# ON A NEW CONSTRUCTION OF PSEUDO EFFECT ALGEBRAS 

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#### Abstract

We define a new class of pseudo effect algebras, called kite pseudo effect algebras, which is connected not necessarily with partially ordered groups, but rather with generalized pseudo effect algebras where the greatest element is not guaranteed. Starting even with a commutative generalized pseudo effect algebra, we can obtain a non-commutative pseudo effect algebra. We show how such kite pseudo effect algebras are tied with different types of the Riesz Decomposition Properties. We find conditions when kite pseudo effect algebras have the least non-trivial normal ideal.


## 1. Introduction

A basic algebraic structure describing events observing during the measuring process in quantum mechanics is an effect algebra introduced in FoBe. Such algebras are partial algebras with a primary notion + which means that $a+b$ denotes the disjunction of mutually excluding events $a$ and $b$. This class was inspired by an algebraic counterpart of so-called POV-measures (positive operator-valued measures). The orthodox example of effect algebras is the class $\mathcal{E}(H)$ of Hermitian operators between the zero, $O$, and identity operator, $I$, acting on a Hilbert space $H$. Effect algebras are intensively studied during the last 20 years because they generalize Boolean algebras, orthomodular lattices and posets, and orthoalgebras.

In many important examples, an effect algebra is an interval $[0, u]$ in the positive cone of an Abelian partially ordered group (=po-group). This is true e.g. if (1) $\mathcal{B}(H)$ is the system of Hermitian operators of a Hilbert space $H$, then $\mathcal{E}(H)$ is the interval $[O, I]$ in $\mathcal{B}(H)$, or (2) if the effect algebra satisfies the Riesz Decomposition Property (RDP for short), c.f. Rav.

A more general structure with a partially defined operation + is a generalized effect algebra where the existence of the greatest element 1 is not a priori guaranteed. Such an example is the set of all positive Hermitian operators $\mathcal{B}(H)^{+}$of a Hilbert space $H$. Every generalized effect $E$ algebra can be embedded into the unitization of $E$, i.e. into the effect algebra $E \uplus \bar{E}$, where $E$ is a lower part and $\bar{E}$

[^0]is a copy of $E$ with the reverse order in the upper part, and $\uplus$ denotes the ordinal sum.

In the Nineties, the assumption that addition + is commutative was canceled in DvVe1, DvVe2, and pseudo effect algebras were introduced as a non-commutative generalization of effect algebras. They have been published in physical journals. Some physical motivation for pseudo effect algebras with possible physical situations in quantum mechanics were presented in DvVe6. In some important examples they are also intervals in po-groups that are not necessarily commutative. Also in physics, there are important structures with non-commutative operations as for example, multiplication of matrices is a non-commutative operation and matrices are frequently used in mathematical physics. In particular, the class of quadratic matrices of the form

$$
A(a, b)=\left(\begin{array}{ll}
a & b \\
0 & 1
\end{array}\right)
$$

for $a>0, b \in(-\infty, \infty)$ with usual multiplication of matrices is a non-commutative linearly ordered group with the neutral element $A(1,0)$ and with the positive cone consisting of matrices $A(a, b)$ with $a>1$ or $a=1$ and $b \geq 0$.

Similarly, generalized pseudo effect algebras were introduced in DvVe4, DvVe5 as partial algebras with partial addition + where the top element 1 can fail. If a pseudo effect algebra satisfies a stronger type of the Riesz Decomposition Property, $\mathrm{RDP}_{1}$, then it is again an interval in a po-group (not necessarily Abelian) with strong unit, DvVe1, DvVe2. If a generalized pseudo effect algebra satisfies also $\mathrm{RDP}_{1}$, by [DvVe5], it can be embedded into the positive cone of a po-group with $\mathrm{RDP}_{1}$. We note that $\mathrm{RDP}_{1}$ for effect algebras coincides with RDP, but for pseudo effect algebras they can be different.

Recently in Dvu4, there was introduced a new construction of pseudo effect algebras starting from a partially ordered group $G$ with the positive cone and negative cone $G^{+}$and $G^{-}$, respectively, using an index set $I$ and two bijective mapping $\lambda, \rho: I \rightarrow I$. The universes of these algebras are of the form $\left(G^{+}\right)^{I}$ down and $\left(G^{-}\right)^{I}$ up and the addition between two sequences $\left\langle x_{i}: i \in I\right\rangle$ and $\left\langle y_{j}: j \in I\right\rangle$ is ruled by properties of $\lambda$ and $\rho$. Therefore, even starting with a commutative po-group, e.g. with the group $\mathbb{Z}$ of integers, the resulting algebra is not necessarily commutative. The basic properties of such pseudo effect algebras, called kite pseudo effect algebras, are presented in Dvu4, DvHo.

In the present paper we generalize the construction of pseudo effect algebras when we change $G^{+}$to a generalized pseudo effect algebra because $G^{+}$is an example of generalized pseudo effect algebras. It is interesting that the original construction works also for this case with necessary specifications, therefore, some present proofs resemble original ones from $\mathrm{Dvu} 4, \mathrm{DvHo}$. If $I$ is a singleton, then our construction corresponds to the so-called unitization studied e.g. in XLGRD, when $E$ is used as a lower part and $\bar{E}$ (a copy of $E$ with the reverse order as original one in $E$ ) is an upper part of the pseudo effect algebra $E \uplus \bar{E}$.

The main aim of the paper is to introduce this new construction of pseudo effect algebras because every theory is as so good as it contains the largest reservoir of important examples. Similarly as in Dvu4, DvHo, we present the basic construction with the fundamental properties of kite pseudo effect algebras. We concentrate to a relation between the Riesz Decomposition Properties of the original generalized pseudo effect algebra and of the resulting kite pseudo effect algebra. Finally,
we show also cases when the kite pseudo effect algebra with $\mathrm{RDP}_{1}$ is subdirectly irreducible (equivalently, it contains the smallest non-trivial normal ideal).

The paper is organized as follows. In Section 1 we gather basic definitions and notions from the theory of pseudo effect algebras and generalized pseudo effect algebras, theory of partially ordered groups. Section 3 introduces the construction of kite pseudo effect algebras starting with a generalized pseudo effect algebra. In addition, it studies also different types of the Riesz Decomposition Property. Section 4 concentrates to a description of subdirect irreducible kite pseudo effect algebras using the subdirect irreducibility of the original generalized pseudo effect algebra.

## 2. Basic Definitions and Properties

In the present section, we gather the necessary notions from theory of pseudo effect algebra and partially ordered groups which we will use in the paper.

According to DvVe1, DvVe2, we say that a pseudo effect algebra is a partial algebra $E=(E ;+, 0,1)$, where + is a partial binary operation and 0 and 1 are constants, such that for all $a, b, c \in E$, the following holds
(i) $a+b$ and $(a+b)+c$ exist if and only if $b+c$ and $a+(b+c)$ exist, and in this case $(a+b)+c=a+(b+c)$;
(ii) there is exactly one $d \in E$ and exactly one $e \in E$ such that $a+d=e+a=1$;
(iii) if $a+b$ exists, there are elements $d, e \in E$ such that $a+b=d+a=b+e$;
(iv) if $1+a$ or $a+1$ exists, then $a=0$.

If we define $a \leq b$ if and only if there exists an element $c \in E$ such that $a+c=b$, then $\leq$ is a partial ordering on $E$ such that $0 \leq a \leq 1$ for all $a \in E$. It is possible to show that $a \leq b$ if and only if $b=a+c=d+a$ for some $c, d \in E$. We write $c=a / b$ and $d=b \backslash a$. Then

$$
(b \backslash a)+a=b=a+(a / b)
$$

and we write $a^{-}=1 \backslash a$ and $a^{\sim}=a / 1$ for all $a \in E$. Then $a^{-}+a=1=a+a^{\sim}$ and $a^{-\sim}=a=a^{\sim-}$ for all $a \in E$.

We recall that a po-group ( = partially ordered group) is a group $G=(G ;+,-, 0)$ endowed with a partial order $\leq$ such that if $a \leq b, a, b \in G$, then $x+a+y \leq x+b+y$ for all $x, y \in G$. We denote by $G^{+}:=\{g \in G: g \geq 0\}$ the positive cone of $G$. If, in addition, $G$ is a lattice under $\leq$, we call it an $\ell$-group ( $=$ lattice ordered group). An element $u \in G^{+}$is said to be a strong unit (or an order unit) if, given $g \in G$, there is an integer $n \geq 1$ such that $g \leq n u$. The pair $(G, u)$, where $u$ is a fixed strong unit of $G$, is said to be a unital po-group. We recall that the lexicographic product of two po-groups $G_{1}$ and $G_{2}$ is the group $G_{1} \times G_{2}$, where the group operations are defined by coordinates, and the ordering $\leq$ on $G_{1} \times G_{2}$ is defined as follows: For $\left(g_{1}, h_{1}\right),\left(g_{2}, h_{2}\right) \in G_{1} \times G_{2}$, we have $\left(g_{1}, h_{1}\right) \leq\left(g_{2}, h_{2}\right)$ whenever $g_{1}<g_{2}$ or $g_{1}=g_{2}$ and $h_{1} \leq h_{2}$. For more information on po-group, see Dar, Fuc.

We denote by $\mathbb{Z}$ the commutative $\ell$-group of integers.
Let $G$ be a po-group and fix an element $u \in G^{+}$. If we set $\Gamma(G, u):=[0, u]=$ $\{g \in G: 0 \leq g \leq u\}$, then $\Gamma(G, u)=(\Gamma(G, u) ;+, 0, u)$ is a pseudo effect algebra, where + is the restriction of the group addition + to $[0, u]$, i.e. $a+b$ is defined in $\Gamma(G, u)$ for $a, b \in \Gamma(G, u)$ iff $a+b \in \Gamma(G, u)$. Then $a^{-}=u-a$ and $a^{\sim}=-a+u$ for all $a \in \Gamma(G, u)$. A pseudo effect algebra which is isomorphic to some $\Gamma(G, u)$ for some po-group $G$ with $u>0$ is said to be an interval pseudo effect algebra.

If + is commutative, i.e. $a+b$ is defined in $E$ iff $b+a$ is defined in $E$ and $a+b=b+a$, then $E$ is an effect algebra in the sense of [FoBe]. For more information on effect algebras, we recommend DvPu .

A pseudo effect algebra $E$ is said to be symmetric if $a^{-}=a^{\sim}$ for all $a \in E$. We note that if $E$ is symmetric, then $E$ is not automatically an effect algebra. Indeed, if $G$ is a po-group that is not Abelian, then for the lexicographic product $\mathbb{Z} \overrightarrow{\times} G$ of the po-group $\mathbb{Z}$ with $G$ we have $E=\Gamma(\mathbb{Z} \overrightarrow{\times} G,(1,0))$ is a symmetric pseudo effect algebra that is not an effect algebra.

A more general structure than pseudo effect algebras is the class of generalized pseudo effect algebras introduced in $\overline{D v V e 4}$, $D v V e 5$. A structure ( $E ;+, 0$ ), where + is a partial binary operation and 0 is a constant, is called a generalized pseudoeffect algebra (or a GPEA for short) if, for all $a, b, c \in E$, the following hold:
(GP1) $a+b$ and $(a+b)+c$ exist if and only if $b+c$ and $a+(b+c)$ exist, and in this case, $(a+b)+c=a+(b+c)$;
(GP2) if $a+b$ exists, there are elements $d, e \in E$ such that $a+b=d+a=b+e$;
(GP3) if $a+b$ and $a+c$ exist and are equal, then $b=c$. If $b+a$ and $c+a$ exist and are equal, then $b=c$;
(GP4) if $a+b$ exists and $a+b=0$, then $a=b=0$;
(GP5) $a+0$ and $0+a$ exist and both are equal to $a$.
A GPEA $E$ is trivial if $E=\{0\}$ and it is non-trivial if $|E| \geq 2$.
In the same way as for pseudo effect algebras, we introduce a binary relation $\leq$ in a GPEA $E$ : For $a, b \in E$, we define $a \leq b$ if and only if there is an element $c \in E$ such that $a+c=b$. Equivalently, there exists an element $d \in E$ such that $d+a=b$. Then $\leq$ is a partial order on $E$.

If the partial operation + on $E$ is commutative, then a GPEA $E$ is said to be a generalized effect algebra, GEA for short.

We introduce also two partial binary operations $\backslash$ and / on a GPEA $E$ in the same way as for pseudo effect algebras: For any $a, b \in E, a / b$ is defined if and only if $b \backslash a$ is defined if and only if $a \leq b$, and in such a case we have $(b \backslash a)+a=b=a+(a / b)$. Then $a=(b \backslash a) / b=b \backslash(a / b)$.

For example, if $G$ is a po-group and $G^{+}:=\{g \in G: g \geq 0\}$ is the positive cone of $G$, then $\left(G^{+} ; 0,+\right)$ is a GPEA where + is the restriction of the group addition + in $G$ to $G^{+}$in the obvious sense. Similarly, let $G_{0}$ be a non-empty subset of $G^{+}$ such that for all $a, b \in G_{0}$, where $b \leq a$, also $a-b,-b+a \in G_{0}$. Then $\left(G_{0} ;+, 0\right)$, where + is the group addition restricted to those pairs of elements of $G_{0}$ whose sum is again in $G_{0}$, is a GPEA, see DvVe5, Ex. 2.3]. In particular, if $u \in G^{+}$, $u>0$, and $[0, u):=\{g \in G: 0 \leq g<u\}$, then $([0, u) ;+0)$ is a GPEA which has no top element, therefore, it is not a pseudo effect algebra. If $E$ is a PEA, then the same is true for $(E \backslash\{1\} ;+, 0)$. (In the latter two cases, + is the restriction of the original addition in the natural sense.)

Let $E, F$ be GPEAs. A mapping $h: E \rightarrow F$ is said to be a homomorphism of PGEAs if $h(a+b)=h(a)+h(b)$ whenever $a+b$ is defined in $E$; we note that $h(0)=0$. If $E, F$ are PEAs, then a mapping $h: E \rightarrow F$ such that (i) $h(a+b)=h(a)+h(b)$ whenever $a+b$ is defined in $E$, and (ii) $h(1)=1$ is said to be a homomorphism of PEAs. If $h$ and $h^{-1}$ are homomorphisms, then $h$ is said to be an isomorphism.

A subset $A$ of a GPEA $E$ is a sub-GPEA of $E$ if (i) $0 \in A$, and (ii) if from three elements $x, y, z \in E$ such that $x+y=z$ at least two are in $A$, then all $x, y, z \in A$.

If $E$ is a PEA, then $A \subseteq E$ is a sub-PEA of $E$ iff (i) $1 \in A$, (ii) $a \in A$ implies $a^{-}, a^{\sim} \in A$, (iii) if $a, b \in A$ and $c=a+b$, then $c \in A$.

We remind that a subset $P_{0}$ of a poset $P$ is called convex if, for any two elements $a, b \in P_{0}$ and any $c \in P$ such that $a \leq c \leq b$, we have $c \in P_{0}$. We note that if $G$ is a po-group and $G_{0}$ is a convex subset of $G^{+}$containing 0 , then $\left(G_{0} ;+, 0\right)$ is a GPE-algebra.

We recall that a poset $P$ is said to be directed (more precisely upwards directed) if, for all $a, b \in P$, there is an element $c \in P$ such that $a, b \leq c$.

By XLGRD, a GPEA or a PEA $E$ is said to be weakly commutative if $x+y$ is defined in $E$ iff $y+x$ is defined in $E$. For example, if $G$ is a po-group that is not Abelian, then $G^{+}$is a weakly commutative GPEA that is not commutative. It is easy to show that a pseudo effect algebra $E$ is weakly commutative if and only if $E$ is symmetric.

We say that a GPEA $E$ satisfies
(i) the Riesz Interpolation Property (RIP for short) if, for $a_{1}, a_{2}, b_{1}, b_{2} \in E$, $a_{1}, a_{2} \leq b_{1}, b_{2}$ implies there exists an element $c \in E$ such that $a_{1}, a_{2} \leq c \leq$ $b_{1}, b_{2}$;
(ii) $\mathrm{RDP}_{0}$ if, for $a, b, c \in E, a \leq b+c$, there exist $b_{1}, c_{1} \in E$, such that $b_{1} \leq b$, $c_{1} \leq c$ and $a=b_{1}+c_{1} ;$
(iii) RDP if, for all $a_{1}, a_{2}, b_{1}, b_{2} \in E$ such that $a_{1}+a_{2}=b_{1}+b_{2}$, there are four elements $c_{11}, c_{12}, c_{21}, c_{22} \in E$ such that $a_{1}=c_{11}+c_{12}, a_{2}=c_{21}+c_{22}$, $b_{1}=c_{11}+c_{21}$ and $b_{2}=c_{12}+c_{22}$; this property will be formally denoted by the following table:

| $a_{1}$ | $c_{11}$ | $c_{12}$ |
| :---: | :---: | :---: |
| $a_{2}$ | $c_{21}$ | $c_{22}$ |
|  | $b_{1}$ | $b_{2}$ |$;$

(iv) $\operatorname{RDP}_{1}$ if, for all $a_{1}, a_{2}, b_{1}, b_{2} \in E$ such that $a_{1}+a_{2}=b_{1}+b_{2}$, there are four elements $c_{11}, c_{12}, c_{21}, c_{22} \in E$ such that $a_{1}=c_{11}+c_{12}, a_{2}=c_{21}+c_{22}$, $b_{1}=c_{11}+c_{21}$ and $b_{2}=c_{12}+c_{22}$, and $0 \leq x \leq c_{12}$ and $0 \leq y \leq c_{21}$ imply $x+y=y+x$
(v) $\mathrm{RDP}_{2}$ if, for all $a_{1}, a_{2}, b_{1}, b_{2} \in E$ such that $a_{1}+a_{2}=b_{1}+b_{2}$, there are four elements $c_{11}, c_{12}, c_{21}, c_{22} \in E$ such that $a_{1}=c_{11}+c_{12}, a_{2}=c_{21}+c_{22}$, $b_{1}=c_{11}+c_{21}$ and $b_{2}=c_{12}+c_{22}$, and $c_{12} \wedge c_{21}=0$.
If, for $a, b \in E$, we have for all $0 \leq x \leq a$ and $0 \leq y \leq b, x+y=y+x$, we denote this property by $a \operatorname{com} b$.

If we change a GPEA $E$ to a positive cone $G^{+}$of a po-group $G$, we say that $G$ satisfies the analogous type of the Riesz Decomposition Property.

By [DvVe1, Prop 4.2] for directed po-groups, we have

$$
\mathrm{RDP}_{2} \Rightarrow \mathrm{RDP}_{1} \quad \Rightarrow \mathrm{RDP} \quad \Rightarrow \mathrm{RDP}_{0} \quad \Leftrightarrow \quad \mathrm{RIP}
$$

but the converse implications do not hold, in general. A directed po-group $G$ satisfies $\mathrm{RDP}_{2}$ iff $G$ is an $\ell$-group, DvVe1, Prop 4.2(ii)].

The fundamental result of theory of pseudo effect algebras, which is a bridge between pseudo effect algebras satisfying $\mathrm{RDP}_{1}$ and the class of unital po-groups satisfying $\mathrm{RDP}_{1}$, is the following representation theorem [DvVe2, Thm 7.2]:

Theorem 2.1. For every pseudo effect algebra $E$ with $\mathrm{RDP}_{1}$, there is a unique (up to isomorphism of unital po-groups) unital po-group ( $G, u$ ) with $\mathrm{RDP}_{1}$ such that $E \cong \Gamma(G, u)$.

In addition, $\Gamma$ defines a categorical equivalence between the category of pseudo effect algebras with $\mathrm{RDP}_{1}$ and the category of unital po-groups with $\mathrm{RDP}_{1}$.

A similar result as Theorem 2.1, Theorem 2.2 below, holds also for directed PEAs, see DvVe5, Thm 4.8, Prop 5.3, Thm 6.4]. To show this, we need the following notions. By the couple $\left(G, G_{0}\right)$ we mean a directed po-group $G$ with a fixed directed convex subset $G_{0}$ of the positive cone such that $0 \in G_{0}$ and $G_{0}$ generates $G$ as a po-group. We note that $G_{0}$ is always a GPEA which is a subalgebra of $G^{+}$. If $u$ is a strong unit of $G$, then $(G,[0, u])$ is such an example. $\left(G, G_{0}\right)$ and $\left(H, H_{0}\right)$ are isomorphic, if there is a po-group isomorphism $f: G \rightarrow H$ such that $f\left(G_{0}\right)=H_{0}$.

By $\mathcal{G} \mathcal{P E} \mathcal{A}$ we understand the category whose objects are directed GPEAs with $\mathrm{RDP}_{1}$ and morphisms are homomorphisms of GPEAs. By $\mathcal{P O G}$ we mean the category whose objects are couples $\left(G, G_{0}\right)$, where $G$ is a directed po-group with $\mathrm{RDP}_{1}$ with a fixed directed convex subset $G_{0} \subseteq G^{+}, 0 \in G_{0}, G_{0}$ generates $G$, and morphisms from $\left(G, G_{0}\right)$ into $\left(H, H_{0}\right)$ are homomorphisms $h: G \rightarrow H$ of po-groups such that $h\left(G_{0}\right) \subseteq H_{0}$. Now by $\Gamma\left(G, G_{0}\right)$ we mean the GPEA $G_{0}:=\left(G_{0} ;+, 0\right)$ as defined above.

Theorem 2.2. For every directed generalized pseudo effect algebra $E$ with $\mathrm{RDP}_{1}$, there is a unique couple $\left(G, G_{0}\right)$ (up to isomorphism), where $G$ is a directed pogroup with $\mathrm{RDP}_{1}$ with a fixed directed convex subset $G_{0} \subseteq G^{+}$generating $G$ such that $E \cong G_{0}$.

In addition, $\Gamma$ defines a categorical equivalence between the category $\mathcal{G} \mathcal{P E} \mathcal{A}$ of pseudo effect algebras with $\mathrm{RDP}_{1}$ and the category $\mathcal{P O G}$ of directed po-groups with $\mathrm{RDP}_{1}$.

We note that in the later theorem, $G_{0}$ is a directed GPEA which is a subalgebra of the GPEA $G^{+}$.

We finish this section with a note that a PEA $E$ satisfies $\mathrm{RDP}_{2}$ iff $E$ is a lattice and, for all $a, b \in E$, we have $a \backslash(a \wedge b)=(a \vee b) \backslash a$ and $a /(a \vee b)=(a \wedge b) / a$, DvVe2, Sect 8]. In addition, there is a unique unital (up to isomorphism of unital $\ell$-groups) $\ell$-group $(G, u)$ such that $E \cong \Gamma(G, u)$, see Dvu1.

## 3. Kite Pseudo Effect Algebras

In this section we define kite pseudo effect algebras starting with a GPEA. We show when this kind of pseudo effect algebras satisfies different types of the Riesz Decomposition Properties.

Let $E$ be a generalized pseudo effect algebra. We denote by $\bar{E}$ an identical copy of $E$ whose elements are of the form $\bar{a}$ for all $a \in E$, that is $\bar{E}:=\{\bar{a}: a \in E\}$. We assume that $\bar{a}=\bar{b}$ iff $a=b$.

Let $I$ be a set. Define an algebra whose universe is the set $E^{I} \uplus(\bar{E})^{I}$, where $\uplus$ denotes the union of disjoint sets. Let $\lambda, \rho: I \rightarrow I$ be bijections. We define two special elements $0=0^{I}:=\left\langle 0_{j}: j \in I\right\rangle$ and $1=\overline{0}^{I}:=\left\langle\overline{0}_{i}: i \in I\right\rangle$, where $0_{j}=0=0_{i}$ for all $j, i \in I$. The elements of $E^{I}$ will be denoted by $\left\langle f_{j}: j \in I\right\rangle$ and ones of $(\bar{E})^{I}$ by $\left\langle\bar{a}_{i}: i \in I\right\rangle$, where $f_{j}, a_{i} \in E$ for all $i, j \in I$.

We say that a GPEA $E$ is $\lambda, \rho$-weakly commutative, where $\lambda, \rho: I \rightarrow I$ are bijections, if given sequences $\left\langle f_{j}: \underline{j} \in I\right\rangle,\left\langle g_{j}: j \in I\right\rangle$ and sequences $\left\langle\bar{a}_{i}: i \in\right.$ $I\rangle,\left\langle\bar{b}_{i}: i \in I\right\rangle$ of elements of $E$ and $\bar{E}$, respectively, we have the equivalences: (i) Given $i \in I, f_{\rho^{-1}(i)}+a_{i}$ is defined in $E$ iff $a_{i}+f_{\lambda^{-1}(i)}$ is defined in $E$, and (ii) given $i \in I, g_{\lambda^{-1}(i)}+b_{i}$ is defined in $E$ iff $b_{i}+g_{\rho^{-1}(i)}$ is defined in $E$. For example, if $E=G^{+}$for some po-group $G$ or $a+b$ is defined in $E$ for all $a, b \in E$ (that is, + is total), then $E$ is $\lambda, \rho$-weakly commutative. The same is true if $\lambda=\rho$ and $E$ is a GEA or a weakly commutative GPEA. In addition, it is possible to show that if $\lambda \neq \rho$, then $E$ is $\lambda$, $\rho$-weakly commutative iff $a+b$ is defined in $E$ for all $a, b \in E$.

Theorem 3.1. Let $\lambda, \rho: I \rightarrow I$ be bijections and $E$ be a $\lambda, \rho$-weakly commutative generalized pseudo effect algebra. Let us endow the set $(E)^{I} \uplus(\bar{E})^{I}$ with $0=0^{I}$, $1=(\overline{0})^{I}$ and with a partial operation + as follows,

$$
\begin{equation*}
\left\langle\bar{a}_{i}: i \in I\right\rangle+\left\langle\bar{b}_{i}: i \in I\right\rangle \tag{I}
\end{equation*}
$$

is not defined;

$$
\begin{equation*}
\left\langle\bar{a}_{i}: i \in I\right\rangle+\left\langle f_{j}: j \in I\right\rangle:=\left\langle\overline{f_{\rho^{-1}(i)} / a_{i}}: i \in I\right\rangle \tag{II}
\end{equation*}
$$

whenever $f_{\rho^{-1}(i)} \leq a_{i}$ for all $i \in I$;

$$
\begin{equation*}
\left\langle f_{j}: j \in I\right\rangle+\left\langle\bar{a}_{i}: i \in I\right\rangle:=\left\langle\overline{a_{i} \backslash f_{\lambda^{-1}(i)}}: i \in I\right\rangle \tag{III}
\end{equation*}
$$

whenever $f_{\lambda^{-1}(i)} \leq a_{i}$ for all $i \in I$,

$$
\begin{equation*}
\left\langle f_{j}: j \in I\right\rangle+\left\langle g_{j}: j \in I\right\rangle:=\left\langle f_{j}+g_{j}: j \in I\right\rangle \tag{IV}
\end{equation*}
$$

whenever $f_{j}+g_{j}$ is defined in $E$ for all $j \in I$.
Then the partial algebra $K_{I}^{\lambda, \rho}(E):=\left((E)^{I} \uplus(\bar{E})^{I} ;+, 0,1\right)$ is a pseudo effect algebra.

In addition, for the negations in the kite $K_{I}^{\lambda, \rho}(E)$, we have

$$
\begin{aligned}
\left\langle\bar{a}_{i}: i \in I\right\rangle^{\sim} & =\left\langle a_{\rho(j)}: j \in I\right\rangle \\
\left\langle\bar{a}_{i}: i \in I\right\rangle^{-} & =\left\langle a_{\lambda(j)}: j \in I\right\rangle \\
\left\langle f_{j}: j \in I\right\rangle^{\sim} & =\left\langle\bar{f}_{\lambda^{-1}(i)}: i \in I\right\rangle \\
\left\langle f_{j}: j \in I\right\rangle^{-} & =\left\langle\bar{f}_{\rho^{-1}(i)}: i \in I\right\rangle,
\end{aligned}
$$

and $K_{I}^{\lambda, \rho}(E)$ is symmetric if and only if $\lambda=\rho$.
Proof. (i) To prove associativity of + , we have eight cases, and it is enough to prove only the following case because all others are simple.

$$
\begin{aligned}
\left(\left\langle f_{j}: j \in I\right\rangle+\left\langle\bar{b}_{i}: i \in I\right\rangle\right) & +\left\langle h_{j}: j \in I\right\rangle=\left\langle\overline{b_{i} \backslash f_{\lambda^{-1}(i)}}: i \in I\right\rangle+\left\langle h_{j}: j \in I\right\rangle \\
& =\left\langle\overline{h_{\rho^{-1}(i)} /\left(b_{i} \backslash f_{\lambda^{-1}(i)}\right)}: i \in I\right\rangle .
\end{aligned}
$$

If the left hand side of the first line exists, then $f_{\lambda^{-1}(i)} \leq b_{i}$ and $h_{\rho^{-1}(i)} \leq$ $b_{i} \backslash f_{\lambda^{-1}(i)}$. The second inequality entails $h_{\rho^{-1}(i)} \leq b_{i} \backslash f_{\lambda^{-1}(i)}^{-1} \leq b_{i}$ for $i \in I$ so that $\left\langle\bar{b}_{i}: i \in I\right\rangle+\left\langle h_{j}: j \in I\right\rangle$ and $\left\langle f_{j}: j \in I\right\rangle+\left(\left\langle\bar{b}_{i}: i \in I\right\rangle+\left\langle h_{j}: j \in I\right\rangle\right)$ are defined, and the left hand side coincides with

$$
\left\langle f_{j}: j \in I\right\rangle+\left(\left\langle\bar{b}_{i}: i \in I\right\rangle+\left\langle h_{j}: j \in I\right\rangle\right)
$$

Conversely, let the later elements be defined, then $h_{\rho^{-1}(i)} \leq b_{i}$ and $f_{\lambda^{-1}(i)} \leq$ $h_{\rho^{-1}(i)} / b_{i}$. Then $h_{\rho^{-1}(i)}+f_{\lambda^{-1}(i)} \leq b_{i}$ and $f_{\lambda^{-1}(i)} \leq h_{\rho^{-1}(i)}+f_{\lambda^{-1}(i)} \leq b_{i}$, we have $h_{\rho^{-1}(i)} \leq b_{i} \backslash f_{\lambda^{-1}(i)}$. This yields that the elements $\left\langle f_{j}: j \in I\right\rangle+\left\langle\bar{b}_{i}: i \in I\right\rangle$ and $\left(\left\langle f_{j}: j \in I\right\rangle+\left\langle\bar{b}_{i}: i \in I\right\rangle\right)+\left\langle h_{j}: j \in I\right\rangle$ are defined, and the last expression coincides with $\left(\left\langle f_{j}: j \in I\right\rangle+\left\langle\bar{b}_{i}: i \in I\right\rangle\right)+\left\langle h_{j}: j \in I\right\rangle$.
(ii) If we define the left and right negations as it is indicated, we have $x^{-}+x=$ $1=x+x^{\sim}$. Their uniqueness can be proved thanks to uniqueness properties in the GPEA $E$.
(iii) Assume $x+y$ is defined. For example, let $\left\langle\bar{a}_{i}: i \in I\right\rangle+\left\langle f_{j}: j \in I\right\rangle:=$ $\left\langle\overline{f_{\rho^{-1}(i)} / a_{i}}: i \in I\right\rangle$ be defined.

Since $a_{i}=f_{\rho^{-1}(i)}+\left(f_{\rho^{-1}(i)} / a_{i}\right) \in E$, from the $\lambda, \rho$-weak commutativity we conclude that the element $b_{i}=\left(f_{\rho^{-1}(i)} / a_{i}\right)+f_{\lambda^{-1}(i)}$ is defined in $E$, too. It is easy to show that the element $h_{j}=\left(f_{\rho^{-1}(\lambda(j))} / a_{\lambda(j)}\right) / a_{\lambda(j)}$ is defined in $E$. Therefore,

$$
\left\langle h_{j}: j \in I\right\rangle+\left\langle\bar{a}_{i}: i \in I\right\rangle=\left\langle\bar{a}_{i}: i \in I\right\rangle+\left\langle f_{j}: j \in I\right\rangle=\left\langle f_{j}: j \in I\right\rangle+\left\langle\bar{b}_{i}: i \in I\right\rangle .
$$

In a dual way, we proceed with the case $\left\langle g_{j}: j \in I\right\rangle+\left\langle\bar{b}_{i}: i \in I\right\rangle$; the third case is evident.
(iv) Let $1+x$ or $x+1$ be defined, then it easy to show that $x=0$.

Summarizing (i)-(iv), we see $K_{I}^{\lambda, \rho}(E)$ is a pseudo effect algebra.
Using the negations, we see that $K_{I}^{\lambda, \rho}(E)$ is symmetric iff $\lambda=\rho$.
Remark 3.2. (1) From Theorem 3.1 and from its proof of (iii), we see that if $E=G^{+}$for some po-group $G$, then $\left\langle f_{j}: j \in J\right\rangle \leq\left\langle\bar{a}_{i}: i \in I\right\rangle$. In a general case, $\left\langle f_{j}: j \in J\right\rangle \leq\left\langle\bar{a}_{i}: i \in I\right\rangle$ if and only if $a_{i}+f_{\lambda^{-1}(i)}$ is defined in $E$ for all $i \in I$, equivalently, $f_{\rho^{-1}(i)}+a_{i}$ is defined in $E$ for all $i \in I$.
(2) $\left\langle f_{j}: j \in J\right\rangle \leq\left\langle\bar{a}_{i}: i \in I\right\rangle$ for all $\left\langle f_{j}: j \in J\right\rangle,\left\langle\bar{a}_{i}: i \in I\right\rangle$ iff + is a total operation. Consequently, e.g. if $E=[0,1],|I|=1$, then $\langle 0,6\rangle \not \leq\langle\overline{0,6}\rangle$.

The resulting pseudo effect algebra $K_{I}^{\lambda, \rho}(E)$ from Theorem 3.1 is said to be a kite pseudo effect algebra or more precisely a kite pseudo effect algebra of $E$. We note that Theorem 3.1 generalizes the construction of kite pseudo effect algebras studied in Dvu4, DvHo. More precisely, if $E=G^{+}$, where $G$ is a po-group, then $G^{+}$is $\lambda, \rho$-weakly commutative, and $K_{I}^{\lambda, \rho}\left(G^{+}\right)$and the kite $K_{I}^{\lambda, \rho}(G)_{e a}$ defined in Dvu4, DvHo coincide.

We present some examples of kites which are connected with the case when $E$ is the positive cone of some po-group. We start with the case $E=\{0\}$, then $E$ is $\lambda, \rho$-weakly commutative and $K_{I}^{\lambda, \rho}(E)$ is a two-element Boolean algebra.
Example 3.3. Let $|I|=n$ for $n \geq 2$, and $\lambda(i)=i, \rho(i)=i-1(\bmod n)$. We put $G_{n}=\mathbb{Z} \overrightarrow{\times}\left(\mathbb{Z}^{n}\right)$ which is ordered lexicographically. We define the addition $*$ on $G_{n}$ as follows
$\left(m_{1}, x_{0}, \ldots, x_{n-1}\right) *\left(m_{2}, y_{0}, \ldots, y_{n-1}\right)=\left(m_{1}+m_{2}, x_{0}+y_{0+m_{1}}, \ldots, x_{n-1}+y_{n-1+m_{1}}\right)$, where addition of the subscripts is performed by $\bmod n$. Then $G_{n}$ with $*$ is an $\ell$ group with the inverse given by $-\left(m, a_{0}, \ldots, a_{n-1}\right)=\left(-m,-a_{-m}, \ldots,-a_{n-1-m}\right)$, and the element $u_{n}=(1,0, \ldots, 0)$ is a strong unit. Then $K_{I}^{\rho, \lambda}\left(\mathbb{Z}^{+}\right)$is isomorphic to the pseudo effect algebra $\Gamma\left(G_{n}, u_{n}\right)$ with $\mathrm{RDP}_{2}$.

Example 3.4. Let $I=\mathbb{Z}$ and put $\lambda(i)=i$ and $\rho(i)=i-1, i \in I$. Then the kite pseudo effect algebra $K_{\mathbb{Z}}^{\lambda, \rho}\left(\mathbb{Z}^{+}\right)$satisfies $\mathrm{RDP}_{2}$.

Define $W(\mathbb{Z}):=\mathbb{Z} \overrightarrow{\times} \mathbb{Z}^{\mathbb{Z}}$, and let multiplication $*$ on it be defined as follows: $\left(m_{1}, x_{i}\right) *\left(m_{2}, y_{i}\right)=\left(m_{1}+m_{2}, x_{i}+y_{i+m_{1}}\right)$. Then $(W(\mathbb{Z}) ;(0), *)$ is an $\ell$-group, called the wreath product of $\mathbb{Z}$ by $\mathbb{Z}$ [Dar, Ex 35.1], with strong unit $u=(1,(0))$, and the kite pseudo effect algebra $K_{\mathbb{Z}}^{\lambda, \rho}\left(\mathbb{Z}^{+}\right)$is isomorphic to $\Gamma(W(\mathbb{Z}), u)$.

The following result describes the Riesz Decomposition Properties of kite pseudo effect algebras depending on original $E$.

Theorem 3.5. Let a set $I$, bijections $\lambda, \rho: I \rightarrow I$ be given and let $E$ be a directed GPEA. (1) If $E$ is $\lambda$, $\rho$-weakly commutative and the kite pseudo effect algebra $K_{I}^{\lambda, \rho}(E)$ satisfies $\mathrm{RDP}\left(\right.$ or $\mathrm{RDP}_{1}$ and $\mathrm{RDP}_{2}$, respectively), then $E$ satisfies RDP (or $\mathrm{RDP}_{1}$ and $\mathrm{RDP}_{2}$, respectively).
(2) If + is total and $E$ satisfies RDP (or $\mathrm{RDP}_{1}$ and $\mathrm{RDP}_{2}$, respectively), then $K_{I}^{\lambda, \rho}(E)$ satisfies RDP (or $\mathrm{RDP}_{1}$ and $\mathrm{RDP}_{2}$, respectively).
Proof. (1) Let $K_{I}^{\lambda, \rho}(E)$ satisfy RDP and let, for $a_{1}, a_{2}, b_{1}, b_{2} \in E$, we have $a_{1}+a_{2}=$ $b_{1}+b_{2}$. Fix an element $i_{0} \in I$ and for $i=1,2$, let us define $A_{i}=\left\langle f_{j}^{i}: j \in I\right\rangle$ by $f_{j}^{i}=a_{i}$ if $j=i_{0}$ and $f_{j}^{i}=0$ otherwise, $B_{i}=\left\langle g_{j}^{i}: j \in I\right\rangle$ by $g_{j}^{i}=a_{i}$ if $j=i_{0}$ and $j_{j}^{i}=0$ otherwise. Then $A_{1}+A_{2}=B_{1}+B_{2}$ so that there are $E_{11}=\left\langle e_{j}^{11}: j \in I\right\rangle$, $E_{12}=\left\langle e_{j}^{12}: j \in I\right\rangle, E_{21}=\left\langle e_{j}^{21}: j \in I\right\rangle, E_{22}=\left\langle e_{j}^{22}: j \in I\right\rangle$, such that $A_{1}=$ $E_{11}+E_{12}, A_{2}=E_{21}+E_{22}, B_{1}=E_{11}+E_{21}$, and $B_{2}=E_{12}+E_{22}$. Using (IV) of Theorem 3.1, from $j=i_{0}$ we conclude that the elements $e_{u v}=e_{i_{0}}^{u v}, u, v=1,2$, form the desired decomposition for $a_{1}, a_{2}, b_{1}, b_{2}$ which proves $E$ satisfies RDP.

To prove that $E$ satisfies $\operatorname{RDP}_{1}$ if $K_{I}^{\lambda, \rho}(E)$ satisfies $\operatorname{RDP}_{1}$, we see that for $j \neq i_{0}$ we have $e_{j}^{12}=0=e_{j}^{21}$. If now $0 \leq x \leq e_{12}$ and $0 \leq y \leq e_{21}$ Putting $X=\left\langle x_{j}: j \in I\right\rangle$ and $Y=\left\langle y_{j}: j \in I\right\rangle$, where $x_{j}=x, y_{j}=y$ if $j=i_{0}$ and $x_{j}=0$ and $y_{j}=0$ otherwise, we have $X+Y=Y+X$, consequently, $x+y=y+x$ which proves $e_{12} \operatorname{com} e_{21}$, and $E$ satisfies $\mathrm{RDP}_{1}$.

In the same way we prove the case with $\mathrm{RDP}_{2}$.
(2) Conversely, suppose $E$ satisfies RDP (or $\mathrm{RDP}_{1}$ ) and + is total. Then $E$ is $\lambda, \rho$-weakly commutative.
(i) If $\left\langle f_{j}: j \in I\right\rangle+\left\langle g_{j}: j \in I\right\rangle=\left\langle h_{j}: j \in I\right\rangle+\left\langle k_{j}: j \in I\right\rangle$ from (IV) of Theorem 3.1 we conclude that for them we can find an RDP decomposition or an $\mathrm{RDP}_{1}$ one.
(ii) Assume $\left\langle\bar{a}_{i}: i \in I\right\rangle+\left\langle f_{j}: j \in I\right\rangle=\left\langle\bar{b}_{i}: i \in I\right\rangle+\left\langle g_{j}: j \in I\right\rangle$. Then $a_{i}, b_{i}, f_{j}, g_{j} \in$ $E$ for all $i, j \in I$.

Now we will use tables so that we will write the elements of the kite in a simpler way: Instead of $\left\langle\bar{a}_{i}: i \in I\right\rangle$ and $\left\langle f_{j}: j \in I\right\rangle$ we use $\left\langle\bar{a}_{i}\right\rangle$ and $\left\langle f_{j}\right\rangle$, respectively.

Since $E$ is directed, for any $i \in I$, there is an element $d_{i} \in E$ such that $a_{i}, b_{i} \leq d_{i}$.
From (II) of Theorem 3.1, we have $f_{\rho^{-1}(i)} / a_{i}=g_{\rho^{-1}(i)} / b_{i}$ for all $i \in I$. Then $d_{i} \backslash\left(f_{\rho^{-1}(i)} / a_{i}\right)$ is defined in $E$. We assert $d_{i} \backslash\left(f_{\rho^{-1}(i)} / a_{i}\right)=\left(d_{i} \backslash a_{i}\right)+f_{\rho^{-1}(i)}$ for all $i \in I$. Indeed, we have $\left(d_{i} \backslash a_{i}\right)+\left(a_{i} \backslash f_{\rho^{-1}(i)}\right)+f_{\rho^{-1}(i)}=d_{i}=\left(d_{i} \backslash b_{i}\right)+$ $\left(b_{i} \backslash g_{\lambda^{-1}(i)}\right)+g_{\lambda^{-1}(i)}$. Therefore, the element $x=\left(d_{i} \backslash a_{i}\right)+f_{\rho^{-1}(i)}$ is defined in $E$. Hence

$$
\begin{aligned}
d_{i} \backslash a_{i} & =x \backslash f_{\rho^{-1}(i)} \\
d_{i}=\left(x \backslash f_{\rho^{-1}(i)}\right)+a_{i}=\left(x \backslash f_{\rho^{-1}(i)}\right) & +f_{\rho^{-1}(i)}+\left(f_{\rho^{-1}(i)} / a_{i}\right)=x+\left(f_{\rho^{-1}(i)} / a_{i}\right) \\
x & =d_{i} \backslash\left(f_{\rho^{-1}(i)} / a_{i}\right)
\end{aligned}
$$

In the same way we can show that $d_{i} \backslash\left(g_{\lambda^{-1}(i)} / b_{i}\right)=\left(d_{i} \backslash b_{i}\right)+g_{\lambda^{-1}(i)}$ and $d_{i} \backslash\left(f_{\rho^{-1}(i)} / a_{i}\right)=d_{i} \backslash\left(g_{\lambda^{-1}(i)} / b_{i}\right)$ so that

$$
\left(d_{i} \backslash a_{i}\right)+f_{\rho^{-1}(i)}=\left(d_{i} \backslash b_{i}\right)+g_{\lambda^{-1}(i)}
$$

for all $i \in I$.
Using RDP for $E$, there are $c_{i u v} \in E, u, v=1,2$, such that we have the decomposition tables:

$$
\begin{array}{c|cc}
d_{i} \backslash a_{i} & c_{i 11} & c_{i 12} \\
f_{\rho^{-1}(i)} & c_{i 21} & c_{i 22} \\
\hline & d_{i} \backslash b_{i} & g_{\rho^{-1}(i)}
\end{array} .
$$

From this table we have

$$
\begin{gathered}
d_{i} \backslash a_{i}=c_{i 11}+c_{i 12}, \quad d_{i} \backslash b_{i}=c_{i 11}+c_{i 21} \\
d_{i}=c_{i 11}+c_{i 12}+a_{i}, \quad d_{i}=c_{i 11}+c_{i 21}+b_{i} \\
c_{i 12}+a_{i}=c_{i 21}+b_{i}
\end{gathered}
$$

This yields that we have an RDP decomposition of the kite for case (ii) as follows

| $\left\langle\bar{a}_{i}\right\rangle$ | $\left\langle\overline{c_{i 12}+a_{i}}\right\rangle$ | $\left\langle c_{\rho(j) 12}\right\rangle$ |
| :---: | :---: | :---: |
| $\left\langle f_{j}\right\rangle$ | $\left\langle c_{\rho(j) 21}\right\rangle$ | $\left\langle c_{\rho(j) 22}\right\rangle$ |
|  | $\left\langle\bar{b}_{i}\right\rangle$ | $\left\langle g_{j}\right\rangle$ |.

It is evident that if $E$ satisfies $\mathrm{RDP}_{1}\left(\mathrm{RDP}_{2}\right)$, the later table gives also an $\mathrm{RDP}_{1}$ decomposition ( $\mathrm{RDP}_{2}$ decomposition).
(iii) Assume $\left\langle f_{j}: j \in I\right\rangle+\left\langle\bar{a}_{i}: i \in I\right\rangle=\left\langle g_{j}: j \in I\right\rangle+\left\langle\bar{b}_{i}: i \in I\right\rangle$. We follows the ideas from the proof of case (ii). For any $i \in I$, there is $d_{i} \in E$ such that $a_{i}, b_{i} \leq d_{i}$. By (III) of Theorem 3.1, we have $a_{i} \backslash f_{\lambda^{-1}(i)}=b_{i} \backslash g_{\lambda^{-1}(i)}$. In the same way as in (ii), we can show that

$$
\left(a_{i} \backslash f_{\lambda^{-1}(i)}\right) / d_{i}=f_{\lambda^{-1}(i)}+\left(a_{i} / d_{i}\right)=\left(b_{i} \backslash f_{\lambda^{-1}(i)}\right) / d_{i}=g_{\lambda^{-1}(i)}+\left(b_{i} / d_{i}\right)
$$

The RDP holding in $E$ entails the decompositions


Therefore, we have $a_{i}+d_{i 21}=b_{i}+d_{i 12}$. This implies an RDP decomposition for case (iii)

$$
\begin{array}{c|cc}
\left\langle f_{j}\right\rangle & \left\langle d_{\lambda(j) 11}\right\rangle & \left\langle d_{\lambda(j) 12}\right\rangle \\
\left\langle\bar{a}_{i}\right\rangle & \left\langle d_{\lambda(j) 21}\right\rangle & \left\langle a_{i}+d_{i 21}\right\rangle \\
\hline & \left\langle g_{j}\right\rangle & \left\langle\bar{b}_{i}\right\rangle
\end{array} .
$$

This decomposition is also an $\mathrm{RDP}_{1}$ decomposition whenever $E$ satisfies $\mathrm{RDP}_{1}$; the same is true for $\mathrm{RDP}_{2}$.
(iv) Assume $\left\langle\bar{a}_{i}: i \in I\right\rangle+\left\langle f_{j}: j \in I\right\rangle=\left\langle g_{j}: j \in I\right\rangle+\left\langle\bar{b}_{i}: i \in I\right\rangle$.

By $(I I)-(I I I)$ of Theorem 3.1, we have $f_{\rho^{-1}(i)} / a_{i}=b_{i} \backslash g_{\lambda^{-1}(i)}$ for all $i \in I$. Due to Remark $3.2(2),\left\langle g_{j}\right\rangle \leq\left\langle\bar{a}_{i}\right\rangle$, and by property (iii) of pseudo effect algebras, there is $\left\langle\bar{k}_{i}\right\rangle \in K_{I}^{\lambda, \rho}(E)$ such that $\left\langle\bar{a}_{i}\right\rangle=\left\langle g_{j}\right\rangle+\left\langle\bar{k}_{i}\right\rangle$ so that $k_{i}=a_{i}+g_{\lambda^{-1}(i)} \in E$ for all $i \in I$. In the similar way, there is $\left\langle\bar{l}_{i}\right\rangle \in K_{I}^{\lambda, \rho}(E)$ such that $\left\langle\bar{l}_{i}\right\rangle+\left\langle f_{j}\right\rangle=\left\langle\bar{b}_{i}\right\rangle$.

Hence, $l_{i}=f_{\rho^{-1}(i)}+b_{i} \in E$ for all $i \in I$. Since $f_{\rho^{-1}(i)} / a_{i}=b_{i} \backslash g_{\lambda^{-1}(i)}$, we have $a_{i}+g_{\lambda^{-1}(i)}=f_{\rho^{-1}(i)}+b_{i}$ for all $i \in I$.

Therefore, we can use the decomposition table

| $\left\langle\bar{a}_{i}\right\rangle$ | $\left\langle g_{j}\right\rangle$ | $\left\langle\overline{a_{i}+g_{\lambda-1}(i)}\right\rangle$ |
| :---: | :---: | :---: |
| $\left\langle f_{j}\right\rangle$ | $\left\langle 0_{j}\right\rangle$ | $\left\langle f_{j}\right\rangle$ |
|  | $\left\langle g_{j}\right\rangle$ | $\left\langle\bar{b}_{i}\right\rangle$ |,

where $0_{j}=0$ for all $j \in I$, to prove RDP. The table gives also an $\operatorname{RDP}_{1}\left(\operatorname{RDP}_{2}\right.$ decomposition) decomposition table whenever $E$ satisfies $\operatorname{RDP}_{1}\left(\mathrm{RDP}_{2}\right)$.

We note that if $E$ satisfies $\mathrm{RDP}_{1}$, then by Theorem 3.5, the kite pseudo effect algebra $K_{I}^{\lambda, \rho}(E)$ satisfies also $\operatorname{RDP}_{1}$, so that by Theorem 2.1 there is a unique (up to isomorphism) unital po-group $(G, u)$ such that $K_{I}^{\lambda, \rho}(E) \cong \Gamma(G, u)$. Please, find it. This was an open problem in Dvu4 even for the case when $E$ is the positive cone of some po-group with $\mathrm{RDP}_{1}$.

Proposition 3.6. Let $E$ be a GPEA such that $a+b$ exists in $E$ for all $a, b \in E$. The kite pseudo effect algebra $K_{I}^{\lambda, \rho}(E)$ satisfies $\mathrm{RDP}_{0}$ if and only if $E$ satisfies $\mathrm{RDP}_{0}$.

Proof. Using (IV) of Theorem 3.1, it is evident that if $K_{I}^{\lambda, \rho}(E)$ satisfies $\operatorname{RDP}_{0}$, so does $E$.

Conversely, let $E$ be with $\mathrm{RDP}_{0}$. We note $\left\langle x_{j}: j \in I\right\rangle \leq\left\langle y_{j}: j \in I\right\rangle$ iff $x_{j} \leq y_{j}$ for all $j \in I$.
(i) Let $\left\langle f_{j}: j \in I\right\rangle \leq\left\langle g_{j}: j \in I\right\rangle+\left\langle h_{j}: j \in I\right\rangle=\left\langle g_{j}+h_{j}: j \in I\right\rangle$ which implies that for all $j \in I$, there are $g_{j 1}, h_{j 1} \in E$ such that $g_{j 1} \leq g_{j}, h_{j 1} \leq h_{j}$ and $f_{j}=g_{j 1}+h_{j 1}$. Then $\left\langle f_{j}: j \in I\right\rangle=\left\langle g_{j 1}: j \in I\right\rangle+\left\langle h_{j 1}: j \in I\right\rangle$ and $\left\langle g_{j 1}: j \in I\right\rangle \leq$ $\left\langle g_{j}: j \in I\right\rangle$ and $\left\langle h_{j 1}: j \in I\right\rangle \leq\left\langle h_{j}: j \in I\right\rangle$.
(ii) Let $\left\langle f_{j}: j \in I\right\rangle \leq\left\langle\bar{a}_{i}: i \in I\right\rangle+\left\langle g_{j}: j \in I\right\rangle$. Then $\left\langle f_{j}: j \in I\right\rangle=\left\langle f_{j}: j \in\right.$ $I\rangle+\left\langle 0_{j}: j \in I\right\rangle$, and $\left\langle f_{j}: j \in I\right\rangle \leq\left\langle\bar{a}_{i}: i \in I\right\rangle$ and $\left\langle 0_{j}: j \in I\right\rangle \leq\left\langle g_{j}: j \in I\right\rangle$.

In the analogous way we can prove the case $\left\langle f_{j}: j \in I\right\rangle \leq\left\langle g_{j}: j \in I\right\rangle+\left\langle\bar{a}_{i}: i \in I\right\rangle$.
(iii) Let $\left\langle\bar{a}_{i}: i \in I\right\rangle \leq\left\langle\bar{b}_{i}: i \in I\right\rangle+\left\langle f_{j}: j \in I\right\rangle$. We note that $\left\langle\bar{x}_{i}: i \in I\right\rangle \leq$ $\left\langle\bar{y}_{i}: i \in I\right\rangle$ iff $x_{i} \geq y_{i}$ for all $i \in I$. For each $i \in I$, we have $a_{i} \geq f_{\rho^{-1}(i)} / b_{i}$ and $b_{i} \leq f_{\rho^{-1}(i)}+a_{i}$. Using $\mathrm{RDP}_{0}$ for $E$, we can find positive elements $a_{i 1} \leq a_{i}$ and $f_{\rho^{-1}(i) 1} \leq f_{\rho^{-1}(i)}$ such that $b_{i}=f_{\rho^{-1}(i) 1}+a_{i 1}$. Then we have $\left\langle\bar{a}_{i}: i \in I\right\rangle=$ $\left\langle\overline{f_{\rho^{-1}(i) 1}+a_{i}}: i \in I\right\rangle+\left\langle f_{j 1}: j \in I\right\rangle$, and $\left\langle\overline{f_{\rho^{-1}(i) 1}+a_{i}}: i \in I\right\rangle \leq\left\langle\bar{b}_{i}: i \in I\right\rangle$ and $\left\langle f_{j 1}: j \in I\right\rangle \leq\left\langle f_{j}: j \in I\right\rangle$.

In a dual way, we proceed also with the case $\left\langle\bar{a}_{i}: i \in I\right\rangle \leq\left\langle f_{j}: j \in I\right\rangle+\left\langle\bar{b}_{i}: i \in I\right\rangle$.
Summing up all cases, we have that $K_{I}^{\lambda, \rho}(E)$ satisfies $\mathrm{RDP}_{0}$.
Let $x$ be an element of the pseudo effect algebra $E$ and $n \geq 0$ be an integer. We define $0 x:=0,1 x:=x$, and $(n+1) x:=n x+x$ whenever $n x$ and $n x+x$ are defined in $E$. An element $x \in E$ is said to be infinitesimal if $n x$ exists in $E$ for any integer $n \geq 1$. We denote by $\operatorname{Infinit}(E)$ the set of infinitesimal elements of $E$.

Perfect pseudo effect algebras are characterized as those PEAs where every element is either infinitesimal or co-infinitesimal, see e.g. DXY, DvKr. Such PEAs are often of the form $\Gamma(\mathbb{Z} \overrightarrow{\times} G,(1,0))$ for some directed po-group $G$, c.f. Dvu4, Thm 5.2].

If $A, B$ are two subsets of a pseudo effect algebra $E, A+B:=\{a+b: a \in A, b \in$ $B, a+b \in E\}$, and we say that $A+B$ is defined in $E$ if $a+b$ exists in $E$ for each
$a \in A$ and each $b \in B$. We write $A \leqslant B$ whenever $a \leq b$ for all $a \in A$ and all $b \in B$. In addition, we write $A^{-}:=\left\{a^{-}: a \in A\right\}$ and $A^{\sim}:=\left\{a^{\sim}: a \in A\right\}$.

We note that an ideal of a generalized pseudo effect algebra $E$ is any non-empty subset $I$ of $E$ such that (i) if $x, y \in I$ and $x+y$ is defined in $E$, then $x+y \in I$, and (ii) $x \leq y \in I$ implies $x \in I$. An ideal $I$ is maximal if it is a proper subset of $E$ and it is not a proper subset of any ideal $J \neq E$. An ideal $I$ is normal if $x+I=I+x$ for any $x \in E$, where $x+I:=\{x+y: y \in I, x+y$ exists in $E\}$ and in the dual way we define $I+x$. The set $\{0\}$ is always a normal ideal, and an ideal $I$ is non-trivial if $I \neq\{0\}$.

We say that a mapping $s: E \rightarrow[0,1]$, is a state, where $E$ is a PEA, if (i) $s(a+b)=s(a)+s(b)$ whenever $a+b$ is defined in $E$, and (ii) $s(1)=1$. A state $s$ is extremal if from $s=\alpha s_{1}+(1-\alpha) s_{2}$, where $s_{1}, s_{2}$ are states on $E$ and $\alpha \in(0,1)$, we conclude $s=s_{1}=s_{2}$. A state models a finitely additive probability measure. We denote by $\mathcal{S}(E)$ the set of all states on $E$. It can happen that $\mathcal{S}(E)$ is empty. In general, $\mathcal{S}(E)$ is either empty, or a singleton or an infinite set. We recall that if $E$ is an effect algebra with RDP, then $E$ has at least one state.

If $s$ is a state on $E$, then the kernel of $s$, i.e. the set $\operatorname{Ker}(s):=\{a \in E: s(a)=0\}$, is a normal ideal of $E$.

We say that a pseudo effect algebra $E$ is perfect if there are two subsets $E_{0}, E_{1}$ of $E$ such that $E_{0} \cap E_{1}=\emptyset$ and $E=E_{0} \cup E_{1}$ such that
(a) $E_{i}^{-}=E_{i}^{\sim}=E_{1-i}, i=0,1$,
(b) if $x \in E_{i}, y \in E_{j}$ and $x+y$ is defined in $E$, then $i+j \leq 1$ and $x+y \in E_{i+j}$ for $i, j=0,1$,
(c) $E_{0}+E_{0}$ is defined in $E$.

In such a case, we write $E=\left(E_{0}, E_{1}\right)$. By Dvu4, Prop 5.1], $E_{0}$ consists of all infinitesimal elements of $E$, i.e. $E_{0}=\operatorname{Infinit}(E), E$ has a unique state, say $s$, and $s\left(E_{0}\right)=0$ and $s\left(E_{1}\right)=1$.

The following result was established in Dvu4, Thm 5.3] for the case when $E=$ $G^{+}$.

Theorem 3.7. Assume that a set $I$, a bijection $\lambda: I \rightarrow I$ and a directed weakly commutative GPEA E with $\mathrm{RDP}_{1}$ are given. Assume that + on $E$ is a total operation. There is a unique (up to isomorphism) directed po-group $G_{I}^{\lambda}$ with $\mathrm{RDP}_{1}$ such that the perfect kite pseudo effect algebra $K_{I}^{\lambda, \lambda}(E)$ is isomorphic to $\Gamma\left(\mathbb{Z} \overrightarrow{\times} G_{I}^{\lambda},(1,0)\right)$. In particular, $K_{I}^{\lambda, \lambda}(E)$ is a perfect PEA.

Proof. It is evident that the kite pseudo effect algebra $K_{I}^{\lambda, \lambda}(E)$ is $\lambda, \lambda$-weakly commutative, and in addition, by negations presented in Theorem3.1, it is symmetric. It is evident that any kite pseudo effect algebra is perfect. Since $E$ is with $\mathrm{RDP}_{1}$, by Theorem [3.5, the kite $K_{I}^{\lambda, \lambda}(E)$ satisfies $\mathrm{RDP}_{1}$. Consequently, all conditions of Dvu4, Thm 5.2] are satisfied, thus there is a unique (up to isomorphism) directed po-group $G_{I}^{\lambda}$ with $\operatorname{RDP}_{1}$ such that $K_{I}^{\lambda, \lambda}(E) \cong \Gamma\left(\mathbb{Z} \overrightarrow{\times} G_{I}^{\lambda},(1,0)\right)$.

Remark 3.8. (1) If the operation + is not total in a GPEA $E$, then Theorem 3.7 is not valid. For example if $E=[0,1)$ is the interval or real numbers, then $K_{I}^{\lambda, \lambda}(E)$ is not a perfect kite pseudo effect algebra.
(2) If the operation + is total in a GPEA $E$, then by [Fuc, Prop X.1], $E$ is the positive cone of a po-group.

## 4. Subdirect Irreducibility of Kite Pseudo Effect Algebras

In this section we generalize the results from Dvu4, DvHo concerning the subdirect irreducibility of the kite pseudo effect algebras. We note that in Dvu4, DvHo these results were established for the special case of a GPEA $E=G^{+}$, which is always $\lambda, \rho$-weakly commutative, the present proofs follow the basic steps and ideas of the original proofs from Dvu4, DvHo. To be self-contained, our proofs are given with fullness with necessary changes.

We remind that by an o-ideal of a directed po-group $G$ we understand any normal directed convex subgroup $H$ of $G$. If $G$ is a po-group, so is $G / H$, where $x / H \leq y / H$ iff $x \leq h_{1}+y$ for some $h_{1} \in H$ iff $x \leq y+h_{2}$ for some $h_{2} \in H$. If $G$ satisfies one of RDP's, then $G / H$ satisfies the same RDP, Dvu2, Prop 6.1].

We say that an equivalence $\sim$ on a GPEA $E$ is a congruence, if for $a_{1}, a_{2}, b_{1}, b_{2}$ such that $a_{1} \sim a_{2}$ and $b_{1} \sim b_{2}$ the existence $a_{1}+b_{1}$ and $a_{2}+b_{2}$ in $E$ implies $a_{1}+b_{1} \sim a_{2}+b_{2}$. If $I$ is a normal ideal of $E$, then the relation $\sim_{I}$ defined on $E$ by $a \sim_{I} b$ iff there are $e, f \in I$ such that $a \backslash e=b \backslash f$. Then $\sim_{I}$ is a congruence on $E$. In the same way as [Dvu2, Prop 4.1], we can show that if the GPEA $E$ satisfies $\mathrm{RDP}_{1}$, then there is a one-to-one correspondence between congruences on $E$ and normal ideals of $E$. We note that if $\sim$ is a congruence, then $I=\{a \in E: a \sim 0\}$ is a normal ideal of $E$.

If a directed GPEA $E$ satisfies $\mathrm{RDP}_{1}$, by Theorem [2.1, there is a unique (up to isomorphism) couple ( $G, G_{0}$ ), where $G$ is a directed po-group with $\mathrm{RDP}_{1}, G_{0}$ is a GPEA which is a subalgebra of $\left(G^{+} ;+, 0\right)$ and $G_{0}$ generates the po-group $G$, such that $E \cong \Gamma\left(G, G_{0}\right)$. In the same way as [Dvu3, Thm 4.2], we can prove the following statement:

Proposition 4.1. Let $E=\Gamma\left(G, G_{0}\right)$ for some directed po-group $G$ with $\mathrm{RDP}_{1}$. If $I$ is an ideal of $E$, then the set

$$
\phi(I)=\left\{x \in G: \exists x_{i}, y_{j} \in I, x=x_{1}+\cdots+x_{n}-y_{1}-\cdots-y_{m}\right\}
$$

is a convex subgroup of $G$ generated by $I$. The set $\phi(I)$ is an o-ideal if and only if $I$ is a normal ideal, and $E / I=\Gamma(G / \phi(I), u / \phi(I))$. The mapping $I \mapsto \phi(I)$ is a one-to-one correspondence between normal ideals of $E$ and o-ideals of $G$ preserving set-theoretical inclusion. If $K$ is an o-ideal, then the restriction $K \cap G_{0}$ is a normal ideal of $E$.

Remark 4.2. We note that in the same way as DvHo, Lem 3.4], we san prove that of a GPEA $E$ satisfies $\mathrm{RDP}_{1}$ and $I$ is a normal ideal of $E$, then $E / I$ is a GPEA with $\mathrm{RDP}_{1}$.

We say that a GPEA $E$ (or a po-group) is a subdirect product of a family $\left(E_{t}: t \in\right.$ $T$ ) of GPEAs (po-groups), and we write $E \leq \prod_{t \in T} E_{t}$ if there is an injective homomorphism $h: E \rightarrow \prod_{t \in T} E_{t}$ such that $\pi_{t} \circ h(E)=E_{t}$ for all $t \in T$, where $\pi_{t}$ is the $t$-th projection from $\prod_{t \in T} E_{t}$ onto $E_{t}$. In addition, $E$ is subdirectly irreducible if whenever $E$ is a subdirect product of $\left(E_{t}: t \in T\right)$, there exists $t_{0} \in T$ such that $\pi_{t_{0}} \circ h$ is an isomorphism of generalized pseudo effect algebras. It is possible to show that a GPEA $E$ is subdirectly irreducible iff $E$ is either trivial or it possesses the least non-trivial normal ideal $I$, or equivalently, the intersection of all non-trivial normal ideals of $E$ is a non-trivial normal ideal.

The following result generalizes an analogous one from Dvu4, Lem 3.2], where it was proved for PEAs with $\mathrm{RDP}_{1}$; the present proof follows the original proof from Dvu4, Lem 3.2].

Lemma 4.3. Every directed generalized pseudo effect algebra with $\mathrm{RDP}_{1}$ is a subdirect product of subdirectly irreducible directed generalized pseudo effect algebras with $\mathrm{RDP}_{1}$.

Proof. If $E$ is a trivial GPEA, i.e. $E=\{0\}, E$ is subdirectly irreducible. Thus, suppose $E$ is not trivial. By Theorem [2.2, $\mathrm{RDP}_{1}$ entails that $E \cong \Gamma\left(G, G_{0}\right)$ for some po-group $G$ satisfying $\mathrm{RDP}_{1}$, and $G_{0}$ is a GPEA which is a subalgebra of $G^{+}$ generating $G$; for simplicity, we assume $E \subseteq \Gamma\left(G, G_{0}\right)$. Given a non-zero element $g \in G$, let $N_{g}$ be an o-ideal of $G$ which is maximal with respect to not containing $g$; by Zorn's Lemma, it exists. Then $\bigcap_{g \neq 0} N_{g}=\{0\}$ and $N_{g} \cap G_{0}$ is a normal ideal of $E$. Therefore, $G \leq \prod_{g \neq 0} G / N_{g}$ and $E \leq \prod_{g \neq 0} E /\left(N_{g} \cap G_{0}\right)$. In addition, for every $g \neq 0$, the o-ideal of $G / N_{g}$ generated by $g / N_{g}$ is the least non-trivial o-ideal of $G / N_{g}$ which proves that every $G / N_{g}$ is subdirectly irreducible. Therefore, every $E /\left(N_{g} \cap G_{0}\right)$ is a subdirectly irreducible pseudo effect algebra, and by Dvu3, Prop 4.1], the quotient pseudo effect algebra $E /\left(N_{g} \cap G_{0}\right)$ also satisfies $\mathrm{RDP}_{1}$.

Let $\alpha$ be a cardinal. An element $\left\langle f_{j}: j \in I\right\rangle$ is said to be $\alpha$-dimensional, if $\left|\left\{j \in I: f_{j} \neq e\right\}\right|=\alpha$. In the same way we define an $\alpha$-dimensional element $\left\langle\bar{a}_{i}: i \in I\right\rangle$.

Proposition 4.4. Let $I$ be a set and $\lambda, \rho: I \rightarrow I$ be bijections. If $H$ is a normal ideal of a directed $\lambda$, $\rho$-weakly commutative GPEA $E$, then $H^{I}:=\left\{\left\langle f_{j}: j \in I\right\rangle\right.$ : $\left.f_{j} \in H, j \in I\right\}$ is a normal ideal of the kite pseudo effect algebra $K_{I}^{\lambda, \rho}(E)$. In addition, if $H_{f}^{I}$ denotes the set of all finite dimensional elements from $H^{I}$, then $H_{f}^{I}$ is a non-trivial normal ideal of the kite pseudo effect algebra. In particular, if $H=E$, then $E^{I}$ is a maximal ideal of $K_{I}^{\lambda, \rho}(E)$.

Conversely, if $J$ is a normal ideal of $K_{I}^{\lambda, \rho}(E), J \subseteq H^{I}$, then $\pi_{j}(J)$ is a normal ideal of $E$ which is a subset of $H$, where $\pi_{j}$ is the $j$-th projection of $\left\langle f_{j}: j \in I\right\rangle \mapsto f_{j}$.

Proof. It is identical with the proof of Dvu4, Prop 6.3], where it was proved for the special case $E=G^{+}$for some directed po-group $G$. We prove only that $E^{I}$ is a maximal ideal of $K_{I}^{\lambda, \rho}(E)$. Take $x=\left\langle\bar{f}_{j}: j \in I\right\rangle$ and let $F$ be an ideal of the kite generated by $A^{I}$ and $x$. By Theorem [3.1] $x^{-} \in A^{I}$, so that $x^{-}+x=1$ therefore, $1 \in F$ which proves that $A^{I}$ is maximal.

Proposition 4.5. Let a kite pseudo effect algebra of a non-trivial directed $\lambda, \rho$ weakly commutative GPEA $E, K_{I}^{\lambda, \rho}(E)$, have the least non-trivial normal ideal. Then $E$ has the least non-trivial normal ideal.

Proof. Suppose the converse, i.e. $E$ has no least non-trivial normal ideal. There exists a set $\left\{H_{t}: t \in T\right\}$ of non-trivial normal ideals of $E$ such that $\bigcap_{t \in T} H_{t}=\{0\}$. By Proposition 4.4 every $H_{t}^{I}$ is a normal ideal of the kite. Hence, $\bigcap_{t \in T} H_{t} \neq\{0\}$ and there is a non-zero element $f=\left\langle f_{j}: j \in I\right\rangle \in \bigcap_{t \in T} H_{t}^{I}$. Then, for every index $j \in I, f_{j} \in H_{t}$ for every $t \in T$ which yields $f_{j}=0$ for each $j \in I$ and $f=\left\langle f_{j}: j \in I\right\rangle=0$, which is a contradiction. Therefore, $E$ has the least nontrivial normal ideal.

According to DvKo Dvu4, DvHo, we say that elements $i, j \in I$ are connected if there is an integer $m \geq 0$ such that $\left(\rho \circ \lambda^{-1}\right)^{m}(i)=j$ or $\left(\lambda \circ \rho^{-1}\right)^{m}(i)=j$; otherwise, $i$ and $j$ are said to be disconnected.

The relation $i$ and $j$ are connected is an equivalence on $I$. We call this equivalence class a connected component of $I$. We denote by $\mathcal{C}(I)$ the set of all connected components of $I$.

It is noteworthy to recall that if $C$ is a connected component of $I$, then $\lambda^{-1}(C)=$ $\rho^{-1}(C)$. Indeed, let $i \in C$ and $k=\lambda^{-1}(i)$. Then $j=\rho(k)=\rho \circ \lambda^{-1}(i) \in C$. Hence, $k=\rho^{-1}(j)$ which proves $\lambda^{-1}(C) \subseteq \rho^{-1}(C)$. In the same way we prove the opposite inclusion. In particular, we have $\lambda\left(\rho^{-1}(C)\right)=\rho\left(\lambda^{-1}(C)\right)=C=\lambda\left(\lambda^{-1}(C)\right)=$ $\rho\left(\rho^{-1}(C)\right)$.

Theorem 4.6. Let $I$ be a set and $\lambda, \rho: I \rightarrow I$ be bijections and let $E$ be a nontrivial directed $\lambda, \rho$-weakly commutative GPEA. For the kite pseudo effect algebra $K_{I}^{\lambda, \rho}(E)$ with $\mathrm{RDP}_{1}$, the following statements are equivalent:
(1) E has the least non-trivial normal ideal and for all $i, j \in I$ there exists an integer $m \geq 0$ such that $\left(\rho \circ \lambda^{-1}\right)^{m}(i)=j$ or $\left(\lambda \circ \rho^{-1}\right)^{m}(i)=j$.
(2) $K_{I}^{\lambda, \rho}(E)$ has the least non-trivial normal ideal.

Proof. (1) $\Rightarrow(2)$. By Theorem 3.5, the pseudo kite effect algebra $K_{I}^{\lambda, \rho}(E)$ satisfies $\mathrm{RDP}_{1}$.

Let $H$ be the least non-trivial normal ideal of $E$. By Proposition4.4, $H^{I}$ and $H_{f}^{I}$ are normal ideals of the kite $K_{I}^{\lambda, \rho}(E)$. Therefore, it is only necessary to show that $H_{f}^{I}$ is the least non-trivial normal ideal of the kite. Equivalently, we have to prove that the normal ideal of the kite generated by any nonzero element $\left\langle f_{j}: j \in I\right\rangle \in H_{f}^{I}$ equals to $H_{f}^{I}$. This is equivalent to show the same for any one-dimensional element from $H_{f}^{I}$. Indeed, let $f=\left\langle f_{j}: j \in I\right\rangle$ be any element from $H^{I} \backslash\{0\}$. There is a onedimensional element $g=\left\langle g_{j}: j \in I\right\rangle \in H^{I}$ such that $0<\left\langle g_{j}: j \in I\right\rangle \leq\left\langle f_{j}: j \in I\right\rangle$. Hence, $H_{f}^{I}=H_{0}(g) \subseteq H_{0}(f) \subseteq H_{f}^{I}$.

Thus let $g$ be any one-dimensional element from $H_{f}^{I}$ and let $N_{0}(g)$ be the normal ideal of the kite generated by the element $g$. Without loss of generality assume $g=\left\langle g_{0}, e, \ldots\right\rangle$, where $g_{0}>e, g_{0} \in E$; this is always possible by a suitable reindexing of $I$, regardless of its cardinality. Then $g_{0}$ generates $H$, that is $H=\{x \in$ $E: x=-x_{1}+g_{1}+x_{1}+\cdots-x_{n}+g_{n}+x_{n}=y_{1}+g_{1}^{\prime}-y_{1}+\cdots y_{m}+g_{m}^{\prime}-y_{m}, 0 \leq$ $\left.g_{i}, g_{j}^{\prime} \leq g_{0}, x_{i}, y_{j} \in E, i=1, \ldots, n, j=1, \ldots, m, n, m \geq 1\right\}$, since $H$ is the least non-trivial normal ideal of $G$. (We note that we can assume that by Theorem 2.2, $E=\Gamma\left(G, G_{0}\right)$ for some po-group $G$ with $\mathrm{RDP}_{1}$, so that the elements of the form $-x_{1}+g_{1}+x_{1}$ and $y_{1}+g_{1}^{\prime}-y_{1}$ exist in $G_{0}$ as well as in $E$.) Doing double negations $m$ times of $\left\langle f_{j}: j \in I\right\rangle$, we obtain that either $\left\langle f_{j}: j \in I\right\rangle^{--m}=\left\langle f_{\left(\rho \circ \lambda^{-1}\right)^{m}(j)}: j \in I\right\rangle$ and it belongs to $N_{0}(g)$ or $\left\langle f_{j}: j \in I\right\rangle^{\sim \sim m}=\left\langle f_{\left(\lambda \circ \rho^{-1}\right)^{m}(j)}: j \in I\right\rangle$ which also belongs to $N_{0}(g)$. Consequently, for any $j \in I$, there is an integer $m$ such that $\left(\rho \circ \lambda^{-1}\right)^{m}(j)=0$ or $\left(\lambda \circ \rho^{-1}\right)^{m}(j)=0$, so that the one-dimensional element whose $j$-th coordinate is $g_{0}$ is defined in $N_{0}(g)$ for any $j \in J$; it is either $g^{--m}$ or $g^{\sim \sim m}$. It is easy to show that $-f+g_{0}+f$ and $k+g_{0}-k$ belong to $N_{0}\left(g_{0}\right)$ for all $g, k \in E$, which implies, for every $g \in H$, the one-dimensional element $\langle g, e, \ldots\rangle$ belongs to $N_{0}\left(g_{0}\right)$, and finally, every one-dimensional element $\langle\ldots, g, \ldots\rangle$ from $H_{f}^{I}$ belongs also to $N_{0}\left(g_{0}\right)$.

Now let $J_{0}=\left\{j_{1}, \ldots, j_{n}\right\}$ be an arbitrary finite subset of $J,\left|J_{0}\right|=m \geq 1$, and choose arbitrary $m$ elements $h_{j_{1}}, \ldots, h_{j_{m}} \in H$. Define an $m$-dimensional element $g_{J_{0}}=\left\langle g_{j}: j \in I\right\rangle$, where $g_{j}=h_{j_{k}}$ if $j=j_{k}$ for some $k=1, \ldots, m$, and $g_{j}=0$ otherwise. In addition, for $k=1, \ldots, m$, let $g_{k}=\left\langle f_{j}: j \in I\right\rangle$, where $f_{j}=h_{j_{k}}$ if $j=j_{k}$ and $f_{j}=e$ if $j \neq j_{k}$. Then $g_{J_{0}}=g_{1}+\cdots+g_{k} \in N_{0}\left(g_{0}\right)$.

Consequently, $N_{0}\left(g_{0}\right)=H_{f}^{I}$.
$(2) \Rightarrow(1)$. By Proposition 4.5, $E$ has the least non-trivial normal ideal, say $H_{0}$. Suppose that (1) does not hold. Then for all $i, j \in I$ and every integer $m \geq 0$, we have $\left(\rho \circ \lambda^{-1}\right)^{m}(i) \neq j$ and $\left(\lambda \circ \rho^{-1}\right)^{m}(i) \neq j$. We can observe that such indices $i$ and $j$ are disconnected. By the assumption, there are two elements $i_{0}, j_{0} \in I$ which are disconnected. Let $I_{0}$ and $I_{1}$ be maximal sets of mutually connected elements containing $i_{0}$ and $j_{0}$, respectively. Then no element of $I_{0}$ is connected to any element of $I_{1}$.

We define $H_{0}^{I_{0}}$ as the set of all elements $\left\langle f_{j}: j \in I\right\rangle$ such that $j \notin I_{0}$ implies $f_{j}=e$. In a similar way we define $H^{I_{1}}$. Then both sets are non-trivial normal ideals of the kite. For example, let $\left\langle f_{j}: j \in I\right\rangle \in N_{0}^{I_{0}}$. Then $\left\langle f_{j}: j \in I\right\rangle+\left\langle\bar{a}_{i}: i \in I\right\rangle=$ $\left\langle\overline{a_{i} \backslash f_{\lambda^{-1}(i)}}: i \in I\right\rangle$. If we set $h_{j}=a_{\rho(j)} \backslash\left(a \backslash f_{\lambda^{-1}(\rho(j))}\right)=a_{\rho(j)}+f_{\lambda^{-1}(\rho(j))}-a_{\rho(j)}^{-1} \in$ $E$, then $\left\langle h_{j}: j \in I\right\rangle \in N_{0}^{I_{0}}$ and $\left\langle\bar{a}_{i}: i \in I\right\rangle+\left\langle h_{j}: j \in I\right\rangle=\left\langle f_{j}: j \in I\right\rangle+\left\langle\bar{a}_{i}: i \in I\right\rangle$. Similarly for the other possibilities. In the same way we can show that $H_{0}^{I_{1}}$ is a non-trivial normal ideal of the kite.

On the other hand, we have $H_{0}^{I_{0}} \cap H_{0}^{I_{1}}=\{0\}$ which contradicts that the kite has the least non-trivial normal ideal.

Theorem 4.6 has important two consequences when $I$ is finite or countable describing subdirectly irreducible kite pseudo effect algebras.

Theorem 4.7. Let $|I|=n$ for some $n \geq 0, \lambda, \rho: I \rightarrow I$ be bijections and $E$ be a non-trivial directed $\lambda$, $\rho$-weakly commutative GPEA. If the kite pseudo effect algebra $K_{I}^{\lambda, \rho}(E)$ with $\mathrm{RDP}_{1}$ has the least non-trivial normal ideal, then $E$ has the least non-trivial normal ideal and $K_{I}^{\lambda, \rho}(E)$ is isomorphic to one of:
(1) $K_{0}^{\emptyset, \emptyset}(E)$, if $n=0, K_{1}^{i d, i d}(E)$ if $n=1$.
(2) $K_{n}^{\lambda, \rho}(E)$ for $n \geq 1$ and $\lambda(i)=i$ and $\rho(i)=i-1(\bmod n)$.

Proof. In any case, $E$ has the least non-trivial normal ideal.
If $I$ is empty, the only bijection from $I$ to $I$ is the empty function. The kite pseudo effect algebra $K_{0}^{\emptyset, \emptyset}(E)$ is a two-element Boolean algebra. For $n \geq 1$, we can assume without loss of generality that $\lambda$ is the identity map and $\rho$ is a permutation of $I=$ $\{0,1, \ldots, n-1\}$. If $\rho$ is not cyclic, then there are $i, j \in I$ such that $j$ does not belongs to the orbit of $i$, which means that $i$ and $j$ are not connected which contradicts Theorem 4.6. So $\rho$ must be cyclic. We can then renumber $I=\{0,1, \ldots, n-1\}$ following the $\rho$-cycle, so that $\rho(i)=i-1(\bmod n), i=0,1, \ldots, n-1$.

The proofs of the following two results are identical as those from Dvu4, Lem 6.8, Thm 6.9] for the case $E=G^{+}$, but to be selfcontained, we present them here in fullness.

Now we show that a necessary condition to be a kite $K_{I}^{\lambda, \rho}(E)$ with $I$ infinite subdirectly irreducible is $I$ has to be at most countable.

Lemma 4.8. Let $I$ be a set and $\lambda, \rho: I \rightarrow I$ be bijections and let $E$ be a non-trivial directed $\lambda, \rho$-weakly commutative GPEA. If the kite pseudo effect algebra $K_{I}^{\lambda, \rho}(E)$ with $\mathrm{RDP}_{1}$ has the least non-trivial normal ideal, then I is at most countable.
Proof. Suppose $I$ is uncountable and choose an element $i \in I$. Consider the set $P(i)=\left\{\left(\rho \circ \lambda^{-1}\right)^{m}(i): m \geq 0\right\} \cup\left\{\left(\lambda \circ \rho^{-1}\right)^{m}(i): m \geq 0\right\}$. The set $P(i)$ is at most countable, so there is $j \in I \backslash P(i)$. But $P(i)$ exhausts all finite paths alternating $\lambda$ and $\rho$ starting from $i$. Then $i$ and $j$ are disconnected which contradicts Theorem 4.6. Hence, $I$ is at most countable.

Finally, we present kite pseudo effect algebras having the least non-trivial normal ideal when $I$ is infinite. In such a case by Lemma 4.8, $I$ has to be countable.
Theorem 4.9. Let $|I|=\aleph_{0}, \lambda, \rho: I \rightarrow I$ be bijections and let $E$ be a non-trivial directed $\lambda, \rho$-directed GPEA. If the kite pseudo effect algebra $K_{I}^{\lambda, \rho}(E)$ with $\mathrm{RDP}_{1}$ has the least non-trivial normal ideal, then $K_{I}^{\lambda, \rho}(E)$ is isomorphic to $K_{\mathbb{Z}}^{i d, \rho}(E)$, where $\rho(i)=i-1, i \in \mathbb{Z}$.

Proof. After re-indexing we can assume that $\lambda$ is the identity on $I$. If $\rho$ is not cyclic, there would be two elements which should be disconnected which is impossible. Therefore, there is an element $i_{0} \in I$ such that the orbit $P\left(i_{0}\right):=\left\{\rho^{m}\left(i_{0}\right): m \in\right.$ $\mathbb{Z}\}=I$. Hence, we can assume that $I=\mathbb{Z}$, and $\rho(i)=i-1, i \in \mathbb{Z}$. Indeed, if we set $j_{m}=\rho^{-m}\left(i_{0}\right), m \in \mathbb{Z}$, then $\rho\left(j_{m}\right)=j_{m-1}$ and we have $\rho(i)=i-1, i \in \mathbb{Z}$.

It is possible to define kite pseudo effect algebras also in the following way. Let $J$ and $I$ be two sets, $\lambda, \rho: J \rightarrow I$ be two bijections, and $E$ be a $\lambda, \rho$-weakly commutative GPEA. We define $K_{J, I}^{\lambda, \rho}(E)=(E)^{J} \uplus(\bar{E})^{I}$, where $\uplus$ denotes a union of disjoint sets. The elements $0=\left\langle e_{j}: j \in J\right\rangle$ and $1=\langle\bar{e}: i \in I\rangle$ are assumed to be the least and greatest elements of $K_{J, I}^{\lambda, \rho}(E)$. If we define a partial operation + by Theorem 3.1, changing in formulas $(I I)-(I V)$ the notation $j \in I$ to $j \in J$, we obtain that $K_{J, I}^{\lambda, \rho}(E)=\left(K_{J, I}^{\lambda, \rho}(E) ;+, 0,1\right)$ with this,+ 0 and 1 is a pseudo effect algebra, called also a kite pseudo effect algebra. In particular, $K_{I}^{\lambda, \rho}(E)=K_{I, I}^{\lambda, \rho}(E)$.

Since $J$ and $I$ are of the same cardinality, there is practically no substantial difference between kite pseudo effect algebras of the form $K_{I}^{\lambda, \rho}(E)$ and $K_{I, J}^{\lambda, \rho}(E)$ and all known results holding for the first kind are also valid for the second one. We note that in DvKo, the "kite" structure used two index sets, $J$ and $I$. The second form, which was used also in the proof of DvHo, Thm 3.6], will be practical also for the following result; its proof here follows the main ideas of the original proof.

Lemma 4.10. Let $K_{I}^{\lambda, \rho}(E)$ be the kite pseudo effect algebra with $\mathrm{RDP}_{1}$ of a directed $\lambda, \rho$-weakly commutative GPEA E. Then $K_{I}^{\lambda, \rho}(E)$ is a subdirect product of the system of kite pseudo effect algebras with $\operatorname{RDP}_{1}\left(K_{I^{\prime}, J^{\prime}}^{\lambda^{\prime}, \rho^{\prime}}(E): I^{\prime} \in \mathcal{C}(I)\right)$, where $I^{\prime}$ is any connected component of $I, J^{\prime}=\lambda^{-1}\left(I^{\prime}\right)=\rho^{-1}\left(I^{\prime}\right)$, and $\lambda^{\prime}, \rho^{\prime}: J^{\prime} \rightarrow I^{\prime}$ are the restrictions of $\lambda$ and $\rho$ to $J^{\prime} \subseteq I$.
Proof. If $E$ is trivial, the statement is evident. Now, let $E$ be non-trivial. By the comments before this lemma, we see that $\lambda^{\prime}, \rho^{\prime}: J^{\prime} \rightarrow I^{\prime}$ are bijections. Let $I^{\prime}$ be a connected component of $I$. Let $N_{I^{\prime}}$ be the set of all elements $f=\left\langle f_{j}: j \in I\right\rangle \in(E)^{I}$ such that $f_{j}=0$ whenever $j \in J^{\prime}$. It is straightforward to see that $N_{I^{\prime}}$ is a normal ideal of $K_{I}^{\lambda, \rho}(E)$.

We note that under our conditions, $E$ is also $\lambda^{\prime}, \rho^{\prime}$-weakly commutative, so that $K_{J^{\prime}, I^{\prime}}^{\lambda^{\prime}, \rho^{\prime}}(E)$ is by Theorem 3.1 again a pseudo effect algebra.

It is also not difficult to see that $K_{I}^{\lambda, \rho}(E) / N_{I^{\prime}}$ is isomorphic to $K_{J^{\prime}, I^{\prime}}^{\lambda^{\prime}, \prime^{\prime}}(E)$ and, by Dvu3, DvVe4, $K_{J^{\prime}, I^{\prime}}^{\lambda^{\prime}, \rho^{\prime}}(E)$ satisfies $\mathrm{RDP}_{1}$.

For each $I^{\prime} \in \mathcal{C}(I)$, let $N_{I^{\prime}}$ be the normal filter defined as above. As connected components are disjoint, we have $\bigcap_{I^{\prime} \in \mathcal{C}(I)} N_{I^{\prime}}=\{0\}$. This proves $K_{I}^{\lambda, \rho}(E)=$ $K_{I, I}^{\lambda, \rho}(E) \leq \prod_{I^{\prime} \in \mathcal{C}(I)} K_{I^{\prime}, J^{\prime}}^{\lambda^{\prime}, \rho^{\prime}}(E)$.

This result entails the following important statement on representability of kite pseudo effect algebras as a subdirect product of other subdirectly irreducible kites with $\mathrm{RDP}_{1}$.
Theorem 4.11. Every kite pseudo effect algebra $K_{I}^{\lambda, \rho}(E)$ with $\operatorname{RDP}_{1}$, where $E$ is a directed $\lambda$, $\rho$-weakly commutative GPEA, is a subdirect product of subdirectly irreducible kite pseudo effect algebras with $\mathrm{RDP}_{1}$.
Proof. If $E$ is trivial, the statement is evident. Now, let $E$ be non-trivial and take the kite $K_{I}^{\lambda, \rho}(E)$. By Theorem 2.2 $E$ satisfies $\mathrm{RDP}_{1}$. If the kite is not subdirectly irreducible, then there are two possible cases: (i) $E$ is not subdirectly irreducible, or (ii) $E$ is subdirectly irreducible but there exist $i, j \in I$ such that, for every $m \in \mathbb{N}$, we have $\left(\rho \circ \lambda^{-1}\right)^{m}(i) \neq j$ and $\left(\lambda \circ \rho^{-1}\right)^{m}(i) \neq j$. Observe that this happens if and only if $i$ and $j$ do not belong to the same connected component of $I$.

Now, using Remark 4.2 we can reduce (i) to (ii). So, suppose $E$ is subdirectly irreducible. Then, using Lemma 4.10, we can subdirectly embed $K_{I}^{\lambda, \rho}(E)$ into $\prod_{I^{\prime}} K_{I^{\prime}, J^{\prime}}^{\lambda^{\prime}, \rho^{\prime}}(E)$, where $I^{\prime}$ ranges over the connected components of $I, J^{\prime}=\lambda^{-1}\left(I^{\prime}\right)$ and $\lambda^{\prime}, \rho^{\prime}$ are restrictions of $\lambda, \rho$ to $J^{\prime}$. But then, each $K_{I^{\prime}, J^{\prime}}^{\lambda, \rho}(E)$ is subdirectly irreducible by Theorem 4.6 and by Theorem [2.2] it satisfies $\mathrm{RDP}_{1}$.

Now let $E$ be a directed GPEA with $\mathrm{RDP}_{1}$. By Theorem 2.2 $E \cong \Gamma\left(G, G_{0}\right)$, where $G$ is a po-group with $\mathrm{RDP}_{1}$ and $G_{0}$ is a directed GPEA that is a subalgebra of the GPEA $G^{+}$. In what follows, we show a relationship between the kite pseudo effect algebras $K_{I}^{\lambda, \rho}(E)$ and $K_{I}^{\lambda, \rho}(G)_{e a}:=K_{I}^{\lambda, \rho}\left(G^{+}\right)$and their relation to the pogroup $G$.
Theorem 4.12. Let $E=\Gamma\left(G, G_{0}\right)$ be a directed $\lambda$, $\rho$-weakly commutative GPEA with $\mathrm{RDP}_{1}$, where $G$ is a directed po-group satisfying and $G_{0}$ is a non-trivial directed GPEA which is a subalgebra of $G^{+}$and $G_{0}$ generates $G$. Then
(i) The kite pseudo effect algebra $K_{I}^{\lambda, \rho}(E)$ is a subalgebra of the kite pseudo effect $K_{I}^{\lambda, \rho}\left(G^{+}\right)$.
(ii) $K_{I}^{\lambda, \rho}(E)$ has the least non-trivial normal ideal if and only if $K_{I}^{\lambda, \rho}\left(G^{+}\right)$has the least non-trivial o-ideal.
(iii) The following are equivalent:
(1) $G$ has the least non-trivial o-ideal and for all $i, j \in I$ there exists an integer $m \geq 0$ such that $\left(\rho \circ \lambda^{-1}\right)^{m}(i)=j$ or $\left(\lambda \circ \rho^{-1}\right)^{m}(i)=j$.
(2) $K_{I}^{\lambda, \rho}(E)$ has the least non-trivial normal ideal.

Proof. (i) It is evident.
(ii) Let $J$ be the least non-trivial normal ideal of $E$ and let $N(J)$ be the normal ideal of $K_{I}^{\lambda, \rho}(E)$ generated by $J$. If $K$ is a non-trivial normal ideal of $K_{I}^{\lambda, \rho}(E)$
such that $K \subseteq N(J)$, then $K_{0}:=K \cap K_{I}^{\lambda, \rho}(E)$ is a normal ideal of $K_{I}^{\lambda, \rho}(E)$. Since $N(J)=\left\{x \in K_{I}^{\lambda, \rho}\left(G^{+}\right): x \leq x_{1}+\cdots+x_{n}, x_{i} \in J, i=1, \ldots, n, n \geq 1\right\}$, we have $K_{0}$ is non-trivial and whence $K_{0}=J$ and $N(J)$ is the least non-trivial normal ideal of $K_{I}^{\lambda, \rho}\left(G^{+}\right)$, alias $K_{I}^{\lambda, \rho}\left(G^{+}\right)$is subdirectly irreducible.

Conversely, let $K_{I}^{\lambda, \rho}\left(G^{+}\right)$be subdirectly irreducible with the least normal ideal $K$ and set $K_{0}=K \cap K_{I}^{\lambda, \rho}(E)$. Then $K_{0}$ is a normal ideal of $K_{I}^{\lambda, \rho}(E)$. Since every element $g \in G^{+}$is the sum of elements of $E$, we see $K_{0}$ is a non-trivial normal ideal. Now assume $J$ be any non-trivial normal ideal of $K_{I}^{\lambda, \rho}(E)$ such that $J \subseteq K_{0}$. We assert that $J=K_{0}$ otherwise the normal ideal $N(J)$ of $K_{I}^{\lambda, \rho}\left(G^{+}\right)$generated by $J$ should be a proper subset of $J$ which is a contradiction. Hence, $K_{I}^{\lambda, \rho}(E)$ is subdirectly irreducible.
(iii) Since $K_{I}^{\lambda, \rho}(G)_{e a}=K_{I}^{\lambda, \rho}\left(G^{+}\right)$, comparing Dvu4, Thm 6.6] and Theorem 4.6, we see that (1) and (2) are equivalent.

Remark 4.13. If $E$ is trivial, the corresponding kite is a two-element Boolean algebra, therefore, Proposition 4.5, Theorem 4.7, Lemmas 4.8-4.9, Theorem 4.12 are not necessarily valid.

## 5. Conclusion

In the paper we have extended the reservoir of interesting examples of pseudo effect algebras which are closely connected with either the positive cones po-groups as well as with generalized pseudo effect algebras and their unitization. These pseudo algebras, called kite pseudo effect algebras, are ordinal sums of the product, $E^{I}$, of a generalized pseudo effect algebra $E$ with the product $(\bar{E})^{I}(\bar{E}$ is a copy of $E$ which is ordered in the reverse order as $E$ ), where operations depend on two bijections of an index set $I$. These algebras can be both either commutative or noncommutative, and starting with a commutative generalized effect algebra, it can happen that the corresponding kite pseudo effect algebra can be non-commutative. The resulting algebra depends on the bijections that are used, Theorem 3.1.

We have shown how kite pseudo effect algebras are connected with different types of the Riesz Decomposition Properties, Theorem 3.5. Since pseudo effect algebras are partial algebras, it is not straightforward how to study some notions of universal algebras. But for pseudo effect algebras with $\mathrm{RDP}_{1}$ it is possible to study subdirect irreducibility in an equivalent way to show when a kite pseudo effect algebra possesses the least non-trivial normal ideal, Section 4. It was shown that this property is closely connected with the existence of the least non-trivial normal ideal in the original GPEA and with special behavior of the bijections that are used. Some representation theorems, Theorem 4.7 and Theorem 4.9, were presented. In addition, an open problem was formulated.

The paper enriched the realm of pseudo effect algebras and one has again shown the importance of po-groups and generalized pseudo effect algebras for the theory of quantum structures which can be useful also for modeling events of quantum measurements when commutativity is not a priori guaranteed.

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