On Carlson's inequality for Sugeno and Choquet integrals

Michał Boczek,[*] Marek Kaluszka

Institute of Mathematics, Lodz University of Technology, 90-924 Lodz, Poland

Abstract

We present a Carlson type inequality for the generalized Sugeno integral and a much wider class of functions than the comonotone functions. We also provide three Carlson type inequalities for the Choquet integral. Our inequalities generalize many known results.

Keywords: Choquet integral; Sugeno integral; Capacity; Semicopula; Carlson inequality.

1   Introduction

The pioneering concept of the fuzzy integral was introduced by Sugeno [31] as a tool for modelling non-deterministic problems. Theoretical investigations of the integral and its generalizations have been pursued by many researchers. Wang and Klir [34] presented an excellent general overview on fuzzy integration theory. On the other hand, fuzzy integrals have also been successfully applied to various fields (see, e.g., [14, 22]).

The study of inequalities for Sugeno integral was initiated by Román-Flores et al. [27]. Since then, the fuzzy integral counterparts of several classical inequalities,

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[*]Corresponding author. E-mail adress: 800401@edu.p.lodz.pl;

including Chebyshev's, Jensen's, Minkowski's and Hölder's inequalities, are given by Flores-Franulič and Román-Flores [9], Agahi et al. [2], L. Wu et al. [35] and others. Furthermore many researchers started to study inequalities for the seminormed Sugeno integral [1, 16, 17, 24].

The Carlson inequality for the Lebesgue integral is of the form

$$\int\limits_0^\infty f(x)\,\mathrm{d}x \leqslant \sqrt{\pi}\Big(\int\limits_0^\infty f^2(x)\,\mathrm{d}x\Big)^{\frac{1}{4}}\Big(\int\limits_0^\infty x^2 f^2(x)\,\mathrm{d}x\Big)^{\frac{1}{4}}, \qquad (1)$$

where f is any non-negative, measurable function such that the integrals on the right-hand side converge. The equality in (1) is attained iff $f(x) = \dfrac{\alpha}{\beta + x^2}$ for some constants $\alpha \geqslant 0$, $\beta > 0$. The modified versions of the Carlson inequality can be found in [3] and [20].

The purpose of this paper is to study the Carlson inequality for the generalized Sugeno as well the Choquet integrals. In Section 2, we provide inequalities for the generalized Sugeno integral. The results are obtained for a rich class of functions, including the comonotone functions as a special case. In Section 3 we present the corresponding results for the Choquet integral.

2   Carlson's type inequalities for Sugeno integral

Let $(X, \mathcal{F})$ be a measurable space and $\mu\colon \mathcal{F} \to Y$ be a monotone measure, i.e., $\mu(\emptyset) = 0$, $\mu(X) > 0$ and $\mu(A) \leqslant \mu(B)$ whenever $A \subset B$. Throughout the paper $Y = [0, 1]$ or $Y = [0, \infty]$. Suppose $\circ\colon Y \times Y \to Y$ is a non-decreasing operator, i.e. $a \circ c \geqslant b \circ d$ for $a \geqslant b$ and $c \geqslant d$. An example of non-decreasing operators is a t-seminorm, also called a semicopula [8, 24]. There are three important t-seminorms: M, Π and $\circ_L$, where $\mathrm{M}(a, b) = a \wedge b$, $\Pi(a, b) = ab$ and $\circ_L(a, b) = (a + b - 1) \vee 0$ usually called the Łukasiewicz t-norm [18]. Hereafter, $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$.

For a measurable function $h\colon X \to Y$, we define the generalized Sugeno integral of h on a set $A \in \mathcal{F}$ with respect to μ and a non-decreasing operator $\circ\colon Y \times Y \to Y$

as

$$\int_A h \circ \mu = \sup_{\alpha \in Y} \left\{ \alpha \circ \mu \big(A \cap \{h \geqslant \alpha\} \big) \right\}, \tag{2}$$

where $\{h \geqslant a\}$ stands for $\{x \in X \colon h(x) \geqslant a\}$. For $\circ = \mathrm{M}$, we get the *Sugeno integral* [31]. If $\circ = \Pi$, then (2) is called the *Shilkret integral* [28]. We denote the Sugeno and the Shilkret integral as $(S) \int_A f \, \mathrm{d}\mu$ and $(N) \int_A f \, \mathrm{d}\mu$, respectively. Moreover, we obtain the *seminormed fuzzy integral* if $\circ$ is a semicopula [30].

Let $f, g \colon X \to Y$ be measurable functions and $A, B \in \mathcal{F}$. The functions $f_{|A}$ and $g_{|B}$ are *positively dependent* with respect to μ and an operator $\triangle \colon Y \times Y \to Y$ if for any $a, b \in Y$

$$\mu\Big(\{f_{|A} \geqslant a\} \cap \{g_{|B} \geqslant b\} \Big) \geqslant \mu\Big(\{f_{|A} \geqslant a\} \Big) \triangle \mu\Big(\{g_{|B} \geqslant b\} \Big), \tag{3}$$

where $h_{|C}$ denotes the restriction of the function $h \colon X \to Y$ to a set $C \subset X$. Obviously, $\{h_{|C} \geqslant a\} = \{x \in C \colon h(x) \geqslant a\} = C \cap \{h \geqslant a\}$. Taking $a \triangle b = a \wedge b$ and $a \triangle b = ab$, we recover two important examples of positively dependent functions, namely comonotone functions and independent random variables. Recall that f and g are comonotone if $(f(x) - f(y))(g(x) - g(y)) \geqslant 0$ for all $x, y \in X$. More examples of positively dependent functions can be found in [16].

Suppose $\star, \square \colon Y \times Y \to Y$ are non-decreasing operators. Let $\Diamond \colon Y \times Y \to Y$ be a non-decreasing and left-continuous operator, i.e. $\lim_{n \to \infty} (x_n \circ y_n) = x \circ y$ for all $x_n \nearrow x$ and $y_n \nearrow y$, where $a_n \nearrow a$ means that $\lim_{n \to \infty} a_n = a$ and $a_n < a_{n+1}$ for all n. Assume $\triangle \colon Y \times Y \to Y$ is an arbitrary operator and $f, f_i, g \colon X \to Y$, $i = 1, 2, 3$, are measurable functions.

We recall two inequalities for generalized Sugeno integral.

Theorem 2.1 ([17])**.** *For $s \geqslant 1$ and $A \in \mathcal{F}$, the Jensen type inequality*

$$\int_A f^s \circ \mu \geqslant \left(\int_A f \circ \mu \right)^s \tag{4}$$

holds if $a^s \circ b \geqslant (a \circ b)^s$ for all $a, b \in Y$.

Remark 1. If $\circ = \wedge$, then (4) is satisfied provided $(S) \int_A f \, d\mu \leqslant 1$ (see [36] and [16], Theorem 3.1).

Theorem 2.2 ([16]). *The Chebyshev type inequality of the form*

$$\int\limits_{A \cap B} (f_1 \square f_2) \circ \mu \geqslant \left(\int\limits_A f_1 \circ \mu \right) \Diamond \left(\int\limits_B f_2 \circ \mu \right) \tag{5}$$

holds for all positively dependent functions $f_{1|A}$, $f_{2|B}$ and $A, B \in \mathcal{F}$ if $(a \square b) \circ (c \bigtriangleup d) \geqslant (a \circ c) \Diamond (b \circ d)$ for all $a, b, c, d \in Y$.

Now we are ready to derive a Carlson-type inequality for the generalized Sugeno integral.

Theorem 2.3. *Suppose $p, q \geqslant 1$ and $r, s > 0$. Then for arbitrary pairs of positively dependent functions $f_{|A}$, $g_{|B}$ and $f_{|A}$, $h_{|B}$, the following inequality*

$$\left(\left(\int\limits_A f \circ \mu \right) \Diamond \left(\int\limits_B g \circ \mu \right) \right)^r \star \left(\left(\int\limits_A f \circ \mu \right) \Diamond \left(\int\limits_B h \circ \mu \right) \right)^s \tag{6}$$

$$\leqslant \left(\int\limits_{A \cap B} (f \square g)^p \circ \mu \right)^{\frac{r}{p}} \star \left(\int\limits_{A \cap B} (f \square h)^q \circ \mu \right)^{\frac{s}{q}}$$

is satisfied if for all $a, b, c, d \in Y$ and $s > 1$,

$$a^s \circ b \geqslant (a \circ b)^s, \quad (a \square b) \circ (c \bigtriangleup d) \geqslant (a \circ c) \Diamond (b \circ d). \tag{7}$$

Proof. Observe that all integrals in (6) are elements of Y. From the Jensen inequality (4), it follows that

$$\int\limits_{A \cap B} (f \square g) \circ \mu \leqslant \left(\int\limits_{A \cap B} (f \square g)^p \circ \mu \right)^{\frac{1}{p}}, \tag{8}$$

$$\int\limits_{A \cap B} (f \square h) \circ \mu \leqslant \left(\int\limits_{A \cap B} (f \square h)^q \circ \mu \right)^{\frac{1}{q}}. \tag{9}$$

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The operator $\star$ is non-decreasing, so by (8) and (9),

$$\left(\int\limits_{A\cap B} (f\,\square\,g)\circ\mu \right)^r \star \left(\int\limits_{A\cap B} (f\,\square\,h)\circ\mu \right)^s$$

$$\leqslant \left(\int\limits_{A\cap B} (f\,\square\,g)^p\circ\mu \right)^{\frac{r}{p}} \star \left(\int\limits_{A\cap B} (f_1\,\square\,h)^q\circ\mu \right)^{\frac{s}{q}}. \qquad (10)$$

From (5) we get

$$\int\limits_{A\cap B} (f\,\square\,\psi)\circ\mu \geqslant \left(\int\limits_A f\circ\mu \right)\Diamond\left(\int\limits_B \psi\circ\mu \right) \quad \text{for } \psi = g, h. \qquad (11)$$

To complete the proof, it is enough to apply (11) to (10). $\qquad\qquad\square$

Theorem 2.3 extends all known (obtained by different methods) Carlson type inequalities for the Sugeno integral. In order to see this, we first put $A = B$, $\triangle = \circ = \wedge$ and $\square = \star = \Diamond = \cdot$ in Theorem 2.3. Putting further $g = 1$, $h = x$, $p = q = 2$, $r = s = 1$ and $A = [0,1]$ yields the result of Caballero et al. [4]. If μ is the Lebesgue measure then

$$(S)\int\limits_A f\,\mathrm{d}\mu \leqslant \sqrt{2}\left((S)\int\limits_A f\,\mathrm{d}\mu \right)^{\frac{1}{4}}\left((S)\int\limits_A x^2 f^2\,\mathrm{d}\mu \right)^{\frac{1}{4}},$$

since $(S)\displaystyle\int\limits_{[0,1]} x\,\mathrm{d}\mu = 0.5$ and if f and g are comonotone, then $f_{|A}$ and $g_{|A}$ are positively dependent with respect to the operator $\wedge$ (see Example 2.1 in [16]).

Setting $r = s = 1$, we obtain Theorem 3.1 of Xu and Ouyang [36]

$$(S)\int\limits_A f\,\mathrm{d}\mu \leqslant \frac{1}{\sqrt{C}}\left((S)\int\limits_A f^p g^p\,\mathrm{d}\mu \right)^{\frac{1}{2p}}\left((S)\int\limits_A f^q h^q\,\mathrm{d}\mu \right)^{\frac{1}{2q}},$$

where $C = \left((S)\displaystyle\int\limits_A g\,\mathrm{d}\mu \right)\left((S)\displaystyle\int\limits_A h\,\mathrm{d}\mu \right)$ (see Remark 1). Taking $r = p/(p+q)$ and $s = 1 - r$, we get Theorem 2.7 from [33]

$$(S)\int\limits_A f\,\mathrm{d}\mu \leqslant \frac{1}{K}\left((S)\int\limits_A f^p g^p\,\mathrm{d}\mu \right)^{\frac{1}{p+q}}\left((S)\int\limits_A f^q h^q\,\mathrm{d}\mu \right)^{\frac{1}{p+q}},$$

where
$$K = \left((S) \int_A g \, \mathrm{d}\mu \right)^{\frac{p}{p+q}} \left((S) \int_A h \, \mathrm{d}\mu \right)^{\frac{q}{p+q}}.$$

Combining the above results with other inequalities for comonotone functions one can also derive (similarly as in [6]) some related Carlson type inequalities for the Sugeno integral.

From Theorem 2.3 one can obtain many other Carlson type inequalities since the conditions (7) are fulfilled by many systems of operators. Examples are:

1. $\triangle = \wedge$ and $\square = \lozenge = \circ$, where $\circ$ is any t-norm satysfying the condition $(a^s \circ b) \geqslant (a \circ b)^s$ for $s \geqslant 1$ since $a \circ b \leqslant a \wedge b$ and any t-norm is an associative and commutative operator [18];

2. $\triangle = \square = \circ = \lozenge = \cdot$ on $Y = [0, 1]$;

3. $\triangle = \square = \lozenge = \cdot$ and $\circ = \wedge$ with $Y = [0, 1]$;

4. $\triangle = \square = \lozenge$, $\circ = \wedge$ and $Y = [0, 1]$;

5. $\triangle = \square = \lozenge = \circ$, where $\circ$ is any t-norm satysfying the condition $(a^s \circ b) \geqslant (a \circ b)^s$ for $s \geqslant 1$, e.g. the Dombi t-norm $a \circ b = ab/(a + b - ab)$;

6. $\square = \lozenge$, $\triangle$ is any operator, $a \circ b = a$ for all $a, b \in Y$ and $Y = [0, 1]$ or $Y = [0, \infty]$.

Example 2.1. The following inequality for the Shilkret integral of a non-decreasing function f is valid:

$$(N) \int_A f \, \mathrm{d}\mu \leqslant \frac{1}{\sqrt{K}} \cdot \left((N) \int_A f^2 \, \mathrm{d}\mu \right)^{\frac{1}{4}} \left((N) \int_A x^2 f^2 \, \mathrm{d}\mu \right)^{\frac{1}{4}},$$

where $K = \mu(A) \cdot \left((N) \int_A x \, \mathrm{d}\mu \right)$; to see this put $\triangle = \wedge$ or $\triangle = \cdot$, $g = 1$, $h = x$, $\lozenge = \star = \square = \circ = \cdot$, $p = q = 2$, $r = s = 1$ and $A = B$ in Theorem 2.3.

Example 2.2. Let $(X, \mathcal{F}, \mathsf{P})$ be a probability space. Put $Y = [0, 1]$, $r = s = 1$, $g = 1$, $A = B = X$, $f = \phi(U)$ and $h = 1 - \psi(U)$, where U has the uniform distribution

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on $[0, 1]$ and $\phi, \psi\colon [0, 1] \to [0, 1]$ are increasing functions. The functions f and h are not comonotone but

$$\mathsf{P}\big(f \geqslant a, h \geqslant b\big) = \big(\psi^{-1}(1 - b) - \phi^{-1}(a)\big)_+ = \mathsf{P}\big(f \geqslant a\big) \circ_L \mathsf{P}\big(h \geqslant b\big),$$

so f and h are positively dependent with respect to P and $\circ_L$. The conditions (7) are satisfied for $\triangle = \square = \Diamond = \circ_L$ and $\star = \circ = \cdot$ (see [16], formula (40)), thus the corresponding Carlson inequality takes the form

$$\big(N(f) \circ_L 1\big) \cdot \big(N(f) \circ_L N(h)\big) \leqslant \big(N(f^p)\big)^{\frac{1}{p}} \Big(N\big((f \circ_L h)^q\big)\Big)^{\frac{1}{q}},$$

where $N(f) = (N) \displaystyle\int_X f \, \mathrm{d}\mathsf{P}$.

3 Carlson's type inequality for Choquet integral

In this section, $\mu\colon \mathcal{F} \to [0, \infty]$ is a monotone measure. Denote by $\mathcal{M}$ the set of monotone measures on $(X, \mathcal{F})$. The Choquet integral of $f\colon X \to [0, \infty)$ on $A \in \mathcal{F}$ is defined as

$$\int_A f \, \mathrm{d}\mu = \int_0^\infty \mu\big(A \cap \{f \geqslant t\}\big) \, \mathrm{d}t,$$

where the integral on the right-hand side is the improper Riemann integral. A function f is said to be *integrable* on a measurable set A if $\displaystyle\int_A f \, \mathrm{d}\mu < \infty$. The importance of the Choquet integral still increases due to many applications in mathematics and economics, see for instance [5, 7, 12, 13, 15].

First, we show that it does not exist a functional $c\colon \mathcal{M} \to [0, \infty]$ such that for any monotone measure μ and any integrable function f

$$\int_X f \, \mathrm{d}\mu \leqslant c(\mu) \bigg(\int_X g f^2 \, \mathrm{d}\mu\bigg)^{\frac{1}{4}} \bigg(\int_X h f^2 \, \mathrm{d}\mu\bigg)^{\frac{1}{4}} \tag{12}$$

provided $\inf\limits_{x \in X} (g(x)h(x)) = 0$. Indeed, put $\mu(A) = 1$ for all $A \neq \emptyset$, $f(x) = 1$ for $x = t$ and $f(x) = 0$ otherwise, where t is any fixed point of X. Since $\displaystyle\int_X \psi \, \mathrm{d}\mu = \sup\limits_{x \in X} \psi(x)$,

from (12) we have $1 \leqslant c(\mu)(g(t)h(t))^{1/4}$, a contradiction with $\inf_{x \in X}(g(x)h(x)) = 0$ and $c(\mu) < \infty$. Therefore, some extra conditions should be imposed on f, g, h or μ.

Now, we present the Carlson type inequality for the Choquet integral of comonotone functions.

Theorem 3.1. *Let $p, q \geqslant 1$ and $r, s > 0$. Suppose $f, g \colon X \to [0, \infty)$ and $f, h \colon X \to [0, \infty)$ are pairs of comonotone functions. If f is integrable on A, then*

$$\int_A f \, \mathrm{d}\mu \leqslant K\left(\mu(A)\right)^d \left(\int_A f^p g^p \, \mathrm{d}\mu\right)^{\frac{r}{p(r+s)}} \left(\int_A f^q h^q \, \mathrm{d}\mu\right)^{\frac{s}{q(r+s)}}, \qquad (13)$$

where $K = \left(\int_A g \, \mathrm{d}\mu\right)^{-\frac{r}{r+s}} \left(\int_A h \, \mathrm{d}\mu\right)^{-\frac{s}{r+s}}$ and $d = 2 - \frac{1}{r+s}\left(\frac{r}{p} + \frac{s}{q}\right)$.

Proof. Without loss of generality, we assume that $0 < \mu(A) < \infty$. Put $m(B) = \mu(A \cap B)/\mu(A)$ for $B \in \mathcal{F}$. For a given $c \geqslant 1$, the following Jensen type inequality

$$\left(\int f \, \mathrm{d}m\right)^c \leqslant \int f^c \, \mathrm{d}m \qquad (14)$$

is satisfied [11, 19, 38]. Hereafter, we write $\int f \, \mathrm{d}m$ instead of $\int_A f \, \mathrm{d}m$. From (14), we have

$$\left(\int fg \, \mathrm{d}m\right)^r \left(\int fh \, \mathrm{d}m\right)^s \leqslant \left(\int f^p g^p \, \mathrm{d}m\right)^{\frac{r}{p}} \left(\int f^q h^q \, \mathrm{d}m\right)^{\frac{s}{q}}. \qquad (15)$$

Since f, g are comonotone functions, the following Chebyshev inequality

$$\int fg \, \mathrm{d}m \geqslant \int f \, \mathrm{d}m \int g \, \mathrm{d}m$$

holds (see [10]). The functions f, h are also comonotone, so from (15) we get

$$\left(\int f \, \mathrm{d}m\right)^{r+s} \left(\int g \, \mathrm{d}m\right)^r \left(\int h \, \mathrm{d}m\right)^s \leqslant \left(\int f^p g^p \, \mathrm{d}m\right)^{\frac{r}{p}} \left(\int f^q h^q \, \mathrm{d}m\right)^{\frac{s}{q}}.$$

Combining this with the equality $\int \phi \, \mathrm{d}m = \left(\mu(A)\right)^{-1} \int_A \phi \, \mathrm{d}\mu$, completes the proof. $\qquad \square$

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Putting $g = 1$ and $r = s$ in Theorem 3.1, we have

$$\left(\int_A f \, d\mu \right)^2 \leqslant \frac{\mu(A)^{3-\left(\frac{1}{p}+\frac{1}{q}\right)}}{\int_A h \, d\mu} \left(\int_A f^p \, d\mu \right)^{\frac{1}{p}} \left(\int_A f^q h^q \, d\mu \right)^{\frac{1}{q}},$$

since $\int_A 1 \, d\mu = \mu(A)$. This result was obtained by Ouyang for $p, q > 1$ as a consequence of Hölder's inequality for the Choquet integral of comonotone functions f, h and Chebyshev's inequality [25, 37].

The inequality (13) is sharp. In fact, if $\mu(B) = 1$ for $B \neq \emptyset$, then $\int_A \phi \, d\mu = s(\phi)$, where $s(\phi)$ denotes the supremum of ϕ on A, so the inequality (13) takes the form

$$s(f) \leqslant s(g)^{-\frac{r}{r+s}} s(h)^{-\frac{s}{r+s}} \left(s(fg) \right)^{\frac{r}{r+s}} \left(s(fh) \right)^{\frac{s}{r+s}}. \tag{16}$$

Since $s(\phi\psi) = s(\phi)s(\psi)$ for comonotone functions ϕ, ψ (see [21]), the equality in (16) is attained.

Now, we provide the Carlson type inequality for the Choquet integral with respect to a submodular monotone measure μ. Recall that μ is *submodular* if

$$\mu(A \cap B) + \mu(A \cup B) \leqslant \mu(A) + \mu(B)$$

for $A, B \in \mathcal{F}$. The Choquet integral is subadditive for all measurable functions f, g iff μ is submodular (see [26], Theorem 7.7). Define

$$H_{pq}(a, b) = (ab)^{\frac{1}{p}} \left(\int_A \frac{1}{(bg + ah)^{q-1}} \, d\mu \right)^{\frac{1}{q}}. \tag{17}$$

Theorem 3.2. *If μ is submodular, $f \colon X \to [0, \infty)$ and $A \in \mathcal{F}$, then*

$$\int_A f \, d\mu \leqslant 2^{1/p} H_{pq} \left(\int_A gf^p \, d\mu, \int_A hf^p \, d\mu \right), \tag{18}$$

where $p > 1$ and $1/p + 1/q = 1$. The equality in (18) is attained if $\left(g \int_A hf^p \, d\mu + h \int_A gf^p \, d\mu \right)^q f^p = \gamma$ for some $\gamma \geqslant 0$ provided μ is modular or gf^p and hf^p are comonotone functions.

Proof. Since μ is submodular, the following Hölder inequality

$$\int_A \phi\psi \, \mathrm{d}\mu \leqslant \left(\int_A \phi^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} \left(\int_A \psi^q \, \mathrm{d}\mu \right)^{\frac{1}{q}} \tag{19}$$

is valid, where $\phi, \psi \geqslant 0$ (see [32], Theorem 3.5). The equality in (19) holds if $\alpha\phi^p = \beta\psi^q$ for $\alpha, \beta \geqslant 0$, $\alpha + \beta > 0$ [23]. By (19) and the subadditivity and positively homogeneity of the Choquet integral, we get

$$\int_A f \, \mathrm{d}\mu = \int_A (bg + ah)^{1/p} f \frac{1}{(bg + ah)^{1/p}} \, \mathrm{d}\mu$$

$$\leqslant \left(\int_A (bg + ah) f^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} \left(\int_A \frac{1}{(bg + ah)^{q/p}} \, \mathrm{d}\mu \right)^{\frac{1}{q}}$$

$$\leqslant \left(b \int_A g f^p \, \mathrm{d}\mu + a \int_A h f^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} \left(\int_A \frac{1}{(bg + ah)^{q-1}} \, \mathrm{d}\mu \right)^{\frac{1}{q}}. \tag{20}$$

Putting $a = \int_A g f^p \, \mathrm{d}\mu$ and $b = \int_A h f^p \, \mathrm{d}\mu$, we obtain (18). Note that if μ is modular then from Theorem 7.7 of [26] it follows that

$$\int_A (bg f^p + ah f^p) \, \mathrm{d}\mu = \int_A bg f^p \, \mathrm{d}\mu + \int_A ah f^p \, \mathrm{d}\mu. \qquad \square$$

If μ is the Lebesgue measure, $g(x) = 1$, $h(x) = x^2$, $p = q = 2$ and $A = [0, \infty]$, we obtain the classical Carlson inequality (1).

Next, we present the Carlson type inequality for the Choquet integral with respect to a subadditive monotone measure μ.

Theorem 3.3. *Suppose $f \colon X \to [0, \infty)$, $A \subset [0, \infty]$ and $\mu \in \mathcal{M}$ such that $\mu(A \cup B) \leqslant \mu(A) + \mu(B)$ for $A, B \in \mathcal{F}$. Then*

$$\int_A f \, \mathrm{d}\mu \leqslant 4^{1/p} \left(\frac{1}{\sqrt{p}} + \frac{1}{\sqrt{q}} \right)^2 H_{pq} \left(\int_A g f^p \, \mathrm{d}\mu, \int_A h f^p \, \mathrm{d}\mu \right),$$

where $H_{pq}(a, b)$ is given by (17), $p > 1$ and $1/p + 1/q = 1$.

Proof. The proof is similar to that of Theorem 3.2, but we use the following inequalities (see [29])

$$\int\limits_A fg \, \mathrm{d}\mu \leqslant \left(\frac{1}{\sqrt{p}} + \frac{1}{\sqrt{q}}\right)^2 \left(\int\limits_A f^p \, \mathrm{d}\mu\right)^{\frac{1}{p}} \left(\int\limits_A g^q \, \mathrm{d}\mu\right)^{\frac{1}{q}},$$

$$\int\limits_A (f+g) \, \mathrm{d}\mu \leqslant 2\left(\int\limits_A f \, \mathrm{d}\mu + \int\limits_A g \, \mathrm{d}\mu\right).$$

instead of those in (20). Since $(1/\sqrt{p} + 1/\sqrt{q})^2 \leqslant 2$, the bounds obtained are better than the bounds of [5]. $\square$

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