Łukasiewicz logic and Riesz spaces

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Abstract

We initiate a deep study of *Riesz MV-algebras* which are MV-algebras endowed with a scalar multiplication with scalars from $[0,1]$. Extending Mundici's equivalence between MV-algebras and ℓ-groups, we prove that Riesz MV-algebras are categorically equivalent with unit intervals in Riesz spaces with strong unit. Moreover, the subclass of norm-complete Riesz MV-algebras is equivalent with the class of commutative unital C*-algebras. The propositional calculus $\mathbb{R}\mathcal{L}$ that has Riesz MV-algebras as models is a conservative extension of Łukasiewicz ∞-valued propositional calculus and it is complete with respect to evaluations in the standard model $[0,1]$. We prove a normal form theorem for this logic, extending McNaughton theorem for Łukasiewicz logic. We define the notions of quasi-linear combination and quasi-linear span for formulas in $\mathbb{R}\mathcal{L}$ and we relate them with the analogue of de Finetti's coherence criterion for $\mathbb{R}\mathcal{L}$.
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1 Introduction

MV-algebras are the algebraic structures of Łukasiewicz ∞-valued logic. The real unit interval $[0,1]$ equipped with the operations

$$x^* = 1 - x \text{ and } x \oplus y = \min(1, x+y)$$

for any $x, y \in [0,1]$, is the standard MV-algebra, i.e. an equation holds in any MV-algebra if and only if it holds in $[0,1]$. In [28] Mundici proved that MV-algebras are categorically equivalent with abelian lattice-ordered groups with strong unit. Consequently, for any MV-algebra there exists a lattice-ordered group with strong unit (G,u) such that $A \simeq [0,u]_G$, where

$$[0,u]_G = ([0,u], \oplus, ^*, 0),$$
$$[0,u] = \{x \in G \mid 0 \leq x \leq u\},$$
$$x \oplus y = (x+y) \wedge u \text{ and } x^* = u - x \text{ for any } x,\, y \in [0,u].$$

If (V, u) is a Riesz space (vector-lattice) [22] with strong unit then the unit interval $[0,u]_V$ is closed to the scalar multiplication with scalars from $[0,1]$. The structure

$$[0,u]_V = ([0,u], \cdot, \oplus, ^*, 0),$$

where $([0,u], \oplus, ^*, 0)$ is the MV-algebra defined as above and $\cdot : [0,1] \times [0,u]_V \to [0,u]_V$ satisfies the axioms of the scalar product is the fundamental example in the theory of *Riesz MV-algebras*, initiated in [11] and further developed in the present paper.

The study of Riesz MV-algebras is related to the problem of finding a complete axiomatization for the variety generated by $([0,1], \cdot, \oplus, ^*, 0)$, where $([0,1], \oplus, ^*, 0)$ is the standard MV-algebra and $\cdot$ is the product of real numbers. The investigations led to the definition of *product MV-algebras* (PMV-algebras), which can be represented as unit intervals in lattice-ordered rings with strong unit [9]. A PMV-algebra is a structure $(P, \cdot)$, where P is an MV-algebra and $\cdot : P \times P \to P$ satisfies the equations of an internal product. PMV-algebras are an equational class, but the standard model $[0,1]$ generates only a quasi-variety which is a proper subclass of PMV-algebras [26]. In this context, it was natural to replace the internal product with an external one: a *Riesz MV-algebra* is a structure $(R, \cdot)$, where R is an MV-algebra and $\cdot : [0,1] \times R \to R$. Since we prove that the variety of Riesz MV-algebras is generated by $[0,1]$, the propositional calculus $\mathbb{R}\mathcal{L}$, that has Riesz MV-algebras as models, is complete with respect to evaluations in $[0,1]$.

The study of Riesz MV-algebras was initiated in [11]. In Section 3 we give an equivalent but more suitable definition of these structures and we prove some of their fundamental properties.

The categorical equivalence between Riesz MV-alge-bras and Riesz spaces with strong unit is proved in Section 4.4. As a consequence, the standard Riesz MV-algebra $[0,1]$ generates the variety of Riesz MV-algebras.

In Section 5, the categorical equivalence is specialized to the class of *norm-complete Riesz MV-algebras*, which is dually equivalent with the category of compact Hausdorff spaces. Using the Gelfand-Naimark duality, this leads us to a connection with the theory of commutative unital C*-algebras.

Section 6 presents the propositional calculus $\mathbb{R}\mathcal{L}$ which simplifies the one introduced in [11]. In Section 7 we prove a normal form theorem for formulas of $\mathbb{R}\mathcal{L}$. Since $\mathbb{R}\mathcal{L}$ is a conservative extension of Łukasiewicz logic $\mathcal{L}$, this theorem is a generalization of McNaughton theorem [24]. Our result asserts that *for any continuous piecewise linear function* $f : [0,1]^n \to [0,1]$ *there exists a formula* φ *of* $\mathbb{R}\mathcal{L}$ *with* n *variables such that* f *is the term function associated to* φ.

In Section 8 we initiate the theory of*quasi-linear combinations* of formulas in $\mathbb{R}\mathcal{L}$. If $f_i : [0,1]^n \to \mathbb{R}$ are continuous piecewise linear functions and c_i are real numbers for any $i \in \{1, \ldots, k\}$, then the normal form theorem guarantees the existence of a formula Φ of $\mathbb{R}\mathcal{L}$, whose term function is equal to

$((\sum_{i=1}^{k} c_i f_i)\vee 0)\wedge 1$ and, in this case, we say that Φ is *quasi-linear combination* of $f_1, \cdots, f_k$. We prove de Finetti's coherence criterion for $\mathbb{R}\mathcal{L}$ and we provide an equivalent characterization by the fact that a quasi-linear span contains only invalid formulas.

Some of results contained in this paper may overlap with the proceeding paper [11]. Other results are proved in a more general setting in [10, 12, 21]. For the sake of completeness we sketched the proofs that we consider important for the present development.

2 Preliminaries on MV-algebras

An *MV-algebra* is a structure $(A, \oplus, ^*, 0)$, where $(A, \oplus, 0)$ is an abelian monoid and the following identities hold for all $x, y \in A$:

(MV1) $(x^*)^* = x$,

(MV2) $0^* \oplus x = 0^*$,

(MV3) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

We refer to [7] for all the unexplained notions concerning MV-algebras and to [31] for advanced topics. On any MV-algebra A the following operations are defined for any $x, y \in A$:

$$1 = 0^*,\ x \odot y = (x^* \oplus y^*)^*,\ x \to y = x^* \oplus y$$
$$0x = 0,\ mx = (m-1)x \oplus x \text{ for any } m \geq 1.$$

We assume in the sequel that the operation $\odot$ is more binding then $\oplus$.

Remark 2.1. *Any MV-algebra A is a bounded distributive lattice, with the partial order defined by*

$$x \leq y \textit{ if and only if } x \odot y^* = 0$$

and the lattice operations defined by

$$x \vee y = x \oplus y \odot x^* \textit{ and } x \wedge y = x \odot (x^* \oplus y)$$

for any x, $y \in A$.

Any MV-algebra A has an internal distance:

$$d(x, y) = (x \odot y^*) \oplus (x^* \odot y) \text{ for any } x,\ y \in A.$$

Lemma 2.2. *[7, Proposition 1.2.5] In any MV-algebra A, the following properties hold for any x, y, $z \in A$:*
(a) $d(x, y) = d(y, x)$,
(b) $d(x, y) = 0$ iff $x = y$,
(c) $d(x, z) \leq d(x, y) \oplus d(y, z)$.

If $(A, \oplus, ^*, 0)$ is an MV-algebra then an *ideal* is a nonempty subset $I \subseteq A$ such that for any x, $y \in A$ the following conditions are satisfied:

(i1) $x \in I$ and $y \leq x$ imply $y \in I$,

(i2) x and $y \in I$ imply $x \oplus y \in I$.

An ideal I of A uniquely defines a congruence $\sim_I$ by

$$x \sim_I y \text{ iff } x \odot y^* \in I \text{ and } y \odot x^* \in I.$$

We denote by A/I the quotient MV-algebra and we refer to [7] for more details.

We recall that an *ℓ-group* is a structure $(G, +, 0, \leq)$ such that $(G, +, 0)$ is a group, $(G, \leq)$ is a lattice and any group translation is isotone [3]. In the following the ℓ-groups are abelian. For an ℓ-group G we denote $G_+ = \{x \in G \mid x \geq 0\}$. An element $u \in G$ is a *strong unit* if $u \geq 0$ and for any $x \in G$ there is a natural number n such that $x \leq nu$. An ℓ-group is *unital* if it posses a strong unit. If (G, u) is a unital ℓ-group, we define $[0, u] = \{x \in G \mid 0 \leq x \leq u\}$ and

$$x \oplus y = (x + y) \wedge u, \; x^* = u - x \text{ for any } x, y \in [0, u].$$

Then $[0, u]_G = ([0, u], \oplus, \neg, 0)$ is an MV-algebra [7, Proposition 2.1.2].

Lemma 2.3. *[7, Lemma 7.1.3] Let (G, u) be a unital ℓ-group, $x \geq 0$ in G and $n \geq 1$ a natural number such that $x \leq nu$. Then $x = x_1 + \cdots + x_n$, where*

$$x_i = ((x - (i - 1)u) \vee 0) \wedge u \in [0, u]$$

for any $i \in \{1, \ldots, n\}$.

We denote by $\mathbb{MV}$ the category of MV-algebras and by $\mathbb{AG}_u$ the category of unital abelian lattice-ordered groups with unit-preserving morphisms. In [27] the functor $\Gamma \colon \mathbb{AG}_u \to \mathbb{MV}$ is defined as follows:
$\Gamma(G, u) = [0, u]_G$ for any unital ℓ-group (G, u),
$\Gamma(f) = f|_{[0,u]}$ for any morphsim $f : (G, u) \to (G', u')$ from $\mathbb{AG}_u$.

Theorem 2.4. *[7, Corollary 7.1.8] The functor Γ yields an equivalence between $\mathbb{AG}_u$ and $\mathbb{MV}$.*

Definition 2.5. *If A and B are MV-algebras then a function $\omega : A \to B$ is called* additive *if*

$$x \odot y = 0 \textit{ implies } \omega(x) \odot \omega(y) = 0 \textit{ and } \omega(x \oplus y) = \omega(x) \oplus \omega(y).$$

Additivity was firstly studied in the context of states defined on MV-algebras [28]. The theory of states generalizes the boolean probability theory and reflects the theory of states defined on ℓ-groups.

Definition 2.6. *[28] If A is an MV-algebra then a function $s : A \to [0, 1]$ is a* state *if the following properties are satisfied for any x, $y \in A$:*

(s1) if $x \odot y = 0$ then $s(x \oplus y) = s(x) + s(y)$,

(s2) $s(1) = 1$.

The following results are proved in [21], but we sketch the proofs for the sake of completeness. We also note that particular instances of these results are proved in [12]. Proposition 2.7 is proved for states in [28].

Proposition 2.7. *Assume (G, u) and (H, v) are unital ℓ-groups, $A = \Gamma(G, u)$ and $B = \Gamma(H, v)$. Then for any additive function $\omega : A \to B$ there exists a unique group morphism $\overline{\omega} : G \to H$ such that $\overline{\omega}(x) = \omega(x)$ for any $x \in [0, u]$.*

Proof. If $x \in G$ and $x \geq 0$ then there are $x_1, \ldots, x_m \in [0, u]$ such that $x = x_1 + \cdots + x_m$. Then we define

$$\overline{\omega}(x) := \omega(x_1) + \cdots + \omega(x_m).$$

The fact that $\overline{\omega}(x)$ is well defined follows by Riesz decomposition property in ℓ-groups [3, 1.2.16]. Hence $\overline{\omega}(x)$ is well defined for $x \in G_+$ and $\overline{\omega}(x + y) = \overline{\omega}(x) + \overline{\omega}(y)$ for any $x, y \in G_+$. By [3, 1.1.7] it follows that $\overline{\omega}$ can be uniquely extended to a group homomorphism defined on G. □

Lemma 2.8. *If A and B are MV-algebras and $\omega : A \to B$ is a function, then the following are equivalent:*
(a) ω is additive,
(b) the following properties hold for any $x, y \in A$:
(b1) $x \leq y$ implies $\omega(x) \leq \omega(y)$,
(b2) $\omega(x \odot (x \wedge y)^*) = \omega(x) \odot \omega(x \wedge y)^*$.

Proof. (a)⇒(b) If $x \leq y$ then $y = x \vee y = x \oplus y \odot x^*$, so $\omega(y) = \omega(x) \oplus \omega(y \odot x^*)$ and $\omega(x) \leq \omega(y)$. Hence, ω is isotone. We remark that $\omega(x \wedge y) \leq \omega(x)$, so

$$\omega(x) \odot \omega(x \wedge y)^* \oplus \omega(x \wedge y) = \omega(x) \vee \omega(x \wedge y) = \omega(x) = \omega(x \vee (x \wedge y)) = \omega(x \odot (x \wedge y)^*) \oplus \omega(x \wedge y).$$

It follows that

$$\begin{aligned}
&\omega(x) \odot \omega(x \wedge y)^* = \\
&(\omega(x) \odot \omega(x \wedge y)^*) \wedge \omega(x \wedge y)^* = \\
&(\omega(x) \odot \omega(x \wedge y)^* \oplus \omega(x \wedge y)) \odot \omega(x \wedge y)^* = \\
&(\omega(x \odot (x \wedge y)^*) \oplus \omega(x \wedge y)) \odot \omega(x \wedge y)^* = \\
&\omega(x \odot (x \wedge y)^*) \wedge \omega(x \wedge y)^* = \\
&\omega(x \odot (x \wedge y)^*)
\end{aligned}$$

(b)⇒(a) We remark that for $x = 1$ in (b2) we get $\omega(y^*) = \omega(1) \odot \omega(y)^*$, so $\omega(y^*) \leq \omega(y)^*$. Assume $x \odot y = 0$, so $x \leq y^*$. Using (b1), we get $\omega(x) \leq \omega(y^*) \leq \omega(y)^*$, so $\omega(x) \odot \omega(y) = 0$. In this case, using (b2) we get

$$\omega(x) = \omega(x \wedge y^*) = \omega((x \oplus y) \odot y^*) = \omega(x \oplus y) \odot \omega(y)^*.$$

It follows that:

$$\omega(x) \oplus \omega(y) = \omega(x \oplus y) \odot \omega(y)^* \oplus \omega(y) = \omega(x \oplus y) \vee \omega(y).$$

Using (b1), $\omega(y) \leq \omega(x \oplus y)$, and we get $\omega(x \oplus y) = \omega(x) \oplus \omega(y)$. □

3 Riesz MV-algebras

Riesz MV-algebras are introduced in [11]. Below we give a simpler and more suitable definition, which provides directly an equational characterization. The equivalence between this definition and the one from [11] is proved in Theorem 3.17.

Definition 3.1. *A* Riesz MV-algebra *is a structure*

$$(R, \cdot, \oplus, ^*, 0),$$

where $(R, \oplus, ^*, 0)$ *is an MV-algebra and the operation* $\cdot : [0,1] \times R \to R$ *satisfies the following identities for any* r, $q \in [0,1]$ *and* x, $y \in R$*:*

(RMV1) $r \cdot (x \odot y^*) = (r \cdot x) \odot (r \cdot y)^*$,

(RMV2) $(r \odot q^*) \cdot x = (r \cdot x) \odot (q \cdot x)^*$,

(RMV3) $r \cdot (q \cdot x) = (rq) \cdot x$,

(RMV4) $1 \cdot x = x$.

In the following we write rx *instead of* $r \cdot x$ *for* $r \in [0,1]$ *and* $x \in R$*. Note that* rq *is the real product for any* r, $q \in [0,1]$.

Example 3.2. *If* X *is a compact Hausdorff space then*

$$C(X)_u = \{f : X \to [0,1] \mid f \text{ continuous}\}$$

is a Riesz MV-algebra, with all the operations defined componentwise. This example will be further investigated in Section 5

Example 3.3. *If* G *is an abelian* ℓ*-group, then* $R = \Gamma(\mathbb{R} \times_{lex} G, (1,0))$ *is a Riesz MV-algebra, where* $\mathbb{R} \times_{lex} G$ *is the lexicographic product of* ℓ*-groups and the scalar multiplication is defined by* $r(q,x) = (rq, x)$ *for any* $r \in [0,1]$ *and* $(q,x) \in R$.

Lemma 3.4. *In any Riesz MV-algebra* R *the following properties hold for any* r, $q \in [0,1]$ *and* x, $y \in R$*:*
(a) $0x = 0$, $r0 = 0$,
(b) $x \leq y$ *implies* $rx \leq ry$,
(c) $r \leq q$ *implies* $rx \leq qx$,
(d) $rx \leq x$.

Proof. (a) follows by (RMV1) and (RMV2) for $x = y$ and, respectively, $r = q$.
(b), (c) follow by Remark 2.1.
(d) follows by (c) and (RMV4).

□

Proposition 3.5. *The function* $\iota : [0,1] \to R$ *defined by* $\iota(r) = r1$ *for any* $r \in [0,1]$ *is an embedding. Consequently, any Riesz MV-algebra* R *contains a subalgebra isomorphic with* $[0,1]$.

Proof. By Lemma 3.4 we get $\iota(0) = 0$. If $r,\ q \in [0,1]$ then

$$\iota(r^*) = r^*1 = (1 \cdot 1) \odot (r1)^* = (r1)^*,$$
$$\iota(r \odot q) = \iota(r \odot q^{**}) = (r \odot q^{**})1 = (r1) \odot (q^*1)^* = (r1) \odot (q1)^{**} = (r1) \odot (q1).$$

$\square$

A *Riesz space* (*vector lattice*) [22] is a structure

$$(V, \cdot, +, 0, \leq)$$

such that $(V, +, 0, \leq)$ is an abelian ℓ-group, $(V, \cdot, +, 0)$ is a real vector space and, in addition,
(RS) $x \leq y$ implies $r \cdot x \leq r \cdot y$,
for any $x,\ y \in V$ and $r \in \mathbb{R}$, $r \geq 0$.
A Riesz space is *unital* if the underlaying ℓ-group is unital.

Lemma 3.6. *If (V, u) is a unital Riesz space, then*

$$[0, u]_V = ([0, u], \cdot, \oplus, ^*, 0)$$

is a Riesz MV-algebra, where rx is the scalar multiplication of V for any $r \in [0,1]$ and $x \in [0, u]$.

Proof. Assume $r, q \in [0,1]$ and $x, y \in [0, u]$.
(RMV1) $r(x \odot y^*) = r((x - y) \vee 0) = (rx - ry) \vee 0 = (rx) \odot (ry)^*$.
(RMV2) If $r \leq q$ then $rx \leq qx$, so $(r \odot q^*)x =$
$((r - q) \vee 0)x = 0 = (rx - qx) \vee 0 = (rx) \odot (qx)^*$.

If $r > q$ then $((r-q) \vee 0)x = (r-q)x = rx - qx = (rx) - (qx) \vee 0 = (rx) \odot (qx)^*$.

We note that (RMV3) and (RMV4) hold in V, therefore they hold in $[0, u]$. $\square$

Remark 3.7. *If $(R, \cdot, \oplus, ^*, 0)$ is a Riesz MV-algebra then we denote its MV-algebra reduct by $\mathrm{U}(R) = (R, \oplus, ^*, 0)$. Assume I is an ideal of $\mathrm{U}(R)$. By Lemma 3.4 (d) we infer that $rx \in I$ whenever $r \in [0,1]$ and $x \in I$. It follows, by (RMV1), that $rx \sim_I ry$ whenever $r \in [0,1]$ and $x \sim_I y$. As consequence, the quotient R/I has a canonical structure of Riesz MV-algebra.*

Remark 3.8. *A Riesz MV-algebra R has the same theory of ideals (congruences) as its reduct $\mathrm{U}(R)$. If R is a Riesz MV-algebra and $P \subseteq R$ an ideal then it is straightforward that the following hold:*
(a) P is prime iff R/P is linearly ordered,
(b) P is maximal iff $R/P \simeq [0,1]$.
Note that (b) holds since, for any maximal ideal P, the quotient R/P is an MV-subalgebra of $[0,1]$. But the only subalgebra of $[0,1]$ which is a Riesz MV-algebra is $[0,1]$ by Proposition 3.5, so $R/P \simeq [0,1]$.

Lemma 3.9. *If R is a Riesz MV-algebra, $I \subseteq R$ an ideal and $x \in R$ such that $rx \in I$ for some $r \in (0,1]$ then $x \in I$.*

Proof. Let $r \in (0,1]$ such that $rx \in I$ and let m be the integer part of $\frac{1}{r}$. Hence $\frac{1}{m+1}x \leq rx$, so $\frac{1}{m+1}x \in I$. Since $x = (m+1)(\frac{1}{m+1}x)$ we get $x \in I$.

$\square$

Corollary 3.10. *Any simple Riesz MV-algebra is isomorphic with* $[0,1]$. *Any semisimple Riesz MV-algebra is a subdirect product of copies of* $[0,1]$.

In the sequel we investigate the morphisms of Riesz MV-algebras.

Corollary 3.11. *If* R_1 *and* R_2 *are Riesz MV-algebras and* $f : U(R_1) \to U(R_2)$ *is a morphism of MV-algebras then*

$$f(rx) = rf(x) \text{ for any } r \in [0,1] \text{ and } x \in R_1.$$

Proof. Assume J is an ideal in R_2. Since f is an morphism of MV-algebras, it follows that $f^{-1}(J)$ is an ideal in R_1. If $x \in R_1$ and $r \in [0,1]$ we have

$$\begin{gathered} rf(x) \in J \Rightarrow f(x) \in J \Rightarrow x \in f^{-1}(J) \Rightarrow \\ rx \in f^{-1}(J) \Rightarrow f(rx) \in J, \\ f(rx) \in J \Rightarrow rx \in f^{-1}(J) \Rightarrow x \in f^{-1}(J) \Rightarrow \\ f(x) \in J \Rightarrow rf(x) \in J. \end{gathered}$$

Note that we used Lemma 3.9 twice. We proved that, for any ideal J of R_2

$$rf(x) \in J \Leftrightarrow f(rx) \in J.$$

Therefore $rf(x) \odot f(rx)^* \in J$ and $f(rx) \odot (rf(x))^*$ for any ideal J of R_2. This means that $rf(x) \odot f(rx)^* = f(rx) \odot (rf(x))^* = 0$, so $f(rx) = rf(x)$. □

Remark 3.12. *The above result asserts that a morphism of Riesz MV-algebra is simply a morphism between the corresponding MV-algebra reducts.*

The following result is similar with Chang's representation theorem for MV-algebras [6, Lemma 3].

Corollary 3.13. *Any Riesz MV-algebra is a subdirect product of linearly ordered Riesz MV-algebras.*

Proof. If R is a Riesz MV-algebra then, by Remark 3.8, $\bigcap\{P \mid P \text{ prime ideal of } R\} = \{0\}$ and R/P is linearly ordered for any prime ideal P. As consequence, R is a subdirect product of the family

$$\{R/P \mid P \text{ prime ideal of } R\}.$$

□

In order to prove that Riesz MV-algebras introduced in Definition 3.1 coincide with the ones defined in [11], we recall some results from [12].

Remark 3.14. *[12] If* Ω *is a set of unary operation symbols, then an* MV-algebra with Ω-operators *is a structure* (A, Ω_A) *where* A *is an MV-algebra and for any* $\omega \in \Omega$ *the operation* $\omega_A : A \to A$ *is additive. An additive function* $\omega : A \to A$ *is an* f-operator *if*

$$x \wedge y = 0 \text{ implies } \omega(x) \wedge y = 0 \text{ for any } x,\, y \in A.$$

If (A, Ω_A) is an MV-algebra with Ω-operators such that ω_A is an f-operator for any $\omega \in \Omega$, then (A, Ω_A) is a subdirect product of linearly ordered MV-algebras with Ω-operators [12, Corollary 5.6.].

Remark 3.15. *Assume that $(R, \oplus, ^*, 0)$ is an MV-algebra and let $\cdot : [0,1] \times R \to R$ such that* (RMV2), (RMV3), (RMV4) *hold and the function*

$$\omega_r : R \to R, \ \omega_r(x) = r \cdot x$$

is additive for any $r \in [0,1]$. By (RMV2) *and* (RMV4) *we get $\omega_r(x) \leq x$ for any $r \in [0,1]$ and $x \in R$, so ω_r is an f-operator for any $r \in [0,1]$. If $\Omega = \{\omega_r \mid r \in [0,1]\}$ then, by Remark 3.14, (R, Ω) is an MV-algebra with Ω-operators that can be represented as subdirect product of linearly ordered MV-algebras with Ω-operators.*

Lemma 3.16. *Assume that $(R, \oplus, ^*, 0)$ is an MV-algebra. If $\cdot : [0,1] \times R \to R$ then the following are equivalent:*
(RMV2) $(r \odot q^*) \cdot x = (r \cdot x) \odot (q \cdot x)^*$
for any r, $q \in [0,1]$ and $x \in R$,
(RMV2′) $r \odot q = 0$ *then* $(r \cdot x) \odot (q \cdot x) = 0$ *and*

$$(r \oplus q) \cdot x = (r \cdot x) \oplus (q \cdot x)$$

for any r, $q \in [0,1]$ and $x \in R$.

Proof. For $x \in R$ define $\omega_x : [0,1] \to R$ by $\omega_x(r) = rx$ for any $r \in [0,1]$. If ω_x satisfies (RMV2) then the condition (b) from Lemma 2.8 is satisfied, so ω_x satisfies also (RMV2′). Conversely, if ω_x satisfies (RMV2′) then, by Lemma 2.8, we also get $\omega_x(0) = 0$. Assume r, $q \in [0,1]$ such that $r \leq q$. Hence $r \odot q^* = 0$ and $rx \leq qx$, so $(r \odot q^*)x = 0x = 0 = (rx) \odot (qx)^*$. If r, $q \in [0,1]$ such that $r > q$ then (RMV2) coincides with the equation (b2) from Lemma 2.8. $\square$

Theorem 3.17. *Assume that $(R, \oplus, ^*, 0)$ is an MV-algebra and $\cdot : [0,1] \times R \to R$. Then $(R, \cdot, \oplus, ^*, 0)$ is a Riesz MV-algebra if and only if the following properties are satisfied for any x, $y \in R$ and r, $q \in [0,1]$:*
(RMV1′) *if* $x \odot y = 0$ *then* $(r \cdot x) \odot (r \cdot y) = 0$ *and*

$$r \cdot (x \oplus y) = (r \cdot x) \oplus (r \cdot y),$$

(RMV2′) *if* $r \odot q = 0$ *then* $(r \cdot x) \odot (q \cdot x) = 0$ *and*

$$(r \oplus q) \cdot x = (r \cdot x) \oplus (q \cdot x),$$

(RMV3) $r \cdot (q \cdot x) = (rq) \cdot x$,
(RMV4) $1 \cdot x = x$.

Proof. By Lemma 3.16, if $(R, \oplus, ^*, 0)$ is an MV-algebra and $\cdot : [0,1] \times R \to R$, then the algebra $(R, \cdot, \oplus, ^*, 0)$ satisfies (RMV2), (RMV3) and (RMV4) if and only if it satisfies (RMV2′), (RMV3) and (RMV4). Assume now that $(R, \cdot, \oplus, ^*, 0)$ satisfies (RMV2), (RMV3) and (RMV4). We have to prove that (RMV1) is satisfied if and only if (RMV1′) is satisfied. By Corollary 4.1 and Remark 3.15, it suffices to prove the equivalence for linearly ordered structures. In this case, by Lemma 2.8, the equivalence of (RMV1) and (RMV1′) is straightforward.

$\square$

Note that in [11] a Riesz MV-algebra is defined by (RMV1′), (RMV2′), (RMV3) and (RMV4), so we proved that Definition 3.1 is equivalent with the initial one.

4 Riesz MV-algebras and Riesz spaces. The completeness theorem.

By Theorem 3.17, Riesz MV-algebras are exactly the MV-modules [10] over $[0,1]$. Hence some basic properties follow from the general theory of MV-modules developed in [10, 21]. One of the most important results is the categorical equivalence between Riesz MV-algebras and unital Riesz spaces. For the sake of completeness, we sketch the proof of this result.

Proposition 4.1. *For any Riesz MV-algebra R there is a unital Riesz space (V,u) such that $R \simeq [0,u]_V$.*

Proof. By Theorem 2.4, there exists a unital ℓ-group (V,u) such that R and $[0,u]_V$ are isomorphic MV-algebras. For any $\lambda \in \mathbb{R}$ and $x \in V$ we have to define the scalar multiplication λx. We can safely assume that $R = [0,u] \subseteq V$.

If $r \in [0,1]$ then $x \mapsto rx$ is an additive function from $[0,u]_V$ to $[0,u]_V$ so, by Proposition 2.7, it can be uniquely extended to a group morphism $\omega_r : V \to V$. Hence we define $rx = \omega_r(x)$ for any $x \in V$. We note that $x \geq 0$ implies $rx \geq 0$.

If $q \in [0,1]$ then $\omega_{rq} = \omega_r \circ \omega_q$ since they coincide on the positive cone, so $r(qv) = (rq)v$.

Note that $v = (v \vee 0) - ((-v) \vee 0)$, so

$$1v = 1(v \vee 0) - 1((-v) \vee 0) = (v \vee 0) - ((-v) \vee 0) = v.$$

If $\lambda \geq 0$ and $v \in V$, then there are $r_1, \ldots, r_m \in [0,1]$ such that $\lambda = r_1 + \cdots + r_m$. Then we define

$$\lambda v = r_1 v + \cdots + r_m v.$$

One can prove that λv is well-defined using the Riesz decomposition property [3, 1.2.16]. If $\mu \geq 0$, then $\mu = q_1 + \cdots + q_n$ for some $q_1, \ldots, q_n \in [0,1]$ and

$$\lambda(\mu v) = \lambda(\textstyle\sum_{j=1}^{n} q_j v) = \sum_{i=1}^{m} r_i \left(\sum_{j=1}^{n} q_j v\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} r_i(q_j v) =$$
$$\textstyle\sum_{i=1}^{m} \sum_{j=1}^{n} (r_i q_j) v = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} (r_i q_j)\right) v = (\lambda\mu)v.$$

If $\lambda \leq 0$ in $\mathbb{R}$ then we set $\lambda v = -(|\lambda| v))$, where $|\lambda|$ is the module of λ in $\mathbb{R}$. It is straightforward that $\lambda(\mu v) = (\lambda\mu)v$ for another $\mu \in \mathbb{R}$.

We know that (V,u) is a unital ℓ-group and we defined the scalar product λv for any $\lambda \in R$ and $v \in V$ such that $\lambda v \geq 0$ whenever $\lambda \geq 0$ and $v \geq 0$. Therefore, (V,u) is a unital vector lattice.

□

We denote by $\mathbb{RMV}$ the category of Riesz MV-algebras and by $\mathbb{RS}_u$ the category of unital Riesz spaces with unit-preserving morphisms.

Following this construction we get a functor

$$\Gamma_{\mathbb{R}} : \mathbb{RS}_u \to \mathbb{RMV}$$

defined as follows:
$\Gamma_{\mathbb{R}}(V, u) = [0, u]_V$ for any unital Riesz space (V, u),
$\Gamma_{\mathbb{R}}(f) = f|_{[0,u]}$ for any morphism $f : (V, u) \to (V', u')$ from $\mathbb{RS}_u$.

Theorem 4.2. *[11, 10] The functor $\Gamma_{\mathbb{R}}$ yields an equivalence between $\mathbb{RS}_u$ and $\mathbb{RMV}$.*

Proof. It follows from Theorem 2.4, Corollary 3.11 and Proposition 4.1. □

It is straightforward that the following diagram is commutative, where U are forgetful functors:

$$\begin{array}{ccc} \mathbb{RS}_u & \xrightarrow{\Gamma_{\mathbb{R}}} & \mathbb{RMV} \\ \mathrm{U}\downarrow & & \downarrow\mathrm{U} \\ \mathbb{AB}_u & \xrightarrow{\Gamma} & \mathbb{MV} \end{array}$$

The standard Riesz MV-algebra is $([0, 1], \cdot, \oplus, ^*, 0)$, where $\cdot : [0, 1] \times [0, 1] \to [0, 1]$ is the product of real numbers and $([0, 1], \oplus, ^*, 0)$ is the standard MV-algebra. In the sequel we prove that the variety of Riesz MV-algebras is generated by $[0, 1]$, i.e. an identity holds in any Riesz MV-algebra if and only if it holds in the standard Riesz MV-algebra $[0, 1]$. Our approach follows closely the proof of Chang's completeness theorem for Łukasiewicz logic [6]. To any sentence in the first-order theory of Riesz MV-algebras we associate a sentence in the first-order theory of Riesz spaces such that the satisfiability is preserved by the $\Gamma_{\mathbb{R}}$ functor. The first-order theory of Riesz MV-algebras, as well as the theory of Riesz spaces, are obtained considering for each scalar r an unary function ρ_r which denotes in a particular model the scalar multiplication by r, i.e. $x \overset{\rho_r}{\mapsto} rx$. In the following, the language of Riesz MV-algebras is $\mathcal{L}_{RMV} = \{\oplus, ^*, 0, \{\rho_r\}_{r\in[0,1]}\}$ and the language of Riesz spaces is $\mathcal{L}_{Riesz} = \{\leq, +, -, \vee, \wedge, 0, \{\rho_r\}_{r\in\mathbb{R}}\}$.

Let $t(v_1, \ldots, v_k)$ be a term of $\mathcal{L}_{RMV}$ and v a propositional variable different from $v_1, \ldots, v_k$. We define $\overline{t}$ as follows:
- if $t = 0$ then $\overline{0}$ is 0,
- if $t = v$ then $\overline{t}$ is v
- if $t = t_1^*$ then $\overline{t}$ is $v - \overline{t_1}$,
- if $t = t_1 \oplus t_2$ then $\overline{t}$ is $(t_1 + t_2) \wedge v$,
- if $t = \rho_r(t_1)$ then $\overline{t}$ is $\rho_r(\overline{t_1})$.

Let $\varphi(v_1, \ldots, v_k)$ be a formula of $\mathcal{L}_{RMV}$ such that all the free and bound variables of φ are in $\{v_1, \ldots, v_k\}$ and v a propositional variable different from $v_1, \ldots, v_k$. We define $\overline{\varphi}$ as follows:
- if φ is $t_1 = t_2$ then $\overline{\varphi}$ is $\overline{t_1} = \overline{t_2}$,
- if φ is $\neg\psi$ then $\overline{\varphi}$ is $\neg\overline{\psi}$,
- if φ is $\psi\vee\chi$ then $\overline{\varphi}$ is $\overline{\psi}\vee\overline{\chi}$ and similarly for

$$\wedge, \to, \leftrightarrow,$$

- if φ is $(\forall v_i)\psi$ then $\overline{\varphi}$ is $\forall v_i((0 \leq v_i) \wedge (v_i \leq v) \rightarrow \overline{\psi})$,
- if φ is $\exists v_i \psi$ then $\overline{\varphi}$ is $\exists v_i((0 \leq v_i) \wedge (v_i \leq v) \rightarrow \overline{\psi})$.

Thus to any formula $\varphi(v_1, \ldots, v_k)$ of $\mathcal{L}_{RMV}$ we associate a formula $\overline{\varphi}(v_1, \ldots, v_k, v)$ of $\mathcal{L}_{Riesz}$. As a consequence, to any sentence σ of $\mathcal{L}_{RMV}$ corresponds a formula with only one free variable $\overline{\sigma}(v)$ of $\mathcal{L}_{Riesz}$.

Proposition 4.3. *Let (V, u) be a Riesz space with strong unit and $R = \Gamma_{\mathbb{R}}(V, u)$. If σ is a sentence in the first-order theory of Riesz MV-algebras then*

$$R \models \sigma \text{ if and only if } V \models \overline{\sigma}[u].$$

Proof. By structural induction on terms it follows that $t[a_1, \ldots, a_n] = \overline{t}[a_1, \ldots, a_n, u]$ whenever $t(v_1, \ldots, v_n)$ is a term of $\mathcal{L}_{RMV}$ and $a_1, \ldots, a_n \in R$. The rest of the proof is straightforward. $\square$

Theorem 4.4. *An equation σ in the theory of Riesz MV-algebras holds in all Riesz MV-algebras if and only if it holds in the standard Riesz MV-algebra $[0, 1]$.*

Proof. One implication is obvious. To prove the other one, let R be a Riesz MV-algebra such that $R \not\models \sigma$. Since $R \simeq \Gamma_{\mathbb{R}}(V, u)$ for some Riesz space with strong unit (V, u), we have that $\Gamma_{\mathbb{R}}(V, u) \not\models \sigma$. Using Proposition 4.3, we infer that $V \not\models \overline{\sigma}[u]$ in the theory of Riesz spaces. Since the order relation in any lattice can be expressed equationally, we note that $\overline{\sigma}(v)$ is a quasi-identity. By [20, Corollary 2.6] a quasi-identity is satisfied by all Riesz spaces if and only if it is satisfied by $\mathbb{R}$. Hence there exists a real number $c \geq 0$ such that $\mathbb{R} \not\models \overline{\sigma}[c]$. Since $\mathbb{R} \models \overline{\sigma}[0]$, we get $c > 0$. If follows that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) \mapsto x/c$ is an automorphism of Riesz spaces. We infer that $\mathbb{R} \not\models \overline{\sigma}[1]$, so $[0, 1] \not\models \sigma$. $\square$

Remark 4.5. *The variety of Riesz MV-algebras is generated by the standard model $[0, 1]$ in the language of MV-algebras enriched with unary operations $x \mapsto rx$ for any $r \in [0, 1]$. This features are reflected by the propositional calculus $\mathbb{R}\mathcal{L}$ presented in Section 6, whose Lindenbaum-Tarski algebra is a Riesz MV-algebra.*

Since the class of Riesz MV-algebras is a variety, free structures exist. The free Riesz MV-algebra with n free generators is characterized in Corollary 7.8.

5 Norm-complete Riesz MV-algebras

Let (V, u) be a unital Riesz space and define

$$\| \cdot \|_u : V \to \mathbb{R} \text{ by}$$
$$\|x\|_u = \inf\{\alpha \geq 0 \mid |x| \leq \alpha u\} \text{ for any } x \in V.$$

Then $\| \cdot \|_u$ is a seminorm [25, Proposition 1.2.13] and

$$|x| \leq |y| \text{ implies } \|x\|_u \leq \|y\|_u \text{ for any } x,\, y \in V.$$

Remark 5.1. *If (V, u) is a unital Riesz space, then*

$$\|x\|_u = \inf\{\alpha \in [0, 1] \mid x \leq \alpha u\} \textit{ for any } x \in [0, u].$$

This fact leads us to the following definition, which was suggested by V. Marra (private communication).

Definition 5.2. *[23] If R is a Riesz MV-algebra then the* unit seminorm $\|\cdot\| : R \to [0,1]$ *is defined by*

$$\|x\| = \inf\{r \in [0,1] \mid x \leq r1\} \text{ for any } x \in R.$$

Remark 5.3. *If R_1 and R_2 are Riesz MV-algebras and $f : R_1 \to R_2$ is a morphism, then $x \leq r1$ in R_1 implies $f(x) \leq r1$ in R_2, so $\|f(x)\| \leq \|x\|$ for any $x \in R_1$. If f is injective then $\|f(x)\| = \|x\|$ for any $x \in R_1$.*

This fact allows us to infer properties of the unit seminorm in Riesz MV-algebras directly from the properties of the unit seminorm in Riesz spaces.

Lemma 5.4. *In any Riesz MV-algebra R, the following properties hold for any x, $y \in R$ and $r \in [0,1]$.*
(a) $\|0\| = 0$, $\|1\| = 1$,
(b) $\|x \oplus y\| \leq \|x\| + \|y\|$,
(c) $x \leq y$ *implies* $\|x\| \leq \|y\|$,
(d) $\|rx\| = r\|x\|$,
(e) if $(m-1)x \leq x^*$ *then* $\|mx\| = m\|x\|$ *for any natural number* $m \geq 1$.

Proof. By Theorem 4.2 and Remark 5.3 we can safely assume that R is $[0,u]_V$ for some unital Riesz space (V,u). Hence (a)-(d) follow from the properties of the unit seminorm in Riesz spaces [13, 25H].
(e) Note that $(m-1)x \leq x^*$ implies

$$\underbrace{x \oplus \cdots \oplus x}_{m} = \underbrace{x + \cdots + x}_{m},$$

where $+$ is the group addition of V, so the desired equality is straightforward. □

Example 5.5. *If X is a compact Hausdorff space, then $C(X)_u = \{f : X \to [0,1] | f \text{ continuous}\}$ is a Riesz MV-algebra and, for any $f \in C(X)_u$, we have*

$$\|f\| = \inf\{r \in [0,1] \mid f(x) \leq r \ \forall x \in X\} = \sup\{f(x) | x \in X\} = \|f\|_\infty.$$

Recall that an *M-space* is a unital Riesz space (V,u) that is norm-complete with respect to the unit norm.

Example 5.6. *If X is a compact Hausdorff space and*

$$C(X) = \{f : X \to \mathbb{R} \mid f \text{ continuous}\},$$

then $(C(X), \mathbf{1})$ is an M-space, where $\mathbf{1}$ is the constant function $\mathbf{1}(x) = 1$ for any $x \in X$.

The above example is fundamental, as proved by Kakutani's representation theorem.

Theorem 5.7. *[17] For any M-space (V,u) there exists a compact Hausdorff space X such that (V,u) is isomorphic with $(C(X), \mathbf{1})$.*

Let us denote by $\mathbb{MU}$ the category of M-spaces with unit-preserving morphisms and by $\mathbb{KH}aus\mathbb{S}p$ the category of compact Hausdorff spaces with continuous maps.

Theorem 5.8. *[2, 16] The category $\mathbb{KH}aus\mathbb{S}p$ is dual to the category $\mathbb{MU}$.*

We characterize in the sequel those Riesz MV-algebras that are, up to isomorphism, unit intervals in M-spaces. Note that, on any Riesz MV-algebra R, we can define

$$\delta_{\|\cdot\|}(x,y) = \|d(x,y)\| \text{ for any } x,\, y \in R.$$

By Lemmas 2.2 and 5.4 it follows that $\delta_{\|\cdot\|}$ is a pseudometric on R.

Definition 5.9. *We say that a Riesz MV-algebra R is* norm-complete *if $(R, \delta_{\|\cdot\|})$ is a complete metric space.*

Theorem 5.10. *If (V,u) is a unital Riesz space then the following are equivalent:*
(i) (V,u) is an M-space,
(ii) $\Gamma_{\mathbb{R}}(V,u)$ is a norm-complete Riesz MV-algebra.

Proof. We denote $R = \Gamma_{\mathbb{R}}(V,u)$.
(i)⇒(ii) By Remark 5.1, $\|x\| = \|x\|_u$ for any $x \in [0,u]$. In consequence, any Cauchy sequence w.r.t $\|\cdot\|$ from $R\,\Gamma_{\mathbb{R}}(V,u)$ is a Cauchy sequence w.r.t $\|\cdot\|_u$ in (V,u) and we use the fact that (V,u) is norm-complete.
(ii)⇒(i) Let $(v_n)_n$ be a Cauchy sequence in V w.r.t. $\|\cdot\|_u$ such that $v_n \geq 0$ for any n. It follows that it is bounded, i.e. there is $y \in V$ and $\|v_n\|_u \leq \|y\|_u$ for any n. We get $v_n \leq \|v_n\|_u u \leq \|y\|_u u$ for any n, so there exists a natural number k such that $v_n \leq ku$ for any n. By Lemma 2.3,

$$v_n = v_{n_1} + \cdots + v_{n_k}, \text{ where } v_{n_i} = ((v_n - (i-1)u) \vee 0) \wedge u \in [0,u]$$

for any $i \in \{1, \ldots, k\}$. Since $v_{n_i} \in [0,u]$ we get

$$\|v_{n_i}\|_u = \|v_{n_i}\| \text{ for any } n \text{ and } i \in \{1, \ldots, k\}.$$

One can easily see that $(v_{n_i})_n$ is a Cauchy sequence in $R = [0,u]$ for any $i \in \{1, \ldots, k\}$. Since R is norm-complete it follows that, for any $i \in \{1, \ldots, k\}$ there is $w_i \in R$ such that $\lim_n d(v_{n_i}, w_i) = \lim_n \|v_{n_i} - w_i\|_u = 0$. If $w = w_1 + \cdots + w_k$ then

$$\|v_n - w\|_u \leq \|v_{n_1} - w_1\|_u + \cdots + \|v_{n_k} - w_k\|_u$$

for any n, so $\lim_n \|v_n - w\| = 0$ and $(v_n)_n$ is convergent w.r.t. $\|\cdot\|_u$ in V.

Recall that $v = (v \vee 0) - ((-v) \vee 0)$ for any $v \in V$, so the convergence of arbitrary Cauchy sequences reduces to the convergence of positive Cauchy sequences. □

Denote by $\mathbb{URMV}$ the category of norm-complete Riesz MV-algebras, which is a full subcategory of $\mathbb{RMV}$. By Remark 5.3, the norm-preserving morphisms coincide with the monomorphisms of $\mathbb{URMV}$.

Using Theorem 5.10, the functor $\Gamma_{\mathbb{R}}$ yields the following categorical equivalence.

Corollary 5.11. *The categories* $\mathbb{URMV}$ *and* $\mathbb{MU}$ *are equivalent.*

Corollary 5.12. *The categories* $\mathbb{URMV}$ *and* $\mathbb{KH}aus\mathbb{S}p$ *are dually equivalent.*

Remark 5.13. *Following [16, Chapter IV] and [2], the functors establishing the above equivalences are defined on objects as follows:*

$$R \mapsto Max(R) \text{ and } X \mapsto C(X)_u$$

for any norm-complete Riesz space R *and compact Hausdorff space* X*, where* $Max(R)$ *is the set of all maximal ideals of* R*.*

In the sequel, we connect our result with the Gelfand-Naimark duality for C*-algebras [14]. Recall that MV-algebras are related with AF C*-algebras in [27], but in this case the K-theory is used.

Denote $\mathbb{C}^*$ the category whose objects are commutative unital C*-algebras and whose morphisms are unital C*-algebra morphisms.

Theorem 5.14. *[18, Chapter 1.1] The categories* $\mathbb{C}^*$ *and* $\mathbb{KH}aus\mathbb{S}p$ *are dually equivalent.*

As a corollary we infer immediately that the categories of commutative unital C*-algebras and norm-complete Riesz MV-algebras are equivalent.

Corollary 5.15. *The categories* $\mathbb{C}^*$ *and* $\mathbb{URMV}$ *are equivalent.*

6 The propositional calculus $\mathbb{R}\mathcal{L}$

We denote by $\mathcal{L}_\infty$ the ∞-valued propositional Łukasiewicz logic. Recall that $\mathcal{L}_\infty$ has $\neg$ (unary) and $\to$ (binary) as primitive connectives and, for any φ and ψ we have:

$\varphi \vee \psi := (\varphi \to \psi) \to \psi$, $\varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi)$,

$\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$.

The language of $\mathbb{R}\mathcal{L}$ contains the language of $\mathcal{L}_\infty$ and a family of unary connectives $\{\nabla_r | r \in [0,1]\}$. We denote by $Form(\mathbb{R}\mathcal{L})$ the set of formulas, which are defined inductively as usual.

Definition 6.1. *An* axiom *of* $\mathbb{R}\mathcal{L}$ *is any formula that is an axiom of* $\mathcal{L}_\infty$ *and any formula that has one of the following forms, where* φ*,* ψ*,* $\chi \in Form(\mathbb{R}\mathcal{L})$ *and* r*,* $q \in [0,1]$*:*

(RL1) $\nabla_r(\varphi \to \psi) \leftrightarrow (\nabla_r\varphi \to \nabla_r\psi)$*;*

(RL2) $\nabla_{(r\odot q^*)}\varphi \leftrightarrow (\nabla_q\varphi \to \nabla_r\varphi)$*;*

(RL3) $\nabla_r\nabla_q\varphi \leftrightarrow \nabla_{(rq)}\varphi$*;*

(RL4) $\nabla_1\varphi \leftrightarrow \varphi$*,*

The deduction rule of $\mathbb{R}\mathcal{L}$ *is* modus ponens *and provability is defined as usual.*

Remark 6.2. *$\{\varphi\} \vdash \nabla_r \varphi$ is a derived deduction rule for any $r \in [0,1]$.*

We recall the usual construction of the Lindenbaum-Tarski algebra. The equivalence relation $\equiv$ is defined on $Form(\mathbb{R}\mathcal{L})$ as follows:

$$\varphi \equiv \psi \text{ iff } \vdash \varphi \to \psi \text{ and } \vdash \psi \to \varphi.$$

We denote by $[\varphi]$ the equivalence class of a formula φ and we define on $Form(\mathbb{R}\mathcal{L})/\equiv$ the following operations:
$[\varphi]^* = [\neg\varphi]$,
$[\varphi] \oplus [\psi] = [\neg\varphi \to \psi]$, $[\varphi] \odot [\psi] = [\neg(\varphi \to \neg\psi)]$,
$0 = [\neg(v_1 \to v_1)]$, $1 = 0^* = [v_1 \to v_1]$.
In order to define the scalar multiplication we introduce new connectives:

$$\Delta_r \varphi := \neg(\nabla_r \neg\varphi)$$

and we set $r[\varphi] = [\Delta_r \varphi]$ for any $r \in [0,1]$ and φ formula of $\mathbb{R}\mathcal{L}$.

Proposition 6.3. *The Lindenbaum-Tarski algebra*

$$RL = (Form(\mathbb{R}\mathcal{L})/\equiv, \cdot, \oplus, ^*, 0)$$

is a Riesz MV-algebra.

Proof. The axioms (RL1)-(RL4) are logical expressions of the duals of (RMV1)-(RMV4). We prove in detail that RL satisfies (RMV1). If φ and ψ are two formulas and $r \in [0,1]$ then, by (RL1), we get

$$[\nabla_r(\neg\varphi \to \neg\psi)] = [\nabla_r \neg\varphi \to \nabla_r \neg\psi].$$

It follows that:

$$\begin{gathered}
[\nabla_r \neg(\neg\varphi \odot \psi)] = [\neg(\nabla_r \neg\varphi \odot \neg\nabla_r \neg\psi)] \\
[\neg\nabla_r \neg(\neg\varphi \odot \psi)] = [\nabla_r \neg\varphi \odot \neg\nabla_r \neg\psi] \\
[\Delta_r(\neg\varphi \odot \psi)] = [\neg\Delta_r \varphi \odot \nabla_r \psi] \\
r([\varphi]^* \odot [\psi]) = (r[\varphi])^* \odot (r[\psi]),
\end{gathered}$$

so (RMV1) holds in RL. □

Let R be an Riesz MV-algebra. An *evaluation* is a function $e : Form(\mathbb{R}\mathcal{L}) \to R$ which satisfies the following conditions for any $\varphi, \psi \in Form(\mathbb{R}\mathcal{L})$ and $r \in [0,1]$:
(e1) $e(\varphi \to \psi) = e(\varphi)^* \oplus e(\psi)$,
(e2) $e(\neg\varphi) = e(\varphi)^*$,
(e3) $e(\nabla_r \varphi) = (re(\varphi)^*)^*$.

As a consequence of Theorem 4.4, the propositional calculus $\mathbb{R}\mathcal{L}$ is complete with respect to $[0,1]$.

Theorem 6.4. *For a formula φ of $\mathbb{R}\mathcal{L}$ the following are equivalent:*
(i) φ is provable in $\mathbb{R}\mathcal{L}$,
(ii) $e(\varphi) = 1$ for any Riesz MV-algebra R and for any evaluation $e : Form(\mathbb{R}\mathcal{L}) \to R$,
(iii) $e(\varphi) = 1$ for any evaluation $e : Form(\mathbb{R}\mathcal{L}) \to [0,1]$.

Remark 6.5. *The system $\mathbb{R}\mathcal{L}$ is a conservative extension of $\mathcal{L}_\infty$, i.e. a formula φ of $\mathcal{L}_\infty$ is a theorem of $\mathcal{L}_\infty$ if and only if it is a theorem of $\mathbb{R}\mathcal{L}$. Since any proof in $\mathcal{L}_\infty$ is also a proof in $\mathbb{R}\mathcal{L}$, one implication is obvious. To prove the other one, assume that φ is a formula of $\mathcal{L}_\infty$ which is not a theorem of $\mathcal{L}_\infty$. Hence there exists an evaluation $e' : Form(\mathcal{L}_\infty) \to [0,1]$ such that $e'(\varphi) \neq 1$. Let $e : Form(\mathbb{R}\mathcal{L}) \to [0,1]$ the unique evaluation in $\mathbb{R}\mathcal{L}$ such that $e(v) = e'(v)$ for any propositional variable v. By structural induction on formulas one can prove that $e(\psi) = e'(\psi)$ for any $\psi \in Form(\mathcal{L}_\infty)$. It follows that $e(\varphi) = e'(\varphi) \neq 1$, so φ is not a theorem of $\mathbb{R}\mathcal{L}$.*

Remark 6.6. *A formula φ with variables from $\{v_1, \dots, v_n\}$ uniquely defines a* term function*:*

$$\widetilde{\varphi} : [0,1]^n \to [0,1],\ \widetilde{\varphi}(x_1, \dots, x_n) = e(\varphi),$$

where e is an evaluation such that $e(v_i) = x_i$ for any $i \in \{1, \dots, n\}$. By Theorem 6.4 it follows that $[\varphi] = [\psi]$ if and only if $\widetilde{\varphi} = \widetilde{\psi}$.

7 Term functions and piecewise linear functions

In the following, we characterize the class of functions that can be defined by formulas in $\mathbb{R}\mathcal{L}$.

Definition 7.1. *Let $n > 1$ be a natural number. A* piecewise linear function *is a function $f : \mathbb{R}^n \to \mathbb{R}$ for which there exists a finite number of affine functions*

$$q_1,\ \dots,\ q_k : \mathbb{R}^n \to \mathbb{R}$$

and for any $(x_1, \dots, x_n) \in \mathbb{R}^n$ there is $i \in \{1, \dots, k\}$ such that $f(x_1, \dots, x_n) = q_i(x_1, \dots, x_n)$. We say that $q_1, \dots, q_k$ are the components *of f.*

We denote byPL_n the set of all continuous functions $f : [0,1]^n \to [0,1]$ that are piecewise linear.

For the rest of the paper, all piecewise linear functions are continuous.

Theorem 7.2. *If φ is a formula of $\mathbb{R}\mathcal{L}$ with propositional variables from $\{v_1, \cdots, v_n\}$ then $\widetilde{\varphi} \in PL_n$.*

Proof. We prove the result by structural induction on formulas.

If φ is v_i for some $i \in \{1, \dots, n\}$ then $\widetilde{\varphi} = \pi_i$ (the i-th projection).

If φ is $\neg\psi$ and $q_1, \dots, q_s$ are the components of $\widetilde{\psi}$, then $1 - q_1, \dots, 1 - q_s$ are the components of $\widetilde{\varphi}$.

Assume φ is $\psi \to \chi$. If $q_1, \dots, q_m$ are the components of $\widetilde{\psi}$ and $p_1, \dots, p_k$ are the components of $\widetilde{\chi}$, then $\widetilde{\varphi}$ is defined by $\{1\} \cup \{s_{ij}\}_{i,j}$, where $s_{ij} = 1 - q_i + p_j$ for any $i \in \{1, \dots, s\}$ and $j \in \{1, \dots, r\}$.

If φ is $\Delta_r\psi$ for some $r \in [0,1]$ and $q_1, \dots, q_s$ are the components of $\widetilde{\psi}$, then $1 - r + rq_1, \dots, 1 - r + rq_s$ are the components of $\widetilde{\varphi}$. $\square$

Remark 7.3. *The continuous piecewise linear functions* $f : [0, 1]^n \to [0, 1]$ *with* integer coefficients *are called* McNaughton functions *and they are in one-one correspondence with the formulas of Łukasiewicz logic by McNaughton theorem [24]. The continuous piecewise linear functions with* rational coefficients *correspond to formulas of Rational Łukasiewicz logic, a propositional calculus developed in [15] that has divisible MV-algebras as models. In Theorem 7.7 we prove that any continuous piecewise linear function with* real coefficients $f : [0, 1]^n \to [0, 1]$ *is the term function of a formula from* $\mathbb{R}\mathcal{L}$.

For now on we define $\varrho : \mathbb{R} \to [0, 1]$ by

$$\varrho(x) = (x \vee 0) \wedge 1 \text{ for any } x \in \mathbb{R}.$$

Lemma 7.4. *For any* $x, y \in \mathbb{R}$ *the following hold:*
(a) $(x \vee 0) + (y \vee 0) \geq (x + y) \vee 0$,
(b) $x \geq 0$ *iff* $\varrho(-x) = 0$,
(c) $\varrho(x) = \varrho(x \vee 0)$.

Proof. (a) $(x \vee 0) + (y \vee 0) = (x + y) \vee x \vee y \vee 0 \geq (x + y) \vee 0$.
(b) $\varrho(-x) = 0$ iff $((-x) \vee 0) \wedge 1 = 0$ iff $(-x) \vee 0 = 0$ iff $-x \leq 0$ iff $x \geq 0$.
(c) $\varrho(x \vee 0) = (x \vee 0 \vee 0) \wedge 1 = (x \vee 0) \wedge 1 = \varrho(x)$. □

In the following we generalize some results from [7, Lemma 3.1.9].

Lemma 7.5. *If* $g : [0, 1]^n \to \mathbb{R}$ *and* $h : [0, 1]^n \to [0, 1]$ *then the following properties hold.*
(a) $\varrho \circ (g + h) = ((\varrho \circ g) \oplus h) \odot (\varrho \circ (g + 1))$.
(b) $\varrho \circ (1 - g) = 1 - (\varrho \circ g)$

Proof. Let $\mathbf{x} = (x_1, \cdots, x_n)$ be an element from $[0, 1]^n$.
(a) If $g(\mathbf{x}) > 1$ then $g(\mathbf{x}) + 1 > 1$ and $g(\mathbf{x}) + h(\mathbf{x}) > 1$. It follows that

$$\varrho(g(\mathbf{x})) = \varrho((g + 1)(\mathbf{x})) = \varrho((g + h)(\mathbf{x})) = 1,$$

so the intended identity is obvious.

If $g(\mathbf{x}) \in [0, 1]$ then $\varrho(g(\mathbf{x})) = g(\mathbf{x})$ and $\varrho((g + 1)(\mathbf{x})) = 1$ for any $\mathbf{x} \in [0, 1]^n$, so

$$\begin{aligned} \varrho((g + h)(\mathbf{x})) &= h(\mathbf{x}) \oplus g(\mathbf{x}) \\ &= h(\mathbf{x}) \oplus \varrho(g(\mathbf{x})) = ((\varrho \circ g) \oplus h)(\mathbf{x}) \\ &= (((\varrho \circ g) \oplus h) \odot 1)(\mathbf{x}) \\ &= (((\varrho \circ g) \oplus h) \odot (\varrho \circ (g + 1)))(\mathbf{x}). \end{aligned}$$

Assume that $g(\mathbf{x}) < 0$, so $\varrho(g(\mathbf{x})) = 0$. We have to prove that $\varrho((g + h)(\mathbf{x})) = h(\mathbf{x}) \odot \varrho((g + 1)(\mathbf{x}))$.

If $g(\mathbf{x}) \leq -1$ then $\varrho(g(\mathbf{x})) = \varrho((g + 1)(\mathbf{x})) = 0$ and $g(\mathbf{x}) + h(\mathbf{x}) \leq -1 + h(\mathbf{x}) \leq 0$, so

$$\varrho((g + h)(\mathbf{x})) = 0 = h(\mathbf{x}) \odot 0 = (h \odot (\varrho \circ (g + 1)))(\mathbf{x}).$$

If $g(\mathbf{x}) \in (-1,0)$ then $\varrho((g+1)(\mathbf{x})) = (g+1)(\mathbf{x}) = g(\mathbf{x})+1$, so we get

$$\begin{aligned}(\varrho \circ (g+h))(\mathbf{x}) \quad &= 0 \vee (1 \wedge (h(\mathbf{x}) + g(\mathbf{x}))) \\ &= 0 \vee (h(\mathbf{x}) + g(\mathbf{x})) \\ &= 0 \vee (h(\mathbf{x}) + g(\mathbf{x}) + 1 - 1) \\ &= h(\mathbf{x}) \odot (g(\mathbf{x}) + 1) \\ &= h(\mathbf{x}) \odot \varrho((g+1)(\mathbf{x})) \\ &= (h \odot (\varrho \circ (g+1)))(\mathbf{x}).\end{aligned}$$

(b) If $g(\mathbf{x}) < 0$ then $\varrho(g(\mathbf{x})) = 0$ and

$$\varrho((1-g)(\mathbf{x})) = 1 = 1 - 0 = 1 - \varrho(g(\mathbf{x})).$$

If $g(\mathbf{x}) \in [0,1]$ then $(1-g)(\mathbf{x}) \in [0,1]$, so

$$\varrho((1-g)(\mathbf{x})) = (1-g)(\mathbf{x}) = 1 - g(\mathbf{x}) = 1 - \varrho(g(\mathbf{x})).$$

If $g(\mathbf{x}) > 1$ then $\varrho(g(\mathbf{x})) = 1$ and

$$\varrho((1-g)(\mathbf{x})) = 0 = 1 - 1 = 1 - \varrho(g(\mathbf{x})).$$

□

Proposition 7.6. *For any affine function $f : [0,1]^n \to \mathbb{R}$ there exists a formula φ of $\mathbb{R}\mathcal{L}$ such that $\varrho \circ f = \widetilde{\varphi}$.*

Proof. Let $f : [0,1]^n \to \mathbb{R}$ be an affine function, i.e. there are $c_0, \ldots, c_n \in \mathbb{R}$ such that

$$f(x_1, \ldots, x_n) = c_n x_n + \cdots + c_1 x_1 + c_0$$

for any $(x_1, \ldots, x_n) \in [0,1]^n$. Note that for any $c \in \mathbb{R}$ there is a natural number m such that $c = r_1 + \cdots + r_m$ where $r_1, \ldots, r_m \in [-1,1]$. Hence we assume that

$$f(x_1, \ldots, x_n) = r_m y_m + \cdots r_{p+1} y_{p+1} + r_p + \cdots + r_1$$

where $m \geq 1$ and $0 \leq p \leq m$ are natural numbers, $r_j \in [-1,1] \setminus \{0\}$ for any $j \in \{1, \ldots, m\}$ and
$y_j \in \{x_1, \ldots, x_n\}$ for any $j \in \{p+1, \cdots, m\}$.

We prove the theorem by induction on $m \geq 1$. Let us denote $\mathbf{x} = (x_1, \cdots, x_n)$ an element from $[0,1]^n$

Initial step $m = 1$. We have $f(\mathbf{x}) = r$ for any $\mathbf{x} \in [0,1]^n$ or $f(\mathbf{x}) = r x_i$ for any $\mathbf{x} \in [0,1]^n$, where $r \in [-1,1] \setminus \{0\}$ and $i \in \{1, \ldots, n\}$. If $r \in [-1,0)$ then $\varrho \circ f = 0$ so $\varrho \circ f = \widetilde{\varphi}$ for $\varphi = v_1 \odot \neg v_1$. If $r \in (0,1]$ then $f = \varrho \circ f$. It follows that $f = \widetilde{\varphi}$ where $\varphi = \nabla_r(v_1 \to v_1)$ if $f(\mathbf{x}) = r$ for any $\mathbf{x} \in [0,1]^n$ and $\varphi = \nabla_r v_i$ if $f(\mathbf{x}) = r x_i$ for any $\mathbf{x} \in [0,1]^n$.

Induction step. We take $f = g + h$ where $\varrho \circ g = \widetilde{\varphi}$ for some formula φ and there are $r \in [-1,1] \setminus \{0\}$ and $i \in \{1, \ldots, n\}$ such that $h(\mathbf{x}) = r$ for any $\mathbf{x} \in [0,1]^n$, or $h(\mathbf{x}) = r x_i$ for any $\mathbf{x} \in [0,1]^n$. We consider two cases.

Case 1. If $r \in (0,1]$ then $h : [0,1]^n \to [0,1]$ so

$$\varrho \circ f = ((\varrho \circ g) \oplus h) \odot (\varrho \circ (1+g))$$

by Lemma 7.5 (a). Following the initial step, there is a formula ψ such that $h = \widetilde{\psi}$. Note that $1+g = 1-(-g)$ and, since the induction hypothesis holds for $(-g)$, there is a formula χ such that $\varrho \circ (-g) = \widetilde{\chi}$. In consequence, by Lemma 7.5 (b), $\varrho \circ (1+g) = 1 - \widetilde{\chi} = \widetilde{\neg\chi}$. We get $\varrho \circ f = \widetilde{\theta}$ where $\theta = (\varphi \oplus \psi) \odot \neg\chi$.

Case 2. If $r \in [-1, 0)$, then $g + h = (g-1) + (1+h)$ and $1 + h : [0,1]^n \to [0,1]$. By Lemma 7.5 (a) we get

$$\varrho \circ f = ((\varrho \circ (g-1)) \oplus (1+h)) \odot (\varrho \circ g).$$

Following the initial step, there is a formula ψ such that

$$-h = \widetilde{\psi}, \text{ so } 1 + h = 1 - (-h) = \widetilde{\neg\psi}.$$

In the sequel we have to find a formula χ that corresponds to $\varrho \circ (g-1)$, where

$$g(\mathbf{x}) = r_m y_m + \cdots r_{p+1} y_{p+1} + r_p + \cdots + r_1$$

with $r_j \in [-1,1] \setminus \{0\}$ for any $j \in \{1, \cdots, m\}$ and $y_j \in \{x_1, \ldots, x_n\}$ for any $j \in \{p+1, \cdots, m\}$.

Case 2.1. If $r_j \leq 0$ for any $j \in \{1, \cdots, m\}$ then
$g - 1 \leq 0$, so $\varrho \circ (g-1) = 0 = \widetilde{\chi}$ with $\chi = v_1 \odot \neg v_1$.

Case 2.2. If there is $j_0 \in \{1, \cdots, p\}$ such that $r_{j_0} > 0$, then it follows that

$$(g-1)(\mathbf{x}) = r_m y_m + \cdots r_{p+1} y_{p+1} + r_p + \cdots + (r_{j_0} - 1) + \cdots + r_1$$

and $r_{j_0} - 1 \in [-1, 0)$, so the induction hypothesis applies to $g-1$. In consequence, there exists a formula χ such that $\varrho \circ (g-1) = \widetilde{\chi}$.

Case 2.3. If there is $j_0 \in \{p+1, \cdots, m\}$ such that $r_{j_0} > 0$, then we set $h_0(\mathbf{x}) = r_{j_0} y_{j_0}$ and

$$g_0(\mathbf{x}) = g(\mathbf{x}) - r_{j_0} y_{j_0} - 1.$$

It follows that $g - 1 = g_0 + h_0$ such that g_0 satisfies the induction hypothesis and $h_0 : [0,1]^n \to [0,1]$. We are in the hypothesis of *Case 1*, so there exists a formula χ such that $\varrho \circ (g-1) = \widetilde{\chi}$.

Summing up, we get $\varrho \circ (g+h) = \widetilde{\theta}$ with $\theta = ((\chi \oplus \neg\psi) \odot \varphi)$. $\square$

Theorem 7.7. *For any $f : [0,1]^n \to [0,1]$ from PL_n there is a formula φ of $\mathbb{R}\mathcal{L}$ such that $f = \widetilde{\varphi}$.*

Proof. Let $f : [0,1]^n \to [0,1]$ be in PL_n. Using the Max-Min representation from [32], there are finite sets I and J such that

$$f = \bigvee_{i \in I} \bigwedge_{j \in J} f_{ij},$$

where $f_{ij} : [0,1]^n \to \mathbb{R}$ are affine functions. We note that

$$f = \varrho \circ f = \bigvee_{i \in I} \bigwedge_{j \in J} (\varrho \circ f_{ij}).$$

By Proposition 7.6, for any $i \in I$ and $j \in J$ there is a formula φ_{ij} such that $\varrho \circ f_{ij} = \widetilde{\varphi_{ij}}$. In consequence, if we set $\varphi = \bigvee_{i \in I} \bigwedge_{j \in J} \varphi_{ij}$ then $f = \widetilde{\varphi}$. $\square$

For any $n \geq 1$, the set PL_n is a Riesz MV-algebra with the operations defined componentwise. If RL_n is the Lindenbaum-Tarski algebra of $\mathbb{R}\mathcal{L}$ defined on formulas with variables from $\{v_1, \ldots, v_n\}$, then RL_n is the free Riesz MV-algebra with n free generators by standard results in universal algebra [4]. Since the function $[\varphi] \mapsto \widetilde{\varphi}$ is obviously an isomorphism between RL_n and PL_n the following corollary is straightforward.

Corollary 7.8. *PL_n is the free Riesz MV-algebra with n free generators.*

8 Linear combinations of formulas and de Finetti's coherence criterion

We recall in the beginning de Finetti's coherence criterion for boolean events. If $S = \{\varphi_1, \ldots, \varphi_k\}$ is a set of classical events then a *book* is a set

$$\{(\varphi_i, r_i) \mid i \in \{1, \ldots, k\}\},$$

where $r_i \in [0, 1]$ is a "betting odd" assigned by a bookmaker for φ_i for any $i \in \{1, \ldots, k\}$. The book is coherent if there is no system of bets $\{c_1, \cdots, c_k\}$ which causes the bookmaker a sure loss. This means that for any real numbers $\{c_1, \cdots, c_k\}$ there exists an evaluation $e : S \to \{0, 1\}$ such that $\sum_{i=1}^{k} c_i(r_i - e(\varphi_i)) \geq 0$. De Finetti's coherence criterion [8] states that the book $\{(\varphi_i, r_i) | i \in \{1, \ldots, k\}\}$ is coherent if there is a boolean probability μ defined on the algebra of events generated by S such that $\mu(\varphi_i) = r_i$ for any $i \in \{1, \ldots, k\}$. When the underlying logic is Łukasiewicz logic [30], the events belongs to an MV-algebra and they are evaluated in $[0, 1]$. Consequently, the coherence criterion uses states instead of boolean probabilities.

We recall the MV-algebraic approach to de Finetti's notion of coherence [19] and we prove a similar coherence criterion for Riesz MV-algebras. We also provide a logical expression of the coherence criterion when the events are represented by formulas $\mathbb{R}\mathcal{L}$. In order to accomplish this task, we initiate the study of linear combinations of formulas in $\mathbb{R}\mathcal{L}$.

The next result characterizes the *states* defined on Riesz MV-algebras.

Lemma 8.1. *If R is a Riesz MV-algebra then any state $s : U(R) \to [0, 1]$ is also homogeneous:*
(s3) $s(r \cdot x) = rs(x)$ for any $r \in [0, 1]$, $x \in R$.

Proof. We can safely assume that $R = [0, u]_V$ for some unital Riesz space (V, u). By [28, Theorem 2.4], there is a state $s' : V \to \mathbb{R}$ such that $s'(x) = s(x)$ for any $x \in [0, u]$.

If $r \in [0,1] \cap \mathbb{Q}$ then $r = \frac{m}{n}$ and $rx = y$ in $[0,u]$ implies $mx = ny$ in V. It follows that $s'(mx) = s'(ny)$, so $ms(x) = ns(y)$ and we get $s(rx) = s(y) = rs(x)$.

If $r \in (0,1)$ there are rational sequences $(r_n)_n$ and $(q_n)_n$ such that $r_n \uparrow r$ and $q_n \downarrow r$. Hence

$$r_n s(x) = s(r_n x) \leq s(rx) \leq s(q_n x) = q_n s(x)$$

for any n and $x \in [0,1]$. The intended result follows by an application of Stolz-Cesàro theorem.

□

Definition 8.2. *If R is a Riesz MV-algebra, a* state *on R is a function $s : R \to [0,1]$ which satisfies the conditions (s1), (s2) and (s3) (additivity, normalization and homogeneity).*

The previous lemma asserts that the states of a Riesz MV-algebra R coincide with the states of its MV-algebra reduct $\mathrm{U}(R)$.

The following definition generalizes de Finetti's notion of coherence and provides an algebraic approach within Łukasiewicz logic [19].

Definition 8.3. *[19] If A is an MV-algebra and $x_1, \ldots, x_k$ are in A then a map $\beta : \{x_1, \ldots, x_n\} \to [0,1]$ is* coherent *if for any $c_1, \cdots, c_k \in \mathbb{R}$ there exists a morphism of MV-algebras $e : A \to [0,1]$ such that*

$$\textstyle\sum_{i=1}^{k} c_i(\beta(x_i) - e(x_i)) \geq 0.$$

Theorem 8.4. *[19, Theorem 3.2] If A is an MV-algebra, $x_1, \ldots, x_k \in A$ and $\beta : \{x_1, \ldots, x_n\} \to [0,1]$ then the following are equivalent:*
(i) the map β is coherent,
(ii) there exists a state $s : A \to [0,1]$ such that

$$s(x_i) = \beta(x_i) \text{ for any } i \in \{1, \ldots, k\},$$

(iii) there exists $e_1, \ldots, e_m : A \to [0,1]$ morphisms of MV-algebras such that $m \leq k+1$ and β is the restriction of a convex combination of $\{e_1, \ldots, e_m\}$.

By Remark 3.12, any morphsim of Riesz MV-algebras is just a morphism between the MV-algebra reducts of its domain and codomain. Therefore, the notion of *coherent map* remains unchanged on Riesz MV-algebras and an analogue of Theorem 8.4 can be proved for Riesz MV-algebras as well.

Corollary 8.5. *If R is a Riesz MV-algebra, $x_1, \ldots, x_k \in R$ and $\beta : \{x_1, \ldots, x_k\} \to [0,1]$ then the following are equivalent:*
(i) the map β is coherent,
(ii) there exists a state $s : R \to [0,1]$ such that

$$s(x_i) = \beta(x_i) \text{ for any } i \in \{1, \ldots, k\},$$

(iii) there exists $e_1, \ldots, e_m : R \to [0,1]$ morphisms such that $m \leq k+1$ and β is the restriction of a convex combination of $\{e_1, \ldots, e_m\}$.

Proof. (i)⇔(iii) and (ii)⇒(i) follow by Theorem 8.4 applied to the MV-algebra reduct of R and by Remark 3.12.
(iii)⇒(ii) There are $\alpha_1, \ldots, \alpha_m \in [0,1]$ such that

$$\alpha_1 + \cdots + \alpha_m = 1 \text{ and}$$
$$\beta(x_i) = \alpha_1 e_1(x_i) + \cdots + \alpha_m e_m(x_i)$$

for any $i \in \{1, \ldots, k\}$. We set $s = \alpha_1 e_1 + \cdots + \alpha_m e_m$ satisfies (s1), (s2) and (s3), so $s : R \to [0,1]$ is the required state. □

The above result is an algebraic version of de Finetti's coherence criterion. In the sequel we provide a logical approach within $\mathbb{R}\mathcal{L}$.

We firstly recall de Finetti's coherence criterion for Łukasiewicz logic $\mathcal{L}_\infty$.

Theorem 8.6. *[30] If $\varphi_1, \ldots, \varphi_k$ are formulas of $\mathcal{L}_\infty$ with variables $v_1, \ldots, v_n$ and $r_1, \ldots, r_k \in [0,1]$ then the following are equivalent:*
(i) the book $\{(\varphi_i, r_i) \mid i \in \{1, \ldots, k\}\}$ is coherent,
(ii) there exists a state $s : L_n \to [0,1]$ such that

$$s([\varphi_i]) = r_i \text{ for any } i \in \{1, \ldots, k\},$$

where L_n is the Lindenbaum-Tarski algebra of the formulas in n variables and $[\varphi]$ is the equivalence class of the formula φ in L_n.

When we consider $\mathbb{R}\mathcal{L}$ instead of $\mathcal{L}_\infty$, we get the following.

Definition 8.7. *If $\varphi_1, \ldots, \varphi_k$ are formulas of $\mathbb{R}\mathcal{L}$ and $r_1, \ldots, r_k \in [0,1]$ then the book $\{(\varphi_i, r_i) \mid i \in \{1, \ldots, k\}\}$ is* coherent *if for any $c_1, \cdots, c_k \in \mathbb{R}$ there exists an evaluation $e : Form(\mathbb{R}\mathcal{L}) \to [0,1]$ such that*

$$\textstyle\sum_{i=1}^k c_i(r_i - e(\varphi_i)) \geq 0.$$

In order to characterize coherence in logical terms within $\mathbb{R}\mathcal{L}$ we introduce the notion of *quasi-linear combination* of piecewise linear functions.

In the sequel we assume that n is a natural number and all the formulas from $\mathbb{R}\mathcal{L}$ have variables from the set $\{v_1, \ldots, v_n\}$. As in the previous section, we use the function

$$\varrho : \mathbb{R} \to [0,1],\ \varrho(x) = (x \vee 0) \wedge 1 \text{ for any } x \in \mathbb{R}.$$

Remark 8.8. *If $f_1, \ldots, f_k : [0,1]^n \to \mathbb{R}$ are continuous piecewise linear functions and $c_1, \ldots, c_k \in \mathbb{R}$ then $\sum_{i=1}^k c_i f_i$ is also a continuous piecewise linear function, so $\varrho \circ (\sum_{i=1}^k c_i f_i)$ is in PL_n. By Theorem 7.7, there exists a formula φ of $\mathbb{R}\mathcal{L}$ such that $\widetilde{\varphi} = \varrho \circ (\sum_{i=1}^k c_i f_i)$. Therefore, we introduce the following definition.*

Definition 8.9. *Let $f_1, \ldots, f_k : [0,1]^n \to \mathbb{R}$ be continuous piecewise linear functions. We say that φ is a* quasi-linear combination *of $f_1, \ldots, f_k$ whenever*

$$\widetilde{\varphi} = \varrho \circ (\sum_{i=1}^{k} c_i f_i)$$

for some $c_1, \ldots, c_k \in \mathbb{R}$. *We define* $qspan(f_1, \ldots, f_k)$ *as the subset of* $Form(\mathbb{R}\mathcal{L})$ *that contains all the quasi-linear combinations of* $f_1, \ldots, f_k$. *If* $\varphi_1, \ldots, \varphi_k$ *are formulas of* $\mathbb{R}\mathcal{L}$ *then* $qspan(\widetilde{\varphi_1}, \ldots, \widetilde{\varphi_k})$ *will be denoted by* $qspan(\varphi_1, \ldots, \varphi_k)$. *If* $\varphi \in qspan(\varphi_1, \ldots, \varphi_k)$ *then we say that* φ *is a* quasi-linear combination of the formulas $\varphi_1, \ldots, \varphi_k$ *in* $\mathbb{R}\mathcal{L}$.

Lemma 8.10. *If* $\varphi, \varphi_1, \ldots, \varphi_k$ *are formulas in* $\mathbb{R}\mathcal{L}$ *such that* $\widetilde{\varphi} = \varrho \circ (\sum_{i=1}^{k} c_i \widetilde{\varphi}_i)$ *for some* $c_1, \ldots, c_k \in \mathbb{R}$ *and* $e : Form(\mathbb{R}\mathcal{L}) \to [0,1]$ *is an evaluation, then*

$$e(\varphi) = \varrho(\sum_{i=1}^{k} c_i e(\varphi_i)).$$

Proof. We have $e(\varphi) = \widetilde{\varphi}(e(v_1), \cdots, e(v_n)) =$
$\varrho(\sum_{i=1}^{k} c_i \widetilde{\varphi}_i(e(v_1), \cdots, e(v_n))) = \varrho(\sum_{i=1}^{k} c_i e(\varphi_i))$. □

Lemma 8.11. *If* $\varphi_1, \ldots, \varphi_k$ *are formulas of* $\mathbb{R}\mathcal{L}$ *and* $r_1, \cdots, r_k \in [0,1]$ *then*

$$\nabla_{r_1}\varphi_1 \oplus \cdots \oplus \nabla_{r_k}\varphi_k \in qspan(\varphi_1, \ldots, \varphi_k).$$

Proof. Under the above hypothesis, we get

$$\varrho(\sum_{i=1}^{k} r_i e(\varphi_i)) = r_1 e(\varphi_1) \oplus \cdots \oplus r_k e(\varphi_k) = e(\nabla_{r_1}\varphi_1) \oplus \cdots \oplus e(\nabla_{r_k}\varphi_k)$$

for any evaluation $e : Form(\mathbb{R}\mathcal{L}) \to [0,1]$. □

Definition 8.12. *We say that a formula* φ *of* $\mathbb{R}\mathcal{L}$ *is* invalid *if there exists an evaluation* $e : Form(\mathbb{R}\mathcal{L}) \to [0,1]$ *such that* $e(\varphi) = 0$

Theorem 8.13. *Let* $\varphi_1, \ldots, \varphi_k$ *are formulas of* $\mathbb{R}\mathcal{L}$ *and* $r_1, \cdots, r_k \in [0,1]$. *The following are equivalent:*
(i) the book $\{(\varphi_i, r_i) \mid i \in \{1, \ldots, k\}\}$ *is coherent,*
(ii) there exists a state $s : RL_n \to [0,1]$ *such that* $s([\varphi_i]) = r_i$ *for any* $i \in \{1, \ldots, k\}$,
(iii) $qspan(\widetilde{\varphi_1} - r_1, \ldots, \widetilde{\varphi_k} - r_k)$ *is a set of invalid formulas of* $\mathbb{R}\mathcal{L}$.

Proof. (i)⇔(ii) Apply Corollary 8.5 to $\beta([\varphi_i]) = r_i$ for any $i \in \{1, \ldots, k\}$.
(i)⇔(iii) The following facts are equivalent:
(1) $qspan(\widetilde{\varphi_1} - r_1, \ldots, \widetilde{\varphi_k} - r_k)$ is a set of invalid formulas,
(2) for any $\Psi \in qspan(\widetilde{\varphi_1} - r_1, \ldots, \widetilde{\varphi_k} - r_k)$ there exists an evaluation $e : Form(\mathbb{R}\mathcal{L}) \to [0,1]$ such that $e(\Psi) = 0$,
(3) for any $c_1, \ldots, c_k \in \mathbb{R}$, if $\widetilde{\Psi} = \varrho \circ (\sum_{i=1}^{k} c_i(\widetilde{\varphi}_i - r_i))$ then there exists an evaluation $e : Form(\mathbb{R}\mathcal{L}) \to [0,1]$ such that $e(\Psi) = 0$,
(4) for any $c_1, \ldots, c_k \in \mathbb{R}$, there exists an evaluation $e : Form(\mathbb{R}\mathcal{L}) \to [0,1]$ such that

$$\varrho \circ (\sum_{i=1}^{k} c_i(e(\varphi_i) - r_i) = 0,$$

(5) for any $c_1, \ldots, c_k \in \mathbb{R}$, there exists an evaluation $e : Form(\mathbb{R}\mathcal{L}) \to [0,1]$ such that

$$\sum_{i=1}^{k} c_i(r_i - e(\varphi_i)) \geq 0.$$

Note that by Lemma 8.10 is used for (3)⇔(4) and Lemma 7.4 (b) is used for (4)⇔(5).

□

If $r \in [0,1]$ and φ a formula then the piecewise linear function $r - \widetilde{\varphi} : [0,1]^n \to \mathbb{R}$ may have negative values, therefore it may not correspond to a formula of $\mathbb{R}\mathcal{L}$. The next result provides a necessary condition for a book to be coherent using quasi-linear combinations of formulas. We also prove a sufficient condition in Corollary 8.16, but using different formulas.

We set $\mathbf{r} = \Delta_r(\varphi \to \varphi)$ and $\varphi \ominus \psi = \varphi \odot \neg\psi$ whenever $r \in [0,1]$ and φ, $\psi \in Form(\mathbb{R}\mathcal{L})$. Note that $\chi = \varphi \ominus \psi$ implies $\widetilde{\chi} = 0 \vee (\widetilde{\varphi} - \widetilde{\psi})$.

Proposition 8.14. *Assume $\varphi_1, \ldots, \varphi_k$ are formulas of $\mathbb{R}\mathcal{L}$ and $r_1, \cdots, r_k \in [0,1]$ such that the book $\{(\varphi_i, r_i) \mid i \in \{1,\ldots,k\}\}$ is coherent. Hence*

$$qspan((\mathbf{r}_1 \ominus \varphi_1), \ldots, (\mathbf{r}_k \ominus \varphi_k))$$

is a set of invalid formulas.

Proof. Let $\chi \in qspan((\mathbf{r}_1 \ominus \varphi_1), \ldots, (\mathbf{r}_k \ominus \varphi_k))$ and $c_1, \cdots, c_k \in \mathbb{R}$ such that

$$\widetilde{\chi} = \varrho \circ (\textstyle\sum_{i=1}^{k} c_i(0 \vee (r_i - \widetilde{\varphi}_i))).$$

Since the book $\{(\varphi_i, r_i) \mid i \in \{1,\ldots,k\}\}$ is coherent there is an evaluation e such that

$\sum_{i=1}^{k}(-c_i)(r_i - e(\varphi_i)) \geq 0$, which implies that $\sum_{i=1}^{k}(-c_i)((r_i - e(\varphi_i)) \vee 0) \geq 0$.

We get $\sum_{i=1}^{k}(-c_i)e(\mathbf{r}_i \ominus \varphi_i) \geq 0$. By Lemma 7.4 (b), $\varrho(\sum_{i=1}^{k} c_i e(\mathbf{r}_i \ominus \varphi_i)) = 0$. Using Lemma 8.10 it follows that $e(\chi) = 0$, so χ is an invalid formula. □

If $\varphi \in Form(\mathbb{R}\mathcal{L})$, $r \in [0,1]$ and $c \in \mathbb{R}$ we denote

$$\psi(\varphi, r, c) = \begin{cases} \varphi \ominus \mathbf{r}, & \text{if } c \geq 0 \\ \mathbf{r} \ominus \varphi, & \text{if } c < 0. \end{cases}$$

Proposition 8.15. *Assume $\varphi_1, \ldots, \varphi_k \in Form(\mathbb{R}\mathcal{L})$ and $r_1, \cdots, r_k \in [0,1]$ such that for any $c_1, \ldots, c_k \in \mathbb{R}$ the formula Φ of $\mathbb{R}\mathcal{L}$ is invalid whenever $\widetilde{\Phi} = \varrho \circ (\sum_{i=1}^{k} |c_i|\widetilde{\psi}_i)$, where $|c_i|$ is the module of c_i and $\psi_i = \psi(\varphi_i, r_i, c_i)$ for any $i \in \{1,\ldots,k\}$. Then the book $\{(\varphi_i, r_i) \mid i \in \{1,\ldots,k\}\}$ is coherent.*

Proof. If $c_1, \cdots, c_k \in \mathbb{R}$ and Φ in $Form(\mathbb{R}\mathcal{L})$ is an invalid formula such that $\widetilde{\Phi} = \varrho \circ (\sum_{i=1}^{k} |c_i|\widetilde{\psi}_i)$, then there exists an evaluation e such that $e(\Phi) = 0$ and, by Lemma 8.10, we get $\varrho(\sum_{i=1}^{k} |c_i| e(\psi_i)) = 0$, so

$$\varrho\left(\begin{array}{l} \sum_{c_i \geq 0} c_i((e(\varphi_i) - r_i) \vee 0) + \\ \sum_{c_i < 0}(-c_i)((r_i - e(\varphi_i)) \vee 0) \end{array} \right) = 0$$

By Lemma 7.4(a),

$$\varrho((\sum_{c_i\geq 0} c_i(e(\varphi_i)-r_i)+\sum_{c_i<0}(-c_i)(r_i-e(\varphi_i)))\vee 0)=0$$

and, by Lemma 7.4(c),

$$\varrho(\sum_{c_i\geq 0} c_i(e(\varphi_i)-r_i)+\sum_{c_i<0}(-c_i)(r_i-e(\varphi_i)))=0.$$

Using Lemma 7.4 (b), we get

$$-(\sum_{c_i\geq 0} c_i(e(\varphi_i)-r_i)+\sum_{c_i<0}(-c_i)(r_i-e(\varphi_i)))\geq 0.$$

It follows that $\sum_{i=1}^{k} c_i(r_i-e(\varphi_i))\geq 0$. $\square$

Corollary 8.16. *Assume* $\varphi_1, \ldots, \varphi_k \in Form(\mathbb{R}\mathcal{L})$ *and* $r_1, \cdots, r_k \in [0,1]$ *such that*

$$qspan(\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_k)$$

is a set of invalid formulas, where

$$\alpha_i=\mathbf{r}_i\ominus\varphi_i \text{ and } \beta_i=\varphi_i\ominus\mathbf{r}_i \text{ for any } i\in\{1,\ldots,k\}.$$

Then the book $\{(\varphi_i,r_i)\mid i\in\{1,\ldots,k\}\}$ *is coherent.*

Proof. For any $c_1, \cdots, c_k \in \mathbb{R}$, if Φ is a formula of $\mathbb{R}\mathcal{L}$ such that $\widetilde{\Phi}=\varrho\circ\sum_{i=1}^{k}|c_i|\psi_i$ as in Proposition 8.15, then $\Phi\in qspan(\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_k)$. In consequence, we can apply Proposition 8.15. $\square$

Remark 8.17. *We initiate the theory of quasi-linear combinations in* $\mathbb{R}\mathcal{L}$ *and we relate it with de Finetti's notion of coherence, which can be expressed by an invalidity condition. The linear combinations of formulas from Łukasiewicz logic were approached in [5] and [1], as representations for a particular class of neural networks. The composition between the function* ϱ *and a linear combination of formulas Łukasiewicz logic can be naturally represented by a formula in our logic* $\mathbb{R}\mathcal{L}$ *and, therefore, the theory of linear combinations can be approached within a simple defined logical system. Note that* $\mathbb{R}\mathcal{L}$ *is a conservative extension of Łukasiewicz logic. It has standard completeness theorem with respect to* $[0,1]$ *and it is supported by the algebraic theory of Riesz MV-algebras which are categorically equivalent with unital Riesz spaces. Hence, our hope for the future is that the system* $\mathbb{R}\mathcal{L}$ *is enough expressive for representing classes of neural networks in a pure logical frame, having in mind the role of classical logic in the synthesis and analysis of boolean circuits.*

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