{M}\right)$ is defined,
(ii) $z_{S}=x_{S} \oplus_{\mathrm{S}(E)} y_{S} \oplus_{\mathrm{S}(E)} S\left(x_{M}, y_{M}\right)$ is defined,
(iii) $z_{M}=\left(x_{M} \ominus_{\mathrm{M}(E)}\left(S\left(x_{M}, y_{M}\right) \wedge x_{M}\right)\right) \oplus_{\mathrm{M}(E)}\left(y_{M} \ominus_{\mathrm{M}(E)}\left(S\left(x_{M}, y_{M}\right) \wedge y_{M}\right)\right)$ is defined,
(iv) $z_{M} \in h\left(z_{S}^{\prime}\right)$.

In this case $\left(z_{S}, z_{M}\right)=\left(x_{S}, x_{M}\right) \oplus_{\mathrm{T}(E)}\left(y_{S}, y_{M}\right)$. Let $0_{\mathrm{T}(E)}=\left(0_{E}, 0_{E}\right)$ and $1_{\mathrm{T}(E)}=\left(1_{E}, 0_{E}\right)$. Then $\mathrm{T}(E)=\left(\mathrm{T}(E), \oplus_{\mathrm{T}(E)}, 0_{\mathrm{T}(E)}, 1_{\mathrm{T}(E)}\right)$ is an effect algebra and the mapping $\varphi: E \rightarrow \mathrm{~T}(E)$ given by $\varphi(x)=\left(\widetilde{x}, x \ominus_{E} \widetilde{x}\right)$ is an isomorphism of effect algebras.
Proof. Evidently, $\varphi$ is correctly defined since, for any $x \in E$, we have that $x=\widetilde{x} \oplus_{E}(x \ominus \widetilde{x})=x_{S} \oplus_{E} x_{M}, x_{S} \in \mathrm{~S}(E)$ and $x_{M} \in \mathrm{M}(E)$. Hence $\varphi(x)=$ $\left(x_{S}, x_{M}\right) \in \mathrm{S}(E) \times \mathrm{M}(E)$ and $x_{M} \in h\left(x_{S}^{\prime}\right)$. Let us check that $\varphi$ is bijective. Assume first that $x, y \in E$ such that $\varphi(x)=\varphi(y)$. We have $x=\widetilde{x} \oplus_{E}\left(x \ominus_{E} \widetilde{x}\right)=$ $\widetilde{y} \oplus_{E}\left(y \ominus_{E} \widetilde{y}\right)=y$. Hence $\varphi$ is injective. Let $\left(x_{S}, x_{M}\right) \in \mathrm{S}(E) \times \mathrm{M}(E)$ and $x_{M} \in h\left(x_{S}^{\prime}\right)$. This yields that $x=x_{S} \oplus_{E} x_{M}$ exists and evidently by Lemma 2.5. (i) $\widetilde{x}=x_{S}$ and $x \ominus_{E} \widetilde{x}=x_{M}$. It follows that $\varphi$ is surjective. Moreover, $\varphi\left(0_{E}\right)=\left(0_{E}, 0_{E}\right)=0_{\mathrm{T}(E)}$ and $\varphi\left(1_{E}\right)=\left(1_{E}, 0_{E}\right)=1_{\mathrm{T}(E)}$.

Now, let us check that, for all $x, y \in E, x \oplus_{E} y$ is defined iff $\varphi(x) \oplus_{\mathrm{T}(E)} \varphi(y)$ is defined in which case $\varphi\left(x \oplus_{E} y\right)=\varphi(x) \oplus_{\mathrm{T}(E)} \varphi(y)$. For any $x, y, z, u \in E$ we obtain

$$
\begin{aligned}
& z=x \oplus_{E} y \Longleftrightarrow z=\left(\widetilde{x} \oplus_{E}\left(x \ominus_{E} \widetilde{x}\right)\right) \oplus_{E}\left(\widetilde{y} \oplus_{E}\left(y \ominus_{E} \widetilde{y}\right)\right) \\
& \Longleftrightarrow z=\left(\widetilde{x} \oplus_{E} \widetilde{y}\right) \oplus_{E}\left(\left(x \ominus_{E} \widetilde{x}\right) \oplus_{E}\left(y \ominus_{E} \widetilde{y}\right)\right) \quad \Longleftrightarrow \quad \text { by Lemma 3.1 } \\
& u=S\left(x \ominus_{E} \widetilde{x}, y \ominus_{E} \widetilde{y}\right) \text { and } \\
& z=\left(\widetilde{x} \oplus_{E} \widetilde{y}\right) \oplus_{E}\left(u \oplus_{E}\left(\left(x \ominus_{E} \widetilde{x}\right) \ominus_{E}\left(u \wedge\left(x \ominus_{E} \widetilde{x}\right)\right)\right)\right. \\
& \left.\oplus_{E}\left(\left(y \ominus_{E} \widetilde{y}\right) \ominus_{E}\left(u \wedge\left(y \ominus_{E} \widetilde{y}\right)\right)\right)\right) \\
& \Longleftrightarrow \quad u=S\left(x \ominus_{E} \widetilde{x}, y \ominus_{E} \widetilde{y}\right) \text { and } \\
& z=\left(\widetilde{x} \oplus_{E} \widetilde{y} \oplus_{E} u\right) \oplus_{E}\left(\left(\left(x \ominus_{E} \widetilde{x}\right) \ominus_{E}\left(u \wedge\left(x \ominus_{E} \widetilde{x}\right)\right)\right)\right. \\
& \left.\oplus_{E}\left(\left(y \ominus_{E} \widetilde{y}\right) \ominus_{E}\left(u \wedge\left(y \ominus_{E} \widetilde{y}\right)\right)\right)\right) \\
& \Longleftrightarrow \quad u=S\left(x \ominus_{E} \widetilde{x}, y \ominus_{E} \widetilde{y}\right) \text { and } \\
& z=\left(\widetilde{x} \oplus_{\mathrm{S}(E)} \widetilde{y} \oplus_{\mathrm{S}(E)} u\right) \oplus_{E}\left(\left(\left(x \ominus_{E} \widetilde{x}\right) \ominus_{\mathrm{M}(E)}\left(u \wedge\left(x \ominus_{E} \widetilde{x}\right)\right)\right)\right. \\
& \left.\oplus_{\mathrm{M}(E)}\left(\left(y \ominus_{E} \widetilde{y}\right) \ominus_{\mathrm{M}(E)}\left(u \wedge\left(y \ominus_{E} \widetilde{y}\right)\right)\right)\right) \\
& \Longleftrightarrow \quad\left(\widetilde{x}, x \ominus_{E} \widetilde{x}\right) \oplus_{\mathrm{T}(E)}\left(\widetilde{y}, y \ominus_{E} \widetilde{y}\right) \text { is defined and } \\
& \varphi(z)=\left(\widetilde{x} \oplus_{\mathrm{S}(E)} \widetilde{y} \oplus_{\mathrm{S}(E)} S\left(x \ominus_{E} \widetilde{x}, y \ominus_{E} \widetilde{y}\right),\left(( x \ominus _ { E } \widetilde { x } ) \ominus \left(S\left(x \ominus_{E} \widetilde{x}, y \ominus_{E} \widetilde{y}\right) \wedge\right.\right.\right. \\
& \left.\left.\left.\left(x \ominus_{E} \widetilde{x}\right)\right)\right) \oplus_{\mathrm{M}(E)}\left(\left(y \ominus_{E} \widetilde{y}\right) \ominus\left(S\left(x \ominus_{E} \widetilde{x}, y \ominus_{E} \widetilde{y}\right) \wedge\left(y \ominus_{E} \widetilde{y}\right)\right)\right)\right) \\
& =\left(\widetilde{x}, x \ominus_{E} \widetilde{x}\right) \oplus_{\mathrm{T}(E)}\left(\widetilde{y}, y \ominus_{E} \widetilde{y}\right)=\varphi(x) \oplus_{\mathrm{T}(E)} \varphi(y) \text {. }
\end{aligned}
$$

Altogether, $\mathrm{T}(E)=\left(\mathrm{T}(E), \oplus_{\mathrm{T}(E)}, 0_{\mathrm{T}(E)}, 1_{\mathrm{T}(E)}\right)$ is an effect algebra and the mapping $\varphi: E \rightarrow \mathrm{~T}(E)$ is an isomorphism of effect algebras.

The Triple Representation Theorem then follows immediately.
Remark 3.3. Recall that our method may be also used in the case of complete lattice effect algebras as a substitute of the method from [8] since we need only Lemma 2.16, (ii) and Lemma 3.1 to show that Theorem 3.2 hols for complete lattice effect algebras. But to show that Lemma 2.16. (ii) and Lemma 3.1 hold for complete lattice effect algebras is an easy task.

Now, using Theorems 2.19 and 3.2 we can prove the following Triple Representation Theorem for $\mathrm{B}(E)$ of sharply dominating atomic Archimedean lattice effect algebras $E$ with atomic center $\mathrm{C}(E)$.

Theorem 3.4. Let $E$ be a sharply dominating atomic Archimedean lattice effect algebra with atomic center $\mathrm{C}(E)$. Let $\mathrm{T}(\mathrm{B}(E))$ be a subset of $\mathrm{C}(E) \times(\mathrm{M}(E) \cap$ $\mathrm{B}(E))$ given by

$$
\mathrm{T}(\mathrm{~B}(E))=\left\{\left(z_{C}, z_{M B}\right) \in \mathrm{C}(E) \times(\mathrm{M}(E) \cap \mathrm{B}(E)) \mid z_{M B} \in h\left(z_{C}^{\prime}\right) \cap \mathrm{B}(E)\right\}
$$

Let us put $\oplus_{\mathrm{T}(\mathrm{B}(E))}:=\oplus_{\mathrm{T}(E) / \mathrm{T}(\mathrm{B}(E)) \times \mathrm{T}(\mathrm{B}(E))}$ and let $0_{\mathrm{T}(\mathrm{B}(E))}=\left(0_{E}, 0_{E}\right)$ and $1_{\mathrm{T}(\mathrm{B}(E))}=\left(1_{E}, 0_{E}\right)$. Then $\mathrm{T}(\mathrm{B}(E))=\left(\mathrm{T}(\mathrm{B}(E)), \oplus_{\mathrm{T}(\mathrm{B}(E))}, 0_{\mathrm{T}(\mathrm{B}(E))}, 1_{\mathrm{T}(\mathrm{B}(E))}\right)$ is an effect algebra and the mapping $\varphi_{\mathrm{B}(E)}: \mathrm{B}(E) \rightarrow \mathrm{T}(\mathrm{B}(E))$ given by $\varphi_{\mathrm{B}(E)}=$ $\varphi_{/ \mathrm{B}(E)}$ is an isomorphism of effect algebras.

Proof. Recall that from Statement 2.13, (ii) and Theorem 2.19 we know that $\mathrm{B}(E)$ is a sharply dominating atomic Archimedean lattice effect algebra. Moreover, $\mathrm{S}(\mathrm{B}(E))=\mathrm{C}(E), \mathrm{M}(\mathrm{B}(E))=\mathrm{M}(E) \cap \mathrm{B}(E), h_{\mathrm{B}(E)}(c)=h(c) \cap \mathrm{B}(E)$ for all $c \in \mathrm{C}(E)$ and, for all $y \in \mathrm{~B}(E)$, we have that by Statement 2.13, (ii) $\widetilde{y} \in \mathrm{C}(E)$ and $\widehat{y} \in \mathrm{C}(E)$. Since $\mathrm{B}(E)$ and $\mathrm{C}(E)$ are sub-lattice effect algebras of $E$ we obtain that the mappings (M1)-(M4) for the triple $\left(\mathrm{C}(E), \mathrm{M}(\mathrm{B}(E)), h_{\mathrm{B}(E)}\right)$ are natural restrictions of the mappings (M1)-(M4) for the triple ( $\mathrm{S}(E), \mathrm{M}(E), h)$. Invoking Theorem 3.2 we obtain the required statement.

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