The Cuntz semigroup and domain theory

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July 1, 2021

Abstract

Domain theory has its origins in Mathematics and Theoretical Computer Science. Mathematically it combines order and topology. Its central concepts have their origin in the idea of approximating ideal objects by their relatively finite or, more generally, relatively compact parts.

The development of domain theory in recent years was mainly motivated by question in denotational semantics and the theory of computation. But since 2008, domain theoretical notions and methods are used in the theory of C*-algebras in connection with the Cuntz semigroup.

This paper is largely expository. It presents those notions of domain theory that seem to be relevant for the theory of Cuntz semigroups and have sometimes been developed independently in both communities. It also contains a new aspect in presenting results of Elliott, Ivanescu and Santiago on the cone of traces of a C*-algebra as a particular case of the dual of a Cuntz semigroup.

1 Introduction

Continuous lattices have emerged in quite distant areas under various disguises, and the equivalence of the different definitions is not straightforward. The two main sources are in topological algebra on the one hand and in semantics of untyped λ-calculus at the other hand.

In 1974, published in 1976 [18], K. H. Hofmann and A. R. Stralka arrived at the characterization that is now adopted generally. In this work on compact semilattices, it was their aim to characterize order theoretically those compact Hausdorff semilattices that admit a separating family of continuous semilattice homomorphism into the unit interval $[0, 1]$. These compact semilattices were also called *Lawson semilattices*. The following relation turned out to be crucial (see [18, p. 27, lines 20ff.]: For elements x, y in a complete lattice they say that x is *relatively compact in* y if every open covering $(u_i)_i$ of y (aka $y \leq \sup_i u_i$) contains a finite subcover $u_{i_1}, \ldots, u_{i_n}$ of x (aka $x \leq u_{i_1} \vee \cdots \vee u_{i_n}$). This terminology was chosen since, in the lattice of open subsets of a locally compact Hausdorff space, this relative compactness notion agrees with the common use of relative compactness in topology: An open subset V is relatively compact in an open subset W if the closure of V is compact and contained in W. The Lawson semilattices were characterized to be those complete lattices, where each element is the supremum of its relatively compact parts, and they called these lattices *relatively algebraic*. Later on, terminology changed: *relatively compact in* was replaced by the shorter *way-below*.

Two years before, in 1972, D. S. Scott's seminal paper [32] with the title *Continuous Lattices* had appeared. In this paper Scott provided the first models for the untyped λ-calculus using what he had called *continuous lattices*. It took some time until the attention of the compact semilattice community was drawn towards Scott's paper. It was only shortly before the appearance of [18] in 1976 that it was discovered that Scott's continuous lattices were precisely the relatively algebraic lattices in the sense of Hofmann and Stralka.

Continuous lattices were mainly used in denotational semantics of programming languages. In view of those applications a generalization from complete lattices to directed complete partially ordered sets (*dcpos*, for short) was needed. Because of the lack of finite suprema, the relation x way-below y had to be defined by saying that every directed family $(u_i)_i$ covering y (aka $y \leq \sup_i u_i$) contains an element u_{i_o} covering x (aka, $x \leq u_{i_o}$), and a dcpo was said to be a *continuous dcpo* (a *domain*, for short), if each of its elements y is the supremum of a directed family of elements x_i way-below y. The term 'domain' has its origin in the use of these structures as semantic domains.

The author recently discovered that domain theoretic notions and constructions are used in the theory of C*-algebras. These developments were initiated in a paper by Coward, Elliot and Ivanescu [5] in 2008. Their aim was to introduce a new invariant for C*-algebras that is finer than the K-groups. This invariant is called the Cuntz semigroup and is a kind of completion of the classical ordered semigroup introduced by J. Cuntz [6] in 1978. In [5] and the follow-up papers domain theory is not used in its classical form. A variant is considered where the set system of directed subsets is replaced by increasing sequences or, equivalently, by countable directed sets. Thus, partially ordered sets are considered in which not all directed sets but only increasing sequences are required to have a least upper bound. An element x is said to be *compactly contained in* y if, every increasing sequence u_n covering y (aka $y \leq \sup_n u_n$) contains an element u_{n_0} already covering x (aka $x \leq u_{n_0}$). The Cuntz semigroup S of a C*-algebra as introduced in [5] has the following properties among others:

(O0) S is a partially ordered commutative monoid with 0 as smallest element,

(O1) every increasing sequence has a least upper bound,

(O2) every element y is the least upper bound of an increasing sequence of elements x_n compactly contained in y,

(O3) if x_i is compactly contained in y_i for $i = 1, 2$, then $x_1 + x_2$ is compactly contained in $y_1 + y_2$,

(O4) addition is continuous in the sense that it preserves suprema of increasing sequences.

A structure with these properties is then called an *abstract Cuntz semigroup*.

A whole series of papers has appeared since that time with further developments. The author of these lines has been working in domain theory for more than 30 years. He discovered the new developments around the Cuntz semigroups through a paper by Antoine, Perera and Thiel [4]. It turns out that domain theoretical concepts and methods play a more important role than expected. Quite some properties have been rediscovered, other developments occur in parallel to developments in domain theory.

This paper is largely expository. Its purpose is to establish a common platform for communication between domain theory and the community working on Cuntz semigroups. But it also pursues a specific purpose: In 2011, Elliott, Robert and Santiago [8] have published results on the space of lower semicontinuous traces and 2-quasi-traces of C*-algebras. The proofs for the two cases seem to follow a common pattern. The same pattern can be found in a paper by Plotkin in 2009 [28] on a Banach-Alaoglu type theorems for continuous directed complete partially ordered cones. Plotkin's results and methods have been refined and generalized by the author just recently [21]. These results when specialized to abstract Cuntz semigroups give a unified proof for the results of Elliott, Robert and Santiago. For this, we show how the positive cone of a C*-algebra can be viewed as an abstract Cuntz semigroup. It is not amazing that the ingredients for our proof can all be found in the paper of Elliott, Robert and Santiago.

In this presentation, I do not adopt the countable variant of domain theory as used in the C*-algebra community. I use dcpos, the way-below relation and domains as in the monograph [16]. The future will show, if I will be convinced to change to the countable point of view. It is well-known that other subset systems can be used instead of directed sets, and quite analogous developments can be carried through. One may consult a survey by Erné [11] on such variants of domain theory.

The authors from the C*-algebra community avoid the term 'way-below' as if it would be contagious. They use 'compactly contained in', sometimes 'far below'. I do not mind other terminologies, but remain with 'way-below' from time to time, and I hope that nobody feels uneasy about it.

ACKNOWLEDGMENT I am grateful to Hannes Thiel for corrections and useful suggestions.

2 Predomains and c-spaces

We want to stress the concept of a predomain. In the same way as Hilbert spaces are completions of pre-Hilbert spaces, domains are obtained from predomains by a completion process, the round ideal completion. Domains can be defined in terms of partial orders but have a strong topological flavor. Similarly, predomains occur under two different but equivalent disguises: as relational and as topological structures.

The notion of a predomain is not new at all. It is motivated by the notion of a basis for domains. This notion has been axiomatized as a relational structure first by M. Smyth [33] (under the name of an R-structure) and it occurs under the name of an *abstract basis* in standard texts on Domain Theory, most prominently in the Handbook article by Abramsky and Jung [1, Section 2.2.6], where abstract bases are used for free constructions [1, Chapter 6]. This aspect has been rediscovered by Antoine, Perera and Thiel [4] for constructing tensor products of abstract Cuntz semigroups. The topological variant is due to Erné [9, 10] under the name of a c-space and independently to Ershov [14, 15] under the name of an α-space. It was Ershov that insisted on omitting the completeness properties required for domains. He had advocated this aspect already in his early work on computable functionals of higher type; his f-spaces and a-spaces are early manifestations (see [12, 13]).

It seems to me that these concepts have not yet attracted the attention that they deserve. The defining properties are amazingly simple and at the same time as powerful as those of domains. For this reason, I propose a new name that stresses the importance by calling them *predomains*.

2.1 Predomains

Let us concentrate first on the relational aspect. A *predomain* is a set P equipped with a binary relation $\prec\!\!\prec$ that is transitive

$$a \prec\!\!\prec b \prec\!\!\prec c \implies a \prec\!\!\prec c \tag{Trans}$$

and satisfies the following *interpolation property* for every finite subset F and every element c:

$$F \prec\!\!\prec c \implies \exists b \in P.\, F \prec\!\!\prec b \prec\!\!\prec c \tag{IP}$$

where $F \prec\!\!\prec c$ is an abbreviation for '$a \prec\!\!\prec c$ for all $a \in F$'.

For F we may choose the empty set and in this case the interpolation property says:

$$\forall c.\, \exists b.\, b \prec\!\!\prec c \tag{IP0}$$

Choosing F to be a singleton, the interpolation property above implies the ordinary interpolation property

$$a \prec\!\!\prec c \implies \exists b.\, a \prec\!\!\prec b \prec\!\!\prec c \tag{IP1}$$

Choosing F to be a two element set, the interpolation property reads:

$$a_i \prec\!\!\prec c\ (i = 1, 2) \implies \exists b.\, a_i \prec\!\!\prec b \prec\!\!\prec c\ (i = 1, 2) \tag{IP2}$$

Clearly (IP0)and (IP2) together are equivalent to (IP). We use the notation:

$$\twoheaddownarrow c = \{b \in P \mid b \prec\!\!\prec c\}, \quad \twoheaduparrow c = \{a \in P \mid c \prec\!\!\prec a\}$$

The following is our basic example:

Example 2.1. *Let X be a locally compact Hausdorff space, $C_0(X)$ the C^*-algebra of all complex valued continuous functions defined on X that vanish at infinity. Its positive cone $C_0(X)_+$ consisting of those $f \in C_0(X)$ with nonnegative real values is a poset with the usual pointwise order $f \leq g$ if $f(x) \leq g(x)$ for all x. There is a natural predomain structure on $C_0(X)_+$ defined by*

$$f \prec\!\!\prec g \text{ if } f \leq (g - \varepsilon)_+$$

where $(g - \varepsilon)_+$ is the function with value $\max(g(x) - \varepsilon, 0)$ for every $x \in X$.

For the relation $\prec\!\!\prec$ on a predomain P we use a terminology borrowed from the partially ordered sets: A subset D of P is said to be *$\prec\!\!\prec$-directed* if, for every finite subset F of D, there is an element $c \in D$ such that $F \prec\!\!\prec c$.

A subset D' of a $\prec\!\!\prec$-directed set D is said to be *$\prec\!\!\prec$-cofinal* if, for every $d \in D$ there is a $d' \in D'$ such that $d \prec\!\!\prec d'$. Such a $\prec\!\!\prec$-cofinal subset D' is also $\prec\!\!\prec$-directed. Indeed, for a finite subset $F \subseteq D' \subseteq D$ there is an $d \in D$ such that $F \prec\!\!\prec d$ and, choosing an element $d' \in D'$ such that $d \prec\!\!\prec d'$ we obtain $F \prec\!\!\prec d'$.

A subset Q of a predomain P is said to be *$\prec\!\!\prec$-dense* if, whenever $a \prec\!\!\prec c$ holds for elements in P, there is an element $b \in Q$ such that $a \prec\!\!\prec b \prec\!\!\prec c$.

Remark 2.2. *A $\prec\!\!\prec$-dense subset Q of a predomain P is a predomain when equipped with the relation $\prec\!\!\prec$ restricted to Q and, for every $c \in P$, the set $\twoheaddownarrow_Q c = \twoheaddownarrow c \cap Q$ is cofinal in $\twoheaddownarrow c$.*

Clearly the restriction of $\prec\!\!\prec$ to Q is transitive. For the interpolation property (IP) consider a finite subset F of Q and suppose $F \prec\!\!\prec c$ for some $c \in Q$, then $F \prec\!\!\prec b \prec\!\!\prec c$ for some $b \in P$ by (IP) and so we can find an element $b' \in Q$ such that $b \prec\!\!\prec b' \prec\!\!\prec c$, whence $F \prec\!\!\prec b' \prec\!\!\prec c$.

2.2 Continuous posets and domains

Let $(P, \leq)$ be a partially ordered set (*poset*, for short). For elements a, b in P we say that a is relatively compact in b (a is *way-below* b, for short) and we write $a \ll b$ if, for every directed subset D such that $b \leq \sup D$, there is an element $d \in D$ with $a \leq d$, whenever D has a least upper bound $\sup D$ in P. We say that P is a *continuous poset* if for every element $b \in P$ the set

$$\twoheaddownarrow b = \{a \in P \mid a \ll b\}$$

is directed and $b = \sup \twoheaddownarrow b$.

In a continuous poset, if $a \ll b$ and if D is a directed subset such that $b \leq \sup D$, there is a $d \in D$ such that even $a \ll d$.

If $(P, \leq)$ is a partially ordered set such that every directed subset has a supremum, we say that P is *directed complete* (a *dcpo* for short). A continuous dcpo is called a *domain*.

The relation $\ll$ in a poset P has the following properties:

$$a \ll c \quad\Longrightarrow\quad a \leq c \tag{1}$$

$$a \ll b \leq c \quad\Longrightarrow\quad a \ll c \tag{2}$$

$$d \leq a \ll b \quad\Longrightarrow\quad d \ll b \tag{3}$$

Remark 2.3. *Every continuous poset is a predomain, when equipped with its relation $\ll$.*

Proof. For transitivity, suppose that $a \ll b \ll c$. Then $b \leq c$ by property (1), whence $a \ll c$ by property (2). For the interpolation property (IP) let $F \ll a$ for a finite subset F. The family of sets $\twoheaddownarrow b$ with $b \ll a$ is directed, and each of the sets $\twoheaddownarrow b$ is directed. Thus $D = \bigcup_{b \ll a} \twoheaddownarrow b$ is directed, too, and $\sup D = a$. For every $f \in F$ we have $f \ll a$. Thus, there is an element $d_f \in D$ with $f \leq d_f$. Since D is directed, we find an element $d \in D$ such that $f \leq d$ for every element f in the finite set F. Since for $d \in D$ there is an element b such that $d \ll b \ll a$, we have $F \ll b \ll a$. $\square$

Any $\ll$-dense subset B of a continuous poset P is called a *basis* of P. By remark 2.2 every basis B is a predomain for the relation $\ll$ restricted to B; for every $c \in P$, the set $\twoheaddownarrow_B c = \twoheaddownarrow c \cap B$ is directed and cofinal in $\twoheaddownarrow c$ so that $c = \sup \twoheaddownarrow_B c$.

But there are important predomain structures for which relation $\prec\!\!\prec$ is not derived from a partial order as above. This is illustrated best by our basic example 2.1 of the cone $C_0(X)_+$ of nonnegative continuous real valued functions vanishing at infinity on a locally compact Hausdorff space X. This cone carries a natural pointwise order $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$. The predomain relation $\prec\!\!\prec$ does not agree with the relation $\ll$ on $C_0(X)_+$ derived from the partial order except for very special cases. Let us choose X to be the unit interval with its usual compact Hausdorff topology where we denote by $\mathbf{1}$ the constant function with value 1. Then $(1-\varepsilon)\mathbf{1} \prec\!\!\prec \mathbf{1}$, but $(1-\varepsilon)\mathbf{1} \not\ll \mathbf{1}$. Indeed, $f_n(x) = x^{1/n}$ is an increasing sequence of continuous functions and $\mathbf{1}$ is the least upper bound of this sequence in the poset $(C_0([0,1])_+, \leq)$ (although not the pointwise supremum) and $(1-\varepsilon)\cdot\mathbf{1}) \not\leq f_n$ for all n. Thus $(1-\varepsilon)\cdot\mathbf{1}) \not\ll \mathbf{1}$. By a similar argument one can show that there is no $f \ll \mathbf{1}$ except for the constant function 0.

The point in the example $(C_0(X)_+, \leq)$ is that there is a difference between least upper bounds in the poset $(C_0(X)_+, \leq)$ and pointwise least upper bounds. We say that a function $f\colon X \to \mathbb{R}_+$ vanishing at infinity is the pointwise supremum of an increasing sequence or a directed family of functions f_i in $C_0(X)_+$ if $f(x) = sup_i f_i(x)$ for every $x \in X$. By Dini's theorem, f is then continuous and the f_i converge to f uniformly. The predomain relation $\prec\!\!\prec$ may be defined by using this strengthened notion of pointwise least upper bound instead of the notion of a least upper bound in the poset $(C_0(X)_+, \leq)$.

It is important to consider predomain structures $\prec\!\!\prec$ not derived from partial orders as $\ll$ in the case of continuous posets. In the contrary, partial orders can be derived from predomain structures as we will see. Predomains are more general and may be more important than continuous posets.

In the same vein, I propose to replace the notion of a preCuntz semigroup as considered in [3, Definition 2.1] by a more appropriate structure: commutative predomain monoids with an additive relation $\prec\!\!\prec$ (see below 3.1).

2.3 The round ideal completion

We have seen that every domain D is a predomain for its way-below relation. More importantly, predomains occur as bases of domains. Let us see that every predomain has a completion which is a domain.

A *round ideal* is a subset J of a predomain P with the following properties: (1) J is $\prec\!\!\prec$-directed and (2) if $a \prec\!\!\prec b \in J$, then $a \in J$. This is equivalent to the requirement that a finite subset F of P is contained in J if and only if there is an element $b \in J$ such that $F \prec\!\!\prec b$.

For every element $b \in P$, the set

$$\Downarrow b = \{a \in P \mid a \prec\!\!\prec b\}$$

is a round ideal.

Proposition 2.4. *The set $\mathcal{RI}(P)$ of all round ideals of a predomain P ordered by inclusion is a domain, called the* round ideal completion *of the predomain P. The way-below relation on $\mathcal{RI}(P)$ is given by $I \ll J$ if there is an element $b \in J$ such that $I \subseteq \Downarrow b$. The round ideals $\Downarrow a$, $a \in P$, form a basis of the round ideal completion.*

Proof. Since the union of a family of round ideals that is directed under inclusion is a round ideal, the collection $\mathcal{RI}(P)$ of all round ideals is directed complete.

Given two round ideals I and J, suppose that $I \ll J$. Since J is the union of the round ideals $\Downarrow c$ with $c \in J$, we obtain $I \subseteq \Downarrow c$ for some $c \in J$. Suppose conversely that this latter condition is satisfied and suppose that J is contained in the union of a directed family of round ideals J_j. Then $c \in J_i$ for some i and consequently $\Downarrow c \subseteq J_i$. Hence $I \ll J$.

By the characterization of the way-below relation, the round ideals of the form $\Downarrow c$, $c \in P$, are $\ll$-dense in $\mathcal{RI}(P)$ and, hence, form a basis. ☐

Since the round ideals $\Downarrow c$, $c \in P$, form a basis for the round ideal completion, they form a predomain, when equipped with the restriction of the relation $\ll$ on $\mathcal{RI}(P)$. One may conjecture that $a \prec\!\!\prec b$ if, and only if, $\Downarrow a \ll \Downarrow b$. It is indeed true that $a \prec\!\!\prec b$ in P implies $\Downarrow a \ll \Downarrow b$ in $\mathcal{RI}(P)$. But the converse is not true in general as the following example shows (thus, not every predomain is the basis of a domain):

Example 2.5. *Let D be the union of $[0,1]^2$ and the segment $\{r(1,1) \mid 1 \leq r \leq 2\}$ in $\mathbb{R}^2$. On D we take the coordinatewise order. Then D is a continuous lattice with the way-below relation: $(a,b) \ll (a',b')$ iff $1 < a' = b', (a,b) \in [0,1]^2$ or $a < a' \leq 1, b < b' \leq 1$ or $a < a' \leq 1, b = b' = 0$ or $a = a' = 0, b < b' \leq 1$ or $a = a' = b = b' = 0$.*

We can weaken this way-below relation to a relation $\prec\!\!\prec$ by strengthening the first set of inequalities to $1 < a' = b', a < 1$ or $b < 1$. Thus, for example $(1,a) \ll (2,2)$ but not $(1,a) \prec\!\!\prec (2,2)$. The round ideal completion of $(D, \prec\!\!\prec)$ is the continuous lattice D.

A predomain is called *stratified* if

$$\Downarrow a \ll \Downarrow b \text{ in } \mathcal{RI}(P) \implies a \prec\!\!\prec b \text{ in } P$$

By the characterization of the relation $\ll$ in Proposition 2.4, this is equivalent to

$$\Downarrow a \subseteq \Downarrow c, c \prec\!\!\prec b \implies a \prec\!\!\prec b$$

Every predomain can be stratified by strengthening the relation $\prec\!\!\prec$ to: $a \prec\!\!\prec_s b$ iff $\Downarrow a \ll \Downarrow b$ in the round ideal completion iff there is a $c \prec\!\!\prec b$ such that $\Downarrow a \subseteq \Downarrow c$.

Replacing $\prec\!\!\prec$ by $\prec\!\!\prec_s$ on a predomain P does not change the round ideal completion. Indeed, a domain is the round ideal completion of any of its bases, and the predomain $(P, \prec\!\!\prec_s)$ may be identified with the basis of all $\Downarrow a$, $a \in P$, of the round ideal completion $\mathcal{RI}(P)$.

The containment order on round ideals induces a *natural preorder* on the predomain P: Define $a \leq a'$ if $\Downarrow a \subseteq \Downarrow a'$, that is, if $c \ll a$ implies $c \ll a'$.

The property

$$a \prec\!\!\prec c \leq b \implies a \prec\!\!\prec b \tag{2}$$

then holds for any predomain. The corresponding property

$$a \leq c \prec\!\!\prec b \implies a \prec\!\!\prec b \tag{3}$$

does not hold for predomains, in general; it holds if and only if the predomain is stratified.

Example 2.6. *Let us return to our basic example $C_0(X)_+$ of nonnegative real-valued functions vanishing at infinity defined on a locally compact Hausdorff space X viewed as predomain as in 2.1. This predomain is stratified. Its round ideal completion can be identified with the domain $\mathrm{LSC}(X)$ of all lower semicontinuous functions g from X to the one point compactification $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ of the nonnegative reals. A round ideal J of $C_0(X)_+$ is identified with the function g defined by $g(x) = \sup_{f \in J} f(x)$. The way-below relation $\ll$ on $\mathrm{LSC}(X)$ is given by $g \ll h$ if there is an $f \in C_0(X)_+$ such that $g \leq f \leq (h - \varepsilon)_+$ for some $\varepsilon > 0$.*

2.4 c-Spaces

Let us turn now to the topological variant of predomains. The topologies occurring in this context are highly non-Hausdorff. This is not a default but an essential feature. Indeed these topologies combine order and topology.

In an arbitrary topological space (X, τ) we use the *specialization preorder*: $a \leq_\tau b$ if a belongs to the closure of the singleton $\{b\}$, which is equivalent to saying that every open neighborhood of a is also a neighborhood of b. For any element x we denote by

$$\uparrow x = \{y \in X \mid x \leq_\tau y\}$$

the *saturation* of x, equivalently, the intersection of all open sets containing x. Continuous functions between topological spaces preserve the respective specialization preorders.

A *c-space* is a topological space X with the property that every element b has a neighborhood basis of sets of the form $\uparrow x$. c-Spaces have the remarkable property that separate continuity is equivalent to joint continuity, a property that has been noticed by Ershov:

Proposition 2.7. *Let X be a c-space and Y and Z arbitrary topological spaces. Then every map $f\colon X \times Y \to Z$ that is continuous separately in each of the two arguments is jointly continuous.*

Proof. Let x_0 and y_0 be elements of X and Y, respectively, and U a neighborhood of $f(x_0, y_0)$. If $x \mapsto f(x, y_0)$ is continuous, there is a neighborhood W of x_0 such that $f(x, y_0) \in U$ for every $x \in W$. Since X is a c-space, we may suppose that $W = \uparrow x_1$ for some $x_1 \in X$. Using that $y \mapsto f(x_1, y)$ is continuous, we find a neighborhood V of y_0 such that $f(x_1, y) \in U$ for all $y \in V$. Since $x \mapsto f(x, y)$ is continuous for every y, these maps preserve the specialization order. Hence, $f(x, y) \in U$ for all $(x, y) \in \uparrow x_1 \times V$ and the latter set is a neighborhood of (x_0, y_0). $\square$

2.5 Predomains and c-spaces

Every predomain $(P,\prec\!\!\prec)$ carries a natural topology $\tau_{\prec\!\!\prec}$ that turns it into a c-space: A subset $U of P$ is declared to be open, if (1) $x \in U$ and $x \prec\!\!\prec y$ imply $y \in U$ and (2) for every $x \in U$ there is an element $z \in U$ with $z \prec\!\!\prec x$, equivalently, if $U = \twoheaduparrow U$ where

$$\twoheaduparrow U = \{x \in P \mid \exists z \in U.\, z \prec\!\!\prec x\}$$

Proposition 2.8. *On a predomain $(P,\prec\!\!\prec)$ the open sets just defined form a c-space topology denoted by $\tau_{\prec\!\!\prec}$ for which the sets $\twoheaduparrow x = \{y \in P \mid x \prec\!\!\prec y\}, x \in P$ form a basis. The specialization preorder agrees with the natural preorder of $(P,\prec\!\!\prec)$.*

Proof. Clearly the union of any family of open sets in the sense just defined is open. The intersection of finitely many open sets is open by the interpolation property (IP). The sets of the form $\twoheaduparrow x$ are open by transitivity and the interpolation property (IP1) and they form a basis for the topology $\tau_{\prec\!\!\prec}$ by the interpolation property (IP2). The specialization preorder for this topology agrees with the natural preorder since $x \leq_s y$ iff every open neighborhood of x contains y iff $z \prec\!\!\prec x$ implies $z \prec\!\!\prec y$ iff $\twoheaddownarrow x \subseteq \twoheaddownarrow y$ iff $x \leq y$. Moreover every open neighborhood U of x contains an element $z \ll x$ so that $\uparrow z$ is a subset of U containing the open basic neighborhood $\twoheaduparrow z$ of x. Thus $(P, \tau_{\prec\!\!\prec})$ is a c-space. $\square$

Conversely, on a c-space (P, τ) we consider the *topological way-below relation* $a \prec\!\!\prec_\tau b$ if $\uparrow a$ is a neighborhood of b.

Proposition 2.9. *A c-space (P, τ) becomes a stratified predomain for the topological way-below relation $\prec\!\!\prec_\tau$. The natural preorder associated with the relation $\prec\!\!\prec_\tau$ agrees with the specialization preorder $\leq_\tau$.*

The topological way-below relation has the property that $a' \leq a \prec\!\!\prec_\tau b \implies a' \prec\!\!\prec_\tau b$, that is, it satisfies property (3). Thus, $(P,\prec\!\!\prec_\tau)$ is a stratified predomain.

The two constructions almost yield a one-to-one correspondence between predomains and c-spaces. Starting with a predomain $(P,\prec\!\!\prec)$, then passing to the c-space topology $\tau_{\prec\!\!\prec}$ and then extracting the topological way-below relation yields the stratification of the original relation $\prec\!\!\prec$. Starting with a c-space, extracting its topological way-below relation and forming then the associated c-space topology gives back the original c-space topology.

On a domain, the c-space topology agrees with the Scott topology. Indeed, on a domain the sets of the form $\twoheaduparrow a$ form a basis of the Scott topology [16, Theorem II-1.14].

2.6 Countability conditions

A predomain P is said to be *first countable* if, for every element b, the round ideal $\twoheaddownarrow b$ has a countable $\prec\!\!\prec$-cofinal subset. This is equivalent to the requirement that there is a sequence $a_1 \prec\!\!\prec a_2 \prec\!\!\prec \dots$ which is $\prec\!\!\prec$-cofinal in $\twoheaddownarrow b$. We say that P is *second countable* or *countably based* if it has a countable $\prec\!\!\prec$-dense subset B.

Our basic example $(C_0(X)_+,\prec\!\!\prec)$ 2.1 is first countable choosing $f_n = (f - \frac{1}{n})_+$, $n \in \mathbb{N}$, but not second countable in general.

The first and second countability conditions for predomains correspond to first and second countability of the corresponding c-space topology $\tau_{\prec\!\!\prec}$, respectively.

The round ideal completion of a countably based predomain is countably based. But the round ideal completion of a first countable predomain need not be first countable:

Example 2.10. *Our basic example $(C_0(X)_+,\prec\!\!\prec)$ 2.1 is first countable. Choosing $f_n = (f - \frac{1}{n})_+$, $n \in \mathbb{N}$, we obtain a countable cofinal subset in the round ideal $\twoheaddownarrow f$. The round ideal completion need not be first countable. As an example, let X be an uncountable discrete space. The round ideal completion of $C_0(X)_+$ is the domain of all maps g from X to $\overline{\mathbb{R}}_+$. The maps $f \ll g$ are those maps f with finite support that satisfy $f(x) < g(x)$ for all x in the support of f. Because of the uncountablity of X, there cannot be a cofinal countable subset among the functions f way-below, for example, the constant function 1.*

A round ideal I will be said to be countably generated or simply a round ω-ideal if it contains a sequence

$$a_1 \prec\!\!\prec a_2 \ll \dots$$

such that, for every $b \in I$, there is an n such that $b \ll a_n$. If P is first countable, then $i(a) = \Downarrow a$ is a round ω-ideal and we may form the ω-completion, the collection $\omega\mathcal{RI}(P)$ of all round ω-ideals which is ω-complete in the sense that the union of every increasing sequence of round ω-deals is a round ω-ideal. The round ω-ideal completion is an *ω-domain*. By this we mean that every element a is the supremum of a chain $a_1 \ll_\omega a_2 \ll_\omega \dots$, where $b \ll_\omega a$ if, for every sequence $b_1 \leq b_2 \leq \dots$ with $a \leq \sup_n b_n$, there is an n such that $b \leq b_n$. In a first countable predomain there may exist round ideals that are not countably generated as we have seen in 2.10. Another example is given by the ordered set Ω of countable ordinals. Here, the set Ω itself is a round ideal that is not countably generated.

In the literature related to the Cuntz semigroup, first countability is always required following Coward, Elliott and Santiago [5].

2.7 Morphisms

For predomains it is natural to consider maps $f\colon P \to Q$ that *preserve the relation* $\prec\!\!\prec$, that is $a \prec\!\!\prec b$ implies $f(a) \prec\!\!\prec f(b)$. For the associated c-space topology that is equivalent to saying that $\uparrow f(U)$ is open for every open subset U. Maps between topological spaces will be called *open*, if they satisfy this property. (In topology, a map is called open if the image of every open set is open. We have modified this definition, but in such a way that for T_1 spaces the new definition agrees with the old one. For T_0-spaces this new definition looks more appropriate.)

As for topological spaces in general, for c-spaces it is natural to consider continuous maps. Continuous maps preserve the respective specialization preorders, but not the topological way-below relations. Accordingly, a map $f\colon P \to Q$ between predomains will be called *continuous*, if it is continuous for the respective c-space topologies. This is equivalent to the condition:

$$\forall b \in P.\, \forall c \in Q.\; c \prec\!\!\prec f(b) \implies \exists a \in P.\; a \prec\!\!\prec b \text{ and } c \prec\!\!\prec f(a)$$

The canonical map $i\colon a \mapsto \Downarrow a$ from a predomain P into its round ideal completion preserves the relation $\prec\!\!\prec$. It also is continuous. Indeed, if I is a round ideal with $I \ll \Downarrow a$, then there is an element $b \in \Downarrow a$ such that $I \subseteq \Downarrow b$. As a consequence, $i\colon a \mapsto \Downarrow a$ is a topological embedding.

On posets and dcpos we use Scott continuity. A map between posets is said to be *Scott continuous* if it is monotone and preserves existing suprema of directed sets[1]. This order theoretic notion of continuity is equivalent to continuity with respect to a topology, the Scott topology. The closed sets of the *Scott topology* of a poset are those lower sets that are closed for suprema of directed subsets, as far as they exist.

On domains one can consider the Scott topology and the associated c-space topology. Fortunately the two topologies agree so that there is no ambiguity when talking about continuity of function from or into domains.

2.8 Universality of the round ideal completion

The canonical map $i\colon a \mapsto \Downarrow a$ from a predomain $(P, \prec\!\!\prec)$ into its round ideal completion is continuous and preserves the relation $\prec\!\!\prec$. Both properties are consequences of the characterization of the way-below relation on the round ideal completion: Indeed, if I is a round ideal with $I \ll \Downarrow a$, then there is an element $c \in \Downarrow a$ such that $I \subseteq \Downarrow c$ which shows continuity. For the preservation of $\prec\!\!\prec$, let $a \prec\!\!\prec b$; interpolate an element $a \prec\!\!\prec c \prec\!\!\prec b$ and we have $\Downarrow a \subseteq \Downarrow c$ and $c \in \Downarrow b$, that is $\Downarrow a \ll \Downarrow b$.

The round ideal completion of a predomain has the desired universal property:

[1] It is interesting to remark that Hofmann and Stralka in their 1976 paper [18, Definition 1.29] had proposed to call *normal* those maps that preserve existing directed suprema in analogy to the terminology used for W^*-algebras

Proposition 2.11. *For every continuous map f from a predomain $(P,\prec\!\!\prec)$ into a dcpo Q (with the Scott topology), there is a unique continuous map $\widehat{f}\colon \mathcal{RI}(P) \to Q$ such that $\widehat{f}(\twoheaddownarrow a) = f(a)$:*

$$
\begin{tikzcd}
P \arrow[r, "\twoheaddownarrow"] \arrow[dr, "f"'] & \mathcal{RI}(P) \arrow[d, "\widehat{f}"] \\
& Q
\end{tikzcd}
$$

If Q is a domain, the continuous extension $\widehat{f}$ preserves $\ll$ if and only if f preserves $\ll$.

Proof. For uniqueness suppose that $\widehat{f}\colon \mathcal{RI}(P) \to Q$ is a continuous map satisfying $\widehat{f}(\twoheaddownarrow a) = f(a)$. Any round ideal J is the union of the directed family of $\twoheaddownarrow a$, $a \in J$. Thus $\widehat{f}(J) = \widehat{f}(\bigcup_{a\in P} \twoheaddownarrow a) = \sup_{a\in P} \widehat{f}(\twoheaddownarrow a)$ (by continuity) $= \sup_{a\in P} f(a)$.

We now define $\widehat{f}$ by $\widehat{f}(J) = \sup_{a\in J} f(a)$ for every round ideal J. We first remark that $\widehat{f}$ is well defined, since for a round ideal J of P the image $f(J)$ is directed. Indeed, a continuous map preserves the specialization order; the specialization order on a predomain is the order $a \leq b$ iff $\twoheaddownarrow \subseteq \twoheaddownarrow b$ and the specialization order for the Scott topology on a dcpo is the given order.

We now check that $\widehat{f}$ is Scott-continuous. If J_i is a directed family of round ideals the $J = \bigcup_i J_i$ is its supremum in the domain of round ideals and $\widehat{f}(J) = \sup_{a\in J} f(a) = \sup_i \sup_{a\in J_i} f(a) = \sup_i \widehat{f}(J_i)$.

Now suppose that Q is a continuous dcpo. If $\widehat{f}$ preserves $\ll$, then $f = \widehat{f} \circ \twoheaddownarrow$ preserves $\ll$. Conversely, suppose that f preserves $\ll$. Let $I \ll J$ in $\mathcal{RI}(A)$. There is an $a \in J$ such that $b \ll a$ for all $b \in I$. Let $a \ll a' \in J$. Then $\widehat{f}(I) = \sup_{b\in I} f(b) \leq f(a) \ll f(a') \leq \bigvee^{\uparrow}_{b\in J} f(b) = \widehat{f}(J)$. Thus, $\widehat{f}$ preserves $\ll$. $\square$

For a continuous map f from a predomain P to a predomain Q, the composition $\twoheaddownarrow \circ f\colon P \to \mathcal{RI}(Q)$ is continuous, too. By the preceding proposition, there is a unique continuous map $\mathcal{RI}(f)\colon \mathcal{RI}(P) \to \mathcal{RI}(Q)$ such that $\mathcal{RI}(f) \circ \twoheaddownarrow = \twoheaddownarrow \circ f$:

$$
\begin{tikzcd}
P \arrow[r, "\twoheaddownarrow"] \arrow[d, "f"'] & \mathcal{RI}(P) \arrow[d, "\mathcal{RI}(f)"] \\
Q \arrow[r, "\twoheaddownarrow"'] & \mathcal{RI}(Q)
\end{tikzcd}
$$

and this map is defined by $\mathcal{RI}(f)(J) = \bigcup_{a\in J} \twoheaddownarrow f(a) = \twoheaddownarrow f(J)$. Moreover, $\mathcal{RI}(f)$ preserves $\ll$ if and only if f does. In this way, $\mathcal{RI}$ becomes a functor from the category of predomains and continuous maps to the category of domains and continuous maps. It restricts to a functor if one restricts to continuous maps preserving $\ll$.

From a topological point of view, the round ideal completion $\mathcal{RI}(X)$ of a c-space X can also be seen to be the D-completion in the sense of [22, Proposition 9.1] and equivalently as the sobrification [22, Proposition 10.2] . Thus it has a more general universal property than the one shown above: For every continuous map f from X into a monotone convergence space Y (in particular, into every sober space), there is a unique continuous map $\widehat{f}\colon \mathcal{RI}(X) \to Y$ such that $\widehat{f}(\twoheaddownarrow x) = f(x)$ for all $x \in X$ [22, Theorem 6.7]. But we will not use this more general point of view in this paper.

Not all continuous maps from $\mathcal{RI}(P)$ to $\mathcal{RI}(Q)$ are induced by continuous maps from P to Q, but only those that map the basis P to the basis Q. As the continuous maps from $\mathcal{RI}(P)$ to $\mathcal{RI}(Q)$ are in one-to-one correspondence with the continuous maps from P to $\mathcal{RI}(Q)$, we may view these maps F as set-valued maps from P to Q, where $F(x)$ is a round ideal of Q for every $x \in P$. Alternatively we may view these maps as a relation $R \subseteq P \times Q$ where $(x, y) \in R$

if $y \in F(x)$. It is not difficult to axiomatize such relations; one has to write down, firstly, that the set of all y such that $(x, y) \in R$ form a round ideal and, secondly, that F is continuous. One finds such axioms in [1, Definition 2.2.27].

3 PreCuntz semigroups and their duals

In this paper, a *monoid* $(C, +, 0)$ will always be understood to be commutative. Thus, $+$ is a commutative associative operation with neutral element 0. A *monoid homomorphism* is a map f between monoids such that $f(0) = 0$ and $f(x + y) = f(x) + f(y)$.

A *cone* is a monoid $(C, +, 0)$ endowed with a scalar multiplication by real numbers $r > 0$ which satisfies the identities: $1 \cdot a = a, (rs)a = r(sa), (r + s)a = ra + sa, r(a + b) = ra + rb$. One may extend the scalar multiplication to $r = 0$ by defining $0 \cdot a = 0$, and the above laws for the scalar multiplication remain valid. A *linear* map is a map f between cones which is a monoid homomorphism and satisfies $f(rx) = rf(x)$.

3.1 PreCuntz and Cuntz semigroups

A *predomain monoid* $(C, +, 0, \prec\!\!\prec)$ is a monoid endowed with the structure of a predomain in such a way that addition is continuous.

We will say that the relation $\prec\!\!\prec$ is *additive* if

$$0 \prec\!\!\prec a \quad \text{and} \quad a \prec\!\!\prec a', b \prec\!\!\prec b' \implies a + b \prec\!\!\prec a' + b'$$

A predomain monoid in which the relation $\prec\!\!\prec$ is additive will also be called a *preCuntz semigroup*.

Since for c-spaces separate continuity implies joint continuity by Proposition 2.7, it suffices to require addition to be separately continuous, that is, the maps $x \mapsto a + x$ to be continuous for every a. But the additivity of the relation $\prec\!\!\prec$ has to be understood jointly as defined above. It is not sufficient to require that $a \prec\!\!\prec a'$ implies $a + b \prec\!\!\prec a' + b$.

A directed complete partially ordered monoid (a dcpo-monoid, for short) is a monoid with a directed complete partial order such that the addition is (Scott-) continuous. If the underlying dcpo is a domain, we say that it is a *domain monoid*. A domain monoid with an additive way-below relation will be called a *Cuntz semigroup*. Of course, a Cuntz semigroup is also preCuntz; we just have to concentrate at the way-below relation.

Let us look at the round ideal completion of a preCuntz semigroup C. For two round ideals I and J define

$$I + J = \bigcup_{a \in I, b \in J} \mathop{\downarrow\!\!\downarrow}(a + b)$$

Then $I + J$ is a round ideal. Indeed, if $c \in I + J$, then $c \ll a + b$ for some $a \in I, b \in J$. There are $a' \in I$ and $b' \in J$ with $a \prec\!\!\prec a'$ and $b \prec\!\!\prec b'$. By the additivity of the way-below relation we obtain $c \prec\!\!\prec a + b \prec\!\!\prec a' + b'$, whence $c \in \mathop{\downarrow\!\!\downarrow}(a' + b')$, that is, $c \in I + J$. Using Proposition 2.4 the following is easily verified:

Proposition 3.1. *The round ideal completion $\mathcal{RI}(C)$ of a preCuntz semigroup C is a Cuntz semigroup. The map $a \mapsto \mathop{\downarrow\!\!\downarrow} a \colon C \to \mathcal{RI}(C)$ is a continuous monoid homomorphism preserving $\ll$.*

Proof. It just remains to verify that the continuous extension $\widehat{f}$ according to 2.11 is a monoid homomorphism: $\widehat{f}(I+J) = \widehat{f}(\bigcup_{a \in I, b \in J} \mathop{\downarrow\!\!\downarrow}(a + b) = \sup_{a \in I, b \in J} \widehat{f}(\mathop{\downarrow\!\!\downarrow}(a + b)) = \sup_{a \in I, b \in J} f(a + b) = \sup_{a \in I, b \in J} f(a) + f(b) = \sup_{a \in I} f(a) + \sup_{b \in J} f(b) = \sup_{a \in I} \widehat{f}(\mathop{\downarrow\!\!\downarrow} a) + \sup_{b \in J} \widehat{f}(\mathop{\downarrow\!\!\downarrow} b) = \widehat{f}(I) + \widehat{f}(J)$. ∎

The round ideal completion of a preCuntz semigroup we has the expected universal property:

Proposition 3.2. *If $f \colon C \to D$ is a continuous monoid homomorphism from a preCuntz semigroup C into a dcpo monoid D, the unique continuous extension $\widehat{f} \colon \mathcal{RI}(C) \to D$ satisfying $\widehat{f}(\mathop{\downarrow\!\!\downarrow} a) = f(a)$ for all $a \in C$ according to the universal property 2.11 is a monoid homomorphism.*

Proof. It just remains to verify that the continuous extension $\widehat{f}$ according to 2.11 is a monoid homomorphism: $\widehat{f}(I+J) = \widehat{f}(\bigcup_{a\in I,b\in J} \Downarrow(a+b) = \sup_{a\in I,b\in J} \widehat{f}(\Downarrow(a+b)) = \sup_{a\in I,b\in J} f(a+b) = \sup_{a\in I,b\in J} f(a) + f(b) = \sup_{a\in I} f(a) + \sup_{b\in J} f(b) = \sup_{a\in I} \widehat{f}(\Downarrow a) + \sup_{b\in J} \widehat{f}(\Downarrow b) = \widehat{f}(I) + \widehat{f}(J)$. ☐

Corollary 3.3. *For every continuous monoid homomorphism $f\colon C \to D$ of preCuntz semigroups there is a unique continuous monoid homomorphism $\mathcal{RI}(f)\colon \mathcal{RI}(C) \to \mathcal{RI}(D)$ such that $\mathcal{RI}(f)(\Downarrow a) = \Downarrow f(a)$ for all $a \in C$, and $\mathcal{RI}(f)$ preserves $\ll$ if and only if f does.*

We now add scalar multiplication, thus passing from monoids to cones. The properties stated before remain valid. A *preCuntz cone* is a preCuntz monoid which is also a cone such that scalar multiplication is continuous as a map $\mathbb{R}_{>0} \times C \to C$, where $\mathbb{R}_{>0}$ is considered as a predomain with $<$ as its approximation relation. In a preCuntz cone the map $x \mapsto rx\colon C \to C$ is continuous and it has a continuous inverse $x \mapsto r^{-1}x$, that is, it is a homeomorphism. It follows that $x \prec\!\!\prec y$ implies $rx \prec\!\!\prec ry$ for $0 < r < +\infty$.

The round ideal completion of a preCuntz cone is a Cuntz cone with a universal property analogous to the universal property of the round ideal completion of a preCuntz semigroup: A dcpo-cone is understood to be a dcpo-monoid D which is also a cone in such a way that the scalar multiplication $\mathbb{R}_{>0} \times D \to D$ is Scott-continuous. For every linear map f from a preCuntz cone C to a dcpo-cone D the unique continuous extension $\widehat{f}\colon \mathcal{RI}(C) \to D$ is linear.

Example 3.4. *The nonnegative real numbers form a preCuntz cone $\mathbb{R}_+$ and $\overline{\mathbb{R}}_+$ is its round ideal completion. In both cases the approximation relation is $r \prec\!\!\prec s$ if $r < s$ or $r = s = 0$. The addition is the usual one, extended by $r + \infty = +\infty$.*

Example 3.5. *Our basic example $C_0(X)_+$ for a locally compact Hausdorff space X is a preCuntz cone with the usual pointwise defined addition of functions. But notice that $f \prec\!\!\prec f'$ does not imply $f + g \prec\!\!\prec f' + g$. For example $(x - \frac{1}{2})_+ \prec\!\!\prec (x - \frac{1}{4})_+$, but $x + (x - \frac{1}{2})_+ \not\prec\!\!\prec x + (x - \frac{1}{4})_+$. Also $r \ll s$ does not imply $rf \ll sf$ (for example if $X = [0, 1]$ and $f(x) = x$, then $rf \not\ll sf$, whenever $0 < r < s$).*

The round ideal completion of $(C_0(X)_+, \ll)$ is $\mathrm{LSC}(X)_+$, the set of all lower semicontinuous maps $f\colon X \to \overline{\mathbb{R}}_+$. Here the way below relation is given by $g \ll h$ if there is a $f \in C_0(X)_+$ and an $\varepsilon > 0$ such that $g \leq (f - \varepsilon)_+, f \leq h$.

As a predomain, $C_0(X)$ is first countable, since $(g - \frac{1}{n})_+ \ll (g - \frac{1}{n+1})_+$ and since for every $f \ll g$ we have $f \ll (g - \frac{1}{n})_+$ for some n. I do not think that the round ω-ideal completion is what one wants to consider here, except for those cases where it agrees with $\mathrm{LSC}(X)_+$.

Remark 3.6. The notion of an abstract Cuntz semigroup has been introduced by Coward, Elliott and Ivanescu [5]. First countable preCuntz semigroups have been introduced by Antoine, Perera and Thiel under the name of a pre-W-semigroup [4, Section 2.1]. They construct their ω-round ideal completions and prove a universal property of this construction [4, Chapter 3].

3.2 Topologies on posets and function spaces

Let L be a poset, the order relation being denoted by $\leq$. Denote by L^{op} the same set with the opposite order $\geq$. Besides the Scott topology σ on L one may consider the dual Scott topology σ^{op}, the Scott topology of L^{op}. We are interested in three other topologies on L that look quite simple at a first glance.

The *upper topology* τ_{up} has the principal ideals $\downarrow b = \{y \in L \mid y \leq b\}$, $b \in L$, as a subbasis for the closed sets.

The *lower topology* τ_{lo} has the principal filters $\uparrow a = \{y \in L \mid y \geq a\}$, $a \in L$, as a subbasis for the closed sets

The *interval topology* τ_{iv} is generated by the upper and the lower topology. The closed intervals $[a, b] = \uparrow a \cap \downarrow b$ are closed sets.

We will use the following general observation: For complete lattices L and M, a map $\beta\colon L \to M$ preserving arbitrary suprema has a lower adjoint $\alpha\colon M \to L$ defined by $\alpha(y) = \sup\{x \in L \mid f(x) \leq y\}$. Then α preserves arbitrary meets and $\beta^{-1}(\downarrow y) = \downarrow\alpha(y)$ which shows that β is continuous for the respective upper topologies. Similarly, α is continuous for the respective upper topologies.

On $\overline{\mathbb{R}}_+$, the extended nonnegative reals, the proper open sets of the upper topology are the intervals $]r, +\infty]$, the proper open sets for the lower topology are the intervals $[0, r[$, and the interval topology is the usual compact Hausdorff topology with the open intervals $]r, s[$ as a basis for the open sets. The analogous statement holds for subsets of $\overline{\mathbb{R}}_+$ as the set of nonnegative reals $\mathbb{R}_+$ and the set $\mathbb{R}_{>0}$ of positive reals.

In agreement with classical analysis, a function from a space X into $\overline{\mathbb{R}}_+$ is lower semicontinuous[2] if and only if it is continuous with respect to the upper topology on $\overline{\mathbb{R}}_+$. We are interested in a special case.

Let $(P, \prec\!\!\prec)$ be a predomain with its c-space topology and its natural preorder. We denote by $\mathrm{LSC}(P)$ the set of all lower semicontinuous functions $f\colon P \to \overline{\mathbb{R}}_+$ ordered pointwise: $f \leq g$ if $f(x) \leq g(x)$ for all $x \in P$. We want to look at the intrinsic upper, lower and interval topology on this function space.

Proposition 3.7. *For a predomain P the function space $\mathrm{LSC}(P)$ has the following properties:*

(a) A subbasis for the upper topology of $\mathrm{LSC}(P)$ is given by:

$$V_{x,r} = \{f \in \mathrm{LSC}(P) \mid f(x) > r\},\ x \in P, r \in \mathbb{R}_+$$

A net $(f_i)_i$ of functions in $\mathrm{LSC}(P)$ converges to $f \in \mathrm{LSC}(P)$ for the upper topology if and only if:

$$f(x) \leq \liminf_i f_i(x) \text{ for all } x \in P \tag{upConv}$$

(b) A subbasis for the lower topology is given by:

$$W_{y,r} = \{f \in \mathrm{LSC}(P) \mid f(x) < r \text{ for some } x \in \Uparrow y\},\ y \in P, r \in \mathbb{R}_+.$$

A net $(f_i)_i$ of functions in $\mathrm{LSC}(P)$ converges to $f \in \mathrm{LSC}(P)$ for the lower topology if and only if:

$$\limsup_i f_i(y) \leq f(x) \ \text{ whenever } y \prec\!\!\prec x \text{ in } P \tag{loConv}$$

(c) For the interval topology and the pointwise order, $\mathrm{LSC}(P)$ is a compact ordered space. A net $(f_i)_i$ of functions in $\mathrm{LSC}(P)$ converges to $f \in \mathrm{LSC}(P)$ for the interval topology if and only conditions (upConv) *and* (upConv) *hold.*

The following developments contain a proof for the proposition.

We begin with a set P and we consider the power $\overline{\mathbb{R}}_+^P$ of all functions $g\colon P \to \overline{\mathbb{R}}_+$. With respect to the pointwise order, $\overline{\mathbb{R}}_+^P$ is a complete lattice. Suprema and infima of arbitrary families of functions are formed pointwise. The lower, upper and interval topology on $\overline{\mathbb{R}}_+^P$ agree with the product topologies of the lower, upper and interval topology on $\overline{\mathbb{R}}_+$, respectively. A net $(f_i)_i$ in $\overline{\mathbb{R}}_+^P$ converges to f for the upper (resp., lower) topology if and only if does so pointwise, that is, if and only if for every $x \in P$,

$$f(x) \leq \liminf_i f_i(x) \text{ (resp., } \limsup_i f_i(x) \leq f(x)) \tag{Conv}$$

As a product of compact ordered spaces, $\overline{\mathbb{R}}_+^P$ is a compact ordered space for the interval topology.

Suppose now that P is a preordered set and consider the collection $\mathrm{MON}(P) \subseteq \overline{\mathbb{R}}_+^P$ of all monotone functions. Since pointwise suprema and infima of monotone functions are monotone, $\mathrm{MON}(P)$ is a complete sublattice of $\overline{\mathbb{R}}_+^P$. Its intrinsic

[2] It looks incoherent to call a function *lower* semicontinuous if it is continuous for the *upper* topology on $\overline{\mathbb{R}}_+$. But this is unavoidable if one want to stay coherent with the use of lower semicontinuity in analysis and the terminology for topologies used in [].

lower, upper and interval topology agree with the subspace topology induced by the lower, upper and interval topology on $\overline{\mathbb{R}}_+^P$. Convergence is characterized as above and $\mathrm{MON}(P)$ is closed in $\overline{\mathbb{R}}_+^P$, hence, a compact ordered space, for the interval topology,

We now specialize further and suppose that P is a topological space with its specialization preorder. The lower semicontinuous functions $f\colon P \to \overline{\mathbb{R}}_+$ form a subset $\mathrm{LSC}(P)$ of $\mathrm{MON}(P)$, since continuous functions preserve the specialization preorder. The pointwise supremum of a family of lower semicontinuous functions is again lower semicontinuous, that is, the canonical injection of $\mathrm{LSC}(P)$ into $\mathrm{MON}(P)$ preserves arbitrary suprema. It follows that $\mathrm{LSC}(P)$ is a complete lattice, too, and that the lower adjoint $env\colon \mathrm{MON}(P) \to \mathrm{LSC}(P)$ that assigns to every order preserving map $g\colon P \to \overline{\mathbb{R}}_+$ its lower semicontinuous envelope $env(g) = \sup\{f \in \mathrm{LSC}(P) \mid f \leq g\}$ preserves arbitrary infima. The lower semicontinuous envelope $env(g)$ is also given by

$$env(g)(x) = \liminf_{\mathfrak{u}_x} f(x) = \sup_{U \in \mathfrak{u}_x} \inf_{z \in U} f(z) \tag{Env}$$

for every $x \in P$, where $\mathfrak{u}_x$ is any neighborhood basis of x. The intrinsic upper topology of the lattice $\mathrm{LSC}(P)$ is the subspace topology induced by the upper topology on $\overline{\mathbb{R}}_+^P$. Indeed, if $g \in \mathrm{MON}(P)$ and $f \in \mathrm{LSC}(P)$, then:

$$f \leq g \text{ if and only if } f \leq env(g) \tag{Adj1}$$

Thus convergence in $\mathrm{LSC}(P)$ with respect to the upper topology is characterized by condition (upConv). This proves claim (a).

Infima in $\mathrm{LSC}(P)$ are not formed pointwise, in general. The infimum in $\mathrm{LSC}(P)$ of a family of functions f_i is the lower semicontinuous envelope of the pointwise infimum. The intrinsic lower topology of the lattice $\mathrm{LSC}(P)$ need no longer be the subspace topology induced by the lower topology on $\overline{\mathbb{R}}_+^P$; it can be strictly finer.

We now suppose that P is a predomain. By definition, a subbasis for the closed sets for the lower topology in $\mathrm{LSC}(P)$ is given by the sets $\uparrow h = \{f \in \mathrm{LSC}(P) \mid h \leq f\}$ where h ranges over $\mathrm{LSC}(P)$. If $f \notin \uparrow h$, there is an $x_0 \in P$ such that $f(x_0) < h(x_0)$. Choose r such that $f(x_0) < r < h(x_0)$. By lower semicontinuity, there is a $y \ll x_0$ such that $r < h(y)$. Thus $f \in W_{y,r)} = \{g \in \mathrm{LSC}(P) \mid g(x) < r \text{ for some } x \in \Uparrow y\}$ and $W_{y,r)}$ is disjoint from $\downarrow h$. Moreover $W_{y,r}$ is open for the lower topology, since it is the complement of the subbasic lower closed set of all $f \in \mathrm{LSC}(P)$ below the simple lower semicontinuous function $r\chi_{\Uparrow y}$ which has value r if $x \in \Uparrow u$ and value 0 else. Thus the sets $W_{y,r}$ form a subbasis for the lower topology of $\mathrm{LSC}(P)$.

Lemma 3.8. *For every monotone map g from a predomain P to $\overline{\mathbb{R}}_+$, the lower semicontinuous envelope*[3] *is given by*

$$env(g)(x) = \sup_{y \ll x} g(y)$$

for all $x \in P$ and the map $env\colon \mathrm{MON}(P) \to \mathrm{LSC}(P)$ preserves not only arbitrary infima but also arbitrary suprema.

Proof. In a predomain, an element x has a neighborhood basis of principal filters $\uparrow y$ with $y \ll x$. If g is monotone, we have that $\inf_{z \in \uparrow y} f(z) = f(y)$ and the above formula for the lower semicontinuous envelope simplifies to $env(g)(x) = \sup_{y \ll x} f(y)$.

We now take a family of monotone functions $g_i\colon P \to \overline{\mathbb{R}}_+$ and we show that $env(\sup_i g_i) = \sup_i env(g_i)$. Using the formula for $env(g)$ just proved we have indeed, $env(\sup_i g_i)(x) = \sup_{y \ll x} \sup_i g_i(y) = \sup_i \sup_{y \ll x} g_i(x) = \sup_i env(g_i)(x)$. ☐

Since the map env maps preserves arbitrary infima and arbitrary suprema, it is continuous for the respective lower, upper and interval topologies. It also has a lower adjoint α characterized by

$$g \leq \alpha(f) \text{ if and only if } env(g) \leq f \tag{Adj2}$$

[3] The lower semicontinuous envelope as given by formula (Env) is standard in analysis. The formula given in the special situation of this lemma is standard in Domain Theory (see, e.g., [16,], cite[]dom). It has been rediscovered in [8, Lemma 4.7], [29, Lemma 2.2.1].

for $f \in \mathrm{LSC}(P)$ and $g \in \mathrm{MON}(P)$. Explicitly, $\alpha(f) = \sup\{g \in \mathrm{MON}(P) \mid env(g) \leq f\}$.

We now finish the proof of claim (b) by considering a net $(f_i)_i$ in $\mathrm{LSC}(P)$. Suppose firstly that the net f_i converges to some $f \in \mathrm{LSC}(P)$ for the lower topology. Since α is a lower adjoint, it is continuous for the lower topologies so that the net $\alpha(f_i)$ converges to $\alpha(f)$ for the lower topology in $\mathrm{MON}(P)$. This means that $\limsup_i \alpha(f_i)(x) \leq \alpha(f)(x)$ for every $x \in P$ by condition (Cond). Passing to the lower semicontinuous envelope on both sides yields $sup_{y \prec\!\!\prec x} \limsup_i \alpha(f_i)(y) \leq f(x)$ hence $\limsup_i \alpha(f_i)(y) \leq f(x)$ whenever $y \prec\!\!\prec x$ as in claim (b). Suppose conversely that the latter property holds. In order to prove that the net $(f_i)_i$ converges to f we take any subbasic neighborhood $W_{y,r}$ of f. Then f satisfies $f(x_0) < r$ for some $x_0 \in \Uparrow y$. Choose any x such that $y \prec\!\!\prec x \prec\!\!\prec x_0$. Since $\limsup_i f_i(z) \leq f(x_0) < r$, there is an index j such that $f_i(z) < r$ for all $i \geq j$ and we conclude that $f_i \in W_{y,r}$ for all $i \geq j$.

In order to prove claim (c) we first observe that $\mathrm{LSC}(P)$ is compact for the interval topology, since $\mathrm{MON}(P)$ is compact for the interval topology and the map $env\colon \mathrm{MON}(P) \to \mathrm{LSC}(P)$ is continuous. The following lemma shows that the order in $\mathrm{LSC}(P)$ is closed for the interval topology so that $\mathrm{LSC}(P)$ is a compact ordered space.

Lemma 3.9. *Let $f \not\leq h$ in $\mathrm{LSC}(P)$. Then there is a subbasic upper open neighborhood V of f disjoint from some subbasic lower open neighborhood W of h.*

Proof. Since $f \not\leq h$. There is an x_0 such that $f(x_0) > h(x_0)$. Choose an r with $f(x_0) > r > h(x_0)$. By lower semicontinuity, there is a $y \prec\!\!\prec x_0$ such that $f(y) > r$. Now let $V_{y,r}$ be the sets of all $f \in \mathrm{LSC}(P)$ such that $f(y) > r$ and $W_{y,r}$ the set of all $f \in \mathrm{LSC}(P)$ such that $f(x) < r$ for some x with $y \prec\!\!\prec x$. Then $V_{y,r}$ and $W_{y,r}$ are disjoint subbasic open sets for the upper and lower topology, respectively, containing f and h, respectively. ☐

On $\overline{\mathbb{R}}_+$ addition is jointly continuous with respect to each of the three topologies (upper, lower and interval topology) as a map $\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$. Multiplication is jointly continuous as a map $\mathbb{R}_{>0} \times \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ for these three topologies[4]. Thus $\overline{\mathbb{R}}_+$ is a topological cone with respect to all of the three topologies (upper, lower and interval topology), where a *topological cone* is a cone C with a topology such that addition and scalar multiplication are jointly continuous as maps $C \times C \to C$ and $\mathbb{R}_{>0} \times C \to C$, respectively. This definition has to be read with caution: The question which topology to use on $\mathbb{R}_{>0}$; one has to use the upper, lower and interval topology, respectively, in agreement with the topology used on C.

Since $\overline{\mathbb{R}}_+$ is a topological cone, the power $\overline{\mathbb{R}}_+^P$ is a topological cone, too, for the pointwise defined addition and multiplication with real numbers $r > 0$, and this for each of the three topologies (lower, upper and interval topology). For a preordered set P, the monotone functions form a subcone $\mathrm{MON}(P)$. For a topological space P, the sum $f + g$ of two lower semicontinuous functions $f, g \in \mathrm{LSC}(P)$ and the scalar multiple rf for $0 < r < +\infty$ are lower semicontinuous, too. Thus $\mathrm{LSC}(P)$ is a subcone of $\mathrm{MON}(P)$. Furthermore, if P is a predomain, the map $env\colon \mathrm{MON}(P) \to \mathrm{LSC}(P)$ is linear. This is easily verified using the formula for the lower semicontinuous envelope in Lemma 3.8; but there is also a general argument that we present after the statement of the next proposition. We conclude:

Proposition 3.10. *Let P be a predomain. $\mathrm{MON}(P)$ and $\mathrm{LSC}(P)$ are ordered topological cones for their intrinsic upper, lower and interval topologies, respectively. The map $env\colon \mathrm{MON}(P) \to \mathrm{LSC}(P)$ is linear, monotone and continuous for each of the three topologies.*

Recall that, in the previous proposition, according to our definition of a topological cone, the set $\mathbb{R}_{>0}$ of positive scalars has to be equipped with the respective upper, lower, and interval topology.

We will use the following observation several times:

Observation 3.11. *Let C and D be cones each with a topology that agrees with the upper topology on the rays $\mathbb{R}_{>0} \cdot a$. Then every continuous monoid homomorphism $f\colon C \to D$ is homogeneous, hence linear.* Indeed, by additivity one obtains $f(qa) = qf(a)$ for every rational number $q > 0$. For a real number $r > 0$ choose an increasing sequence q_n

[4]There is no way to extend the multiplication to all of $\overline{\mathbb{R}}_+$ in such a way that it remains continuous for the interval topology.This fact had been overlooked in [8] and had led to misleading statements in [8]. If we extend multiplication by $+\infty \cdot 0 = 0 = 0 \cdot (+\infty)$, it remains continuous for the upper topology, if we extend it by $+\infty \cdot 0 = +\infty = 0 \cdot (+\infty)$, it remains continuous for the lower topology.

of rational numbers with supremum r. Then $a = \sup q_n a$ and $rf(a) = \sup_n q_n f(a)$ since $r \mapsto rx$ is supposed to be continuous for the respective upper topologies. Since f is continuous for the respective upper topologies, we finally obtain $f(ra) = f(\sup_n q_n a) = \sup_n f(q_n a) = \sup_n q_n f(a) = rf(a)$.

One may ask, why we restrict scalar multiplication to $\mathbb{R}_{>0}$ and why we do not extend it to $r = 0$ and $r = +\infty$. The reason is that we have to treat the three cases differently concerning such an extension. While there is no continuous extension of scalar multiplication to $\overline{\mathbb{R}}_+$ for the interval topology, we can proceed as follows for the two other cases.

Using the upper topology, we may define $0 \cdot r = 0 = r \cdot 0$ for all $r \in \overline{\mathbb{R}}_+$ (including $r = +\infty$) and $r \cdot (+\infty) = +\infty = (+\infty) \cdot r$ for $r > 0$. This multiplication is continuous on $\overline{\mathbb{R}}_+$ for the upper topology and can be extended pointwise to a multiplication of functions $g \in \mathrm{MON}(P)$ and $f \in \mathrm{LSC}(P)$ with scalars $r \in \overline{\mathbb{R}}_+$ which remains continuous for the upper topologies and which satisfies all defining laws of scalar multiplication in cones.

Using the lower topology, we may define $0 \cdot r = 0 = r \cdot 0$ for all $r < +\infty$ and $r \cdot (+\infty) = +\infty = (+\infty) \cdot r$ for all $r \in \overline{\mathbb{R}}_+$ (including $r = +\infty$. This multiplication is continuous on $\overline{\mathbb{R}}_+$ for the lower topology and can be extended pointwise to a multiplication of functions $g \in \mathrm{MON}(P)$ and $f \in \mathrm{LSC}(P)$ with scalars $r \in \overline{\mathbb{R}}_+$ which remains continuous for the lower topologies and which satisfies all defining laws of scalar multiplication in cones.

Remark 3.12. In domain theory one usually stresses the Scott topology. In the context of the this section, the Scott topology agrees with the upper topology τ_{up}. This is the case for $\overline{\mathbb{R}}_+^P$, $\mathrm{MON}(P)$ and, in case of a predomain P, also for $\mathrm{LSC}(P)$. The same holds for the dual Scott topology and the lower topology τ_{lo} in all of these cases. The reason is that this phenomenon occurs in complete completely distributive lattices in general (see, e,g,, [16, Section VII-3]). We have preferred to use the lower and upper topology since their definition is simpler.

3.3 Compact ordered and stably compact spaces

Let us point out that in the cases under consideration each one of the three topologies (upper, lower and interval topology) determines the other two uniquely.

According to L. Nachbin [26], a compact space (X, τ) endowed with a partial order $\leq$ the graph $G_{\leq} = \{(x, y) \mid x \leq y\}$ of which is closed in $X \times X$ is called a *compact ordered space*. Such a space is always Hausdorff, since the diagonal in $X \times X$ is closed.

To any compact ordered space $(X, \tau, \leq)$ we associate two other topologies, the lower topology τ^{lo} and the upper topology τ^{up}. The closed sets of the upper (resp., lower) topology are the τ-open upper (resp., lower) sets. Thus, the open sets of the upper (resp., lower) topology are the τ-open upper (resp., lower) sets. We will use the following characterization of these two derived topologies:

Lemma 3.13. *Let $(X, \tau, \leq)$ be a compact ordered space. Suppose that τ_1 (resp., τ_2) are topologies on X that consists of τ-open upper (resp., lower) sets which are separating in the following sense: Whenever $x \not\leq y$, there are disjoint sets $U \in \tau_1$ and $V \in \tau_2$ such that $x \in U$ and $y \in V$. Then τ_1 is the upper and τ_2 the lower topology.*

Proof. Let W be an arbitrary τ-open upper set. We have to show that W belongs to τ_1. For this choose any $x \in W$. It suffices to show that there is a $U \in \tau_1$ such that $x \in U \subseteq W$ (since then W is the union of open sets belonging to τ_1). Thus take any $y \notin U$. Then $x \not\leq y$ and we can find disjoint sets $U_y \in \tau_1$ and $V_y \in \tau_2$ such that $x \in U_y$ and $y \in V_y$. The open sets V_y cover the complement of W which is a closed hence compact set. Thus, finitely many of the V_y cover the complement of W. Take the intersection U of the corresponding finitely many U_y. Then $x \in U \subseteq W$ and $U \in \tau_1$. $\square$

There is an equivalent way to look at this situation. A topological space (X, ω) is called *stably compact* if it is compact, locally compact, sober and coherent. By *coherent* we mean that the intersection of any two compact saturated subsets is compact.

The relation between stably compact spaces and compact ordered spaces is the following (see, e.g., [2] or [16, Section VI-7]):

To every stably compact space (X,ω) we associate a compact ordered space $(X,\omega^p,\leq_\omega)$ in the following way: $\leq_\omega$ is the specialization order associated with the topology ω. The topology ω^p is the coarsest refinement of the given topology ω and the associated co-compact topology ω^{cc} the closed sets of which are the ω-compact saturated subsets of X. Moreover, the original topology ω is the upper topology associated with ω^p and the co-compact topology ω^{cc} is the lower topology.

Conversely, Let $(X,\tau,\leq)$ be a compact ordered space. Then the upper topology τ^{up} is stably compact. Its associated co-compact topology is the lower topology and τ is the coarsest common refinement of the associated upper and lower topologies. The order $\leq$ agrees with the specialization order associated with the upper topology.

This setting allows an alternative proof of Proposition 3.7. We use:

Lemma 3.14. *If X is a stably compact space and Y a retract of X, that is, if there are continuous maps $\rho\colon X\to Y$ and $i\colon Y\to X$ such that $\rho\circ i$ is the identity in Y, then Y is stably compact, too.*

We now let P be a predomain. We recall that $(\mathrm{MON}(P),\tau,\leq)$ is a compact ordered space. Thus, its upper topology τ^{up} is stably compact. For its intrinsic upper topology, $\mathrm{LSC}(P)$ is a subspace of $\mathrm{MON}(P)$ and even a retract under the map $env\colon \mathrm{MON}(P)\to\mathrm{LSC}(P)$ which is continuous for the upper topologies, since env preserves arbitrary suprema by Lemma 3.8. Thus $\mathrm{LSC}(P)$ is stably compact for its intrinsic upper topology τ^{up} by Lemma 3.14. By Lemma 3.13 and Lemma 3.9, the intrinsic lower topology on $\mathrm{LSC}(P)$ agrees with the co-compact topology $(\tau^{up})^{cc}$ and, hence, the compact Hausdorff topology $(\tau^{up})^p$ agrees with the intrinsic interval topology of $\mathrm{LSC}(P)$. We summarize:

Proposition 3.15. *Let P be a predomain. Then $\mathrm{LSC}(P)$ is stably compact for its upper topology. The associated co-compact topology is the lower topology and the associated patch topology is the interval topology.*

3.4 The dual M^* of a preCuntz semigroup

For a predomain monoid $(M,+,0,\ll)$ its *dual* M^* is defined to be the set of all lower semicontinuous monoid homomorphisms $\varphi\colon M\to\overline{\mathbb{R}}_+$. Since the sum $\varphi+\psi$ of monoid homomorphisms φ and ψ and also the scalar multiple $r\varphi$, $0<r<*\infty$ are monoid homomorphisms, M^* is a subcone of $\mathrm{LSC}(M)$. Since the pointwise supremum of a directed family of lower semicontinuous monoid homomorphisms is again not only lower semicontinuous but also a monoid homomorphism, M^* is a dcpo-monoid. But M^* is not a domain. Let us investigate its topological structure.

As in Section 3.2 we will use the set M' of all monotone monoid homomorphisms $\gamma\colon P\to\overline{\mathbb{R}}_+$. Clearly, M' is a subcone of the cone $\mathrm{MON}(P)$ of all monotone maps from P to $\overline{\mathbb{R}}_+$.

The central observation is:

Lemma 3.16. *For a preCuntz semigroup M, the lower semicontinuous envelope $env(\gamma)$ of a monotone monoid homomorphism $\gamma\colon M\to\overline{\mathbb{R}}_+$ is also a monoid homomorphism.*

Proof. Given a monotone monoid homomorphism γ, recall that $env(\gamma)(x)=\sup_{x'\lll x}\gamma(x')$. Thus, clearly $env(\gamma)(0)=0$. In order to show additivity, let $x,y\in M$. Then $env(\gamma)(x)+env(\gamma)(y)=\sup_{x'\lll x}\gamma(x')+\sup_{y'\lll y}\gamma(y')=\sup_{x'\lll x,y'\lll y}\gamma(x')+\gamma(y')=\sup_{x'\lll x,y'\lll y}\gamma(x'+y')\leq\sup_{z\lll x+y}\gamma(z)=env(\gamma)(x+y)$, where we have used that the relation $\lll$ is additive in M for the inequality in the chain of equalities above. The reverse inequality follows from the continuity of addition in M which implies that, if $z\lll x+y$, then there are $x'\lll x$ and $y'\lll y$ such that $z\leq x'+y'$. Thus $\sup_{x'\lll x,y'\lll y}\gamma(x'+y')\geq\sup_{z\lll x+y}\gamma(z)$. This allows to read the above chain of equalities in the with $\leq$ replaced by $\geq$. $\square$

Thus env maps M' onto M^* and we have the following situation where all the arrows denote linear maps:

$$
\begin{tikzcd}
\overline{\mathbb{R}}_+^M & \mathrm{MON}(M) \arrow[l, hook'] \arrow[d, two heads, shift right, "env"'] & M' \arrow[l, hook'] \arrow[d, two heads, shift right, "env"'] \\
 & \mathrm{LSC}(M) \arrow[u, hook, shift right] & M^* \arrow[l, hook'] \arrow[u, hook, shift right]
\end{tikzcd}
$$

We consider the restrictions to M^* of our three topologies on $\mathrm{LSC}(M)$:

The *weak***upper topology* τ^*_{up}, the restriction of the upper topology on $\mathrm{LSC}(P)$. It is the weakest topology for which all the point evaluations $\delta_x\colon \varphi \mapsto \varphi(x)\colon M^* \to \overline{\mathbb{R}}_+$ are lower semicontinuous,

the restriction τ^*_{lo} to M^* of the lower topology τ_{lo} on $\mathrm{LSC}(M)$,

the restriction τ^*_{iv} to M^* of the interval topology τ_{iv}.

We now are ready for our main result:

Theorem 3.17. *Let M be a preCuntz semigroup and M^* its dual cone.*
(a) For the topology τ^*_{iv} and the pointwise order $\leq$, M^* is a compact ordered topological cone.
(b) For the weak*upper topology τ^*_{up}, and similarly for the topology τ^*_{lo}, M^* is a stably compact topological cone.

Proof. Lt us show that M' is closed in $\mathrm{MON}(M)$ for the interval topology. Convergence for the interval topology in $\mathrm{MON}(P)$ is pointwise convergence in $\overline{\mathbb{R}}_+$. Thus if γ_i is a net in M' that converges to some $\gamma \in \mathrm{MON}(M)$, then for $x, y \in M$, $\gamma_i(x)$ converges to $\gamma(x)$, $\gamma_i(y)$ converges to $\gamma(y)$ and $\gamma_i(x+y)$ converges to $\gamma(x+y)$. At the other hand, $\gamma_i(x+y) = \gamma_i(x) + \gamma_i(y)$ converges to $\gamma(x) + \gamma(y)$ by the continuity of addition on $\overline{\mathbb{R}}_+$. Thus $\gamma(x) + \gamma(y) = \gamma(x+y)$.

As a closed subcone of $\mathrm{MON}(M)$, M' is a compact ordered cone for the interval topology. Forming the lower semicontinuous envelope maps M' onto M^* by Lemma 3.16. By Proposition 3.10, the map env is continuous for the respective interval topologies. Hence, M^* is also compact for the topology τ^*_{iv} hence a compact ordered space, and closed in $\mathrm{LSC}(M)$ for the interval topology. We infer that (M^*, τ^*_{iv}) is a compact ordered topological cone.

M^* is also a topological cone for the weak*upper topology τ^*_{up} and the topology τ^*_{lo} which are stably compact according to Proposition 3.15, being the topologies of open upper and lower sets, respectively, for the topology τ^*_{iv}. □

From Proposition 3.7 we also deduce:

(a) A subbasis for the weak*upper topology τ^*_{up} of M^* is given by:

$$V_{x,r} = \{f \in M^* \mid f(x) > r\},\ x \in M, r \in \mathbb{R}_+$$

A subbasis for the topology τ^*_{lo} by:

$$W_{y,r} = \{f \in M^* \mid f(x) < r \text{ for some } x \in \Uparrow y\},\ y \in M, r \in \mathbb{R}_+.$$

Together these subbases constitute a subbasis for the topology τ^*_{iv}.

(b) A net $(f_i)_i$ of functions in M^* converges to f for the weak*upper upper topology τ^*_{up} if and only if:

$$f(x) \leq \liminf_i f_i(x) \text{ for all } x \in M \tag{upConv}$$

for the topology τ^*_{lo} if and only if:

$$\limsup_i f_i(y) \leq f(x) \ \text{ whenever } y \prec\!\!\prec x \text{ in } M \tag{loConv}$$

for the topology τ^*_{iv} if and only conditions (upConv) and (upConv) hold.

These result hold in particular for the dual of our basic example, the preCuntz semigroup $C_0(X)_+$.

Remark 3.18. The main proof technique for the results in this subsection consists in considering first the cone M' of order preserving linear functionals $\lambda\colon M \to \overline{\mathbb{R}}_+$; for those the compactness properties follow from the Tychonoff Theorem on the compactness of product spaces. Taking the lower semicontinuous envelope yields a continuous retraction on the the lower semicontinuous monoid homomorphisms. This technique has first been applied by Jung [2] and is heavily used in [28, 21]. In [8] it is mentioned that in the proof of Theorem 3.7 on the compactness of the space of traces the same idea has been communicated to the authors by E. Kirchberg. In [8, Theorem 4.8] claim (a) of Theorem 3.17 has been proved for Cuntz semigroups.

3.5 The bidual M^{**}

Let M be a preCuntz semigroup and M^* its dual. By the universal property of the round ideal completion (see 2.11), the dual $\mathcal{RI}(M)^*$ of $\mathcal{RI}(M)$ is canonically isomorphic (algebraically and topologically) to the dual M^* of M (and also to the dual of the round ω-ideal completion of M if M is first countable.

We may form the bidual M^{**}, the cone of all linear functionals $\Lambda\colon M^* \to \overline{\mathbb{R}}_+$ that are lower semicontinuous with respect to the weak*upper topology τ^*_{up}; this is equivalent to requiring that these maps are monotone and lower semicontinuous with respect to the patch topology τ^*_p; indeed, by Proposition 3.2 the patch open upper sets agree with the weak*upper open sets. We endow M^{**} with the pointwise order, addition and multiplication by scalars $r > 0$. We note that M^{**} is directed complete (under pointwise suprema). There is a natural map from M into its bidual M^{**}: to very $x \in M$ we assign the point evaluation $\widehat{x}\colon \varphi \mapsto \varphi(x)$. This map from M to M^{**} clearly is a monoid homomorphism, linear and monotone. We would like this map to be an order embedding, that is, $x \not\leq y$ in M implies $\widehat{x} \not\leq \widehat{y}$. For this it suffices to have the following separation property:

Separation Property 3.19. *Whenever $x \not\leq y$ in M, there is a $\varphi \in M^*$ such that $\varphi(x) > \varphi(y)$.*

This separation property will not be true for Cuntz semigroups in general. We provide a proof under the hypothesis that M is a preCuntz cone:

Lemma 3.20. *Whenever $x \not\leq y$ in a preCuntz cone M, there is a $\varphi \in M^*$ such that $\varphi(x) > \varphi(y)$.*

Proof. Consider elements $x \not\leq y$. Then $\twoheaddownarrow x \not\subseteq \twoheaddownarrow y$, that is, there is an element $z \ll x$ with $z \not\ll y$. By interpolation we find an element z' with $z \ll z' \ll x$. Then $\twoheaddownarrow z' \not\subseteq \twoheaddownarrow y$, that is, $z' \not\leq y$. Using interpolation we recursively find a sequence $x \gg x_1 \gg x_2 \gg \cdots \gg z'$. The set U of all $u \in M$ such that $u \gg x_n$ for some n is a $\tau_{\ll}$-open neighborhood of x contained in $\uparrow z'$ whence $y \notin U$. Moreover, U is convex. Indeed, for elements $u, v \in U$ there is an n such that $u, v \gg x_n$. It follows for every r in the open unit interval, $ru + (1-r)v \gg rx_n + (1-r)x_n = x_n$, that is $ru + (1-r)v \in U$. We now can apply [19, Corollary 9.2] which tells us that for every open convex set U in a semitopological cone and every element y not contained in U, there is a lower semicontinuous linear functional φ such that $\varphi(y) < 1$ but $\varphi(u) > 1$ for all $u \in U$, in particular, $\varphi(x) > \varphi(y)$. $\square$

For every round ideal I of M, let $\widehat{I} = \sup\{\widehat{x} \mid x \in I\}$. Clearly, $\widehat{I} \in M^{**}$. Thus, we obtain a map from $\mathcal{RI}(M)$ to M^{**} which is Scott-continuous. Moreover this map preserves $\ll$:

Lemma 3.21. *For round ideals I and J in a Cuntz cone, $I \ll J$ implies $\widehat{I} \ll \widehat{J}$.*

Proof. We first consider elements $x \lll y$ in M. As $y = \sup_{r<1} ry$, there is an $r < 1$ such that $x \lll ry$. Let U_x be the set of all $\varphi \in M^*$ such that $\varphi(x) > 1$, and similarly for U_{ry} and U_y. By definition, U_x, U_{ry} and U_y are weak*upper open and $U_x \subseteq U_{ry} \subset U_y$. We want to show that there is a compact saturated set K such that $U_x \subseteq K \subseteq U_y$. Indeed, let φ_i be a net in U_x converging to some φ for the topology τ^*_{lo}. Let us show that $\varphi \in U_y$. Indeed, $\varphi(ry) \geq \limsup_i \varphi_i(z)$ for all $z \ll ry$, in particular, $\varphi(ry) \geq limsup_i \varphi_i(x) \geq 1$. Thus $r\varphi(y) \geq 1$ whence $\varphi(y) = \frac{1}{r} > 1$, that is $\varphi \in U_y$). From this we conclude that $\widehat{x} \ll \widehat{y}$ in M^{**}. The claim for ideals is a direct consequence, for if $I \ll J$ there are elements $x \lll y$ in J such that $I \subseteq \twoheaddownarrow x$. $\square$

The following question arises:

Question 3.22. *If M is a preCuntz cone, is M^{**} isomorphic to the round ideal completion $\mathcal{RI}(M)$? More precisely, given any $\Lambda \in M^{**}$, is there a round ideal J in M such that $\Lambda = \widehat{J}$.*

The answer to this question is 'yes' in the case of our basic example, the cone $C_0(X)_+$ for a locally compact Hausdorff space X: In this case, $(C_0(X)_+)^{**}$ is naturally isomorphic to the cone $\mathrm{LSC}(X)$ of all lower semicontinuous functions $f\colon X \to \overline{\mathbb{R}}_+$, which is the round ideal completion of $C_0(X)_+$ according. Indeed, $(C_0(X)_+)^*$ corresponds to the cone of all continuous valuations (a topological variant of measures) on X and the claim is a special case of the Schröder-Simpson Theorem (see [20, Theorem 2.15] or [17] for a short proof).

In the search for an affirmatively answer to the question above for a preCuntz cone M, one can use [20, Corollary 4.5] which tells us that every lower semicontinuous linear functional Λ on M^* is the pointwise supremum of functionals of the form $\widehat{x}_i$, $x_i \in M$. If we can show that we can choose this set of x_i to be directed, then we have a positive answer to our question. Indeed, in this case the $y \prec\!\!\prec x_i$ for some i form a round ideal J of M such that $\widehat{J} = \Lambda$.

Robert [30] has investigated the relation between M and the double dual M^{**} for Cuntz semigroups that are not already cones. Here the problem is to embed M into a cone which he succeeds by a kind of tensor product construction but under additional hypotheses on the Cuntz semigroup.

4 Traces on C*-algebras

We now turn to C*-algebras. Let A be a C^*-algebra. The elements of the form $a = xx^*$, $x \in A$, are called *positive*. These elements form a cone denoted A_+. On A_+ we use the topology induced by the norm of the C*-algebra and the natural order $a \leq b$ if $b - a \in A_+$. We refer to standard references for background material.

A *lower semicontinuous trace* is a lower semicontinuous monoid homomorphism $t\colon A_+ \to \overline{\mathbb{R}}_+$ such that $t(xx^*) = t(x^*x)$ for all $x \in A$. We denote by $T(A)$ the set of all traces. $T(A)$ becomes an ordered cone for the pointwise defined order, addition and multiplication by real numbers $r > 0$. We would like to view A_+ as a predomain in such a way that $T(A)$ is its dual. We let us guide by the basic example 2.1 $C_0(X)_+$ 2.1.

We remark that a lower semicontinuous trace satisfies $t(ra) = rt(a)$ for $r \in Rp$ and $a \in A_+$ and, hence, is a linear map on A_+. This follows from the properties of being a monoid homomorphism and lower semicontinuity.

4.1 A_+ as a preCuntz semigroup

Every element $a \in A_+$ generates a commutative C^*-subalgebra $C^*(a)$ of A. By Gelfand's representation theorem, there is an isometrical isomorphism $i_a\colon C^*(a) \to C_0(X)$ for some locally compact Hausdorff space X. We denote by $(a-\varepsilon)_+$ the element of $C^*(a)$ that corresponds to the function $(i_a(a) - \varepsilon)_+$ in $C_0(X)$.

As a first try we define $a \ll b$ for elements $a, b \in A_+$ if $a \leq (b - \varepsilon)_+$ for some $\varepsilon > 0$. In this way A_+ becomes a predomain. We first check the interpolation property. We have indeed $0 \ll b$ for every $b \in A_+$ and if $a_i \ll c$ for $i = 1, 2$, then $a_i \leq (b - \varepsilon)_+$ for some $\varepsilon > 0$. For $c = b - \frac{\varepsilon}{2}$ we then have $a_i \ll c \ll b$ for $i = 1, 2$. Transitivity follows from the fact that $a \ll b$ implies $a \leq b$ and that $a \leq b \ll c$ implies $a \ll c$.

The relation $\ll$ just defined will not have the desired properties. Following Cuntz and Pedersen [7], one should take in account an equivalence relation that identifies elements that are identified by every lower semicontinuous trace. Since traces identify the elements xx^* and x^*x, we consider xx^* and x^*x to be equivalent. For a sequence $(x_i)_i$ of elements in A, if the sums $\sum_i x_i x_i^*$ and $\sum_i x_i^* x_i$ both converge, a lower semicontinuous trace will also identify these two sums.

According to [7], two elements a and a' in A_+ are *Cuntz-Pedersen equivalent* and we write $a \sim a'$ if there is a sequence x_n in A such that $a = \sum_n x_n x_n^*$ and $a' = \sum_n x_n^* x_n$.

The relation $\sim$ is indeed an equivalence relation (transitivity is by no means straightforward). Moreover, $\sim$ is countably additive, that is, $a_n \sim b_n$ implies $\sum_n a_n \sim \sum_n b_n$ provided that the respective infinite sums converge. We refer to [7, Section 2] for proofs. Clearly, $\sim$ is a congruence relation, that is, for all $a, a', b \in A_+$ and $r \in \mathbb{R}_+$ one has:

$$a \sim a' \text{ implies } a + b \sim a' + b, ra \sim ra'$$

The *Cuntz-Pedersen preorder* on A_+ is defined by:

$$a \precsim b \text{ if there is an } a' \in A \text{ such that } a \sim a' \leq b \tag{CPP}$$

Note that $a \leq b$ implies $a \precsim b$.

We want to replace the Cuntz-Pedersen preorder by a relation that we like to call the *Cuntz-Pedersen approximation relation* $\prec\!\!\prec$ defined as follows:

$$a \prec\!\!\prec b \text{ if there is a } a' \in A_+ \text{ and an } \varepsilon > 0 \text{ such that } a \sim a' \leq (b-\varepsilon)_+$$

Equivalently:

$$a \prec\!\!\prec b \text{ if there are an } \varepsilon > 0 \text{ such that } a \precsim (b-\varepsilon)_+$$

We note that in particular, $(b-\varepsilon)_+ \prec\!\!\prec (b-\frac{\varepsilon}{2})_+ \prec\!\!\prec b$.

Proposition 4.1. *For every C^*-algebra A, $(A_+, +, 0, \prec\!\!\prec)$ is a first countable preCuntz semigroup.*

For the proof we first observe that $d \prec\!\!\prec c$ implies $d \precsim (c-\varepsilon)_+ \leq c$ for some $\varepsilon > 0$ whence $d \precsim c$. We now show that $\prec\!\!\prec$ endows A_+ with the structure of a predomain.

For transitivity, let $d \prec\!\!\prec c \prec\!\!\prec a$. Then $d \precsim c$ as we just noticed and $c \precsim (a-\varepsilon)_+$. We infer $d \precsim (a-\varepsilon)_+$ from the transitivity of $\precsim$ whence $d \prec\!\!\prec a$.

For interpolation we notice that $0 \prec\!\!\prec a$ for every $a \in A_+$ so that we have (Int0). For (Int2), suppose that $c_i \prec\!\!\prec a$ for $i = 1, 2$. Then there is an $\varepsilon > 0$ such that $c_i \precsim (a-\varepsilon)_+$ for $i = 1, 2$. Since $(a-\varepsilon)_+ \prec\!\!\prec (a-\frac{\varepsilon}{2})_+ \prec\!\!\prec a$, we may choose $c = (a-\frac{\varepsilon}{2})_+$ and we have $c_i \prec\!\!\prec c \prec\!\!\prec a$ for $i = 1, 2$.

It remains to show that addition preserves $\prec\!\!\prec$ and is continuous. For this we use a result by Elliott, Robert and Santiago [8, Proposition 2.3]: Given $a, b \in A_+$ and $\varepsilon > 0$, there is a $\delta > 0$ such that

$$(a-\varepsilon)_+ + (b-\varepsilon)_+ \quad \precsim \quad (a+b-\delta)_+ \tag{4}$$

$$(a+b-\varepsilon)_+ \quad \precsim \quad (a-\delta)_+ + (b-\delta)_+ \tag{5}$$

Indeed, these two inequalities are equivalent to the following properties which express the additivity and the continuity of the relation $\prec\!\!\prec$, respectively:

$$a' \prec\!\!\prec a, b' \prec\!\!\prec b \quad \Longrightarrow \quad a' + b' \prec\!\!\prec a + b \tag{6}$$

$$c \prec\!\!\prec a + b \quad \Longrightarrow \quad \exists a' \prec\!\!\prec a, b' \prec\!\!\prec b.\ c \prec\!\!\prec a' + b' \tag{7}$$

This finishes the proof of Proposition 4.1.

The natural preorder of the predomain $(A_+, \prec\!\!\prec)$ according to 2.3 is defined by $a \precsim_{CP} b$ if $\twoheaddownarrow a \subseteq \twoheaddownarrow b$. More explicitly, $a \precsim_{CP} b$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $(a-\varepsilon)_+ \precsim (b-\delta)_+$. This preorder has already been considered by Robert [29]. Thus, if $c' \precsim_{CP} c \prec\!\!\prec a \precsim_{CP} a'$, then $c' \prec\!\!\prec a'$. From results due to Robert [29] it follows that the converse is not true, that is $\twoheaddownarrow a \subseteq \twoheaddownarrow b$ does not imply $a \precsim b$, in general.

It is natural to ask whether the natural preorder $\precsim_{CP}$ agrees with the Cuntz-Pedersen preorder $\precsim$. L. Robert [29, Proposition 2.1(iii)] has proved the implication:

$$a \precsim b \implies a \precsim_{CP} b \tag{8}$$

But in the same paper, Robert [29] exhibits an example that shows that the converse does not hold, in general.

The proof for the implication (8) is surprisingly sophisticated. One refers to a lemma due to Kirchberg and Rørdam [24, Lemma 2.2]: If $\varepsilon > \|a-b\|$ then there is a contraction d in A such that $(a-\varepsilon)_+ = dbd^*$. From this, one deduces [8, Lemma 2.2]:

$$\|a-b\| < \varepsilon \implies (a-\varepsilon)_+ \precsim b \tag{9}$$

One then shows the following refinement:

$$\|a-b\| < \varepsilon \implies \exists \delta > 0.\ (a-\varepsilon)_+ \precsim (b-\delta)_+ \tag{10}$$

Suppose indeed $\|a-b\|<\varepsilon$. Since $(b-\delta)_+$ converges (in norm) to b for $\delta\to 0$, there is some $\delta>0$ such that still $\|a-(b-\delta)_+\|<\varepsilon$. Now (10) follows from (9).

One further uses from [8, Proposition 2.3]

$$(xx^*-\varepsilon)_+ \quad\sim\quad (x^*x-\varepsilon)_+ \tag{11}$$

for every element x of the C*-algebra A and $\varepsilon>0$. From (5) (see also [29, proof of Proposition 2.1(i)] we deduce:

$$a\le b \quad\Longrightarrow\quad \forall\varepsilon>0.\,\exists\delta>0.\,(a-\varepsilon)_+\precsim(b-\delta)_+ \tag{12}$$

Indeed, if $a\le b$, then $b=a+(b-a)$ and $b-a\in A_+$. Thus, for $\varepsilon>0$ we can find a $\delta>0$ such that $(a-\varepsilon)_+ + ((b-a)-\varepsilon)_+\precsim(b-\delta)_+$. It follows that $(a-\varepsilon)_+\precsim(b-\delta)_+$.

We are now ready for the proof of the implication (8). Suppose $a\precsim b$. There is a sequence x_n of elements in A such that $a=\sum_{n=1}^\infty x_nx_n^*$ and $a'=\sum_{n=1}^\infty x_n^*x_n\le b$. Consider any $\varepsilon>0$. There is an N such that $\|a-\sum_{n=1}^N x_nx_n^*\|<\varepsilon$. The following chain of arguments shows that $a\precsim_{CP} b$:

$$\begin{array}{rcll}
(a-\varepsilon)_+ & \precsim & (\sum_{n=1}^N x_nx_n^*-\delta)_+ & \text{for some } \delta>0 \text{ by (10)}\\
& \precsim & \sum_{n=1}^N (x_nx_n^*-\delta_1)_+ & \text{for some } \delta_1>0 \text{ by (5)}\\
& \sim & \sum_{n=1}^N (x_n^*x_n-\delta_1)_+ & \text{by (11)}\\
& \precsim & (\sum_{n=1}^N x_n^*x_n-\delta_2)_+ & \text{for some } \delta_2>0 \text{ by (4)}\\
& \precsim & (b-\delta_3)_+ & \text{for some } \delta_3>0 \text{ by (12)}
\end{array}$$

4.2 The cone $T(A)$ of traces

We are ready now to apply our results on the dual of a preCuntz semigroup to the the preCuntz semigroup $(A_+,+,0,\prec\!\!\prec)$ on the positive cone of a C*-algebra A. We first show that the cone $T(A)$ of traces is the dual of the preCuntz semigroup $(A_+,+,0,\prec\!\!\prec)$:

Lemma 4.2. *The lower semicontinuous traces on A_+ agree with the lower semicontinuous monoid homomorphisms from the preCuntz semigroup $(A_+,+,0,\prec\!\!\prec)$ to $\overline{\mathbb{R}}_+$.*

Proof. Consider a monoid homomorphism $t\colon A_+\to\overline{\mathbb{R}}_+$ satisfying $t(a)=t(a')$ whenever $(a\sim a'$. We want to show that λ is lower semicontinuous for the norm topology on A_+ if and only if it is lower semicontinuous for the predomain structure $\prec\!\!\prec$. Thus let r be a nonnegative real number and look at the set $U=\{a\in A_+\mid t(a)>0\}$. We have to show that U is open for the norm topology if and only if it is open for the c-space topology $\tau_{\prec\!\!\prec}$ associated with $\prec\!\!\prec$.

Suppose first that U is open for the norm topology and look at any element $a\in U$. Since $(a-\varepsilon)_+)$ converges to a with respect to the norm, when ε goes to 0, we have $(a-\varepsilon)_+)\in U$ for ε small enough. The we have found an element $b=(a-\varepsilon)_+)\in U$ such that $b\prec\!\!\prec a$. We secondly look at any element $c\in A_+$ with $a\prec\!\!\prec c$. Then there is an a' such that $a\sim a'\le(c-\varepsilon)_+$ for some $\varepsilon>$). Then $t(a)=t(a')\le t(c-\varepsilon)_+\le t(a)$ since a monoid homomorphism on A_+ preserves the order $\le$. Hence $r<t(a)\le t(c)$, that is, $c\in U$. Thus, U is open for the c-space topology $\tau_{\prec\!\!\prec}$.

Suppose conversely that U is open for the c-space topology $\tau_{\prec\!\!\prec}$ and choose any $a\in U$. We want to show that there is an $\varepsilon>0$ such that $b\in U$ for every b such that $\|a-b\|<\varepsilon$. There is an $\varepsilon>0$ such that $(a-\varepsilon)_+\in U$. For every b with $\|a-b\|<\varepsilon$ there is a $\delta>0$ such that $(a-\varepsilon)_+\precsim b$ by (9), whence $(a-\varepsilon)_+\precsim_{CP} b$ by the previous lemma. And the latter implies $b\in U$. ☐

We now may apply Theorem 3.17 and we obtain the following improvement of results by Elliott, Robert, Santiago [8]:

Corollary 4.3. *Let A be a C*-algebra and $T(A)$ the cone of lower semicontinuous traces.*

(a) Equipped with the topology τ_{iv}^*, $T(A)$ is an ordered compact topological cone, that is, addition and scalar multiplication are order preserving and jointly continuous, where $\mathbb{R}_{>0}$ is endowed with the usual Hausdorff topology.

*(b) Equipped with the weak*upper topology, $T(A)$ is a stably compact topological cone, that is, addition and scalar multiplication are continuous, where $\mathbb{R}_{>0}$ is endowed with the upper topology.*

(c) Equipped with the lower topology τ^*_{lo}, $T(A)$ is a stably compact topological cone, that is, addition and scalar multiplication are continuous, where $\mathbb{R}_{>0}$ is endowed with the lower topology.

Subbases and convergence for the three topologies involved in the above corollary can be described as in the text following the proof of Theorem 3.17.

The dual $T(A)^*$ of the cone of traces consisting of the lower semicontinuous linear functionals from $T(A)$ to $\overline{\mathbb{R}}_+$ contains the round ideal completion $\mathcal{RI}(A_+)$ of $(A_+, +, 0, \prec\!\!\prec)$ as a subcone via the map $J \mapsto \widehat{J}$, where $\widehat{J}\colon T(A) \to \overline{\mathbb{R}}_+$ is defined by $\widehat{J}(\varphi) = \sup_{x\in J} \varphi(x)$. This map is also an order embedding by 3.20. Our general question 3.22 can be reformulated in this special case:

Question 4.4. *Is the the dual $T(A)^*$ equal to the round ideal completion $\mathcal{RI}(A_+)$ of $(A_+, +, 0, \prec\!\!\prec)$? More precisely, given any lower semicontinuous linear map $Lambda\colon T(A) \to \overline{\mathbb{R}}_+$, is there a round ideal J in $(A_+, +, 0, \prec\!\!\prec)$ such that $\Lambda = \widehat{J}$.*

The answer to this question is 'yes' for commutative C*-algebras as we have indicated after 3.22.

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