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Preideals in EQ-algebras

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Abstract

EQ-algebras were introduced by Novák in [15] as an algebraic structure of truth values for fuzzy type theory (FFT). Novák and De Baets in [17] introduced various kinds of EQ-algebras such as good, residuated, and IEQ-algebras. In this paper, we define the notion of (pre)ideal in bounded EQ-algebras (BEQ-algebras) and investigate some properties. Then we introduce a congruence relation on good BEQ-algebras by using ideals, and then we solve an open problem in [18]. Moreover, we show that in IEQ-algebras, there is an one-to-one corresponding between congruence relations and the set of ideals. In the follows, we characterize the generated preideal in BEQ-algebras and by using this, we prove that the family of all preideals of a BEQ-algebra, is a complete lattice. Then we show that the family of all preideals of a prelinear IEQ-algebras, is a distributive lattice and become a Heyting algebra. Finally, we show that we can construct an MV-algebra form the family of all preideals of a prelinear IEQ-algebra.

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Keywords: *Bounded EQ-algebra, (pre)ideal, generated preideal, complete lattice, distributive lattice, Heyting algebra, MV-algebra.*

1 Introduction

Fuzzy type theory was developed as a counterpart of the classical higher-order logic. Since the algebra of truth values is no longer a residuated lattice, a specific algebra called an EQ-algebra was proposed by Novák [15, 16, 17] and it continued in [2, 3, 6, 10, 19, 21]. The main primitive operations of EQ-algebras are meet, multiplication, and fuzzy equality. Implication is derived from the fuzzy equality and it is not a residuation with respect to multiplication. Consequently, EQ-algebras overlap with residuated lattices but are not identical with them. Novák and De Baets in [17] introduced various kinds of EQ-algebras. Novák and El-Zekey in [9], proved that the class of EQ-algebras is a variety. El-Zekey in [8] introduced prelinear good EQ-algebras and proved that a prelinear good EQ-algebra is a distributive lattice. Novák and De Baets in [17] defined the concept of prefilter on EQ-algebras which is the same as filter of other algebraic structures such as residuated lattices, MTL-algebras, and etc. But the binary relation has been introduced by prefilters is not a congruence relation. For solving this problem, they added another condition to the definition of prefilter so filter of EQ-algebras is defined. In studying logical algebras, filter theory or ideal theory is very important. From logic point of view, various filters have natural interpretation as various sets of provable formulas. At present, the filter theory of EQ-algebras has been widely studied and some important results are obtained. In particular, Liu and Zhang in [13], introduced positive implicative and implicative (pre)filters of EQ-algebras and showed that these two concepts are the same in IEQ-algebras. Xin et al. [19], have studied fantastic (pre)filters of good EQ-algebras. In [14], the family of prefilter of an EQ-algebras was studied. The notion of ideals has been introduced in many algebraic structures such as lattices, rings, MV-algebras. Ideals theory is a very effective tool for studying various algebraic and logical systems. In some logical algebras such as equality, hoop, and

MV-algebras, filters and ideals are dual notions [1, 18]. While in BL-algebra, with the lack of a suitable algebraic addition, the focus is shifted filters. So the notion of ideals is missing in BL-algebras. To fill this gap the paper [12], introduced the notion of ideals in BL-algebras, which generalized in a natural sense the existing notion in MV-algebras and subsequently all the results about ideals in MV-algebras. The paper also constructed some examples to show that, unlike in MV-algebras, ideals and filters are dual but behave quite differently in BL-algebra. So the notion of ideal from a purely algebraic point of view has a proper meaning in BL-algebras.

In this paper, we define the notion of (pre)ideal in bounded EQ-algebras and investigate the relation is induced by an ideal in good EQ-algebras, is a congruence relation. Also, we show that in IEQ-algebras, any congruence relation introduce an ideal. In the rest of paper, we define the generated preideal by a subset. By this means we prove that the family of all preideals of an EQ-algebra, is a complete lattice. Also, we prove that in prelinear IEQ-algebras, the family of all preideals forms an MV-algebra.

2 Preliminaries

In this section, we recollect some definitions and results which will be used in this paper [8, 9, 13].

An *EQ-algebra* is an algebraic structure $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ of type $(2, 2, 2, 0)$, where for any $a, b, c, d \in E$, the following statements hold:
$(E1)$ $(E, \wedge, 1)$ is a $\wedge$-semilattice with top element 1. For any $a, b \in E$, we set $a \leqslant b$ if and only if $a \wedge b = a$.
$(E2)$ $(E, \otimes, 1)$ is a (commutative) monoid and $\otimes$ is isotone with respect to $\leqslant$.
$(E3)$ $a \sim a = 1$.
$(E4)$ $((a \wedge b) \sim c) \otimes (d \sim a) \leqslant c \sim (d \wedge b)$.
$(E5)$ $(a \sim b) \otimes (c \sim d) \leqslant (a \sim c) \sim (b \sim d)$.
$(E6)$ $(a \wedge b \wedge c) \sim a \leqslant (a \wedge b) \sim a$.
$(E7)$ $a \otimes b \leqslant a \sim b$.

The operations " $\wedge$ ", " $\otimes$ ", and " $\sim$ " are called *meet*, *multiplication*, and *fuzzy equality*, respectively. For any $a, b \in E$, we defined the binary operation *implication* on E by, $a \to b = (a \wedge b) \sim a$. Also, in particular $1 \to a = 1 \sim a = \tilde{a}$. If E contains a bottom element 0, we say E is bounded and denote it by BEQ-algebra. Then an unary operation $\neg$ is defined on E by $\neg a = a \sim 0 = a \to 0$.

Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ-algebra and $a, b, c \in E$ are arbitrary elements. Then $\mathcal{E}$ is called
(i) *spanned*, if $\mathcal{E}$ is a BEQ-algebra and $\tilde{0} = 0$,
(ii) *separated*, if $a \sim b = 1$, then $a = b$,
(iii) *good*, if $a \sim 1 = a$,
(iv) an *involutive (IEQ-algebra)*, if $\mathcal{E}$ is a BEQ-algebra and for any $a \in E$, $\neg\neg a = a$,
(v) *residuated*, where $(a \otimes b) \wedge c = a \otimes b$ if and only if $a \wedge ((b \wedge c) \sim b) = a$,
(vi) *lattice-ordered EQ-algebra*, if it has a lattice reduct,[1]
(vii) *prelinear EQ-algebra*, if the set $\{(a \to b), (b \to a)\}$ has the unique upper bound 1,
$(viii)$ *lattice EQ-algebra (or ℓEQ-algebra)*, if it is a lattice-ordered EQ-algebra and

$$((a \vee b) \sim c) \otimes (d \sim a) \leqslant (d \vee b) \sim c.$$

Proposition 2.1. [9] *Let $\mathcal{E}$ be an EQ-algebra. Then, for all $a, b, c \in E$, the following properties hold:*
(i) $a \sim b = b \sim a$.
(ii) $b \leqslant a \to b$.
(iii) $a \sim b \leqslant a \to b$.
(iv) $a \sim b \leqslant (a \sim c) \sim (b \sim c)$.
(v) $a \sim b \leqslant (a \wedge c) \sim (b \wedge c)$.
(vi) $a \to b \leqslant (c \to a) \to (c \to b)$ *and* $a \to b \leqslant (b \to c) \to (a \to c)$.

[1]Given an algebra $< E, F >$, where F is a set of operations on E and $F' \subseteq F$, then the algebra $< E, F' >$ is called the F'-reduct of $< E, F >$.

(vii) If $a \leqslant b$, then $c \to a \leqslant c \to b$ and $b \to c \leqslant a \to c$.
(viii) If $\mathcal{E}$ is separated, then $a \to b = 1$ if and only if $a \leqslant b$.
(ix) If $\mathcal{E}$ is a good BEQ-algebra, then $\neg a = \neg\neg\neg a$.
(x) If $\mathcal{E}$ is a BEQ-algebra, then $a \to b \leqslant \neg b \to \neg a$ and if $\mathcal{E}$ is involutive, then $a \to b = \neg b \to \neg a$.
(xi) If $\mathcal{E}$ is prelinear, then $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$.

An EQ-algebra $\mathcal{E}$ has *exchange principle condition*, if for any $a, b, c \in E$, $a \to (b \to c) = b \to (a \to c)$.

Proposition 2.2. [13] *Let $\mathcal{E}$ be an EQ-algebra. Then, for all $a, b, c \in E$, the following statements are equivalent:*
(i) $\mathcal{E}$ is good,
(ii) $\mathcal{E}$ is separated and satisfies exchange principle condition,
(iii) $\mathcal{E}$ is separated and $a \leq (a \to b) \to b$.

Theorem 2.3. [17] *Every involutive EQ-algebra is a good ℓEQ-algebra.*

Let $\mathcal{E}$ be an EQ-algebra, $a, b, c \in E$ and $\emptyset \neq F \subseteq E$. Then;
• F is called a *prefilter* of $\mathcal{E}$, if $1 \in F$ and if $a \in F$ and $a \to b \in F$, then $b \in F$.
• a prefilter F of $\mathcal{E}$ is called a *filter* of $\mathcal{E}$, if $a \to b \in F$, then $(a \otimes c) \to (b \otimes c) \in F$.
The set of all prefilters of $\mathcal{E}$ is denoted by $\mathcal{PF}(\mathcal{E})$.

Remark 2.4. [17] Let F be a (pre)filter of EQ-algebra $\mathcal{E}$. If $a \in F$ and $a \leqslant b$, then $b \in F$.

Remark 2.5. [9] Let $\mathcal{E}$ be a separated EQ-algebra. The singleton subset $\{1\} \subseteq E$ is a filter of $\mathcal{E}$.

Theorem 2.6. [9] *Let F be a filter of EQ-algebra $\mathcal{E}$. A binary relation $\approx_F$ on E which is defined by $a \approx_F b$ if and only if $a \sim b \in F$, is a congruence relation on $\mathcal{E}$ and $\mathcal{E}/F = (E/F, \wedge_F, \otimes_F, \sim_F, F)$ is a separated EQ-algebra, where, for any $a, b \in E$, we have,*

$$[a] \wedge_F [b] = [a \wedge b] \quad , \quad [a] \otimes_F [b] = [a \otimes b] \quad , \quad [a] \sim_F [b] = [a \sim b] \quad , \quad [a] \to_F [b] = [a \to b].$$

A binary relation $\leqslant_F$ on E/F which is defined by $[a] \leqslant_F [b]$ if and only if $[a] \wedge_F [b] = [a]$ is a partial order on E/F and for any $[a], [b] \in \mathcal{E}/F$, $[a] \leqslant_F [b]$ if and only if $a \to b \in F$ if and only if $[a] \to_F [b] = [1]$.

Notation. From now on, in this paper, $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ or simply $\mathcal{E}$ is a BEQ-algebra, unless otherwise state.

3 (Pri)Ideals in EQ-algebras

In this section, we introduce the notion of *(pre)ideals* in BEQ-algebras and investigate some properties of them. Also, we prove that the ideals introduce a congruence relation on $\mathcal{E}$.

Definition 3.1. Let I be a nonempty subset of E. Then I is called a *preideal* of $\mathcal{E}$, if for any $a, b, c \in E$, it satisfies the following conditions:

(I_1) If $a \leqslant b$ and $b \in I$, then $a \in I$,

(I_2) If $a, b \in I$, then $\neg a \to b \in I$.

A preideal of $\mathcal{E}$ is an *ideal* of $\mathcal{E}$ if it satisfies the following condition:

(I_3) If $\neg(a \to b) \in I$, then $\neg((a \otimes c) \to (b \otimes c)) \in I$.

The set of all preideals of $\mathcal{E}$ is denoted by $\mathcal{PI}(\mathcal{E})$ and the set of all ideals of $\mathcal{E}$ is denoted by $\mathcal{I}(\mathcal{E})$. It is clear that $\mathcal{I}(\mathcal{E}) \subseteq \mathcal{PI}(\mathcal{E})$.

Example 3.2. (i) Let $E = \{0, a, b, c, d, 1\}$ be a chain where $0 \leqslant a \leqslant b \leqslant c \leqslant d \leqslant 1$. For any $x, y \in E$, we define the operations $\otimes$ and $\sim$ on E as Table 1 and Table 2:

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	0	a	b
c	0	0	0	a	a	c
d	0	0	a	a	a	d
1	0	a	b	c	d	1

Table 1

$\sim$	0	a	b	c	d	1
0	1	c	b	a	0	0
a	c	1	b	a	a	a
b	b	b	1	b	b	b
c	a	a	b	1	c	c
d	0	a	b	c	1	d
1	0	a	b	c	d	1

Table 2

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	1	1	1	1
b	b	b	1	1	1	1
c	a	a	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

Table 3

By routine calculations, we can see that $\mathcal{E} = (E, \wedge, \otimes, \sim, 0, 1)$ is a BEQ-algebra and the operation $\to$ is as Table 3. Also, we can see that $I = \{0, a\}$ is a preideal of $\mathcal{E}$. But I is not an ideal of $\mathcal{E}$. Because, $\neg(1 \to d) = \neg d = 0 \in I$ but $\neg((1 \otimes d) \to (d \otimes d)) = \neg(d \to a) = c \notin I$.

(ii) Let $E = \{0, a, b, c, d, e, f, 1\}$ be a lattice with the following digram (Figure 1), and the operations $\otimes$ and $\sim$ are defined on E as Table 4 and Table 5.

$\otimes$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	a
b	0	0	0	0	0	0	0	b
c	0	0	0	0	0	0	0	c
d	0	0	0	0	d	d	d	d
e	0	0	0	0	d	e	d	e
f	0	0	0	0	d	d	d	f
1	0	a	b	c	d	e	f	1

Table 4

$\sim$	0	a	b	c	d	e	f	1
0	1	e	f	d	c	a	b	0
a	e	1	d	f	c	a	c	a
b	f	d	1	e	c	c	b	b
c	d	f	e	1	c	c	c	c
d	c	c	c	c	1	f	e	d
e	a	a	c	c	f	1	d	e
f	b	c	b	c	e	d	1	f
1	0	a	b	c	d	e	f	1

Table 5

$\to$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	e	1	e	1	1	1	1	1
b	f	f	1	1	1	1	1	1
c	d	f	e	1	1	1	1	1
d	c	c	c	c	1	1	1	1
e	a	a	c	c	f	1	f	1
f	b	c	b	c	e	e	1	1
1	0	a	b	c	d	e	f	1

Table 6

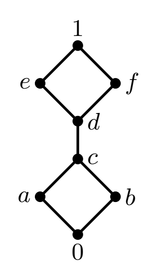

Figure 1

Then $\mathcal{E} = (E, \wedge, \otimes, \sim, 0, 1)$ is a BEQ-algebra [17] and the operation $\to$ is as Table 6. Let $I = \{0, a, b, c\}$. It is easy to see that I is an ideal of $\mathcal{E}$.

Remark 3.3. For the sake of brevity, from now on we define for any $a, b \in E$, $a \oplus b = \neg a \to b$. Also, we consider $a \oplus (a \oplus \cdots (a \oplus a) \cdots) = na$ and $0a = 0$. By Proposition 2.1(ii), we know that for any $n \in \mathbb{N}^+$, $a \leqslant 2a \leqslant \cdots \leqslant na$.

Proposition 3.4. *If $I \in \mathcal{I}(\mathcal{E})$, then for any $a \in E$, the following statements hold.*
(i) $0 \in I$.
(ii) *If $a \in I$, then $\neg\neg a \in I$.*
(iii) *If $\mathcal{E}$ is good, then $\neg\neg a \in I$ implies that $a \in I$.*

(iv) *For any* $n \in \mathbb{N}$, $na \in I$ *if and only if* $a \in I$.
(v) *If* I *and* J *are two (pre)ideals of* $\mathcal{E}$, *then* $I \cap J$ *is a (pre)ideal of* $\mathcal{E}$.

Proof. (i) Let $I \in (\mathcal{P})\mathcal{I}(\mathcal{E})$. Since $I \neq \emptyset$, there exists an elemenet $a \in I$ such that $0 \leqslant a$. Then by (I_1), $0 \in I$.
(ii) Suppose $a \in I$. Since $0 \in I$, by (I_2), $\neg\neg a = \neg a \to 0 \in I$.
(iii) If $\neg\neg a \in I$, since $\mathcal{E}$ is good, by Proposition 2.2(iii), we have $a \leqslant \neg\neg a$ and by (I_1), we get $a \in I$.
(iv) If $na \in I$, then since $a \leqslant na$ by (I_1), we have $a \in I$. Conversely, if $a \in I$, then by (I_2), we obtain for any $n \in \mathbb{N}$, $na \in I$.
(v) The proof is clear. □

Remark 3.5. It is obvious that all of the properties in Proposition 3.4, for $I \in \mathcal{PI}(\mathcal{E})$ hold, too.

Proposition 3.6. *Let* $\varphi : \mathcal{E} \to \mathcal{G}$ *be an EQ-homomorphism. Then the following statements hold.*
(i) *If* $I \in \mathcal{PI}(\mathcal{G})$, *then* $\varphi^{-1}(I) \in \mathcal{PI}(\mathcal{E})$. *Also, if* $I \in \mathcal{I}(\mathcal{G})$, *then* $\varphi^{-1}(I) \in \mathcal{I}(\mathcal{E})$.
(ii) *If* φ *is an isomorphism and* $I \in \mathcal{PI}(\mathcal{E})$, *then* $\varphi(I) \in \mathcal{PI}(\mathcal{G})$. *Also, if* $I \in \mathcal{I}(\mathcal{E})$, *then* $\varphi(I) \in \mathcal{I}(\mathcal{G})$.
(iii) *If* $\mathcal{E}$ *is spanned, then* $J = \{a \in E | \varphi(a) = 0\} \in \mathcal{PI}(\mathcal{E})$. *Also, if* $\mathcal{G}$ *is involutive, then* J *is an ideal of* $\mathcal{E}$.

Proof. (i) Suppose $a, b \in E$, $a \leqslant b$, and $b \in \varphi^{-1}(I)$. Since φ is a homomorphism, we have $\varphi(a) \leqslant \varphi(b)$. Since I is a preideal of $\mathcal{G}$ and $\varphi(b) \in I$, by (I_1), we get $\varphi(a) \in I$ and so, $a \in \varphi^{-1}(I)$. Now, let $a, b \in \varphi^{-1}(I)$. Then $\varphi(a), \varphi(b) \in I$ and so, $\varphi(\neg a \to b) = \neg\varphi(a) \to \varphi(b) \in I$. Hence, (I_2) holds. Therefore, $\varphi^{-1}(I)$ is a preideal of $\mathcal{E}$. Now, suppose I is an ideal of $\mathcal{G}$. Let $\neg(a \to b) \in \varphi^{-1}(I)$. Then $\varphi(\neg(a \to b)) \in I$ and for any $c \in E$, we have

$$\varphi\big(\neg((a \otimes c) \to (b \otimes c))\big) = \neg\big((\varphi(a) \otimes \varphi(c)) \to (\varphi(b) \otimes \varphi(c))\big).$$

Since I is an ideal of $\mathcal{G}$, for any $c \in E$, we have $\varphi\big(\neg((a \otimes c) \to (b \otimes c))\big) \in I$ and so

$$\big(\neg((a \otimes c) \to (b \otimes c))\big) \in \varphi^{-1}(I)$$

Hence, $\varphi^{-1}(I)$ is an ideal of $\mathcal{E}$.
(ii) Let $x, y \in G$. Suppose $x \leqslant y$ and $y \in \varphi(I)$. Then there exist $b \in I$ and $a \in E$ such that $a \leqslant b$, $\varphi(a) = x$, and $\varphi(b) = y$. Since I is a preideal of $\mathcal{E}$, we obtain $a \in I$ and so $x = \varphi(a) \in \varphi(I)$. Now, let $x, y \in \varphi(I)$. Then there exist $a, b \in I$ such that $\varphi(a) = x$ and $\varphi(b) = y$. Since I is a preideal of $\mathcal{E}$, we have $\neg a \to b \in I$. Hence, $\neg x \to y = \varphi(\neg a \to b) \in \varphi(I)$ and so $\varphi(I)$ is a preideal of $\mathcal{G}$. Now, suppose I is an ideal of $\mathcal{G}$. Let $\neg(x \to y) \in \varphi(I)$ and $z \in G$. Since φ is an isomorphism, there exists $c \in E$ such that $\varphi(c) = z$. Then

$$\neg\big((x \otimes z) \to (y \otimes z)\big) = \varphi\big(\neg((a \otimes c) \to (b \otimes c))\big) \in \varphi(I).$$

Thus, $\varphi(I)$ is an ideal of $\mathcal{G}$.
(iii) Since φ is an EQ-homomorphism and $\varphi(0) = 0$, it is clear that $J \neq \emptyset$. Let $a \leqslant b$ and $b \in J$. Since φ is a homomorphism, $\varphi(a) \leqslant \varphi(b) = 0$. Thus, $a \in J$. Suppose $a, b \in J$. Then $\varphi(\neg a \to b) = \neg\varphi(a) \to \varphi(b) = \neg 0 \to 0 = 0$ and so $\neg a \to b \in J$. Hence $J \in \mathcal{PI}(\mathcal{E})$. Now, suppose $\mathcal{G}$ is involutive. Let $\neg(a \to b) \in J$. Then $\varphi(\neg(a \to b)) = 0$ and so $\neg(\varphi(a) \to \varphi(b)) = 0$. Since $\mathcal{E}$ is involutive, we obtain that $\varphi(a) \to \varphi(b) = 1$ and so $\varphi(a) \leqslant \varphi(b)$. Thus, for any $c \in E$,

$$\varphi(a) \otimes \varphi(c) = \varphi(a \otimes c) \leqslant \varphi(b \otimes c) = \varphi(b) \otimes \varphi(c)$$

Hence, $\varphi((a \otimes c) \to (b \otimes c)) = 1$ and so $\varphi(\neg((a \otimes c) \to (b \otimes c))) = 0$. Therefore, $\neg((a \otimes c) \to (b \otimes c)) \in J$ and $J \in \mathcal{I}(\mathcal{E})$. □

In the following example, we show that the spanned condition in Proposition 3.6(iii) is necessary.

Example 3.7. Let $E = \{0, a, b, 1\}$ be a chain where $0 \leqslant a \leqslant b \leqslant 1$. For any $x, y \in E$, we define the operations $\otimes$ and $\sim$ on E as Table 7 and Table 8:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Table 7

$\sim$	0	a	b	1
0	1	a	a	a
a	a	1	b	b
b	a	b	1	1
1	a	b	1	1

Table 8

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	a	b	1	1
1	a	b	1	1

Table 9

Then $\mathcal{E} = (E, \wedge, \otimes, \sim, 0, 1)$ is a BEQ-algebra and the operation $\rightarrow$ is as Table 9 [13]. Since $1 \sim 0 = a \neq 0$, we get $\mathcal{E}$ is not spanned. Let $id : E \rightarrow E$ be the identity homomorphism. Then $\{x \in E | id(x) = 0\} = \{0\}$. But $\{0\}$ is not a preideal of $\mathcal{E}$. Because $0 \oplus 0 = \neg 0 \rightarrow 0 = 1 \rightarrow 0 = a \notin \{0\}$.

Proposition 3.8. *Let $\mathcal{E}, \mathcal{G}$ be two BEQ-algebras. Then K is an (pre)ideal of $\mathcal{E} \times \mathcal{G}$ if and only if there exist $I \in \mathcal{I}(\mathcal{E})(I \in \mathcal{PI}(\mathcal{E}))$ and $J \in \mathcal{I}(\mathcal{G})(J \in \mathcal{PI}(\mathcal{G}))$ such that $K = I \times J$.*

Proof. (i) Let $K \in \mathcal{PI}(\mathcal{E} \times \mathcal{G})$. We consider $I = \{a \in E | (a, b) \in K \text{ for some } b \in G\}$ and $J = \{b \in E | (a, b) \in K \text{ for some } a \in E\}$. It is clear that $K = I \times J$. Now, we prove that I is a preideal of $\mathcal{E}$. Since $(0, 0) \in K$, we have $0 \in I$ and so I is non-empty. Suppose $a_1 \leqslant a_2$ and $a_2 \in I$. Then there exists $b \in G$, such that $(a_2, b) \in K$. Since $(a_1, b) \leqslant (a_2, b)$ and K is a preideal of $\mathcal{E} \times \mathcal{G}$, $(a_1, b) \in K$ and so $a_1 \in I$. Now, let $a_1, a_2 \in I$. Then there exist $b_1, b_2 \in G$ such that $(a_1, b_1), (a_2, b_2) \in K$. Since $(a_1, b_1) \oplus (a_2, b_2) = (a_1 \oplus a_2, b_1 \oplus b_2) \in K$ and $b_1 \oplus b_2 \in G$, we obtain $a_1 \oplus a_2 \in I$. Hence $I \in \mathcal{PI}(\mathcal{E})$.

Now, let K be an ideal. We show that I is an ideal, too. Suppose $\neg(a_1 \rightarrow a_2) \in I$. Then there exists $b \in G$, such that $(\neg(a_1 \rightarrow a_2), b) \in K$. Form the definition of J, we get $b \in J$. By Proposition 3.4(ii), we have $\neg\neg b \in J$. Thus,

$$\neg((a_1, b) \rightarrow (a_2, 0)) = \neg((a_1 \rightarrow a_2), \neg b) = (\neg(a_1 \rightarrow a_2), \neg\neg b) \in K.$$

Since K is ideal, for any $c \in E$, we have

$$\big(\neg((a_1 \otimes c) \rightarrow (a_2 \otimes c)), \neg\neg b\big) = \neg\big(((a_1, b) \otimes (c, 1)) \rightarrow ((a_2, 0) \otimes (c, 1))\big) \in K.$$

Hence, $\neg((a_1 \otimes c) \rightarrow (a_2 \otimes c)) \in I$ and I is an ideal of $\mathcal{E}$. By the similar way, we can prove that $J \in (\mathcal{P})\mathcal{I}(\mathcal{G})$.

Let $I \in \mathcal{PI}(\mathcal{E})$ and $J \in \mathcal{PI}(\mathcal{G})$. We show $K = I \times J$ is a preideal of $\mathcal{E} \times \mathcal{G}$. It is obvious that K is non-empty. Suppose $a_1, a_2 \in E$ and $b_1, b_2 \in G$ such that $(a_1, b_1) \leqslant (a_2, b_2)$ and $(a_2, b_2) \in K$. Then $a_2 \in I$ and $b_2 \in J$. Since $a_1 \leqslant a_2$ and $b_1 \leqslant b_2$, we get $a_1 \in I$ and $b_1 \in J$ and so $(a_1, b_1) \in K$. Now, let $(a_1, b_1), (a_2, b_2) \in K$. Then $a_1, a_2 \in I$ and $b_1, b_2 \in J$. Since $a_1 \oplus a_2 \in I$ and $b_1 \oplus b_2 \in J$, we get $(a_1, b_1) \oplus (a_2, b_2) = (a_1 \oplus a_2, b_1 \oplus b_2) \in K$. Therefore, $K \in \mathcal{PI}(\mathcal{E} \times \mathcal{G})$.

Now, suppose I is an ideal of $\mathcal{E}$ and J is an ideal of $\mathcal{G}$. Let $\neg\big((a_1, b_1) \rightarrow (a_2, b_2)\big) \in K$. Then $\neg(a_1 \rightarrow a_2) \in I$ and $\neg(b_1 \rightarrow b_2) \in J$. Thus, for any $(c, d) \in E \times G$, we get

$$\neg\big(((a_1, b_1) \otimes (c, d)) \rightarrow ((a_2, b_2) \otimes (c, d))\big) = \big(\neg((a_1 \otimes c) \rightarrow (a_2 \otimes c)), \neg((b_1 \otimes d) \rightarrow (b_2 \otimes d))\big) \in I \times J = K.$$

Hence, K is an ideal of $\mathcal{E} \times \mathcal{G}$. □

Corollary 3.9. *Let $\prod_{i=1}^{n} \mathcal{E}_i$ be a finite product of BEQ-algebras. Then K is an (pre)ideal of $\prod \mathcal{E}_i$ if and only if for any i, there exists $I_i \in \mathcal{I}(\mathcal{E}_i)(I_i \in \mathcal{PI}(\mathcal{E}_i))$ such that $K = \prod_{i=1}^{n} I_i$.*

Theorem 3.10. *Let $\mathcal{E}$ be good and I be a non-empty subset of E. Then $I \in \mathcal{PI}(\mathcal{E})$ if and only if it satisfies the following conditions:*
(i) $0 \in I$,
(ii) For any $a, b \in E$, if $\neg(\neg a \rightarrow \neg b) \in I$ and $a \in I$, then $b \in I$.

Proof. Let $I \in \mathcal{PI}(\mathcal{E})$. Then by Proposition 3.4(i), $0 \in I$. Now, suppose for any $a, b \in E$, $\neg(\neg a \to \neg b) \in I$ and $a \in I$. By (I_2),
$$\neg a \to \neg(\neg a \to \neg b) = (\neg a \to \neg b) \to \neg\neg a \in I.$$
On the other hand,
$$b \to ((\neg a \to \neg b) \to \neg\neg a) = (\neg a \to \neg b) \to (b \to \neg\neg a) = (\neg a \to \neg b) \to (\neg a \to \neg b) = 1.$$
Since $\mathcal{E}$ is separated, we obtain $b \leqslant \neg a \to \neg(\neg a \to \neg b)$ and by (I_1), we get $b \in I$.
Conversely, let $a \leqslant b$ and $b \in I$. By Proposition 2.1(vii), we have $\neg b \leqslant \neg a$ and so $\neg b \to \neg a = 1$. Since $\mathcal{E}$ is good, $\neg(\neg b \to \neg a) = \neg 1 = 0 \in I$. Since $b \in I$, by (ii), we have $a \in I$. Now, suppose $a, b \in I$. By Propositions 2.2(iii) and 2.1(vi), we have $\neg a \leqslant (\neg a \to b) \to b \leqslant \neg b \to \neg(\neg a \to b)$. Hence, by Proposition 2.1(vii), $\neg(\neg b \to \neg(\neg a \to b)) \leqslant \neg\neg a$. Since $\neg\neg a \in I$, by Proposition 3.4(iii), we obtain $\neg(\neg b \to \neg(\neg a \to b)) \in I$. Therefore by (ii), $\neg a \to b \in I$. □

Corollary 3.11. *Let $\mathcal{E}$ be good and $I \in \mathcal{PI}(\mathcal{E})$. For any $a, b \in E$ if $\neg(\neg a \sim \neg b) \in I$ and $a \in I$, then $b \in I$.*

Proof. Suppose $\neg(\neg a \sim \neg b) \in I$ and $a \in I$ for any $a, b \in E$. By Proposition 2.1(iii) and (vii), $\neg a \sim \neg b \leqslant \neg a \to \neg b$ and so $\neg(\neg a \to \neg b) \leqslant \neg(\neg a \sim \neg b)$. Since $I \in \mathcal{PI}(\mathcal{E})$, by (I_1), $\neg(\neg a \to \neg b) \in I$ and by Theorem 3.10, $b \in I$. □

In the following example, we show that the good condition in Theorem 3.10, is necessary.

Example 3.12. Let $\mathcal{E}$ be the BEQ-algebra as in Example 3.7. Since $1 \sim a = b \neq a$, we obtain $\mathcal{E}$ is not good. By some calculations, we can see $I = \{0, a\}$ is a preideal of $\mathcal{E}$. On the other hand $\neg(\neg 0 \to \neg 1) = \neg(1 \to a) = \neg b = a \in I$. But $1 \notin I$.

Definition 3.13. Let X be a subset of E. The set of all complement elements (with respect to X) is denoted by $N(X)$ and is defined by $N(X) = \{x \in E | \neg x \in X\}$.

Example 3.14. Let $\mathcal{E}$ be a BEQ-algebra as in Example 3.2(ii) and $X = \{e, f\}$. It is easy to see that $N(X) = \{a, b\}$.

Proposition 3.15. *Let $\mathcal{E}$ be good. For any $a, b \in E$, $\neg a \to \neg b = \neg\neg(\neg a \to \neg b)$.*

Proof. Since $\mathcal{E}$ is good, by Proposition 2.2(iii), we have $\neg a \to \neg b \leqslant \neg\neg(\neg a \to \neg b)$. On the other hand, by Proposition 2.1(ix), we get
$$\begin{aligned}
\neg\neg(\neg a \to \neg b) \to (\neg a \to \neg b) =& \neg a \to (\neg\neg(\neg a \to \neg b) \to \neg b) \\
=& \neg a \to (b \to \neg\neg\neg(\neg a \to \neg b)) \\
=& \neg a \to (b \to \neg(\neg a \to \neg b)) \\
=& \neg a \to ((\neg a \to \neg b) \to \neg b) \\
=& (\neg a \to \neg b) \to (\neg a \to \neg b) \\
=& 1.
\end{aligned}$$
Since $\mathcal{E}$ is separated, $\neg\neg(\neg a \to \neg b) \leqslant \neg a \to \neg b$. Thus, $\neg a \to \neg b = \neg\neg(\neg a \to \neg b)$. □

Proposition 3.16. *Let $\mathcal{E}$ be good. If $I \in \mathcal{PI}(\mathcal{E})$ and $F \in \mathcal{PF}(\mathcal{E})$, then the following statements hold:*
(i) $I = NN(I)$.
(ii) $F \subseteq NN(F)$.
(iii) $N(F) = NNN(F)$.
(iv) $N(F) \in \mathcal{PI}(\mathcal{E})$.
(v) $N(I) \in \mathcal{PF}(\mathcal{E})$, also if $I \in \mathcal{I}(\mathcal{E})$, then $N(I) \in \mathcal{F}(\mathcal{E})$.
(vi) Let $\mathcal{E}$ be involutive. If $F \in \mathcal{F}(\mathcal{E})$, then $N(F) \in \mathcal{I}(\mathcal{E})$.

Proof. (i) Let $a \in I$. Since $NN(I) = \{x \in E | \neg\neg x \in I\}$, by Proposition 3.4$(ii)$, we get $a \in NN(I)$. Conversely, if $a \in NN(I)$, then $\neg\neg a \in I$ and so by Proposition 3.4(iii), we obtain $a \in I$. It is clear that if $I \in \mathcal{I}(\mathcal{E})$, then $NN(I) = I$, too.
(ii) Let $a \in F$. By Proposition 2.2(iii), we have $a \leqslant \neg\neg a$ and so $\neg\neg a \in F$. Thus, $a \in NN(F)$.
(iii) By Proposition 2.1(ix), the proof is clear.
(iv) Since F is a (pre)filter of $\mathcal{E}$, $\neg 0 = 1 \in F$ and so $0 \in N(F)$. Now, suppose $\neg(\neg a \to \neg b) \in N(F)$ and $a \in N(F)$. Then $\neg\neg(\neg a \to \neg b) \in F$ and $\neg a \in F$. By Proposition 3.15, $\neg a \to \neg b \in F$. Since $\neg a \in F$ and F is a (pre)filter of $\mathcal{E}$, we obtain $\neg b \in F$. Hence, $b \in N(F)$ and so $N(F) \in \mathcal{PI}(\mathcal{E})$.
(v) Since $\mathcal{E}$ is good, $\neg 1 = 0 \in I$ and so $1 \in N(I)$. Suppose $a \to b \in N(I)$ and $a \in N(I)$. Then $\neg(a \to b) \in I$ and $\neg a \in I$. By Proposition 2.1(vi) and (vii), $a \to b \leqslant \neg b \to \neg a$ and so $\neg(\neg b \to \neg a) \leqslant \neg(a \to b)$. Thus $\neg(\neg b \to \neg a) \in I$. By Propositions 2.1$(ix)$, and 2.2$(ii)$, we have

$$\neg(\neg\neg a \to \neg\neg b) = \neg(\neg b \to \neg\neg\neg a) = \neg(\neg b \to \neg a) \in I.$$

Since $\neg a \in I$ and $I \in \mathcal{PI}(\mathcal{E})$, by Theorem 3.10$(ii)$, $\neg b \in I$ and so $b \in N(I)$. Hence $N(I) \in \mathcal{PI}(\mathcal{E})$. Now, let I be an ideal of $\mathcal{E}$. Suppose $a \to b \in N(I)$. Then $\neg(a \to b) \in I$ and so for any $c \in E$, we obtain $\neg((a \otimes c) \to (b \otimes c)) \in I$. Thus, $(a \otimes c) \to (b \otimes c) \in N(I)$ an $N(I)$ is a filter of $\mathcal{E}$.
(vi) Let $\mathcal{E}$ be involutive and F be a filter of $\mathcal{E}$. By (iv), we have $N(F) \in \mathcal{PI}(\mathcal{E})$. Now, suppose $\neg(a \to b) \in N(F)$. Then $a \to b = \neg\neg(a \to b) \in F$. Since F is a filter of $\mathcal{E}$, for any $c \in E$, we get

$$(a \otimes c) \to (b \otimes c) = \neg\neg((a \otimes c) \to (b \otimes c)) \in F.$$

Hence, $\neg((a \otimes c) \to (b \otimes c)) \in N(F)$ and $N(F) \in \mathcal{I}(\mathcal{E})$. □

In the following example, we show that the good condition in Proposition 3.16, is necessary.

Example 3.17. (i) Let $\mathcal{E}$ be the BEQ-algebra as in Example 3.7. Since $1 \sim a = b \neq a$, we know that $\mathcal{E}$ is not good. By some calculations, we can see $I = \{0, a\}$ is a preideal of $\mathcal{E}$. Then $N(I) = \{a, b, 1\}$. But $N(I)$ is not a prefilter of $\mathcal{E}$. Because $1 \in N(I)$ and $1 \to 0 = a \in N(I)$, but $0 \notin N(I)$.
(ii) Let $\mathcal{E}$ be the BEQ-algebra as in Example 3.2(i). Since $\mathcal{E}$ is good, by Remark 2.5, $F = \{1\}$ is a filter of $\mathcal{E}$. But as we see in Example 3.2(i), $N(F) = \{0\} \notin \mathcal{I}(\mathcal{E})$.

Although, we proved in good EQ-algebras preideals and prefilters are dual of each others, but the most properties of (pre)ideals will be proved in a different ways.

In [18], the notion of ideals in equality algebras was introduced. But the author could not prove the binary relation introduced by ideals is a congruence relation and an open problem was stated. In the following theorem, we prove the binary relation introduce by ideals of good BEQ-algebras is a congruence relation. Since every good BEQ-algebra is an equality algebra [21], the open problem in [18] is solved.

Theorem 3.18. *Let $\mathcal{E}$ be good and $I \in \mathcal{PI}(\mathcal{E})$. Then for any $a, b \in E$, a binary relation "$\approx_I$" on E can be defined as follows:*

$$a \approx_I b \quad \textit{if and only if} \quad \neg(a \sim b) \in I.$$

(i) *The binary relation "$\approx_I$" is an equivalence relation.*

(ii) *If I is an ideal of $\mathcal{E}$, then "$\approx_I$" is a congruence relation.*

(iii) *If I is an ideal of $\mathcal{E}$, then $\mathcal{E}/I = (E/I, \wedge_I, \otimes_I, \sim_I)$ is a good BEQ-algebra where, for any $a, b \in E$, we have,*

$$[a] \wedge_I [b] = [a \wedge b] \quad , \quad [a] \otimes_I [b] = [a \otimes b] \quad , \quad [a] \sim_I [b] = [a \sim b] \quad , \quad [a] \to_I [b] = [a \to b].$$

Proof. (i) For any $a \in E$, $a \sim a = 1$ and since $\mathcal{E}$ is good, $\neg(a \sim a) = 0 \in I$ and $\approx_I$ is reflexive. By Proposition 2.1(i), it is clear that $\approx_I$ is symmetric. Suppose $a \approx_I b$ and $b \approx_I c$. Then $\neg(a \sim b), \neg(b \sim c) \in I$. By Proposition 2.1$(iv)$, we have

$$a \sim b \leqslant (a \sim c) \sim (b \sim c) \leqslant \neg(a \sim c) \sim \neg(b \sim c) \leqslant \neg\neg(a \sim c) \sim \neg\neg(b \sim c).$$

By Proposition 2.1(vii), $\neg(\neg\neg(a \sim c) \sim \neg\neg(b \sim c)) \leqslant \neg(a \sim b)$. Since $I \in \mathcal{PI}(\mathcal{E})$ and $\neg(a \sim b) \in I$, we obtain $\neg(\neg\neg(a \sim c) \sim \neg\neg(b \sim c)) \in I$. By Corollary 3.11, we get $\neg(a \sim c) \in I$ and so $\approx_I$ is transitive.
(ii) Suppose $a \approx_I b$ and $c \approx_I d$. Then $\neg(a \sim b), \neg(c \sim d) \in I$. By Proposition 2.1$(v)$, we have $a \sim b \leqslant (a \wedge c) \sim (b \wedge c)$ and $c \sim d \leqslant (c \wedge b) \sim (d \wedge b)$. Thus, by Proposition 2.1(vii), $\neg((a \wedge c) \sim (b \wedge c)) \in I$ and $\neg((c \wedge b) \sim (d \wedge b)) \in I$. Since $\approx_I$ is an equivalence relation, we obtain $(a \wedge c) \approx_I (b \wedge d)$. By the similar way, we can see that $(a \sim c) \approx_I (b \sim d)$. Since $I \in \mathcal{I}(\mathcal{E})$, we have $\neg((a \otimes c) \sim (b \otimes c)) \in I$ and $\neg((b \otimes c) \sim (b \otimes d)) \in I$. Since $\approx_I$ is an equivalence relation on $\mathcal{E}$, we get $a \otimes c \approx_I b \otimes d$. Therefore $\approx_I$ is a congruence relation.
(iii) By (ii), it is clear that $\mathcal{E}/I$ is a good BEQ-algebra. □

Corollary 3.19. *Let $\mathcal{E}$ be good and $I \in \mathcal{I}(\mathcal{E})$. Then for any $a, b \in E$, we define an order on E/I as follows,*

$$[a] \leqslant [b] \quad \textit{if and only if} \quad \neg(a \to b) \in I.$$

In the following example, we show that the converse of Theorem 3.18 may not be true in general.

Example 3.20. Let $\mathcal{E}$ be the EQ-algebra as in Example 3.2(i). By routine calculations, we can see that $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ is a good and non-involutive EQ-algebra. Since $\mathcal{E}$ is good, $\{1\}$ is a filter of $\mathcal{E}$ and $\theta = \{(a, b) \in E \times E | a = b\}$ is a congruence relation on $\mathcal{E}$. But $I = \{0\}$ is not an ideal of $\mathcal{E}$. Because, $\neg(a \to d) = \neg d = 0 \in I$ but $\neg((1 \otimes d) \to (d \otimes d)) = \neg(d \to a) = c \notin I$.

Theorem 3.21. *Let $\mathcal{E}$ be good. If θ is a congruence relation on E, then the following statements hold:*

(i) *$[0]_\theta$ is a preideal of $\mathcal{E}$.*

(ii) *If $\mathcal{E}$ is involutive, then $[0]_\theta$ is an ideal of $\mathcal{E}$.*

Proof. (i) Let θ be a congruence relation on E. It is clear that $[0]_\theta$ is non-empty. Suppose $a \leqslant b$ and $b \in [0]_\theta$, then $(b, 0) \in \theta$. Since for any $a \in E$, $(a, a) \in \theta$, we obtain $(a, 0) = (a \wedge b, a \wedge 0) \in \theta$. Thus, (I_1) is satisfied. Let $a, b \in [0]_\theta$. Then $(\neg a, 1) = (a \sim 0, 0 \sim 0) \in \theta$ and so $(\neg a \to b, 0) = (\neg a \to b, 1 \to 0) \in \theta$. Hence, $[a \oplus b]_\theta = [0]_\theta$ and so (I_2) holds and $[0]_\theta \in \mathcal{PI}(\mathcal{E})$.
(ii) Let $\neg(a \to b) \in [0]_\theta$. Then $[\neg(a \to b)]_\theta = [0]_\theta$ and so $[a]_\theta \to [b]_\theta = [1]_\theta$. Since $\mathcal{E}/\theta$ is separated, by Proposition 2.1$(viii)$, we have $[a]_\theta \leqslant [b]_\theta$ and by (E_2), for any $c \in E$, $[a \otimes c]_\theta \leqslant [b \otimes c]_\theta$. Thus, $[\neg((a \otimes c) \to (b \otimes c))]_\theta = [0]_\theta$ and so $\neg((a \otimes c) \to (b \otimes c)) \in [0]_\theta$. Therefore, $[0]_\theta \in \mathcal{I}(\mathcal{E})$. □

4 Generated preideals

In this section, we characterize the generated preideal by a subset of $\mathcal{E}$ and by using this we show that the family of all preideals of $\mathcal{E}$ is a complete lattice. Also, we prove that under some conditions, $\mathcal{PI}(\mathcal{E})$ forms an MV-algebra.

Definition 4.1. Let S be a nonempty subset of E. The smallest preideal of $\mathcal{E}$ containing S is called the *generated preideal* by S and it is denoted by $(S]_P$. It is also the intersection of all preideals of $\mathcal{E}$ containing S.

Theorem 4.2. *Let S be a nonempty subset of E. Then*

$$(S]_P = \{a \in E | a \leqslant s_1 \oplus (s_2 \oplus \cdots (s_{n-1} \oplus s_n) \cdots), \textit{ for some } n \in \mathbb{N} \textit{ and } s_1, \cdots, s_n \in S\}.$$

Proof. Let

$$I = \{a \in E | a \leqslant s_1 \oplus (s_2 \oplus \cdots (s_{n-1} \oplus s_n) \cdots), \text{ for some } n \geqslant 1 \text{ and } s_1 \cdots s_n \in S\}.$$

We should prove that I is the smallest preideal of $\mathcal{E}$ contains S. First, we show that $I \in \mathcal{PI}(\mathcal{E})$. Let $a, b \in E$ such that $a \leqslant b$ and $b \in I$. There exists $n \in \mathbb{N}$ such that for $s_1, s_2, \cdots, s_n \in S$, $b \leqslant s_1 \oplus (s_2 \oplus \cdots (s_{n-1} \oplus s_n) \cdots)$. From $a \leqslant b$, we get $a \leqslant s_1 \oplus (s_2 \oplus \cdots (s_{n-1} \oplus s_n) \cdots)$, and so $a \in I$. Hence, (I_1) holds. Now, suppose $a, b \in I$. Then there exist $n, m \in \mathbb{N}$, $s_1, s_2, \cdots, s_n \in S$ and $r_1, r_2, \cdots, r_m \in S$ such that $a \leqslant s_1 \oplus (s_2 \oplus \cdots (s_{n-1} \oplus s_n) \cdots)$ and $b \leqslant r_1 \oplus (r_2 \oplus \cdots (r_{m-1} \oplus r_m) \cdots)$. By Proposition 2.1$(vii)$, $\neg(s_1 \oplus (s_2 \oplus \cdots (s_{n-1} \oplus s_n) \cdots)) \leqslant \neg a$, and so

$$\neg a \to b \leqslant \neg(s_1 \oplus (s_2 \oplus \cdots (s_{n-1} \oplus s_n) \cdots)) \to b.$$

Since $b \leqslant r_1 \oplus (r_2 \oplus \cdots (r_{m-1} \oplus r_m) \cdots)$, by Proposition 2.1$(vii)$,

$$\neg(s_1 \oplus (s_2 \oplus \cdots (s_{n-1} \oplus s_n) \cdots)) \to b \leqslant \neg(s_1 \oplus (s_2 \oplus \cdots (s_{n-1} \oplus s_n) \cdots)) \to (r_1 \oplus (r_2 \oplus \cdots (r_{m-1} \oplus r_m) \cdots)).$$

Then

$$\neg a \to b \leqslant \neg(s_1 \oplus (s_2 \oplus \cdots (s_{n-1} \oplus s_n) \cdots)) \to (r_1 \oplus (r_2 \oplus \cdots (r_{m-1} \oplus r_m) \cdots))$$

and so

$$a \oplus b \leqslant (s_1 \oplus (s_2 \oplus \cdots (s_{n-1} \oplus s_n) \cdots)) \oplus r_1 \oplus (r_2 \oplus \cdots (r_{m-1} \oplus r_m) \cdots).$$

Thus, $a \oplus b \in I$. Hence, I is a preideal of $\mathcal{E}$.
For any $a, b \in S$, by Proposition 2.1(ii), $a \leqslant \neg b \to a = b \oplus a$ and so $a \in I$. Now, suppose there exists a preideal J such that $S \subseteq J$. It is enough to prove that $I \subseteq J$. Let $a \in I$. Then there exists $n \in \mathbb{N}$ and $s_1, s_2, \cdots s_n \in S$, such that $a \leqslant s_1 \oplus (s_2 \oplus \cdots (s_{n-1} \oplus s_n) \cdots)$. Since $S \subseteq J$ and $J \in \mathcal{PI}(\mathcal{E})$, by (I_2), $s_1 \oplus (s_2 \oplus \cdots (s_{n-1} \oplus s_n) \cdots) \in J$, and so by (I_1), $a \in J$. Hence, I is the smallest preideal of $\mathcal{E}$ contains S. Therefore, $I = (S]_P$. □

In the following example, we show that the generated preideal of a set is not an ideal, in general.

Example 4.3. Let $\mathcal{E}$ be the BEQ-algebra as in Example 3.2(i). Let $S = \{0, a\}$. By routine calculations, we can see that $(S]_P = \{0, a\}$. But as we see in Example 3.2(i), $(S]_P$ is not an ideal of $\mathcal{E}$.

Open problem. *What is the form of generated ideals of subset?*

Proposition 4.4. *If $\mathcal{E}$ is involutive, then $\oplus$ is associative and commutative.*

Proof. Let $a, b, c \in E$. By Propositions 2.1(x) and 2.2 (iii), we have

$$\begin{aligned}(a \oplus b) \oplus c = \neg(\neg a \to b) \to c =& \neg c \to \neg\neg(\neg a \to b)\\ =& \neg c \to (\neg a \to b)\\ =& \neg a \to (\neg c \to b)\\ =& \neg a \to (\neg b \to c)\\ =& a \oplus (b \oplus c).\end{aligned}$$

Hence, $\oplus$ is associative. Also, by Proposition 2.1(x), we get $a \oplus b = \neg a \to b = \neg b \to a = b \oplus a$. Thus, $\oplus$ is commutative. □

In the following example, we show that the involutive condition in Proposition 4.4 is necessary.

Example 4.5. (i) Let $E = \{0, a, b, 1\}$ be a chain where $0 \leqslant a \leqslant b \leqslant 1$. For any $x, y \in E$, we define the operations $\otimes$ and $\sim$ on E as Table 10 and Table 11:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
1	0	a	b	1

Table 10

$\sim$	0	a	b	1
0	1	a	0	0
a	a	1	a	a
b	0	a	1	b
1	0	a	b	1

Table 11

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Table 12

It is easy to see that $\mathcal{E} = (E, \wedge, \otimes, \sim, 0, 1)$ is a non-involutive BEQ-algebra and the operation $\rightarrow$ is as Table 12. We can see that $\oplus$ is not commutative, because $0 \oplus b = 1 \rightarrow b = b$. But $b \oplus 0 = 0 \rightarrow 0 = 1$.

(ii) Let $\mathcal{E}$ be the BEQ-algebra as in Example 3.7. We can see that $\oplus$ is not associative because, $0 \oplus (0 \oplus 0) = b$ but $(0 \oplus 0) \oplus 0 = a$.

Proposition 4.6. *If $\mathcal{E}$ is involutive and prelinear, then for any $a, b, c \in E$, the following statements hold:*

(i) $a \wedge (b \oplus c) \leqslant (a \wedge b) \oplus (a \wedge c)$.

(ii) *For any $n \in \mathbb{N}$, $na \wedge mb \leqslant (n + m)(a \wedge b)$.*

Proof. (i) Let $a, b, c \in E$. By Propositions 2.1(x), (xi), and 2.2(iii), we have

$$\begin{aligned}
(a \wedge (b \oplus c)) \rightarrow ((a \wedge b) \oplus (a \wedge c)) =&(a \wedge (b \oplus c)) \rightarrow (\neg(a \wedge b) \rightarrow (a \wedge c))\\
=&(a \wedge (b \oplus c)) \rightarrow ((\neg(a \wedge b) \rightarrow a) \wedge (\neg(a \wedge b) \rightarrow c))\\
=&\Big((a \wedge (b \oplus c)) \rightarrow ((\neg(a \wedge b) \rightarrow a)\Big) \wedge \Big((a \wedge (b \oplus c)) \rightarrow (\neg(a \wedge b) \rightarrow c)\Big)\\
=&\Big((\neg(a \wedge b)) \rightarrow ((a \wedge (b \oplus c)) \rightarrow a)\Big) \wedge \Big((a \wedge (b \oplus c)) \rightarrow (\neg(a \wedge b) \rightarrow c)\Big)\\
=&1 \wedge \Big((a \wedge (b \oplus c)) \rightarrow (\neg(a \wedge b) \rightarrow c)\Big)\\
=&\big((a \wedge (b \oplus c)) \rightarrow (\neg c \rightarrow (a \wedge b))\big)\\
=&(a \wedge (b \oplus c)) \rightarrow ((\neg c \rightarrow a) \wedge (\neg c \rightarrow b))\\
=&\big((a \wedge (b \oplus c)) \rightarrow (\neg c \rightarrow a)\big) \wedge \big((a \wedge (b \oplus c)) \rightarrow (\neg c \rightarrow b)\big)\\
=&\big(\neg c \rightarrow ((a \wedge (b \oplus c)) \rightarrow a)\big) \wedge \big((a \wedge (b \oplus c)) \rightarrow (\neg b \rightarrow c)\big)\\
=&\big(\neg c \rightarrow ((a \wedge (b \oplus c)) \rightarrow a)\big) \wedge \big((a \wedge (b \oplus c)) \rightarrow (b \oplus c)\big)\\
=&1.
\end{aligned}$$

Therefore, by Proposition 2.1$(viii)$, $a \wedge (b \oplus c) \leqslant (a \wedge b) \oplus (a \wedge c)$.

(ii) First we show $2a \wedge 2b \leqslant 4(a \wedge b)$. By (i), we have

$$\begin{aligned}
(a \oplus a) \wedge (b \oplus b) \leqslant&((a \oplus a) \wedge b) \oplus ((a \oplus a) \wedge b)\\
\leqslant&((a \wedge b) \oplus (a \wedge b)) \oplus ((a \wedge b) \oplus (a \wedge b))\\
=&4(a \wedge b).
\end{aligned}$$

By induction on n and m, the proof is complete. $\square$

In the following example, we show that the prelinear condition in Proposition 4.6 is necessary.

Example 4.7. Let $E = \{0, a, c, d, m, 1\}$ be a lattice with a Hesse diagram as Figure 2. For any $x, y \in E$, we define the operations $\otimes$ and $\sim$ on E as Table 13 and Table 14:

$\otimes$	0	a	c	d	m	1
0	0	0	0	0	0	0
a	0	a	0	0	a	a
c	0	0	c	c	c	c
d	0	0	c	c	c	d
m	0	a	c	c	m	m
1	0	a	c	d	m	1

Table 13

$\sim$	0	a	c	d	m	1
0	1	d	a	a	0	0
a	d	1	0	0	a	a
c	a	0	1	m	d	c
d	a	0	m	1	d	d
m	0	a	d	d	1	m
1	0	a	c	d	m	1

Table 14

$\to$	0	a	c	d	m	1
0	1	1	1	1	1	1
a	d	1	d	d	1	1
c	a	a	1	1	1	1
d	a	a	m	1	1	1
m	0	a	d	d	1	1
1	0	a	c	d	m	1

Table 15

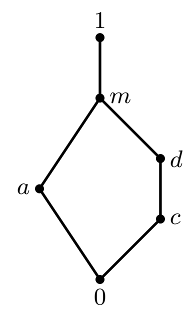

Figure 2

Then $\mathcal{E} = (E, \wedge, \otimes, \sim, 0, 1)$ is a BEQ-algebra and the operation $\to$ is as Table 15. We can see that $\mathcal{E}$ is not prelinear because, $a \to d = d$ and $d \to a = a$ but $a \vee d = m \neq 1$. Also, we can see $d \wedge (a \oplus c) = d \nleqslant (d \wedge a) \oplus (d \wedge c) = 0 \oplus c = c$.

Proposition 4.8. *$(\mathcal{PI}(\mathcal{E}), \subseteq)$ is a complete lattice where "$\wedge$" is the common intersection and for any $I_1, I_2 \in \mathcal{PI}(\mathcal{E})$, $I_1 \vee I_2 = (I_1 \cup I_2]_P$.*

Proof. By Proposition 3.4(iv) and Theorem 4.2, the proof is clear. □

Proposition 4.9. *Let $x, a, b \in E$ and $I, I_1, I_2 \in \mathcal{PI}(\mathcal{E})$. Then the following statements hold:*

- (i) $(x]_P = \{a \in E | \exists n \in \mathbb{N} \text{ such that } a \leqslant nx\}$.
- (ii) If $a \leqslant b$, then $(a]_P \subseteq (b]_P$.
- (iii) If $a \in I$, then $(a]_P \subseteq I$.
- (iv) $I = \bigvee_{a \in I} (a]_P$.
- (v) If $\mathcal{E}$ is involutive, then $(I \cup \{a\}]_P = \{x \in E | x \leqslant na \oplus i, \text{ for some } i \in I \text{ and } n \in \mathbb{N}\}$.
- (vi) If $\mathcal{E}$ is involutive, then $I_1 \vee I_2 = (I_1 \cup I_2]_P = \{x \in E | x \leqslant i_1 \oplus i_2, \text{ for some } i_1 \in I_1 \text{ and } i_2 \in I_2\}$.
- (vii) If $\mathcal{E}$ is involutive, then $(a]_P \vee (b]_P = (a \oplus b]_P$.
- ($viii$) If $\mathcal{E}$ is involutive and prelinear, $(a]_P \wedge (b]_P = (a \wedge b]_P$.

Proof. (i) Let $I = \{a \in E | \exists n \in \mathbb{N} \text{ such that } a \leqslant nx\}$. We show that I is the smallest preideal of $\mathcal{E}$ contains x. Suppose $a \leqslant b$ and $b \in I$. Then there exists $n \in \mathbb{N}$ such that $b \leqslant nx$. From $a \leqslant b$, we have $a \leqslant nx$ and so $a \in I$. Hence, (I_1) holds. Now, suppose $a, b \in I$. Then there exist $n, m \in \mathbb{N}$, such that $a \leqslant nx$ and $b \leqslant mx$. By Proposition 2.1(ii) and (vii), we have $\neg nx \leqslant \neg a$ and $\neg a \to b \leqslant (\neg nx) \to b$. Also, we have

$$a \oplus b = \neg a \to b \leqslant (\neg nx) \to b \leqslant (\neg nx) \to (mx) = (nx) \oplus (mx) = (n+m)x.$$

Thus $a \oplus b \in I$ and $I \in \mathcal{PI}(\mathcal{E})$. By Proposition 2.1($ii$), $x \leqslant \neg x \to x = 2x$ and so $x \in I$. Let $J \in \mathcal{PI}(\mathcal{E})$ such that $x \in J$. Suppose $a \in I$. Then there exists $n \in \mathbb{N}$ such that $a \leqslant nx$. Since J is a preideal of $\mathcal{E}$ and $x \in J$, by (I_2), for any $n \in \mathbb{N}$, we have $nx \in J$. Thus by (I_1), we obtain $a \in J$ and so $(a]_P = I$.

(ii) First, we show that for any $m \in \mathbb{N}$, $ma \leqslant mb$. Since $a \leqslant b$, by Proposition 2.1(vii), we have $\neg b \leqslant \neg a$ and so $\neg a \to a \leqslant \neg b \to b$. Thus by induction on m, we can see that $ma \leqslant mb$. Now, suppose $x \in (a]_P$. Then there exists $m \in \mathbb{N}$ such that $x \leqslant ma$ and so $x \leqslant mb$. Hence, $(a]_P \subseteq (b]_P$.

(iii) Suppose $a \in I$ and $x \in (a]_P$. There exists $n \in \mathbb{N}$ such that $x \leqslant na$. For any $n \in \mathbb{N}$, $na \in I$ and by (I_1), $x \in I$.

(iv) For any $a \in I$, by (i), we have $(a]_P \subseteq I$ and so $\bigvee\limits_{a\in I} (a]_P \subseteq I$. Conversely, if $a \in I$, then $a \in (a]_P$. Thus $a \in \bigvee\limits_{a\in I} (a]_P$. Hence $I = \bigvee\limits_{a\in I} (a]_P$.

(v) Let $J = \{x \in E | x \leqslant na \oplus i, \text{ for some } i \in I \text{ and } n \in \mathbb{N}\}$. By Proposition 2.1($ii$), for any $i \in I$ we have $i \leqslant na \oplus i$ and so $I \subseteq J$. Since $\mathcal{E}$ is good, by Propositions 2.2(iii) and 2.1(vii), $a \leqslant a \oplus i$. Thus, $I \cup \{a\} \subseteq J$. Now, we prove that J is an (pre)ideal of $\mathcal{E}$. Clearly, (I_1) holds. Suppose $x, y \in J$. Then there exist $m, n \in \mathbb{N}$ and $i, j \in I$ such that $x \leqslant na \oplus i$ and $y \leqslant ma \oplus j$. By Proposition 2.1(vi) and (vii) we have

$$x \oplus y = \neg x \to y \leqslant \neg(na \oplus i) \to y \leqslant \neg(na \oplus i) \to (ma \oplus j) = (na \oplus i) \oplus (ma \oplus j).$$

Since $\mathcal{E}$ is involutive, by Proposition 4.4, we have $x \oplus y \leqslant (n+m)a \oplus i \oplus j$. Since I is an (pre)ideal of $\mathcal{E}$, $i \oplus j \in I$ and so $x \oplus y \in J$. Hence, J is an (pre)ideal of $\mathcal{E}$. Now, let $A \in \mathcal{PI}(\mathcal{E})$ such that $I \cup \{a\} \subseteq A$. By (I_2), we get for any $n \in \mathbb{N}$, $na \in A$. Suppose $x \in J$. Then there exist $n \in \mathbb{N}$ and $i \in I$ such that $x \leqslant na \oplus i$. Since $I \subseteq A$, we have $na \oplus i \in A$ and by (I_1), $x \in A$. Therefore, $J \subseteq A$.

(vi) Let $B = \{x \in E | x \leqslant i_1 \oplus i_2 \text{ for some } i_1 \in I_1 \text{ and } i_2 \in I_2\}$. By Propositions 2.1($vii$) and 2.2($iii$), for any $i_1 \in I_1$ and $i_2 \in I_2$, we have $i_1 \leqslant i_1 \oplus i_2$ and $i_2 \leqslant i_1 \oplus i_2$. Thus $I_1 \cup I_2 \subseteq B$. Now, we show that $B \in \mathcal{PI}(\mathcal{E})$. Obviously, (I_1) holds. Let $x, y \in B$. Then there exist $i_1, j_1 \in I_1$ and $i_2, j_2 \in I_2$ such that $x \leqslant i_1 \oplus i_2$ and $y \leqslant j_1 \oplus j_2$. By Proposition 2.1($vi$) and ($vii$), we have

$$x \oplus y \leqslant \neg x \to y = \neg(i_1 \oplus i_2) \to (j_1 \oplus j_2) = (i_1 \oplus i_2) \oplus (j_1 \oplus j_2).$$

Since $\mathcal{E}$ is involutive, by Proposition 4.4, $x \oplus y \leqslant (i_1 \oplus j_1) \oplus (i_2 \oplus j_2)$ and so $B \in \mathcal{PI}(\mathcal{E})$. Let $D \in \mathcal{PI}(\mathcal{E})$ such that $I_1 \cup I_2 \subseteq D$. Suppose $x \in B$. There exist $i_1 \in I_1$ and $i_2 \in I_2$ such that $x \leqslant i_1 \oplus i_2$. By (I_1) and (I_2), we obtain $x \in D$ and so $B \subseteq D$.

(vii) Since $a, b \leqslant a \oplus b$, by ($ii$), we get $(a]_P, (b]_P \subseteq (a \oplus b]_P$. For the converse, suppose $x \in (a \oplus b]_P$. By (i), there exists $n \in \mathbb{N}$ such that $x \leqslant n(a \oplus b)$. Since $\mathcal{E}$ is involutive, by Proposition 4.4, we have $x \leqslant n(a \oplus b) = na \oplus nb$. Thus, by ($iv$), $x \in (a]_P \vee (b]_P$ and so $(a \oplus b]_P \subseteq (a]_P \vee (b]_P$.

($viii$) Since $a \wedge b \leqslant a$ and $a \wedge b \leqslant b$, by (ii), we have $(a \wedge b]_P \subseteq (a]_P \cap (b]_P$. For the converse, let $x \in (a]_P \cap (b]_P$. Then there exist $n, m \in \mathbb{N}$, such that $x \leqslant na$ and $x \leqslant mb$ and so $x \leqslant na \wedge mb$. By Proposition 4.6(ii), we get $x \leqslant na \wedge mb \leqslant (n+m)(a \wedge b)$. Thus, $x \in (a \wedge b]_P$ and $(a]_P \cap (b]_P \subseteq (a \wedge b]_P$. $\square$

Theorem 4.10. *If $\mathcal{E}$ is involutive and prelinear, then $(\mathcal{PI}(\mathcal{E}), \wedge, \vee)$ is a distributive lattice.*

Proof. Let $I_1, I_2, I_3 \in \mathcal{PI}(\mathcal{E})$. Similar to any lattice, we should prove $I_1 \wedge (I_2 \vee I_3) \subseteq (I_1 \wedge I_2) \vee (I_1 \wedge I_3)$. Suppose $a \in I_1 \cap (I_2 \wedge I_3)$. Then $a \in I_1$ and $a \in I_2 \wedge I_3$. By Proposition 4.9(vi), there exist $i_2 \in I_2$ and $i_3 \in I_3$ such that $a \leqslant i_2 \oplus i_3$. By Proposition 4.6, we have $a = a \wedge a \leqslant a \wedge (i_2 \oplus i_3) \leqslant (a \wedge i_2) \oplus (a \wedge i_3)$. Since $a \wedge i_2 \in I_1 \cap I_2$ and $a \wedge i_3 \in I_1 \cap I_3$, we have $a \in (I_1 \wedge I_2) \vee (I_1 \wedge I_3)$. Therefore, $(\mathcal{PI}(\mathcal{E}), \wedge, \vee)$ is a distributive lattice. $\square$

Proposition 4.11. *Let $\mathcal{E}$ be involutive and prelinear. For any $I_1, I_2 \in \mathcal{PI}(\mathcal{E})$, we define a binary operation as $I_1 \to I_2 = \{a \in E | I_1 \cap (a]_P \subseteq I_2\}$. Then $I_1 \to I_2 \in \mathcal{PI}(\mathcal{E})$.*

Proof. Let $B = \{a \in E | I_1 \cap (a]_P \subseteq I_2\}$. Since $0 \in (0]_P$ and $0 \in I_1$, then $0 \in B$ and B is non-empty. Now, suppose $b \leqslant a$ and $a \in B$. By Proposition 4.9(ii), we have $(b]_P \subseteq (a]_P$ and so $I_1 \cap (b]_P \subseteq I_1 \cap (a]_P \subseteq I_2$.

Thus (I_1) holds.
Let $a, b \in B$. Then $I_1 \cap (a]_P \subseteq I_2$ and $I_2 \cap (b]_P \subseteq I_2$. By Proposition 4.8, we get $(I_1 \cap (a]_P) \vee (I_1 \cap (b]_P) \subseteq I_2$. By Theorem 4.10 and Proposition 4.9(vii), we obtain $I_1 \cap (a \oplus b]_P = I_1 \cap ((a]_P \vee (b]_P) = (I_1 \cap (a]_P) \vee (I_1 \cap (b]_P) \subseteq I_2$. Hence, $a \oplus b \in B$ and (I_2) is satisfied. □

A *Heyting algebra* [4] is an algebraic structure $(H, \wedge, \vee, \rightarrow, 0, 1)$ of type $(2, 2, 2, 0, 0)$ which for any $x, y, z \in H$, satisfies the following conditions:

(H_1) $(H, \wedge, \vee)$ is a distributive lattice.

(H_2) $x \wedge 0 = 0$ and $x \vee 1 = 1$.

(H_3) $x \rightarrow x = 1$.

(H_4) $(x \rightarrow y) \wedge y = y$ and $x \wedge (x \rightarrow y) = x \wedge y$.

(H_5) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$ and $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$.

Theorem 4.12. *If $\mathcal{E}$ is involutive and prelinear, then $(\mathcal{PI}(\mathcal{E}), \wedge, \vee, \rightarrow, \{0\}, E)$ is a Heyting algebra where $\wedge$ and $\vee$ are the same as* Proposition 4.8.

Proof. From Theorem 4.10, (H_1) holds. It is clear that (H_2) is satisfied.
Let $I_1, I_2, I_3 \in \mathcal{PI}(\mathcal{E})$. For any $a \in E$, $I_1 \cap (a]_P \subseteq I_1$. Thus, $I_1 \rightarrow I_1 = E$ and (H_3) holds.
Obviously, $(I_1 \rightarrow I_2) \wedge I_2 \subseteq I_2$. Let $a \in I_2$. Then $(a]_P \subseteq I_2$ and so $I_1 \cap (a]_P \subseteq (a]_P \subseteq I_2$. Thus, $a \in I_1 \rightarrow I_2$. Hence, $I_2 \subseteq (I_1 \rightarrow I_2) \wedge I_2$. Now, let $a \in I_1 \wedge (I_1 \rightarrow I_2)$. Then $a \in I_1$ and $a \in I_1 \rightarrow I_2$. By Proposition 4.9$(iii)$, $(a]_P \subseteq I_1$ and so $I_1 \cap (a]_P = (a]_P \subseteq I_2$. Thus, $a \in I_2$ and so $a \in I_1 \cap I_2$. Conversely, suppose $a \in I_1 \wedge I_2$. Then $a \in I_1$ and $a \in I_2$. Thus, $a \in I_1 \cap (a]_P \subseteq (a]_P \subseteq I_2$ and so $a \in I_1 \rightarrow I_2$. Hence, $a \in (I_1 \rightarrow I_2) \wedge I_1$. Therefore, $I_1 \wedge (I_1 \rightarrow I_2) = I_1 \wedge I_2$ and so (H_4) is satisfied.
Let $a \in I_1 \rightarrow (I_2 \wedge I_3)$. Then $I_1 \cap (a]_P \subseteq I_2 \wedge I_3$ and so $I_1 \cap (a]_P \subseteq I_2$ and $I_1 \cap (a]_P \subseteq I_3$. Thus $a \in I_1 \rightarrow I_2$ and $a \in I_1 \rightarrow I_3$. Hence, $I_1 \rightarrow (I_2 \wedge I_3) \subseteq (I_1 \rightarrow I_2) \wedge (I_1 \rightarrow I_3)$. Conversely, suppose $a \in (I_1 \rightarrow I_2) \wedge (I_1 \rightarrow I_3)$. Then $I_1 \cap (a]_P \subseteq I_2$ and $I_1 \cap (a]_P \subseteq I_3$. Since $(\mathcal{PI}(\mathcal{E}), \wedge, \vee)$ is a lattice, we get $I_1 \cap (a]_P \subseteq I_2 \wedge I_3$. Thus, $a \in I_1 \rightarrow (I_2 \wedge I_3)$ and so $(I_1 \rightarrow I_2) \wedge (I_1 \rightarrow I_3) \subseteq I_1 \rightarrow (I_2 \wedge I_3)$.
Let $a \in (I_1 \vee I_2) \rightarrow I_3$. Then $(I_1 \vee I_2) \cap (a]_P \subseteq I_3$. By Proposition 4.10, $(I_1 \cap (a]_P) \vee (I_2 \cap (a]_P) = (I_1 \vee I_2) \cap (a]_P \subseteq I_3$. Thus, $I_1 \cap (a]_P \subseteq I_3$ and $I_2 \cap (a]_P \subseteq I_3$. Hence, $a \in I_1 \rightarrow I_3$ and $a \in I_2 \rightarrow I_3$ and so $a \in (I_1 \rightarrow I_3) \cap (I_2 \rightarrow I_3)$. Conversely, suppose $a \in (I_1 \rightarrow I_3) \cap (I_2 \rightarrow I_3)$. Then $I_1 \cap (a]_P \subseteq I_3$ and $I_2 \cap (a]_P \subseteq I_3$. By Proposition 4.8, we obtain $(I_1 \cap (a]_P) \vee (I_2 \cap (a]_P) \subseteq I_3$. From Proposition 4.10, $(I_1 \vee I_2) \cap (a]_P \subseteq I_3$. Hence, (H_5) holds. Therefore, $(\mathcal{PI}(\mathcal{E}), \wedge, \vee, \rightarrow, \{0\}, E)$ is a Heyting algebra. □

Corollary 4.13. *If $\mathcal{E}$ is involutive and prelinear, then for any $I, J \in \mathcal{PI}(\mathcal{E})$, we have:*

(i) $(I \wedge J) \rightarrow K = I \rightarrow (J \rightarrow K)$.

(ii) $I \wedge (I \rightarrow J) = J \wedge (J \rightarrow I)$.

Proof. (i) Let $a \in (I \wedge J) \rightarrow K$. Then $(a]_P \cap I \cap J \subseteq K$. Now, suppose $x \in (a]_P \cap I$, then $(x]_P \subseteq (a]_P \cap I$ and so $(x]_P \cap J \subseteq (a]_P \cap I \cap J \subseteq K$. Thus, we get that $(a]_P \cap I \subseteq J \rightarrow K$ and so $a \in I \rightarrow (J \rightarrow K)$. Conversely, let $a \in I \rightarrow (J \rightarrow K)$. Then $(a]_P \cap I \subseteq J \rightarrow K$. Thus we have $(a]_P \cap I \cap J \subseteq (a]_P \cap I \subseteq J \rightarrow K$. For any $x \in (a]_P \cap I \cap J$, we have $(x]_P = (x]_P \cap J \subseteq K$ and so $x \in K$. Hence, we obtain $(a]_P \cap I \cap J \subseteq K$ and so $a \in (I \cap J) \rightarrow K$. Therefore, $(I \wedge J) \rightarrow K = I \rightarrow (J \rightarrow K)$.
(ii) Let $a \in I \wedge (I \rightarrow J)$. Then $a \in I$ and so $(a]_P \subseteq I$. Also, since $a \in I \rightarrow J$, we have $(a]_P \cap I = (a]_P \subseteq J$. Thus, we get $a \in J$ and also $a \in J \rightarrow I$. By the similar way, the proof of converse is clear. □

Notation. For any $I \in \mathcal{PI}(\mathcal{E})$, we denote $I^c = I \rightarrow \{0\}$.

Proposition 4.14. *Let $I \in \mathcal{PI}(\mathcal{E})$. If $\mathcal{E}$ is involutive and prelinear, then*

$$I^c = \{a \in E \mid a \wedge i = 0, \text{ for any } i \in I\}.$$

Proof. Let $B = \{a \in E |\ a \wedge i = 0, \text{ for any } i \in I\}$ and $a \in B$. Then for any $i \in I$, $a \wedge i = 0$. By Propositions 4.9(iv), $(viii)$, and 4.10, we have

$$I \cap (a]_P = \bigvee_{i \in I} (i]_P \cap (a]_P = \bigvee_{i \in I} ((i]_P \cap (a]_P) = \bigvee_{i \in I} ((i \wedge a]_P) = \bigvee_{i \in I} (0]_P = \{0\}. \tag{4.1}$$

Thus, $a \in I^c$, and so $B \subseteq I^c$. Conversely, let $a \in I^c$. Then $I \cap (a]_P = \{0\}$. By (4.1), we get for any $i \in I$, $i \wedge a = 0$ and so $I^c \subseteq B$. Therefore, $I^c = B$. □

Corollary 4.15. *Let $\mathcal{E}$ be involutive and prelinear. Then for any $a \in E$, $((a]_P)^c = \{x \in E | x \wedge a = 0\}$.*

Proposition 4.16. *Let $I \in \mathcal{PI}(\mathcal{E})$. If $\mathcal{E}$ is involutive and prelinear, then $(I^c)^c = I$.*

Proof. Let $a \in (I^c)^c$. Then by Proposition 4.14, for any $x \in I^c$, we have $a \wedge x = 0$. On the other hand for any $i \in I$, $x \wedge i = 0$, too. Thus, $a \in I$ and so $(I^c)^c \subseteq I$. Conversely, let $a \in I$. Then for any $x \in I^c$, we have $a \wedge x = 0$. Hence we get $a \in (I^c)^c$ and so $I \subseteq (I^c)^c$. Therefore, $I = I^{cc}$. □

An *MV-algebra* [6] is an algebraic structure $(M, *, ^c, 0)$ of type $(2, 1, 0)$ which for any $a, b \in M$, it satisfies in the following conditions:
$(MV1)$ $(M, *, 0)$ is a commutative monoid.
$(MV2)$ $(a^c)^c = a$.
$(MV3)$ $0^c * a = 0^c$.
$(MV4)$ $(a^c * b)^c * b = (b^c * a)^c * a$.

Theorem 4.17. *If $\mathcal{E}$ is involutive and prelinear, then $(\mathcal{PI}(\mathcal{E}), *, ^c, \{0\})$ is an MV-algebra where for any $I, J \in \mathcal{PI}(\mathcal{E})$,*

$$I * J = (I^c \wedge J^c)^c = (I^c \cap J^c)^c.$$

Proof. First we show $(\mathcal{PI}(\mathcal{E}), *, E)$ is a commutative monoid. Let $I, J, K \in \mathcal{PI}(\mathcal{E})$. Then $(I * J) * K = ((I^c \wedge J^c) \wedge K^c)^c$ and $I * (J * K) = (I^c \wedge (J^c \wedge K^c))^c$. Since $\wedge$ is associative, we have $*$ is associative. Also, we can see that $I * J = (I^c \cap J^c)^c = (J^c \cap I^c)^c = J * I$ and so $*$ is commutative. Since $\{0\}^c = E$, by Proposition 4.16, $I * \{0\} = (I^c)^c = I$ and so $\{0\}$ is the identity element of $\mathcal{PI}(\mathcal{E})$. Thus $(MV1)$ is satisfied. Also, by Proposition 4.16, we can see that $(MV2)$ holds. Since $\{0\}^c = E$, we get $\{0\}^c * I = E * I = (E^c \wedge I^c)^c = (\{0\} \cap I^c)^c = \{0\}^c = E$. Thus $(MV3)$ holds.
From Corollary 4.13, we have

$$(I^c \wedge J)^c = (I^c \wedge J) \to \{0\} = I^c \to (J \to \{0\}) = I^c \to J^c$$

and so

$$(I^c * J)^c * J = ((I \wedge J^c)^c \wedge J^c)^c = (J^c \wedge (J^c \to I^c))^c = (I^c \wedge (I^c \to J^c))^c = (I^c \wedge (I^c \wedge J)^c)^c = (J^c * I)^c * I.$$

Therefore, $(MV4)$ holds and proof is complete. □

Corollary 4.18. *If $\mathcal{E}$ is involutive and prelinear, then $(\mathcal{PI}(\mathcal{E}), \vee, \wedge, \to, \{0\}, E)$ is BL, BE, MTL, and hoop-algebras.*

In the following example, we show that for an involutive and prelinear EQ-algebra, $(\mathcal{PI}(\mathcal{E}), \vee, \wedge, ^c, \{0\}, E)$ is not a Boolean algebra.

Example 4.19. Let $\mathcal{E}$ be an EQ-algebra as in Example 3.2(ii). By some calculations, we can see $(a]_P = \{0, a\}$, $((a]_P)^c = (b]_P = \{0, b\}$, and $(a]_P \wedge (b]_P = \{0\}$. But $(a]_P \vee (b]_P = \{0, a, b, c\} \neq E$.

5 Conclusions and future works

In this paper, the notion of (pre)ideal in BEQ-algebras was defined and proved that the equivalence relation induced by an ideal in a good BEQ-algebra is a congruence relation. The generated preideal by a subset was defined and proved that the family of all preideals of an EQ-algebra is a complete lattice, distributive lattice and Hyting algebra. Also, it proved that for a prelinear IEQ-algebra, the family of all preideals forms an MV-algebra. Since every good EQ-algebra is an equality algebra, most results of this paper hold for equality algebras, too. In [1, 5, 12] different kinds of ideals in hoop, basic algebras and BL-algebras were studied. In the future works, we will study the notions of some kinds of ideals in EQ-algebras and we will try to characterize the generated ideals in EQ-algebras.

6 Compliance with ethical standards

Conflict of interest: The authors declare that there is no conflict of interest.
Human and animal rights: This article does not contain any studies with human participants or animals performed by any of the authors.

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