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FOUNDATION, ALGEBRAIC, AND ANALYTICAL METHODS IN SOFT COMPUTING

Generalization and ranking of fuzzy numbers by relative preference relation

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Abstract

We define $2n + 1$ and $2n$ fuzzy numbers, which generalize triangular and trapezoidal fuzzy numbers, respectively. Then, we extend the fuzzy preference relation and relative preference relation to rank $2n + 1$ and $2n$ fuzzy numbers. When the data is representable in terms of $2n + 1$ fuzzy number, we generalize the FMCDM (fuzzy multi-criteria decision making) model constructed with TOPSIS and relative preference relation. Lastly, we give an example from telecommunications to present the proposed FMCDM model and validate the results obtained.

Keywords Fuzzy number · Fuzzy preference relation · Decision making

1 Introduction

Zadeh (1965) introduced the concept of fuzzy set, and it is widely used to characterize vague or imprecise settings (conditions). Fuzzy sets have applications in automata theory, systems theory, decision theory, switching theory, pattern recognition, image thresholding, etc. [Lalotra and Singh (2020); Singh et al. (2019); Singh and Sharma (2019); Singh et al. (2020)]. Fuzzy numbers generalize real numbers and are very useful to represent data corresponding to uncertain situations. There are several methods to rank or order fuzzy numbers. Lee and Li (1988) utilized the concept of probability measure to determine the order of fuzzy numbers by considering the mean and dispersion of alternatives. Choobineh and Li (1993) proposed an indexing method to order or rank the fuzzy numbers. Dias (1993) proposed a computational approach to rank the alternatives using fuzzy numbers. Fortemps and Roubens (1996) presented a method to compare fuzzy numbers using the area compensation procedure. Cheng (1998) proposed the distance method and coefficient of variation (CV) index method to rank the fuzzy numbers. Chu and Tsao (2002) proposed a method using the area between centroid point and original point of the fuzzy numbers to facilitate ranking. Wang and Lee (2008) later revised this method. Lee (2005b) introduced the 'comparable' property for fuzzy preference relation and showed that only $O(n)$ comparisons of fuzzy numbers are sufficient if a fuzzy preference relation satisfies certain conditions. Asady and Zendehnam (2007) proposed a ranking method for the fuzzy numbers by obtaining the nearest point of support function with respect to fuzzy quantity. Wang (2015b) proposed a fuzzy relation with membership function representing preference degree to compare two fuzzy numbers. A relative preference relation was defined using fuzzy preference relation to compare a set of fuzzy numbers. The relative preference relation expresses preference degrees of several fuzzy numbers over average and facilitates easy and quick ranking of fuzzy numbers.

Decision-making methods often apply fuzzy sets in their computations. Jain (1976) presented a decision method that represented uncertain quantities as fuzzy sets and subsequently obtained an optimal alternative. Jain (1977) also developed a procedure for decision making using fuzzy sets by assigning quantitative numbers to qualitative terms. Wang (2014, 2015a, 2020a, b) proposed various methods using relative preference relation to solve FMCDM problems. In the

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multi-granulation decision-theoretic rough set, Mandal and Ranadive (2019) introduced the optimistic and pessimistic fuzzy preference relation models.

In a multi-criteria decision-making problem with multiple data points, few data points, referred to as fuzzy numbers, are utilized to arrive at a decision. In this regard, different fuzzy numbers, such as triangular, trapezoidal, pentagonal and hexagonal fuzzy numbers, have been reported. These fuzzy numbers consider only a few data points to arrive at a decision. For example, the number of data points in triangular fuzzy numbers is 3, in trapezoidal fuzzy numbers it is 4, and in hexagonal fuzzy numbers it is 6. However, using a few data points to represent data leads to the loss of information. To address this situation, we generalize fuzzy numbers that encompass more data points to represent the data, thus minimizing the loss of information. Practically, when the decision problem is highly sensitive to the number of data points, it is reasonable to choose a larger value of n. The flexibility in implementing this idea is apparent in the case of data representation by $2n+1$ (or $2n$) fuzzy numbers due to the choice for n. Thus, we get a natural advancement to the existing FMCDM methods.

In this paper, we define $2n$ and $2n+1$ fuzzy numbers. Clearly, $2n+1$ fuzzy numbers yield triangular and pentagonal cases when $n = 1, 2$, and $2n$ fuzzy numbers coincide with trapezoidal and hexagonal fuzzy numbers when $n = 2, 3$, respectively. We extend the fuzzy preference relation and relative preference relation given by Wang (2015b) to rank $2n$ and $2n+1$ fuzzy numbers. Then, we compare the results obtained by fuzzy preference relation and relative preference relation with Wang and Lee (2008) method. Wang (2014) developed the FMCDM model with TOPSIS under fuzzy environment and relative preference relation on fuzzy numbers. We present an extension to the FMCDM model when the given data is representable in terms of $2n+1$ fuzzy numbers. We illustrate the suitability of the proposed method in solving FMCDM problems using an example. Subsequently, the proposed method results are validated and compared with VIKOR, MOORA and ELECTRE methods.

The rest of the paper is organized as follows. In Sect. 2, we present the basic definitions and related primary results. In Sect. 3, we give the definition of $2n+1$ fuzzy number and the extension of fuzzy preference and relative preference relations on $2n+1$ fuzzy numbers. In Sect. 4, we define $2n$ fuzzy number and the extension of fuzzy preference and relative preference relations on $2n$ fuzzy numbers. In Sect. 5, we present the proposed FMCDM model along with a telecommunication example. In Sect. 6, we validate the proposed method with popularly used multi-criteria decision-making methods.

2 Definitions and preliminaries

For the following definitions, we refer (Zadeh 1965; Zimmermann 1987, 1991).

Definition 2.1 A fuzzy subset A on the universe U is a set defined by a membership function μ_A representing a mapping $\mu_A : U \longrightarrow [0, 1]$.

Definition 2.2 $A_\alpha = \{x \mid \mu_A(x) \geq \alpha\}$ is called an α-cut of the fuzzy set A.

Definition 2.3 Let X be a fuzzy number. Then, X^L_α and X^U_α are, respectively, defined as

$$X^L_\alpha = \inf_{\mu_X(z)\geq\alpha}(z) \text{ and } X^U_\alpha = \sup_{\mu_X(z)\geq\alpha}(z).$$

Definition 2.4 (Lee 2005a, b; Epp 1990) A fuzzy preference relation R is a fuzzy subset of $\mathbb{R}\times\mathbb{R}$ with membership function $\mu_R(A, B)$ representing preference degree of fuzzy numbers A over B.

1. R is reciprocal if and only if $\mu_R(A, B) = 1 - \mu_R(B, A)$ for all fuzzy numbers A and B.
2. R is transitive if and only if $\mu_R(A, B) \geq \frac{1}{2}$ and $\mu_R(B, C) \geq \frac{1}{2} \Rightarrow \mu_R(A, C) \geq \frac{1}{2}$ for all fuzzy numbers A, B and C.
3. R is a fuzzy total ordering if and only if R is both reciprocal and transitive.

A is preferred to B if and only if $\mu_R(A, B) > \frac{1}{2}$ and A is equal to B if and only if $\mu_R(A, B) = \frac{1}{2}$.

Definition 2.5 (Wang 2015b) Let $\succ$ be a binary relation on fuzzy numbers defined by $A \succ B$ if and only if A is preferred to B (That is, $\mu_R(A, B) > \frac{1}{2}$).

Wang revised the extended fuzzy preference relation defined by Lee (2005b) as follows.

Definition 2.6 (Wang 2015b) Let A and B be two fuzzy numbers, where A is an interval $[a_l, a_r]$ and B is an interval $[b_l, b_r]$. A fuzzy preference relation P is a subset of $\mathbb{R}\times\mathbb{R}$ with membership function $\mu_P(A, B)$ representing preference degree of A over B.

Define

$$\mu_P(A, B) = \frac{1}{2}\left(\frac{\int_0^1 (A-B)^L_\alpha + (A-B)^U_\alpha}{||T||} + 1\right),$$

where

$$||T|| = \int_0^1 ((T^+ - T^-)^L_\alpha + (T^+ - T^-)^U_\alpha) d\alpha \text{ if } t^+_l \geq t^-_r.$$

$$= \int_0^1 ((T^+ - T^-)^L_\alpha + (T^+ - T^-)^U_\alpha + 2(t_r^- - t_l^+))d\alpha \text{ if } t_l^+ < t_r^-.$$

T^+ is an interval $[t_l^+, t_r^+]$, T^- is an interval $[t_l^-, t_r^-]$, $t_l^+ = max\{a_l, b_l\}$, $t_r^+ = max\{a_r, b_r\}$, $t_l^- = min\{a_l, b_l\}$ and $t_r^- = min\{a_r, b_r\}$.

Similarly, fuzzy preference relation on triangular and trapezoidal fuzzy numbers have also been defined.

For the examples on decision-making problems, we refer (Koppula et al. 2019, 2020; Riaz et al. 2020; Chen and Huang 2021).

Definition 2.7 (Wang 2015b) Let $S = \{X_1, X_2, \ldots, X_n\}$ denote a set composed of n fuzzy numbers. A fuzzy number $X_i = [x_{il}, x_{ir}]$ belongs to the set S, where $i = 1, 2, \ldots, n$. Assume $\overline{X} = \frac{\sum_i X_i}{n}$ derived by extension principle is average of the n fuzzy numbers in S. A relative preference relation P^* with membership function $\mu_{P^*}(X_i, \overline{X})$ represents preference degree of X_i over $\overline{X}$ in S.

We define

$$\mu_{P_*}(X_i, \overline{X}) = \frac{1}{2}\left(\frac{\int_0^1 (X_i - \overline{X})^L_\alpha + (X_i - \overline{X})^U_\alpha}{||T_s||} + 1\right),$$

where

$$\| T_s \| = \int_0^1 ((T_s^+ - T_s^-)^L_\alpha + (T_s^+ - T_s^-)^U_\alpha)d\alpha \text{ if } t_{sl}^+ \geq t_{sr}^-;$$
$$= \int_0^1 ((T_s^+ - T_s^-)^L_\alpha + (T_s^+ - T_s^-)^U_\alpha + 2(t_{sr}^- - t_{sl}^+))d\alpha \text{ if } t_{sl}^+ < t_{sr}^-;$$

T_s^+ is an interval $[t_{sl}^+, t_{sr}^+]$, T_s^- is an interval $[t_{sl}^-, t_{sr}^-]$, $t_{sl}^+ = \max_i\{X_{il}\}$, $t_{sr}^+ = \max_i\{X_{ir}\}$, $t_{sl}^- = \min_i\{X_{il}\}$ and $t_{sr}^- = \min_i\{X_{ir}\}$.

Clearly, $0 < \mu_{P_*}(X_i, \overline{X}) < 1$, where $i = 1, 2, \ldots, n$. $\mu_{P_*}(X_i, \overline{X}) < \frac{1}{2}$ expresses that $\overline{X}$ is preferred to X_i. On the other hand, $\mu_{P_*}(X_i, \overline{X}) > \frac{1}{2}$ expresses that X_i is preferred to $\overline{X}$.

Similarly, relative preference relation is defined on triangular and trapezoidal fuzzy numbers.

3 Generalized $2n + 1$ fuzzy number

3.1 Generalized linear $2n + 1$ fuzzy number

Let $\{a_1, a_2, a_3, \ldots, a_{2n+1}\}$ be real numbers such that $a_1 < a_2 < a_3 < \cdots < a_{2n+1}$, $n = 1, 2, 3, \ldots$ and n is finite, $k \geq 2^{n-1}$. Then, we denote

$$P(x) := \frac{1}{k}\left(\frac{x - a_1}{a_2 - a_1}\right); \quad a_1 \leq x \leq a_2$$
$$S_n(x) := \frac{2^{n-2}}{k} + \frac{2^{n-2}}{k}\left(\frac{x - a_n}{a_{n+1} - a_n}\right); \quad a_n \leq x \leq a_{n+1}$$
$$T_{(n,r)}(x) := \frac{2^{n-2r}}{k} + \frac{2^{n-2r}}{k}\left(\frac{a_{n+2} - x}{a_{n+2} - a_{n+1}}\right); \quad a_{n+1} \leq x \leq a_{n+2}$$
$$Q_n(x) := \frac{1}{k}\left(\frac{a_{2n+1} - x}{a_{2n+1} - a_{2n}}\right); \quad a_{2n} \leq x \leq a_{2n+1}.$$

Now, $[P(x), Q_1(x)]$ gives fuzzy membership function of the generalized triangular fuzzy number $(a_1, a_2, a_3; \frac{1}{k})$.

That is,

$$f(a_1, a_2, a_3; x) = \begin{cases} \frac{1}{k}\left(\frac{x - a_1}{a_2 - a_1}\right); & a_1 \leq x \leq a_2 \\ \frac{1}{k}\left(\frac{a_3 - x}{a_3 - a_2}\right); & a_2 \leq x \leq a_3. \end{cases}$$

If $k = 1$ in the above, we get a triangular fuzzy number.

Now,

$$f(a_1, a_2, a_3, \ldots, a_{2n+1}; x) = [P(x), S_2(x), S_3(x), \cdots, S_{n-1}(x), S_n(x), T_{(n,1)}(x), T_{(n+1,2)}(x), T_{(n+2,3)}(x), \ldots, T_{(2n-2,n-1)}(x), Q_n(x)] \ldots (1)$$

gives fuzzy membership function of the generalized fuzzy number $(a_1, a_2, a_3, \ldots, a_{2n+1}; \frac{2^{n-1}}{k})$, where $n \geq 2$ and $k \geq 2^{n-1}$.

In particular, if $k = 2^{n-1}$ then (1) gives fuzzy membership function of the fuzzy number $(a_1, a_2, a_3, \ldots, a_{2n+1}; 1)$.

For example, substitute $n = 2$ in (1), then $[P(x), S_2(x), T_{(2,1)}(x), Q_2(x)]$ gives fuzzy membership function of the generalized pentagonal fuzzy number $(a_1, a_2, a_3, a_4, a_5; \frac{2}{k})$.

That is,

$$f(a_1, a_2, a_3, a_4, a_5; x) = \begin{cases} \frac{1}{k}\left(\frac{x - a_1}{a_2 - a_1}\right); & a_1 \leq x \leq a_2 \\ \frac{1}{k} + \frac{1}{k}\left(\frac{x - a_2}{a_3 - a_2}\right); & a_2 \leq x \leq a_3 \\ \frac{1}{k} + \frac{1}{k}\left(\frac{a_4 - x}{a_4 - a_3}\right); & a_3 \leq x \leq a_4 \\ \frac{1}{k}\left(\frac{a_5 - x}{a_5 - a_4}\right); & a_4 \leq x \leq a_5. \end{cases}$$

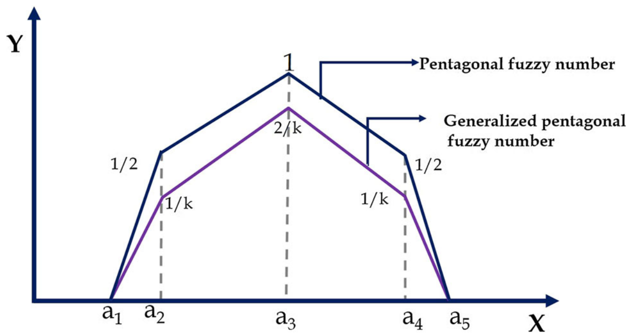

Fig. 1 Pentagonal and generalized pentagonal fuzzy number

If $k = 2$ in the above, we get a pentagonal fuzzy number (Fig. 1).

Similarly, substitute $n = 3$ in (1) then $[P(x), S_2(x), S_3(x), T_{(3,1)}(x), T_{(4,2)}(x), Q_3(x)]$ gives fuzzy membership function of the generalized heptagonal fuzzy number $(a_1, a_2, a_3, a_4, a_5, a_6, a_7; \frac{4}{k})$.

That is,

$$f(a_1, a_2, a_3, a_4, a_5, a_6, a_7; x) = \begin{cases} \frac{1}{k}\left(\frac{x-a_1}{a_2-a_1}\right); & a_1 \le x \le a_2 \\ \frac{1}{k}+\frac{1}{k}\left(\frac{x-a_2}{a_3-a_2}\right); & a_2 \le x \le a_3 \\ \frac{2}{k}+\frac{2}{k}\left(\frac{x-a_3}{a_4-a_3}\right); & a_3 \le x \le a_4 \\ \frac{2}{k}+\frac{2}{k}\left(\frac{a_5-x}{a_5-a_4}\right); & a_4 \le x \le a_5 \\ \frac{1}{k}+\frac{1}{k}\left(\frac{a_6-x}{a_6-a_5}\right); & a_5 \le x \le a_6 \\ \frac{1}{k}\left(\frac{a_7-x}{a_7-a_6}\right); & a_6 \le x \le a_7. \end{cases}$$

If $k = 4$ in the above, we get a heptagonal fuzzy number.

3.2 Generalized nonlinear $2n + 1$ fuzzy number

Let $\{a_1, a_2, a_3, \ldots, a_{2n+1}\}$ be real numbers such that $a_1 < a_2 < a_3 < \cdots < a_{2n+1}$, $m, n = 1, 2, 3, \ldots$ and m, n are finite. Now, take $k \ge 2^{n-1}$. Then, we denote

$$P^m(x) := \frac{1}{k}\left(\frac{x-a_1}{a_2-a_1}\right)^m; \quad a_1 \le x \le a_2$$

$$S_n^m(x) := \frac{2^{n-2}}{k}+\frac{2^{n-2}}{k}\left(\frac{x-a_n}{a_{n+1}-a_n}\right)^m; \quad a_n \le x \le a_{n+1}$$

$$T_{(n,r)}^m(x) := \frac{2^{n-2r}}{k}+\frac{2^{n-2r}}{k}\left(\frac{a_{n+2}-x}{a_{n+2}-a_{n+1}}\right)^m; \quad a_{n+1} \le x \le a_{n+2}$$

$$Q_n^m(x) := \frac{1}{k}\left(\frac{a_{2n+1}-x}{a_{2n+1}-a_{2n}}\right)^m; \quad a_{2n} \le x \le a_{2n+1}.$$

Then, $[P^m(x), Q_1^m(x)]$ gives fuzzy membership function of the generalized nonlinear triangular fuzzy number $(a_1, a_2, a_3; \frac{1}{k})$. Now,

$$f^m(a_1, a_2, a_3, \ldots, a_{2n+1}; x) = [P^m(x), S_2^m(x), S_3^m(x), \ldots, S_{n-1}^m(x), S_n^m(x), T_{(n,1)}^m(x), T_{(n+1,2)}^m(x), T_{(n+2,3)}^m(x), \ldots, T_{(2n-2,n-1)}^m(x), Q_n^m(x)] \ldots (3)$$

gives fuzzy membership function of the generalized nonlinear fuzzy number $(a_1, a_2, a_3, \ldots, a_{2n+1}; \frac{2^{n-1}}{k})$, where $n \ge 2$ and $k \ge 2^{n-1}$.

In particular, if $k = 2^{n-1}$ then (3) gives fuzzy membership function of the nonlinear fuzzy number $(a_1, a_2, a_3, \ldots, a_{2n+1}; 1)$.

Note 3.3 If $m = 1$ in the fuzzy membership function of the generalized nonlinear $2n + 1$ fuzzy number, then we get a fuzzy membership function of the generalized linear $2n + 1$ fuzzy number.

3.4 α-cut of a Generalized linear $2n + 1$ fuzzy number

Let $\{a_1, a_2, a_3, \ldots, a_{2n+1}\}$ be real numbers such that $a_1 < a_2 < a_3 < \cdots < a_{2n+1}$, $n = 1, 2, 3, \ldots$ and n is finite. Now, take $k \ge 2^{n-1}$. Then, we denote

$$E(\alpha) := a_1 + k\alpha(a_2 - a_1); \quad \alpha \in \left[0, \frac{1}{k}\right]$$

$$F_n(\alpha) := a_n + \left(\frac{\alpha k}{2^{n-2}} - 1\right)(a_{n+1} - a_n); \quad \alpha \in \left[\frac{2^{n-2}}{k}, \frac{2^{n-1}}{k}\right]$$

$$G_{(n,r)}(\alpha) := a_{n+2} - \left(\frac{\alpha k}{2^{n-2r}} - 1\right)(a_{n+2} - a_{n+1}); \quad \alpha \in \left[\frac{2^{n-2r}}{k}, \frac{2^{n-2r+1}}{k}\right]$$

$$H_n(\alpha) := a_{2n+1} - k\alpha(a_{2n+1} - a_{2n}); \quad \alpha \in \left[0, \frac{1}{k}\right].$$

Then, $[E(\alpha), H_1(\alpha)]$ gives α-cut of the generalized triangular fuzzy number $(a_1, a_2, a_3; \frac{1}{k})$. That is,

$$h(a_1, a_2, a_3; \alpha) = \begin{cases} a_1 + k\alpha(a_2 - a_1); & \alpha \in \left[0, \frac{1}{k}\right] \\ a_3 - k\alpha(a_3 - a_2); & \alpha \in \left[0, \frac{1}{k}\right]. \end{cases}$$

If $k = 1$ in the above, we get an α-cut of a triangular fuzzy number.

Now,

$$h(a_1, a_2, a_3, \ldots, a_{2n+1}; \alpha) = [E(\alpha), F_2(\alpha), F_3(\alpha), \ldots, F_{n-1}(\alpha), F_n(\alpha), G_{(n,1)}(\alpha), G_{(n+1,2)}(\alpha), G_{(n+2,3)}(\alpha), \ldots, G_{(2n-2,n-1)}(\alpha), H_n(\alpha)] \ldots (5)$$

gives α-cut of the generalized fuzzy number $(a_1, a_2, a_3, \ldots, a_{2n+1}; \frac{2^{n-1}}{k})$, where $n \geq 2$ and $k \geq 2^{n-1}$.

For example, if $n = 3$ in (5) then $[E(\alpha), F_2(\alpha), F_3(\alpha), G_{(3,1)}(\alpha), G_{(4,2)}(\alpha), H_3(\alpha)]$ gives α-cut of the generalized heptagonal fuzzy number $(a_1, a_2, a_3, a_4, a_5, a_6, a_7; \frac{4}{k})$. That is,

$$h(a_1, a_2, a_3, a_4, a_5, a_6, a_7; \alpha) = \begin{cases} a_1 + k\alpha(a_2 - a_1); & \alpha \in \left[0, \frac{1}{k}\right] \\ a_2 + (\alpha k - 1)(a_3 - a_2); & \alpha \in \left[\frac{1}{k}, \frac{2}{k}\right] \\ a_3 + \left(\frac{\alpha k}{2} - 1\right)(a_4 - a_3); & \alpha \in \left[\frac{2}{k}, \frac{4}{k}\right] \\ a_5 - \left(\frac{\alpha k}{2} - 1\right)(a_5 - a_4); & \alpha \in \left[\frac{2}{k}, \frac{4}{k}\right] \\ a_6 - (\alpha k - 1)(a_6 - a_5); & \alpha \in \left[\frac{1}{k}, \frac{2}{k}\right] \\ a_7 - k\alpha(a_7 - a_6); & \alpha \in \left[0, \frac{1}{k}\right]. \end{cases}$$

If $k = 4$ (That is $k = 2^{n-1} = 2^{3-1}$) in the above, we get an α-cut of a heptagonal fuzzy number $(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$.

Infimum and Supremum of α-cut of a $2n+1$ fuzzy number:

Let B be any fuzzy number and $\mu_B(x)$ is the fuzzy membership function of B. Then $B^L_\alpha = \inf_{\mu_B(x) \geq \alpha}(x)$ and $B^U_\alpha = \sup_{\mu_B(x) \geq \alpha}(x)$.

For example, if $B = (a_1, a_2, a_3; \frac{1}{k})$ is a triangular fuzzy number. Then, $B^L_\alpha = E(\alpha)$ and $B^U_\alpha = H_1(\alpha)$.

For a fuzzy number $C = (a_1, a_2, a_3, \ldots, a_{2n+1}; \frac{2^{n-1}}{k})$, $C^L_\alpha = (E(\alpha), F_2(\alpha), F_3(\alpha), \ldots, F_{n-1}(\alpha), F_n(\alpha))$ and

$C^U_\alpha = (G_{(n,1)}(\alpha), G_{(n+1,2)}(\alpha), G_{(n+2,3)}(\alpha), \ldots, G_{(2(n-1),n-1)}(\alpha), H_n(\alpha))$.

3.5 Extension of fuzzy preference relation on $2n + 1$ fuzzy numbers

We extend the fuzzy preference relation given by Wang (2015b) to rank 2n+1 fuzzy numbers as follows.

Definition 3.6 Let X_1 and X_2 be two fuzzy numbers, where $X_1 = [a_1, a_2, \ldots, a_{2n+1}]$ and $X_2 = [b_1, b_2, \ldots, b_{2n+1}]$. An extended fuzzy preference relation R is a subset of $\mathbb{R} \times \mathbb{R}$ with membership function $\mu_R(X_1, X_2)$ representing preference degree of X_1 over X_2. Then,

$$\mu_R(X_1, X_2) = \frac{1}{2}\left(\int_0^{\frac{2^{n-1}}{k}} \left(\frac{(X_1 - X_2)^\alpha_L + (X_1 - X_2)^\alpha_U}{\parallel T \parallel}\right) d\alpha + 1\right) \quad \ldots..(I)$$

where

$$\parallel T \parallel = \begin{cases} \int_0^{\frac{2^{n-1}}{k}} ((T^+ - T^-)^\alpha_L + (T^+ - T^-)^\alpha_U) d\alpha; & if \quad t_1^+ \geq t_{2n-1}^- \\ \int_0^{\frac{2^{n-1}}{k}} ((T^+ - T^-)^\alpha_L + (T^+ - T^-)^\alpha_U + 2(t_{2n+1}^- - t_1^+)) d\alpha; & if \quad t_1^+ < t_{2n-1}^-, \end{cases}$$

T^+ is an interval $[t_1^+, t_2^+, \ldots, t_{2n+1}^+]$, T^- is an interval $[t_1^-, t_2^-, \ldots, t_{2n+1}^-]$ and t_1^+=max $\{a_1, b_1\}$, t_2^+=max $\{a_2, b_2\}$, $\ldots$, t_{2n+1}^+=max$\{a_{2n+1}, b_{2n+1}\}$, t_1^-=min $\{a_1, b_1\}$, t_2^-=min $\{a_2, b_2\}$,…,t_{2n+1}^-=min $\{a_{2n+1}, b_{2n+1}\}$.

In (I), if $\int_0^{\frac{2^{n-1}}{k}} ((T^+ - T^-)^\alpha_L + (T^+ - T^-)^\alpha_U) d\alpha \geq 0$, then $\mu_R(X_1, X_2) \geq \frac{1}{2}$ and if $X_1 = X_2$, then $\mu_R(X_1, X_2) = \frac{1}{2}$.

Lemma 3.7 The extended fuzzy preference relation R is reciprocal. That is, $\mu_R(X_1, X_2) = 1 - \mu_R(X_2, X_1)$ for all $2n + 1$ fuzzy numbers X_1 and X_2.

Lemma 3.8 The extended fuzzy preference relation R is transitive. That is, if $\mu_R(X_1, X_2) \geq \frac{1}{2}$ and $\mu_R(X_2, X_3) \geq \frac{1}{2}$ then $\mu_R(X_1, X_3) \geq \frac{1}{2}$, where X_1, X_2 and X_3 are $2n + 1$ fuzzy numbers.

From Lemmas 3.7 and 3.8, the extended fuzzy preference relation R is a total ordering relation (Epp (1990) and Lee (2005b)).

Lemma 3.9 *Let X_1 and X_2 be two $2n+1$ fuzzy numbers. By the extended fuzzy preference relation R, X_1 is preferred to X_2 if and only if $\mu_R(X_1, X_2) > \frac{1}{2}$.*

Lemma 3.10 *$X_1 \succ X_2$ if and only if $\mu_R(X_1, X_2) > \frac{1}{2}$, where $\succ$ is a binary relation.*

Result 3.11 Let $X_1 = \left(a_1, a_2, a_3; \dfrac{1}{k}\right)$ and $X_2 = \left(b_1, b_2, b_3; \dfrac{1}{k}\right)$ be two generalized triangular fuzzy numbers. Then, $\int_0^{\frac{1}{k}} [(X_1 - X_2)_\alpha^L + (X_1 - X_2)_\alpha^U] d\alpha = \dfrac{(a_1 - b_3) + 2(a_2 - b_2) + (a_3 - b_1)}{2k}$.

Now,

$$\mu_R(X_1, X_2) = \frac{1}{2}\left(\int_0^{\frac{1}{k}}\left(\frac{(X_1 - X_2)_L^\alpha + (X_1 - X_2)_U^\alpha}{\| T \|}\right) d\alpha + 1\right) = \frac{1}{2}\left(\frac{(a_1 - b_3) + 2\,(a_2 - b_2) + (a_3 - b_1)}{2k \| T \|} + 1\right).$$

If $k = 1$ in the above, we get

$$\int_0^1 [(X_1 - X_2)_\alpha^L + (X_1 - X_2)_\alpha^U] d\alpha = \frac{(a_1 - b_3) + 2(a_2 - b_2) + (a_3 - b_1)}{2}.$$

Then, $\mu_R(X_1, X_2) = \frac{1}{2}\left(\dfrac{(a_1 - b_3) + 2(a_2 - b_2) + (a_3 - b_1)}{2 \| T \|} + 1\right)$ (Wang (2015b)).

Lemma 3.12 Let $X_1 = \left(a_1, a_2, a_3, a_4, a_5; \dfrac{2}{k}\right)$ and $X_2 = \left(b_1, b_2, b_3, b_4, b_5; \dfrac{2}{k}\right)$ be two generalized pentagonal fuzzy numbers. Then, $\mu_R(X_1, X_2) = \frac{1}{2}\{[[(a_1 - b_5) + 2(a_2 - b_4) + 2(a_3 - b_3) + 2(a_4 - b_2) + (a_5 - b_1)]/(2k \| T \|)] + 1\}$.

Proof Now,
$X_1 - X_2 = ((a_1 - b_5), (a_2 - b_4), (a_3 - b_3), (a_4 - b_2), (a_5 - b_1))$.

Then,

$$\begin{aligned}\int_0^{\frac{2}{k}} (X_1 - X_2)_\alpha^L d\alpha &= \int_0^{\frac{1}{k}} ((a_1 - b_5) + k\alpha((a_2 - b_4) - (a_1 - b_5)))d\alpha \\ &\quad + \int_{\frac{1}{k}}^{\frac{2}{k}} ((a_2 - b_4) + (k\alpha - 1)((a_3 - b_3) - (a_2 - b_4)))d\alpha \\ &= \frac{(a_1 - b_5)}{k} + \frac{(a_2 - b_4) - (a_1 - b_5)}{2k} \\ &\quad + \frac{2(a_2 - b_4)}{k} + \frac{3((a_3 - b_3) - (a_2 - b_4))}{2k} - \frac{(a_3 - b_3)}{k}.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_0^{\frac{2}{k}} (X_1 - X_2)_\alpha^U d\alpha &= \int_0^{\frac{1}{k}} ((a_5 - b_1) - k\alpha((a_5 - b_1) - (a_4 - b_2))d\alpha) \\ &\quad + \int_{\frac{1}{k}}^{\frac{2}{k}} ((a_4 - b_2) - (k\alpha - 1)((a_4 - b_2) - (a_3 - b_3))d\alpha) \\ &= \frac{(a_5 - b_1)}{k} - \frac{(a_5 - b_1) - (a_4 - b_2)}{2k} + \frac{2(a_4 - b_2)}{k} \\ &\quad - \frac{3((a_4 - b_2) - (a_3 - b_3))}{2k} - \frac{(a_3 - b_3)}{k}.\end{aligned}$$

Then,

$$\begin{aligned}\int_0^{\frac{2}{k}} (X_1 - X_2)_\alpha^L d\alpha + \int_0^{\frac{2}{k}} (X_1 - X_2)_\alpha^U d\alpha &= \frac{1}{2k}[(a_1 - b_5) + 2(a_2 - b_4) \\ &\quad + 2(a_3 - b_3) + 2(a_4 - b_2) \\ &\quad + (a_5 - b_1)].\end{aligned}$$

Now,

$$\begin{aligned}\mu_R(X_1, X_2) &= \frac{1}{2}\left(\int_0^{\frac{2}{k}}\left(\frac{(X_1 - X_2)_L^\alpha + (X_1 - X_2)_U^\alpha}{\| T \|}\right) d\alpha + 1\right) \\ &= \frac{1}{2}\{[[(a_1 - b_5) + 2(a_2 - b_4) + 2(a_3 - b_3) + 2(a_4 - b_2) \\ &\quad + (a_5 - b_1)]/(2k \| T \|)] + 1\}.\end{aligned}$$

$\square$

Note 3.13 If $k = 2$ in Lemma 3.12, then $\mu_R(X_1, X_2) = \frac{1}{2}\{[[(a_1 - b_5) + 2(a_2 - b_4) + 2(a_3 - b_3) + 2(a_4 - b_2) + (a_5 - b_1)]/(4 \| T \|)] + 1\}$.

Lemma 3.14 Let $X_1 = (a_1, a_2, \ldots, a_n, a_{n+1}, a_{n+2}, \ldots, a_{2n}, a_{2n+1}; \frac{2^{n-1}}{k})$ and $X_2 = (b_1, b_2, \ldots, b_n, b_{n+1}, \ldots, b_{2n}, b_{2n+1}; \frac{2^{n-1}}{k})$ be two fuzzy numbers, where $n \geq 3$.

Then,

$$\begin{aligned}\mu_R(X_1, X_2) &= \frac{1}{2}\{[[(a_1 - b_{2n+1}) + 2(a_2 - b_{2n}) \\ &\quad + 3[(a_3 - b_{2n-1}) + 2(a_4 - b_{2n-2}) + 4(a_5 - b_{2n-3}) \\ &\quad + \cdots + 2^{n-3}(a_n - b_{n+2})] + 2^{n-1}(a_{n+1} - b_{n+1}) \\ &\quad + 3[2^{n-3}(a_{n+2} - b_n) + 2^{n-4}(a_{n+3} - b_{n-1}) \\ &\quad + \ldots + 2(a_{2n-2} - b_4) + (a_{2n-1} - b_3)] + 2(a_{2n} - b_2) \\ &\quad + (a_{2n+1} - b_1)]/(2k \| T \|)] + 1\},\end{aligned}$$

where

$$\begin{aligned}\parallel T \parallel &= [(t_1^+ - t_{2n+1}^-) + 2(t_2^+ - t_{2n}^-) + 3[(t_3^+ - t_{2n-1}^-)\\ &+2(t_4^+ - t_{2n-2}^-) + 4(t_5^+ - t_{2n-3}^-) + \cdots + 2^{n-3}(t_n^+ - t_{n+2}^-)]\\ &+2^{n-1}(t_{n+1}^+ - t_{n+1}^-) + 3[2^{n-3}(t_{n+2}^+ - t_n^-)\\ &+2^{n-4}(t_{n+3}^+ - t_{n-1}^-) + \ldots + 2(t_{2n-2}^+ - t_4^-) + (t_{2n-1}^+ - t_3^-)]\\ &+2(t_{2n}^+ - t_2^-) + (t_{2n+1}^+ - t_1^-)]/2k \qquad \text{if} \quad (t_1^+ - t_{2n+1}^-) \geq 0.\\ &= \{[(t_1^+ - t_{2n+1}^-) + 2(t_2^+ - t_{2n}^-) + 3[(t_3^+ - t_{2n-1}^-)\\ &+2(t_4^+ - t_{2n-2}^-) + 4(t_5^+ - t_{2n-3}^-) + \ldots + 2^{n-3}(t_n^+ - t_{n+2}^-)]\\ &+2^{n-1}(t_{n+1}^+ - t_{n+1}^-) + 3[2^{n-3}(t_{n+2}^+ - t_n^-) + 2^{n-4}(t_{n+3}^+ - t_{n-1}^-)\\ &+ \ldots + 2(t_{2n-2}^+ - t_4^-)\\ &+(t_{2n-1}^+ - t_3^-)] + 2(t_{2n}^+ - t_2^-) + (t_{2n+1}^+ - t_1^-)]/2k\}\\ &+2(t_{2n+1}^- - t_1^+) \qquad \text{if} \quad (t_1^+ - t_{2n+1}^-) < 0,\end{aligned}$$

where t_1^+=max $\{a_1, b_1\}$, t_2^+=max $\{a_2, b_2\}$,…,t_{2n+1}^+ =max$\{a_{2n+1}, b_{2n+1}\}$, t_1^-=min $\{a_1, b_1\}$, t_2^-=min $\{a_2, b_2\}$,…, t_{2n+1}^-=min $\{a_{2n+1}, b_{2n+1}\}$.

Proof Now, $X_1 - X_2 = [(a_1 - b_{2n+1}), (a_2 - b_{2n}), \ldots, (a_n - b_{n+2}), (a_{n+1} - b_{n+1}), (a_{n+2} - b_n), \ldots, (a_{2n-1} - b_3), (a_{2n} - b_2), (a_{2n+1} - b_1)]$.

Consider

$$\begin{aligned}&\int_0^{\frac{2^{n-1}}{k}} (X_1 - X_2)_\alpha^L + (X_1 - X_2)_\alpha^U d\alpha\\ &= \int_0^{\frac{1}{k}} [(a_1 - b_{2n+1}) + k\alpha((a_2 - b_{2n}) - (a_1 - b_{2n+1}))]d\alpha\\ &+ \int_{\frac{1}{k}}^{\frac{2}{k}} [(a_2 - b_{2n}) + (k\alpha - 1)((a_3 - b_{2n-1}) - (a_2 - b_{2n}))]d\alpha\\ &+ \int_{\frac{2}{k}}^{\frac{4}{k}} [(a_3 - b_{2n-1}) + (\frac{k\alpha}{2} - 1)((a_4 - b_{2n-2})\\ &-(a_3 - b_{2n-1}))]d\alpha + \cdots + \int_{\frac{2^{n-3}}{k}}^{\frac{2^{n-2}}{k}} [(a_{n-1} - b_{n+3})\\ &+(\frac{k\alpha}{2^{n-3}} - 1)((a_n - b_{n+2}) - (a_{n-1} - b_{n+3}))]d\alpha\\ &+ \int_{\frac{2^{n-2}}{k}}^{\frac{2^{n-1}}{k}} [(a_n - b_{n+2}) + (\frac{k\alpha}{2^{n-2}} - 1)((a_{n+1} - b_{n+1})\\ &-(a_n - b_{n+2}))]d\alpha + \int_{\frac{2^{n-2}}{k}}^{\frac{2^{n-1}}{k}} [(a_{n+2} - b_n) - \left(\frac{k\alpha}{2^{n-2}} - 1\right)\\ &((a_{n+2} - b_n) - (a_{n+1} - b_{n+1}))]d\alpha + \int_{\frac{2^{n-3}}{k}}^{\frac{2^{n-2}}{k}} [(a_{n+3} - b_{n-1})\\ &-(\frac{k\alpha}{2^{n-3}} - 1)((a_{n+3} - b_{n-1}) - (a_{n+2} - b_n))]d\alpha\\ &+ \cdots + \int_{\frac{2}{k}}^{\frac{4}{k}} [(a_{2n-1} - b_3) - (\frac{k\alpha}{2} - 1)((a_{2n-1} - b_3)\\ &-(a_{2n-2} - b_4))]d\alpha + \int_{\frac{1}{k}}^{\frac{2}{k}} [(a_{2n} - b_2) - (k\alpha - 1)\\ &((a_{2n} - b_2) - (a_{2n+1} - b_1))]d\alpha + \int_0^{\frac{1}{k}} [(a_{2n+1} - b_1)\\ &-k\alpha((a_{2n+1} - b_1) - (a_{2n} - b_2))]d\alpha\\ &= \frac{1}{2k}[(a_1 - b_{2n+1}) + 2(a_2 - b_{2n}) + 3[(a_3 - b_{2n-1})\\ &+2(a_4 - b_{2n-2}) + 4(a_5 - b_{2n-3}) + \cdots + 2^{n-3}(a_n - b_{n+2})]\\ &+2^{n-1}(a_{n+1} - b_{n+1}) + 3[2^{n-3}(a_{n+2} - b_n)\\ &+2^{n-4}(a_{n+3} - b_{n-1}) + \ldots + 2(a_{2n-2} - b_4)\\ &+(a_{2n-1} - b_3)] + 2(a_{2n} - b_2) + (a_{2n+1} - b_1)].\end{aligned}$$

Then,

$$\begin{aligned}&\mu_R(X_1, X_2)\\ &= \frac{1}{2}\left(\int_0^{\frac{2^{n-1}}{k}} \left(\frac{(X_1 - X_2)_L^\alpha + (X_1 - X_2)_U^\alpha}{\parallel T \parallel}\right) d\alpha + 1\right)\\ &= \frac{1}{2}\{[[(a_1 - b_{2n+1}) + 2(a_2 - b_{2n}) + 3[(a_3 - b_{2n-1})\\ &+2(a_4 - b_{2n-2}) + 4(a_5 - b_{2n-3}) + \cdots + 2^{n-3}(a_n - b_{n+2})]\\ &+2^{n-1}(a_{n+1} - b_{n+1}) + 3[2^{n-3}(a_{n+2} - b_n)\\ &+2^{n-4}(a_{n+3} - b_{n-1}) + \ldots + 2(a_{2n-2} - b_4) + (a_{2n-1} - b_3)]\\ &+2(a_{2n} - b_2) + (a_{2n+1} - b_1)]/(2k \parallel T \parallel)] + 1\}\end{aligned}$$

and clearly, we get the value of $\parallel T \parallel$.

□

3.15 Extension of relative preference relation on $2n + 1$ fuzzy numbers

Ranking n fuzzy numbers by fuzzy preference relation is time-consuming due to pair-wise comparisons. To reduce the time complexity, Wang (2015b) proposed relative preference relation. For example, to rank n fuzzy numbers by a preference relation, we require $n_{C_2} \in O(n^2)$ fuzzy pair-wise comparisons. In contrast, it is sufficient to use relative preference relation $O(n)$ times to rank the fuzzy numbers. We extend the relative preference relation given by Wang (2015b) to rank $2n + 1$ fuzzy numbers by having the time complexity same as Wang's method, i.e., the time complexity is $O(n)$ for ranking n fuzzy numbers. The method is as follows:

Definition 3.16 Let $A = \{X_1, X_2, \ldots, X_m\}$ be m fuzzy numbers, where each $X_i = \{x_{i1}, x_{i2}, \ldots, x_{i(2n+1)}\}, i = 1, 2, \ldots, m$ and $\overline{X} = \frac{\sum_i X_i}{m}$ is the average of the m fuzzy numbers of A. An extended relative preference relation R_* with the fuzzy membership function $\mu_{R_*}(X_i, \overline{X})$ represents the preference degree of X_i over $\overline{X}$ in A. Now, we define

$$\begin{aligned}&\mu_{R_*}(X_i, \overline{X})\\ &= \frac{1}{2}\left(\int_0^{\frac{2^{n-1}}{k}} \left(\frac{(X_i - \overline{X})_L^\alpha + (X_i - \overline{X})_U^\alpha}{\parallel T_q \parallel}\right) d\alpha + 1\right),\end{aligned}$$

where

$$\parallel T_q \parallel = \int_0^{\frac{2^{n-1}}{k}} [(T_q^+ - T_q^-)_L^\alpha + (T_q^+ - T_q^-)_U^\alpha] d\alpha;$$
$$if \;\; t_{q1}^+ \geq t_{q(2n+1)}^-,$$
$$\int_0^{\frac{2^{n-1}}{k}} [(T_q^+ - T_q^-)_L^\alpha + (T_q^+ - T_q^-)_U^\alpha$$
$$+2(t_{q(2n+1)}^- - t_{q1}^+)] d\alpha;$$
$$if \;\; t_{q1}^+ < t_{q(2n+1)}^-,$$

where T_q^+ is an interval $[t_{q1}^+, t_{q2}^+, \ldots, t_{q(2n+1)}^+]$, T_q^- is an interval $[t_{q1}^-, t_{q2}^-, \ldots, t_{q(2n+1)}^-]$ and $t_{q1}^+ = \max_i\{x_{i1}\}$,

$$t_{q2}^+ = \max_i\{x_{i2}\}, \ldots, t_{q(2n+1)}^+ = \max_i\{x_{i(2n+1)}\},$$
$$t_{q1}^- = \min_i\{x_{i1}\}, t_{q2}^- = \min_i\{x_{i2}\}, \ldots,$$
$$t_{q(2n+1)}^- = \min_i\{x_{i(2n+1)}\}, i = 1, 2, \ldots, m.$$

If $\mu_{R_*}(X_i, \overline{X}) > \frac{1}{2}$ then X_i is preferred to $\overline{X}$ and $\mu_{R_*}(X_i, \overline{X}) < \frac{1}{2}$ then $\overline{X}$ is preferred to X_i.

Lemma 3.17 The extended relative preference relation R_* is a total ordering relation.

Lemma 3.18 Let X_i and X_j be two fuzzy numbers in A. Then, X_i is preferred to X_j if and only if $\mu_{R_*}(X_i, \overline{X}) > \mu_{R_*}(X_j, \overline{X})$.

Lemma 3.19 $X_i \succ X_j$ if and only if $\mu_{R_*}(X_i, \overline{X}) > \mu_{R_*}(X_j, \overline{X})$, where $\succ$ is a binary relation as defined in Definition 2.5.

Lemma 3.20 Let $A = \{X_1, X_2, \ldots, X_m\}$ be a set of fuzzy numbers, where each $X_i = \{x_{i1}, x_{i2}, x_{i3}, \ldots, x_{i(2n+1)}\}$, $i = 1, 2, \ldots, m$ and $\overline{X} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_{2n+1})$ be the average of m fuzzy numbers. The extended relative preference relation R_* with membership function $\mu_{R_*}(X_i, \overline{X})$ represents preference degree of X_i over $\overline{X}$ in A. Then,

$$\mu_{R_*}(X_i, \overline{X}) = \frac{1}{2}\{[[(x_{i1} - \overline{x}_{2n+1}) + 2(x_{i2} - \overline{x}_{2n})$$
$$+3[(x_{i3} - \overline{x}_{2n-1}) + 2(x_{i4} - \overline{x}_{2n-2}) + 4(x_{i5} - \overline{x}_{2n-3})$$
$$+ \cdots + 2^{n-3}(x_{in} - \overline{x}_{n+2})] + 2^{n-1}(x_{i(n+1)} - \overline{x}_{n+1})$$
$$+3[2^{n-3}(x_{i(n+2)} - \overline{x}_n) + 2^{n-4}(x_{i(n+3)} - \overline{x}_{n-1})$$
$$+ \cdots + 2(x_{i(2n-2)} - \overline{x}_4) + (x_{i(2n-1)} - \overline{x}_3)] + 2(x_{i2n} - \overline{x}_2)$$
$$+(x_{i(2n+1)} - \overline{x}_1)]/(2k \parallel T_q \parallel)] + 1\},$$

where

$$\parallel T_q \parallel = [(t_{q1}^+ - t_{q(2n+1)}^-) + 2(t_{q2}^+ - t_{q(2n)}^-) + 3[(t_{q3}^+ - t_{q(2n-1)}^-)$$
$$+2(t_{q4}^+ - t_{q(2n-2)}^-) + 4(t_{q5}^+ - t_{q(2n-3)}^-)$$
$$+ \cdots + 2^{n-3}(t_{qn}^+ - t_{q(n+2)}^-)] + 2^{n-1}(t_{q(n+1)}^+ - t_{q(n+1)}^-)$$
$$+3[2^{n-3}(t_{q(n+2)}^+ - t_{qn}^-) + 2^{n-4}(t_{q(n+3)}^+ - t_{q(n-1)}^-)$$
$$+ \cdots + 2(t_{q(2n-2)}^+ - t_{q4}^-) + (t_{q(2n-1)}^+ - t_{q3}^-)] + 2(t_{q(2n)}^+ - t_{q2}^-)$$
$$+(t_{q(2n+1)}^+ - t_{q1}^-)]/2k \text{ if } (t_{q1}^+ - t_{q(2n+1)}^-) \geq 0$$
$$= [[(t_{q1}^+ - t_{q(2n+1)}^-) + 2(t_{q2}^+ - t_{q(2n)}^-) + 3[(t_{q3}^+ - t_{q(2n-1)}^-)$$
$$+2(t_{q4}^+ - t_{q(2n-2)}^-) + 4(t_{q5}^+ - t_{q(2n-3)}^-)$$
$$+ \cdots + 2^{n-3}(t_{qn}^+ - t_{q(n+2)}^-)] + 2^{n-1}(t_{q(n+1)}^+ - t_{q(n+1)}^-)$$
$$+3[2^{n-3}(t_{q(n+2)}^+ - t_{qn}^-) + 2^{n-4}(t_{q(n+3)}^+ - t_{q(n-1)}^-)$$
$$+ \cdots + 2(t_{q(2n-2)}^+ - t_{q4}^-) + (t_{q(2n-1)}^+ - t_{q3}^-)] + 2(t_{q2n}^+ - t_{q2}^-)$$
$$+(t_{q(2n+1)}^+ - t_{q1}^-)]/(2k)] + 2(t_{q(2n+1)}^- - t_{q1}^+)$$
$$if (t_{q1}^+ - t_{q(2n+1)}^-) < 0.$$

where

$$t_{q1}^+ = \max_i\{x_{i1}\}, t_{q2}^+ = \max_i\{x_{i2}\}, \ldots, t_{q(2n+1)}^+ = \max_i\{x_{i(2n+1)}\},$$
$$t_{q1}^- = \min_i\{x_{i1}\}, t_{q2}^- = \min_i\{x_{i2}\}, \ldots, t_{q(2n+1)}^- = \min_i\{x_{i(2n+1)}\},$$
$$i = 1, 2, \ldots, m.$$

Here, we provide an example.

Example 3.21 Let $X_1 = (2, 3, 5, 6, 8; \frac{2}{3})$ and $X_2 = (1, 2, 4, 7, 9; \frac{2}{3})$ be two pentagonal fuzzy numbers ($K = 3$).

Then,

$$X_1 - X_2 = (-7, -4, 1, 4, 7).$$

Now,

$$t_1^+ = \max\{2, 1\} = 2, \quad t_1^- = \min\{2, 1\} = 1;$$

Similarly,

$$t_2^+ = 3, \quad t_2^- = 2;$$
$$t_3^+ = 5, \quad t_3^- = 4;$$
$$t_4^+ = 7, \quad t_4^- = 6;$$
$$t_5^+ = 9, \quad t_5^- = 8.$$
$$\parallel T \parallel = \frac{1}{6}(-6 - 6 + 2 + 10 + 8) + 2(6) = 13.3333$$

Then,

$$\mu_R(X_1, X_2) = \frac{1}{2}\left[\frac{-7 - 8 + 2 + 8 + 7}{6(13.3333)} + 1\right] = 0.5125 > 1/2.$$

This implies $X_1 > X_2$.

$$\overline{X} = (\overline{x_1}, \overline{x_2}, \overline{x_3}, \overline{x_4}, \overline{x_5}) = (1.5, 2.5, 4.5, 6.5, 8.5)$$

Now,

$$t_{q1}^{+}=2 \quad t_{q1}^{-}=1;$$
$$t_{q2}^{+}=3 \quad t_{q2}^{-}=2;$$
$$t_{q3}^{+}=5 \quad t_{q3}^{-}=4;$$
$$t_{q4}^{+}=7 \quad t_{q4}^{-}=6;$$
$$t_{q5}^{+}=9 \quad t_{q5}^{-}=8.$$
$$\parallel T_q \parallel = \frac{1}{6}[(2-8)+2(3-6)+2(5-4) +2(7-2)+(9-1)]+2(6)=\frac{40}{3}$$

Now,

$$\mu_{R_*}(X_1,\overline{X})=\frac{1}{2}\left[\frac{(2-8.5)+2(3-6.5)+2(5-4.5)+2(6-2.5)+(8-1.5)}{6\left(\frac{40}{3}\right)}+1\right]=0.5063.$$

Similarly,

$$\mu_{R_*}(X_2,\overline{X})=0.49038.$$

As $\mu_{R_*}(X_1,\overline{X}) > \mu_{R_*}(X_2,\overline{X})$, we get $X_1 > X_2$. We compared the above result with Wang and Lee (2008)'s area method.

By Wang and Lee (2008)'s method, we get $\overline{x}(X_1)=4.75$ and $\overline{x}(X_2)=4.6481$. Therefore $X_1 > X_2$.

4 Generalized 2*n* fuzzy number

4.1 Generalized linear 2*n* fuzzy number

Let $\{a_1,a_2,a_3,\ldots,a_{2n}\}$ be real numbers such that $a_1 < a_2 < a_3 < \cdots < a_{2n}$, $n = 2,3,4,\ldots$ and n is finite, $k \geq 2^{n-2}$. Now, we denote

$$P(x):=\frac{1}{k}\left(\frac{x-a_1}{a_2-a_1}\right); \quad a_1 \leq x \leq a_2$$

$$S_n(x):=\frac{2^{n-2}}{k}+\frac{2^{n-2}}{k}\left(\frac{x-a_{n-1}}{a_n-a_{n-1}}\right); \quad a_{n-1} \leq x \leq a_n$$

$$R_n(x):=\frac{2^{n-2}}{k}; \quad a_n \leq x \leq a_{n+1}$$

$$U_{(n,r)}(x):=\frac{2^{n-(2r+1)}}{k}+\frac{2^{n-(2r+1)}}{k}\left(\frac{a_{n+2}-x}{a_{n+2}-a_{n+1}}\right); \quad a_{n+1} \leq x \leq a_{n+2}$$

$$V_n(x):=\frac{1}{k}\left(\frac{a_{2n}-x}{a_{2n}-a_{2n-1}}\right); \quad a_{2n-1} \leq x \leq a_{2n}.$$

Then, $[P(x), R_2(x), V_2(x)]$ gives fuzzy membership function of the generalized trapezoidal fuzzy number $(a_1,a_2,a_3,a_4;\frac{1}{k})$. That is,

$$g(a_1,a_2,a_3,a_4;x)=\begin{cases}\frac{1}{k}\left(\frac{x-a_1}{a_2-a_1}\right); & a_1 \leq x \leq a_2\\ \frac{1}{k}; & a_2 \leq x \leq a_3\\ \frac{1}{k}\left(\frac{a_4-x}{a_4-a_3}\right); & a_3 \leq x \leq a_4.\end{cases}$$

If $k=1$ in the above, we get a trapezoidal fuzzy number. Now,

$$g(a_1,a_2,a_3,\ldots,a_{2n};x) =[P(x),S_2(x),S_3(x),\ldots,S_{n-1}(x),R_n(x),U_{(n,1)}(x), U_{(n+1,2)}(x),U_{(n+2,3)(x)},\ldots, U_{(2n-3,n-2)}(x),V_n(x)].....(2)$$

gives fuzzy membership function of the generalized fuzzy number $(a_1,a_2,a_3,\ldots,a_{2n};\frac{2^{n-2}}{k})$, where $n \geq 3$ and $k \geq 2^{n-2}$.

In particular, if $k=2^{n-2}$ then (2) gives fuzzy membership function of the fuzzy number $(a_1,a_2,a_3,\ldots,a_{2n};1)$, where $n \geq 3$.

For example, substitute $n=3$ in (2) then $[P(x),S_2(x),R_3(x),U_{(3,1)}(x),V_3(x)]$ gives fuzzy membership function of the generalized hexagonal fuzzy number $(a_1,a_2,a_3,a_4,a_5,a_6;\frac{2}{k})$.

That is,

$$g(a_1,a_2,a_3,a_4,a_5,a_6;x) =\begin{cases}\frac{1}{k}\left(\frac{x-a_1}{a_2-a_1}\right); & a_1 \leq x \leq a_2\\ \frac{1}{k}+\frac{1}{k}\left(\frac{x-a_2}{a_3-a_2}\right); & a_2 \leq x \leq a_3\\ \frac{2}{k}; & a_3 \leq x \leq a_4\\ \frac{1}{k}+\frac{1}{k}\left(\frac{a_5-x}{a_5-a_4}\right); & a_4 \leq x \leq a_5\\ \frac{1}{k}\left(\frac{a_6-x}{a_6-a_5}\right); & a_5 \leq x \leq a_6.\end{cases}$$

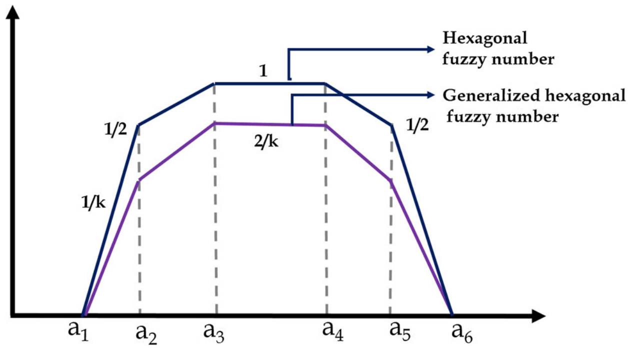

Fig. 2 Hexagonal and generalized hexagonal fuzzy number

If $k = 2$ in the above, we get a hexagonal fuzzy number (Fig. 2).

Similarly, substitute $n = 4$ in (2) then

$$[P(x), S_2(x), S_3(x), R_4(x), U_{(4,1)}(x), U_{(5,2)}(x), V_4(x)]$$

gives fuzzy membership function of the generalized octagonal fuzzy number $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8; \frac{4}{k})$.

That is,

$$g(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8; x) = \begin{cases} \frac{1}{k}\left(\frac{x-a_1}{a_2-a_1}\right); & a_1 \le x \le a_2 \\ \frac{1}{k} + \frac{1}{k}\left(\frac{x-a_2}{a_3-a_2}\right); & a_2 \le x \le a_3 \\ \frac{2}{k} + \frac{2}{k}\left(\frac{x-a_3}{a_4-a_3}\right); & a_3 \le x \le a_4 \\ \frac{4}{k}; & a_4 \le x \le a_5 \\ \frac{2}{k} + \frac{2}{k}\left(\frac{a_6-x}{a_6-a_5}\right); & a_5 \le x \le a_6 \\ \frac{1}{k} + \frac{1}{k}\left(\frac{a_7-x}{a_7-a_6}\right); & a_6 \le x \le a_7 \\ \frac{1}{k}\left(\frac{a_8-x}{a_8-a_7}\right); & a_7 \le x \le a_8. \end{cases}$$

If $k = 4$ in the above, we get an octagonal fuzzy number.

4.2 Generalized nonlinear 2*n* fuzzy number

Let $\{a_1, a_2, a_3, \ldots, a_{2n}\}$ be real numbers such that $a_1 < a_2 < a_3 < \cdots < a_{2n}$, $n = 2, 3, 4, \ldots$ and n is finite, $m = 1, 2, \ldots$ and m is finite. Now, take $k \ge 2^{n-2}$. Then, we denote

$$P^m(x) : \frac{1}{k}\left(\frac{x-a_1}{a_2-a_1}\right)^m; \quad a_1 \le x \le a_2$$

$$S_n^m(x) : \frac{2^{n-2}}{k} + \frac{2^{n-2}}{k}\left(\frac{x-a_{n-1}}{a_n-a_{n-1}}\right)^m; \quad a_{n-1} \le x \le a_n$$

$$R_n(x) : \frac{2^{n-2}}{k}; \quad a_n \le x \le a_{n+1}$$

$$U^m_{(n,r)}(x) : \frac{2^{n-(2r+1)}}{k} + \frac{2^{n-(2r+1)}}{k}\left(\frac{a_{n+2}-x}{a_{n+2}-a_{n+1}}\right)^m; \quad a_{n+1} \le x \le a_{n+2}$$

$$V_n^m(x) : \frac{1}{k}\left(\frac{a_{2n}-x}{a_{2n}-a_{2n-1}}\right)^m; \quad a_{2n-1} \le x \le a_{2n}.$$

Now, $[P^m(x), R_2(x), V_2^m(x)]$ gives the fuzzy membership function of the generalized nonlinear trapezoidal fuzzy number $(a_1, a_2, a_3, a_4; \frac{1}{k})$.

Now,

$$[P^m(x), S_2^m(x), S_3^m(x), \ldots, S_{n-1}^m(x),$$
$$R_n(x), U^m_{(n,1)}(x), U_{(n+1,2)}U^m_{(n+2,3)}(x), \ldots, m(x)$$
$$U^m_{(2n-3,n-2)}(x), V_n^m(x)]\ldots.(4)U^m_{(n+1,2)}(x),$$

gives fuzzy membership function of the generalized nonlinear fuzzy number $(a_1, a_2, a_3, \ldots, a_{2n}; \frac{2^{n-2}}{k})$, where $n \ge 3$ and $k \ge 2^{n-2}$.

In particular, if $k = 2^{n-2}$ then (4) gives fuzzy membership function of the nonlinear fuzzy number $(a_1, a_2, a_3, \ldots, a_{2n}; 1)$, where $n \ge 3$.

Note 4.3 If $m = 1$ in the fuzzy membership function of the generalized nonlinear $2n$ fuzzy number, then we get fuzzy membership function of the generalized linear $2n$ fuzzy number.

4.4 α-cut of a generalized linear 2*n* fuzzy number

Let $\{a_1, a_2, a_3, \ldots, a_{2n}\}$ be real numbers such that $a_1 < a_2 < a_3 < \cdots < a_{2n}$, $n = 1, 2, 3, \ldots$ and n is finite. Now, take $k \ge 2^{n-2}$. We denote

$$E(\alpha) := a_1 + k\alpha(a_2 - a_1);$$
$$\alpha \in \left[0, \frac{1}{k}\right]$$
$$F_n(\alpha) := a_n + \left(\frac{\alpha k}{2^{n-2}} - 1\right)(a_{n+1} - a_n);$$
$$\alpha \in \left[\frac{2^{n-2}}{k}, \frac{2^{n-1}}{k}\right]$$
$$I_{(n,r)}(\alpha) := a_{n+2} - \left(\frac{\alpha k}{2^{n-(2r+1)}} - 1\right)(a_{n+2} - a_{n+1});$$
$$\alpha \in \left[\frac{2^{n-(2r+1)}}{k}, \frac{2^{n-2r}}{k}\right]$$
$$J_n(\alpha) := a_{2n} - k\alpha(a_{2n} - a_{2n-1});$$
$$\alpha \in \left[0, \frac{1}{k}\right].$$

Now, $[E(\alpha), J_1(\alpha)]$ gives α-cut of the generalized trapezoidal fuzzy number $(a_1, a_2, a_3, a_4; \frac{1}{k})$. That is,

$$l(a_1, a_2, a_3, a_4; \alpha) = \begin{cases} a_1 + k\alpha(a_2 - a_1); & \alpha \in \left[0, \frac{1}{k}\right] \\ a_4 - k\alpha(a_4 - a_3); & \alpha \in \left[0, \frac{1}{k}\right]. \end{cases}$$

If $k = 1$ in the above, we get an α-cut of a trapezoidal fuzzy number $(a_1, a_2, a_3, a_4; 1)$.

Now,

$$l(a_1, a_2, a_3, \ldots, a_{2n}; \alpha)$$
$$= [E(\alpha), F_2(\alpha), F_3(\alpha), \ldots, F_{n-1}(\alpha), I_{(n,1)}(\alpha),$$
$$I_{(n+1,2)}(\alpha), I_{(n+2,3)}(\alpha), \ldots, I_{(2n-3,n-2)}(\alpha), J_n(\alpha)] \ldots (6)$$

gives α-cut of the generalized fuzzy number $(a_1, a_2, a_3, \ldots, a_{2n}; \frac{2^{n-2}}{k})$, where $n \geq 3$ and $k \geq 2^{n-2}$.

For example, if $n = 4$ in (6) then

$$[E(\alpha), F_2(\alpha), F_3(\alpha), I_{(4,1)}(\alpha), I_{(5,2)}(\alpha), J_4(\alpha)]$$

gives α-cut of the generalized octagonal fuzzy number $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8; \frac{4}{k})$.

That is,

$$l(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8; \alpha)$$
$$= \begin{cases} a_1 + k\alpha(a_2 - a_1); & \alpha \in \left[0, \frac{1}{k}\right] \\ a_2 + (\alpha k - 1)(a_3 - a_2); & \alpha \in \left[\frac{1}{k}, \frac{2}{k}\right] \\ a_3 + \left(\frac{\alpha k}{2} - 1\right)(a_4 - a_3); & \alpha \in \left[\frac{2}{k}, \frac{4}{k}\right] \\ a_6 - \left(\frac{\alpha k}{2} - 1\right)(a_6 - a_5); & \alpha \in \left[\frac{2}{k}, \frac{4}{k}\right] \\ a_7 - (\alpha k - 1)(a_7 - a_6); & \alpha \in \left[\frac{1}{k}, \frac{2}{k}\right] \\ a_8 - k\alpha(a_8 - a_7); & \alpha \in \left[0, \frac{1}{k}\right]. \end{cases}$$

If $k = 4$ (That is, $k = 2^{n-2} = 2^{4-2}$) in the above, we get an α-cut of octagonal fuzzy number $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8; 1)$.

Infimum and Supremum of α-cut of a $2n$ fuzzy number:

Let $B = (a_1, a_2, a_3, a_4; \frac{1}{k})$ be a trapezoidal fuzzy number and $\mu_B(x)$ is fuzzy membership function of B.

Then,

$$B_\alpha^L = E(\alpha)$$

and

$$B_\alpha^U = J_1(\alpha).$$

For a fuzzy number $(a_1, a_2, a_3, \ldots, a_{2n}; \frac{2^{n-2}}{k})$, $B_\alpha^L = (E(\alpha), F_2(\alpha), F_3(\alpha), \ldots, F_{n-1}(\alpha))$ and $B_\alpha^U = (I_{(n,1)}(\alpha), I_{(n+1,2)}(\alpha), I_{(n+2,3)}(\alpha), \ldots, I_{(2n-3,n-2)}(\alpha), J_n(\alpha))$.

4.5 preference relation on 2*n* fuzzy numbers

We extend the fuzzy preference relation given by Wang (2015b) to rank 2n fuzzy numbers.

Definition 4.6 Let A and B be two $2n$ fuzzy numbers, where $A = (a_1, a_2, \ldots, a_{2n})$ and $B = (b_1, b_2, \ldots, b_{2n})$. An extended fuzzy preference relation R_{2n} is a subset of $\mathbb{R} \times \mathbb{R}$ with membership function $\mu_{R_{2n}}(A, B)$ representing preference degree of A over B. Then,

$$\mu_{R_{2n}}(A, B)$$
$$= \frac{1}{2}\left(\int_0^{\frac{2^{n-2}}{k}} \left(\frac{(A-B)_L^\alpha + (A-B)_U^\alpha}{\| T \|}\right) d\alpha + 1\right)$$
$$\ldots..(\text{II}),$$

where

$$\| T \|= \int_0^{\frac{2^{n-2}}{k}} ((T^+ - T^-)^\alpha_L + (T^+ - T^-)^\alpha_U)d\alpha;$$
$$if \quad t_1^+ \geq t_{2n}^-$$
$$\int_0^{\frac{2^{n-2}}{k}} ((T^+ - T^-)^\alpha_L + (T^+ - T^-)^\alpha_U + 2(t_{2n}^- - t_1^+))d\alpha;$$
$$if\ t_1^+ < t_{2n}^-,$$

T^+ is an interval $[t_1^+, t_2^+, \ldots, t_{2n}^+]$, T^- is an interval $[t_1^-, t_2^-, \ldots, t_{2n}^-]$ and $t_1^+ = \max\{a_1, b_1\}$, $t_2^+ = \max\{a_2, b_2\}$, $\ldots, t_{2n}^+ = \max\{a_{2n}, b_{2n}\}$, $t_1^- = \min\{a_1, b_1\}$, $t_2^- = \min\{a_2, b_2\}, \ldots, t_{2n}^- = \min\{a_{2n}, b_{2n}\}$.

In (II), if $\int_0^{\frac{2^{n-2}}{k}} ((T^+ - T^-)^\alpha_L + (T^+ - T^-)^\alpha_U)d\alpha \geq 0$, then $\mu_{R_{2n}}(A, B) \geq \frac{1}{2}$ and if $A = B$, then $\mu_{R_{2n}}(A, B) = \frac{1}{2}$.

Lemma 4.7 The extended fuzzy preference relation R_{2n} is reciprocal. That is, $\mu_{R_{2n}}(A, B) = 1 - \mu_{R_{2n}}(B, A)$ for all $2n$ fuzzy numbers A and B.

Lemma 4.8 The extended fuzzy preference relation R_{2n} is transitive. That is, if $\mu_{R_{2n}}(A, B) \geq \frac{1}{2}$ and $\mu_{R_{2n}}(B, C) \geq \frac{1}{2}$ then $\mu_{R_{2n}}(A, C) \geq \frac{1}{2}$, where A, B and C are $2n$ fuzzy numbers.

From Lemmas 4.7 and 4.8, the extended fuzzy preference relation R_{2n} is a total ordering relation (Epp (1990) and Lee (2005b).

Lemma 4.9 Let A and B be two $2n + 1$ fuzzy numbers. By extended fuzzy preference relation R_{2n}, A is preferred to B if and only if $\mu_{R_{2n}}(A, B) > \frac{1}{2}$.

Lemma 4.10 $A \succ B$ if and only if $\mu_{R_{2n}}(A, B) > \frac{1}{2}$, where $\succ$ is a binary relation.

Lemma 4.11 Let $A = (a_1, a_2, a_3, a_4; \frac{1}{k})$ and $B = (b_1, b_2, b_3, b_4; \frac{1}{k})$ be two trapezoidal fuzzy numbers. Then, $\int_0^{\frac{1}{k}} [(A-B)^L_\alpha + (A-B)^U_\alpha]d\alpha = \frac{1}{2k}[(a_1-b_4)+(a_2-b_3)+(a_3-b_2)+(a_4-b_1)]$ and $\mu_{R_4}(A, B) = \frac{1}{2}(\int_0^{\frac{1}{k}} (\frac{(A - B)^\alpha_L + (A - B)^\alpha_U}{\| T \|}) d\alpha + 1) = \frac{1}{2}(\frac{(a_1-b_4)+(a_2-b_3)+(a_3-b_2)+(a_4 - b_1)}{2k \| T \|} + 1)$.

where

$$\| T \|= \frac{1}{2k}[(t_1^+ - t_4^-) + (t_2^+ - t_3^-) + (t_3^+ - t_2^-) + (t_4^+ - t_1^-)];$$
$$if \quad t_1^+ \geq t_4^-$$
$$\frac{1}{2k}[(t_1^+ - t_4^-) + (t_2^+ - t_3^-) + (t_3^+ - t_2^-) + (t_4^+ - t_1^-)]$$
$$+2(t_4^- - t_1^+); \quad if \quad t_1^+ < t_4^-.$$

T^+ is an interval $[t_1^+, t_2^+, t_3^+, t_4^+]$, T^- is an interval $[t_1^-, t_2^-, t_3^-, t_4^-]$ and $t_1^+ = \max\{a_1, b_1\}$, $t_2^+ = \max\{a_2, b_2\}$, $t_3^+ = \max\{a_3, b_3\}$, $t_4^+ = \max\{a_4, b_4\}$, $t_1^- = \min\{a_1, b_1\}$, $t_2^- = \min\{a_2, b_2\}$, $t_3^- = \min\{a_3, b_3\}$ and $t_4^- = \min\{a_4, b_4\}$.

Lemma 4.12 *Let* $A = (a_1, a_2, a_3, a_4, a_5, a_6; \frac{2}{k})$ *and* $B = (b_1, b_2, b_3, b_4, b_5, b_6; \frac{2}{k})$ *be two hexagonal fuzzy numbers. Then,*

$$\int_0^{\frac{2}{k}} (A - B)^L_\alpha + (A - B)^U_\alpha d\alpha = \frac{1}{2k}[(a_1 - b_6) + 2(a_2 - b_5)$$
$$+(a_3 - b_4) + (a_4 - b_3) + 2(a_5 - b_2) + (a_6 - b_1)]$$

and

$$\mu_{R_6}(A, B) = \frac{1}{2}\{[[(a_1 - b_6) + 2(a_2 - b_5) + (a_3 - b_4)$$
$$+(a_4 - b_3) + 2(a_5 - b_2) + (a_6 - b_1)]/(2k \| T \|)] + 1\}.$$

where

$$\| T \|= \frac{1}{2k}[(t_1^+ - t_6^-) + 2(t_2^+ - t_5^-)$$
$$+(t_3^+ - t_4^-)$$
$$+(t_4^+ - t_3^-) + 2(t_5^+ - t_2^-) + (t_6^+ - t_1^-)]; \quad if \quad t_1^+ \geq t_6^-$$
$$= \frac{1}{2k}[(t_1^+ - t_6^-) + 2(t_2^+ - t_5^-) + (t_3^+ - t_4^-) + (t_4^+ - t_3^-)$$
$$+2(t_5^+ - t_2^-) + (t_6^+ - t_1^-)] + 2(t_6^- - t_1^+);$$
$$if \quad t_1^+ < t_6^-\ and$$

T^+ is an interval $[t_1^+, t_2^+, t_3^+, t_4^+, t_5^+, t_6^+]$, T^- is an interval $[t_1^-, t_2^-, t_3^-, t_4^-, t_5^-, t_6^-]$ and t_1^+=max $\{a_1, b_1\}$, t_2^+=max $\{a_2, b_2\}$, t_3^+=max$\{a_3, b_3\}$, t_4^+=max$\{a_4, b_4\}$, t_5^+=max$\{a_5, b_5\}$, t_6^+=max$\{a_6, b_6\}$, t_1^-=min$\{a_1, b_1\}$, t_2^-=min$\{a_2, b_2\}$, t_3^-=min $\{a_3, b_3\}$, t_4^-=min$\{a_4, b_4\}$, t_5^-=min$\{a_5, b_5\}$, t_6^-=min$\{a_6, b_6\}$.

Lemma 4.13 *Let* $A = (a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{2n-1}, a_{2n}; \frac{2^{n-2}}{k})$ *and* $B = (b_1, b_2, \ldots, b_n, b_{n+1}, \ldots, b_{2n-1}, b_{2n}; \frac{2^{n-2}}{k})$ *be two fuzzy numbers, where* $n > 3$.
Then,

$$\mu_{R_{2n}}(A, B) = \frac{1}{2}\{[[((a_1 - b_{2n}) + 2(a_2 - b_{2n-1})$$
$$+3[(a_3 - b_{2n-2}) + 2(a_4 - b_{2n-3}) + 4(a_5 - b_{2n-4})$$
$$+ \cdots + 2^{n-4}(a_{n-1} - b_{n+2})] + 2^{n-3}(a_n - b_{n+1})$$
$$+2^{n-3}(a_{n+1} - b_n) + 3[2^{n-4}(a_{n+2} - b_{n-1})$$
$$+2^{n-5}(a_{n+3} - b_{n-2}) + \cdots + 2(a_{2n-3} - b_4)$$
$$+(a_{2n-2} - b_3)]$$
$$+2(a_{2n-1} - b_2) + (a_{2n} - b_1))/(2k \| T \|)] + 1\},$$

where

$$\| T \|= [(t_1^+ - t_{2n}^-) + 2(t_2^+ - t_{2n-1}^-) + 3[(t_3^+ - t_{2n-2}^-)$$

$$+2(t_4^+ - t_{2n-3}^-)$$
$$+4(t_5^+ - t_{2n-4}^-) + \cdots + 2^{n-4}(t_{n-1}^+ - t_{n+2}^-)] + 2^{n-3}$$
$$(t_n^+ - t_{n+1}^-) + 2^{n-3}(t_{n+1}^+ - t_n^-) + 3[2^{n-4}(t_{n+2}^+ - t_{n-1}^-)$$
$$+2^{n-5}(t_{n+3}^+ - t_{n-2}^-) + \cdots + 2(t_{2n-3}^+ - t_4^-) + (t_{2n-2}^+ - t_3^-)]$$
$$+2(t_{2n-1}^+ - t_2^-) + (t_{2n}^+ - t_1^-)]/2k \quad if \quad (t_1^+ - t_{2n}^-) \geq 0$$
$$= [((t_1^+ - t_{2n}^-) + 2(t_2^+ - t_{2n-1}^-) + 3[(t_3^+ - t_{2n-2}^-)$$
$$+2(t_4^+ - t_{2n-3}^-) + 4(t_5^+ - t_{2n-4}^-) + \cdots + 2^{n-4}(t_{n-1}^+ - t_{n+2}^-)]$$
$$+2^{n-3}(t_n^+ - t_{n+1}^-) + 2^{n-3}(t_{n+1}^+ - t_n^-) + 3[2^{n-4}(t_{n+2}^+ - t_{n-1}^-)$$
$$+2^{n-5}(t_{n+3}^+ - t_{n-2}^-) + \cdots + 2(t_{2n-3}^+ - t_4^-) + (t_{2n-2}^+ - t_3^-)]$$
$$+2(t_{2n-1}^+ - t_2^-) + (t_{2n}^+ - t_1^-))/2k] + 2(t_{2n}^- - t_1^+)$$
$$if \quad (t_1^+ - t_{2n}^-) < 0.$$

4.14 Extension of relative preference relation on 2*n* fuzzy numbers

Definition 4.15 Let $A = \{X_1, X_2, \ldots, X_m\}$ be a set of fuzzy numbers, where each $X_i = \{x_{i1}, x_{i2}, x_{i3}, \ldots, x_{i(2n)}\}, i = 1, 2, \ldots, m$. and $\overline{X} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_{2n})$ be the average of m fuzzy numbers. The relative preference relation R_{2n}^* with membership function $\mu_{R_{2n}^*}(X_i, \overline{X})$ represents preference degree of X_i over $\overline{X}$ in A.

Then,

$$\mu_{R_{2n}^*}(X_i, \overline{X}) = \frac{1}{2}\{[((x_{i1} - \overline{x}_{2n})$$
$$+2(x_{i2} - \overline{x}_{2n-1}) + 3[(x_{i3} - \overline{x}_{2n-2})$$
$$+2(x_{i4} - \overline{x}_{2n-3}) + 4(x_{i5} - \overline{x}_{2n-4}) + \cdots +$$
$$2^{n-4}(x_{i(n-1)} - \overline{x}_{n+2})] + 2^{n-3}(x_{in} - \overline{x}_{n+1})$$
$$+2^{n-3}(x_{i(n+1)} - \overline{x}_n) + 3[2^{n-4}(x_{i(n+2)} - \overline{x}_{n-1})$$
$$+2^{n-5}(x_{i(n+3)} - \overline{x}_{n-2}) + \cdots + 2(x_{i(2n-3)} - \overline{x}_4)$$
$$+(x_{i(2n-2)} - \overline{x}_3)] + 2(x_{i(2n-1)} - \overline{x}_2)$$
$$+(x_{i2n} - \overline{x}_1))/(2k \parallel T_q \parallel)] + 1\},$$

where

$$\parallel T_q \parallel = [(t_{q1}^+ - t_{q2n}^-) + 2(t_{q2}^+ - t_{q(2n-1)}^-) + 3[(t_{q3}^+ - t_{q(2n-2)}^-)$$
$$+2(t_{q4}^+ - t_{q(2n-3)}^-) + 4(t_{q5}^+ - t_{q(2n-4)}^-) + \cdots +$$
$$2^{n-4}(t_{q(n-1)}^+ - t_{q(n+2)}^-)] + 2^{n-3}(t_{qn}^+ - t_{q(n+1)}^-)$$
$$+2^{n-3}(t_{q(n+1)}^+ - t_{qn}^-) + 3[2^{n-4}(t_{q(n+2)}^+ - t_{q(n-1)}^-)$$
$$+2^{n-5}(t_{q(n+3)}^+ - t_{q(n-2)}^-) + \cdots + 2(t_{q(2n-3)}^+ - t_4^-)$$
$$+(t_{q(2n-2)}^+ - t_{q3}^-)] + 2(t_{q(2n-1)}^+ - t_{q2}^-) + (t_{q2n}^+ - t_{q1}^-)]/2k$$
$$if \quad (t_{q1}^+ - t_{q2n}^-) \geq 0$$
$$= [((t_{q1}^+ - t_{q2n}^-) + 2(t_{q2}^+ - t_{q2n-1}^-) + 3[(t_{q3}^+ - t_{q2n-2}^-)$$
$$+2(t_{q4}^+ - t_{q2n-3}^-) + 4(t_{q5}^+ - t_{q2n-4}^-) + \cdots +$$
$$2^{n-4}(t_{qn-1}^+ - t_{qn+2}^-)] + 2^{n-3}(t_{qn}^+ - t_{qn+1}^-)$$
$$+2^{n-3}(t_{qn+1}^+ - t_{qn}^-) + 3[2^{n-4}(t_{qn+2}^+ - t_{qn-1}^-)$$
$$+2^{n-5}(t_{qn+3}^+ - t_{qn-2}^-) + \cdots + 2(t_{q2n-3}^+ - t_{q4}^-)$$
$$+(t_{q2n-2}^+ - t_{q3}^-)] + 2(t_{q2n-1}^+ - t_{q2}^-) + (t_{q2n}^+ - t_{q1}^-))/2k]$$
$$+2(t_{q2n}^- - t_{q1}^+) \quad if (t_{q1}^+ - t_{q2n}^-) < 0.$$

Example 4.16 Let

$$X_1 = \{0.12, 0.2, 0.24, 0.28, 0.3, 0.35, 0.4, 0.46; 1\},$$
$$X_2 = \{0.1, 0.15, 0.25, 0.27, 0.32, 0.36, 0.38, 0.48; 1\}$$
$$\text{and } X_3 = \{0.1, 0.17, 0.2, 0.3, 0.35, 0.38, 0.41, 0.5; 1\}$$

be octagonal fuzzy numbers.

Then, $\parallel T \parallel$ of X_1 and X_2 is

$$\frac{1}{8}[(0.12 - 0.46) + 2(0.2 - 0.38) + 3(0.25 - 0.35)$$
$$+2(0.28 - 0.3) + 2(0.32 - 0.27) + 3(0.36 - 0.24)$$
$$+2(0.4 - 0.15) + (0.48 - 0.1)] + 2(0.46 - 0.12)$$
$$= 0.7175,$$

where

$$t_1^+ = \max\{0.12, 0.1\} = 0.12; \quad t_1^- = \min\{0.12, 0.1\} = 0.1$$
$$t_2^+ = \max\{0.2, 0.15\} = 0.2; \quad t_2^- = \min\{0.2, 0.15\} = 0.15$$
$$t_3^+ = \max\{0.24, 0.25\} = 0.25; \quad t_3^- = \min\{0.24, 0.25\} = 0.24$$
$$t_4^+ = \max\{0.28, 0.27\} = 0.28; \quad t_4^- = \min\{0.28, 0.27\} = 0.27$$
$$t_5^+ = \max\{0.3, 0.32\} = 0.32; \quad t_5^- = \min\{0.3, 0.32\} = 0.3$$
$$t_6^+ = \max\{0.35, 0.36\} = 0.36; \quad t_6^- = \min\{0.35, 0.36\} = 0.35$$
$$t_7^+ = \max\{0.4, 0.38\} = 0.4 \quad t_7^- = \min\{0.4, 0.38\} = 0.38$$
$$t_8^+ = \max\{0.46, 0.48\} = 0.48 \quad t_8^- = \min\{0.46, 0.48\} = 0.46.$$

Then,

$$\mu_{R_8}(X_1, X_2) = \frac{1}{2}\{[(0.12 - 0.48) + 2(0.2 - 0.38)$$
$$+3(0.24 - 0.36) + 2(0.28 - 0.32) + 2(0.3 - 0.27)$$
$$+3(0.35 - 0.25) + 2(0.4 - 0.15) + (0.46 - 0.1)]$$
$$/(8 \times 0.7175)] + 1\} = 0.5052 > \frac{1}{2}.$$

Therefore X_1 is preferred to X_2.

Similarly, $\mu_{R_8}(X_1, X_3) = 0.4924 < \frac{1}{2}$.

This implies X_3 is preferred to X_1 and $\mu_{R_8}(X_2, X_3) = 0.4885 < \frac{1}{2}$. This implies X_3 is preferred to X_2. Therefore $X_3 \succ X_1 \succ X_2$.

By extended relative preference relation,

$$\overline{X} = (0.1067, 0.1733, 0.23, 0.2833, 0.3233, 0.3633,$$
$$0.3967, 0.48).$$
$$t_{q1}^+ = \max\{0.12, 0.1, 0.1\} = 0.12$$

$t_{q1}^{-} = \min\{0.12, 0.1, 0.1\} = 0.1$
$t_{q2}^{+} = \max\{0.2, 0.15, 0.17\} = 0.2$
$t_{q2}^{-} = \min\{0.2, 0.15, 0.17\} = 0.15$

Similarly,

$$t_{q3}^{+} = 0.25; \quad t_{q3}^{-} = 0.2$$
$$t_{q4}^{+} = 0.3; \quad t_{q4}^{-} = 0.27$$
$$t_{q5}^{+} = 0.35; \quad t_{q5}^{-} = 0.3$$
$$t_{q6}^{+} = 0.38; \quad t_{q6}^{-} = 0.35$$
$$t_{q7}^{+} = 0.41; \quad t_{q7}^{-} = 0.38$$
$$t_{q8}^{+} = 0.5; \quad t_{q8}^{-} = 0.46$$

and

$$\begin{aligned}\| T_q \| = [[&(0.12 - 0.46) + 2(0.2 - 0.38) + 3(0.25 - 0.35)\\ &+2(0.3 - 0.3) + 2(0.35 - 0.27) + 3(0.38 - 0.2)\\ &+2(0.41 - 0.15) + (0.5 - 0.1)]/(2 * 4)]\\ &+2(0.46 - 0.12) = 0.7175.\end{aligned}$$

Now,

$$\begin{aligned}\mu^*_{R_8}(X_1, \overline{X}) = \frac{1}{2}\{[[&(0.12 - 0.48) + 2(0.2 - 0.3967)\\ &+3(0.24 - 0.3633)\\ &+2(0.28 - 0.3233) + 2(0.3 - 0.2833)\\ &+3(0.35 - 0.23) + 2(0.4 - 0.1733)\\ &+(0.46 - 0.1067)]/(8 * 0.7175)] + 1\} = 0.4991.\end{aligned}$$

Similarly, we get $\mu^*_{R_8}(X_2, \overline{X}) = 0.4939$ and $\mu^*_{R_8}(X_3, \overline{X}) = 0.5070$.

As $\mu^*_{R_8}(X_3, \overline{X}) > \mu^*_{R_8}(X_1, \overline{X}) > \mu^*_{R_8}(X_2, \overline{X})$, we get $X_3 \succ X_1 \succ X_2$.

The above result is compared with Wang and Lee (2008)'s method and according to this method the values are $\overline{x}(X_1) = 0.2954$, $\overline{x}(X_2) = 0.2853$ and $\overline{x}(X_3) = 0.2976$.

Hence we get $X_3 \succ X_1 \succ X_2$.

5 Fuzzy multi-criteria decision-making (FMCDM) model

In this section, we extend FMCDM model given by Wang (2014) using the relative preference relation on $2n+1$ fuzzy numbers.

In this algorithm, we consider the generalized $2n + 1$ fuzzy numbers with $k = 2^{n-1}$ and we denote them as $(a_1, a_2, \ldots, a_{2n+1})$ instead of $(a_1, a_2, \ldots, a_{2n+1}; 1)$.

We take $E_1, E_2, \ldots, E_r$ as the experts who provide their opinion on criteria $C_1, C_2, \ldots, C_t$ of the alternatives $A_1, A_2, \ldots, A_p$. Let $B_{ijl}=(b_{ijl1}, b_{ijl2}, b_{ijl3}, \ldots, b_{ijl(2n+1)})$ be the evaluation rating given by the expert E_l for alternative A_i on criterion C_j, where $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, t$, $l = 1, 2, \ldots, r$.

Then,

$$B_{ij} = (b_{ij1}, b_{ij2}, b_{ij3}, \ldots, b_{ij(2n+1)}),$$

where

$$b_{ij1} = \frac{1}{r}\sum_{l=1}^{r} b_{ijl1}$$

$$b_{ij2} = \frac{1}{r}\sum_{l=1}^{r} b_{ijl2}$$

$$b_{ij3} = \frac{1}{r}\sum_{l=1}^{r} b_{ijl3}$$

$$\vdots$$

$$b_{ij(2n+1)} = \frac{1}{r}\sum_{l=1}^{r} b_{ijl(2n+1)}, \ i = 1, 2, \ldots, p,$$

$j = 1, 2, \ldots, t$, $n = 1, 2, 3, \ldots$ and n is finite.

The normalized value of B_{ij} is denoted by $\widetilde{B}_{ij}$ and it is classified as follows.

If B_{ij} belongs to cost criteria then

$$\widetilde{B}_{ij} = \left(\frac{b^-_{j1}}{b_{ij(2n+1)}}, \frac{b^-_{j1}}{b_{ij(2n)}}, \frac{b^-_{j1}}{b_{ij(2n-1)}}, \ldots, \frac{b^-_{j1}}{b_{ij2}}, \frac{b^-_{j1}}{b_{ij1}}\right),$$

where

$$b^-_{j1} = \min_i\{b_{ij1}\}, \forall\ j$$

If B_{ij} belongs to benefit criteria then

$$\widetilde{B}_{ij} = \left(\frac{b_{ij1}}{b^+_{j(2n+1)}}, \frac{b_{ij2}}{b^+_{j(2n+1)}}, \frac{b_{ij3}}{b^+_{j(2n+1)}}, \ldots, \frac{b_{ij(2n)}}{b^+_{j(2n+1)}}, \frac{b_{ij(2n+1)}}{b^+_{j(2n+1)}}\right),$$

where

$$b^+_{j(2n+1)} = \max_i\{b_{ij(2n+1)}\}, \forall\ j.$$

Let $W_{jl} = (w_{jl1}, w_{jl2}, \ldots, w_{jl(2n+1)})$ be the weight of the criterion C_j given by the expert E_l, where

$$j = 1, 2, \ldots, t; \quad l = 1, 2, \ldots, r.$$

Then,

$$W_j = (w_{j1}, w_{j2}, \ldots, w_{j(2n+1)}),$$

where

$$w_{j1} = \frac{1}{r}\sum_{l=1}^{r} w_{jl1}$$
$$w_{j2} = \frac{1}{r}\sum_{l=1}^{r} w_{jl2}$$
$$\vdots$$
$$w_{j(2n+1)} = \frac{1}{r}\sum_{l=1}^{r} w_{jl(2n+1)},\ \ j = 1, 2, \ldots, t.$$

The normalized values of i^{th} alternative on t criteria are presented as follows.

$$A_i = [\widetilde{B}_{i1}, \widetilde{B}_{i2}, \ldots, \widetilde{B}_{it}],\ \ i = 1, 2, \ldots, p.$$

Then, we find ideal and anti-ideal solutions according to as Wang et al. (2003).

The ideal solution is $A^+ = [\widetilde{B}_1^+, \widetilde{B}_2^+, \ldots, \widetilde{B}_t^+]$, where $\widetilde{B}_j^+ = (t_{qj1}^+, t_{qj2}^+, \ldots, t_{qj(2n+1)}^+)$ is the best rating on jth criterion for all normalized values, $j = 1, 2, \ldots, t$.

The anti-ideal solution is $A^- = [\widetilde{B}_1^-, \widetilde{B}_2^-, \ldots, \widetilde{B}_t^-]$, where $B_j^- = (t_{qj1}^-, t_{qj2}^-, \ldots, t_{qj(2n+1)}^-)$ is the worst rating on jth criterion for all normalized values, $j = 1, 2, \ldots, t$.

Now, $\mu_{R_*}(\widetilde{B}_j^+, \widetilde{B}_{ij})$ indicates the relative preference degree of A^+ over the alternative A_i to the jth criterion,

$$i = 1, 2, \ldots, p,\ \ j = 1, 2, \ldots, t.$$

Then,

$$\begin{aligned}\mu_{R_*}(\widetilde{B}_j^+, \widetilde{B}_{ij}) = \frac{1}{2}\{[[(t_{qj1}^+ - \widetilde{b}_{ij(2n+1)}) + 2(t_{qj2}^+ - \widetilde{b}_{ij(2n)})\\
+3[(t_{qj3}^+ - \widetilde{b}_{ij(2n-1)}) + 2(t_{qj4}^+ - \widetilde{b}_{ij(2n-2)})\\
+4(t_{qj5}^+\widetilde{b}_{ij(2n-3)}) + \ldots + 2^{n-3}(t_{qjn}^+ - \widetilde{b}_{ij(n+2)})]\\
+2^{n-1}(t_{qj(n+1)}^+ - \widetilde{b}_{ij(n+1)}) + 3[2^{n-3}(t_{qj(n+2)}^+ - \widetilde{b}_{ijn})\\
+2^{n-4}(t_{qj(n+3)}^+ - \widetilde{b}_{ij(n-1)}) + \ldots + 2(t_{qj(2n-2)}^+ - \widetilde{b}_{ij4})\\
+(t_{qj(2n-1)}^+ - \widetilde{b}_{ij3})] + 2(t_{qj(2n)}^+ - \widetilde{b}_{ij2})\\
+(t_{qj(2n+1)}^+ - \widetilde{b}_{ij1})]/(2 \times 2^{n-1} \parallel T_{q_j} \parallel)] + 1\}\end{aligned}$$

and

$$\begin{aligned}\mu_{R_*}(\widetilde{B}_{ij}, \widetilde{B}_j^-) = \frac{1}{2}\{[[(\widetilde{b}_{ij1} - t_{qj(2n+1)}^-) + 2(\widetilde{b}_{ij2} - t_{qj(2n)}^-)\\
+3[(\widetilde{b}_{ij3} - t_{qj(2n-1)}^-) + 2(\widetilde{b}_{ij4} - t_{qj(2n-2)}^-)\\
+4(\widetilde{b}_{ij5} - t_{qj(2n-3)}^-) + \cdots + 2^{n-3}(\widetilde{b}_{ijn} - t_{qj(n+2)}^-)]\\
+2^{n-1}(\widetilde{b}_{ij(n+1)} - t_{qj(n+1)}^-) + 3[2^{n-3}(\widetilde{b}_{ij(n+2)} - t_{qjn}^-)\\
+2^{n-4}(\widetilde{b}_{ij(n+3)} - t_{qj(n-1)}^-) + \cdots + 2(\widetilde{b}_{ij(2n-2)} - t_{qjj4}^-)\\
+(\widetilde{b}_{ij(2n-1)} - t_{qj3}^-)] + 2(\widetilde{b}_{ij(2n)} - t_{qj2}^-)\\
+(\widetilde{b}_{ij(2n+1)} - t_{qj1}^-)]/(2 \times 2^{n-1} \parallel T_{q_j} \parallel)] + 1\}\end{aligned}$$

indicates relative preference degree of alternative A_i over A^- to the jth criterion, where

$$\begin{aligned}\parallel T_{q_j} \parallel = \frac{1}{2}[(t_{qj1}^+ - t_{qj(2n+1)}^-) + 2(t_{qj2}^+ - t_{qj2n}^-)\\
+3[(t_{qj3}^+ - t_{qj(2n-1)}^-) + 2(t_{qj4}^+ - t_{qj(2n-2)}^-)\\
+4(t_{qj5}^+ - t_{qj(2n-3)}^-) + \ldots + 2^{n-3}(t_{qjn}^+ - t_{qj(n+2)}^-)]\\
+2^{n-1}(t_{qj(n+1)}^+ - t_{qj(n+1)}^-) + 3[2^{n-3}(t_{qj(n+2)}^+ - t_{qjn}^-)\\
+2^{n-4}(t_{qj(n+3)}^+ - t_{qj(n-1)}^-) + \cdots + 2(t_{qj(2n-2)}^+ - t_{qj4}^-)\\
+(t_{qj(2n-1)}^+ - t_{qj3}^-)] + 2(t_{qj(2n)}^+ - t_{qj2}^-)\\
+(t_{qj(2n+1)}^+ - t_{qj1}^-)]/2k \text{ if } (t_{qj1}^+ - t_{qj(2n+1)}^-) \geq 0.\end{aligned}$$

$$\begin{aligned}\frac{1}{2}[[(t_{qj1}^+ - t_{qj(2n+1)}^-) + 2(t_{qj2}^+ - t_{qj2n}^-) + 3[(t_{qj3}^+ - t_{qj(2n-1)}^-)\\
+2(t_{qj4}^+ - t_{qj(2n-2)}^-) + 4(t_{qj5}^+ - t_{qj(2n-3)}^-)\\
+\cdots + 2^{n-3}(t_{qjn}^+ - t_{qj(n+2)}^-)] + 2^{n-1}(t_{qj(n+1)}^+ - t_{qj(n+1)}^-)\end{aligned}$$

Table 1 Data given by experts about various criteria with respect to alternatives

Criteria	Internet provider	E_1	E_2	E_3	E_4
c_1	A_1	VG	VG	G	A
	A_2	A	G	VG	A
	A_3	P	A	A	G
	A_4	A	G	G	A
c_2	A_1	A	G	A	G
	A_2	VG	A	G	G
	A_3	G	A	P	A
	A_4	A	G	A	VG
c_3	A_1	P	A	G	A
	A_2	G	A	VG	G
	A_3	A	G	A	G
	A_4	G	G	G	A
c_4	A_1	A	G	A	A
	A_2	VG	G	G	A
	A_3	G	G	A	G
	A_4	A	VG	G	A
c_5	A_1	60	65	70	60
	A_2	70	75	70	80
	A_3	90	100	80	95
	A_4	65	60	70	65

Table 2 Linguistic terms of feedback and corresponding fuzzy numbers

Linguistic terms	Fuzzy numbers
Very poor	(0, 0, 10, 20, 30)
Poor	(10, 20, 30, 40, 50)
Average	(30, 40, 50, 60, 70)
Good	(50, 60, 70, 80, 90)
Very Good	(70, 80, 90, 100, 100)

Table 3 Linguistic terms of weights of criteria and corresponding fuzzy numbers

Linguistic terms	Fuzzy numbers
Very Low	(0, 0, 0.1, 0.2, 0.3)
Low	(0.1, 0.2, 0.3, 0.4, 0.5)
Medium	(0.3, 0.4, 0.5, 0.6, 0.7)
High	(0.5, 0.6, 0.7, 0.8, 0.9)
Very High	(0.7, 0.8, 0.9, 1,1)

Table 4 Linguistic terms of weights of criteria

	E_1	E_2	E_3	E_4
C_1	M	H	H	VH
C_2	H	L	M	M
C_3	M	H	VH	H
C_4	M	H	M	H
C_5	H	M	H	M

$$
\begin{aligned}
&+3[2^{n-3}(t^{+}_{qj(n+2)} - t^{-}_{qjn}) + 2^{n-4}(t^{+}_{qj(n+3)} - t^{-}_{qj(n-1)}) \\
&+ \cdots + 2(t^{+}_{qj(2n-2)} - t^{-}_{qj4}) \\
&+(t^{+}_{qj(2n-1)} - t^{-}_{qj3})] + 2(t^{+}_{qj(2n)} - t^{-}_{qj2}) \\
&+(t^{+}_{qj(2n+1)} - t^{-}_{qj1})]/2k] + 2(t^{-}_{qj(2n+1)} - t^{+}_{qj1}) \\
&\ \text{if } (t^{+}_{qj1} - t^{-}_{qj(2n+1)}) < 0,
\end{aligned}
$$

Table 4 represents linguistic terms of weights of criteria given by experts Tables 5, 6 and 7 provide related calculations.
where

$$
\begin{aligned}
t^{+}_{qj1} &= \max_i \{\widetilde{b}_{ij1}\} \\
t^{+}_{qj2} &= \max_i \{\widetilde{b}_{ij2}\} \\
&\vdots \\
t^{+}_{qj(2n+1)} &= \max_i \{\widetilde{b}_{ij(2n+1)}\}
\end{aligned}
$$

and

Table 5 Fuzzy group decision matrix

Criteria	A_1	A_2	A_3	A_4
C_1	(55, 65,75,85,90)	(45, 55,65,75,82.5)	(30, 40,50,60,70)	(40, 50,60,70,80)
C_2	(40, 50,60,70,80)	(50,60,70,80,87.5)	(30, 40,50,60,70)	(45, 55,65,75,82.5)
C_3	(30, 40,50,60,70)	(50,60,70,80,87.5)	(40,50,60,70, 80)	(45, 55,65,75,85)
C_4	(35, 45,55,65,75)	(50,60,70,80,87.5)	(45, 55,65,75,85)	(45, 55,65,75,82.5)
C_5	(63.75, 63.75,63.75,63.75,63.75)	(73.75,73.75,73.75,73.75,73.75)	(91.25, 91.25,91.25,91.25,91.25)	(65, 65,65,65,65)

Table 6 Normalized group decision matrix

Criteria	A_1	A_2	A_3	A_4
C_1	(0.6111, 0.7222,0.8333,0.9444,1)	(0.5, 0.6111, 0.7222,0.8333,0.9167)	(0.3333, 0.4444,0.5556,0.6667,0.7778)	(0.4444,0.5556,0.6667,0.7778, 0.8889)
C_2	(0.4571, 0.5714,0.6857,0.8,0.9143)	(0.5714,0.6857,0.8,0.9143,1)	(0.3429, 0.4571, 0.5714,0.6857,0.8)	(0.5143, 0.6286, 0.7429,0.8571,0.9429)
C_3	(0.3429, 0.4571, 0.5714,0.6857,0.8)	(0.5714,0.6857,0.8,0.9143,1)	(0.4571, 0.5714,0.6857,0.8,0.9143)	(0.5143, 0.6286, 0.7429,0.8571,0.9714)
C_4	(0.4, 0.5143,0.6286,0.7429,0.8571)	(0.5714,0.6857,0.8,0.9143,1)	(0.5143,0.6286,0.7429,0.8571,0.9714)	(0.5143,0.6286,0.7429,0.8571,0.9429)
C_5	(1, 1, 1, 1, 1)	(0.8644, 0.8644, 0.8644, 0.8644, 0.8644)	(0.6986, 0.6986,0.6986,0.6986,0.6986)	(0.9808, 0.9808, 0.9808, 0.9808,0.9808)

Table 7 Fuzzy weights of criteria

Criteria	Fuzzy weights
C_1	(0.5, 0.6, 0.7, 0.8, 0.8750)
C_2	(0.3, 0.4, 0.5, 0.6, 0.7)
C_3	(0.5, 0.6, 0.7, 0.8, 0.8750)
C_4	(0.4, 0.5, 0.6, 0.7, 0.8)
C_5	(0.4, 0.5, 0.6, 0.7, 0.8)

$$t^-_{qj1} = \min_i\{\widetilde{b}_{ij1}\}$$
$$t^-_{qj2} = \min_i\{\widetilde{b}_{ij2}\}$$
$$\vdots$$
$$t^-_{qj(2n+1)} = \min_i\{\widetilde{b}_{ij(2n+1)}\}, \quad i = 1, 2, \ldots, p.$$

D_i^+ is derived by using $\mu_{R_*}(\widetilde{B}_j^+, \widetilde{B}_{ij})$ and W_j, $j = 1, 2, \ldots, t$ and indicates the weighted preference degree of A^+ over the alternative A_i and D_i^- is derived by using $\mu_{R_*}(\widetilde{B}_{ij}, \widetilde{B}_j^-)$ and W_j, $j = 1, 2, \ldots, t$ and indicates the weighted preference degree of A_i over A^- for $i = 1, 2, \ldots, p$.

Now,

$$D_i^+ = (W_1\mu_{R_*}(\widetilde{B}_1^+, \widetilde{B}_{i1})) + (W_2\mu_{R_*}(\widetilde{B}_2^+, \widetilde{B}_{i2})) + \cdots + (W_t\mu_{R_*}(\widetilde{B}_t^+, \widetilde{B}_{it}))$$

and $D_i^- = (W_1\mu_{R_*}(\widetilde{B}_{i1}, \widetilde{B}_1^-)) + (W_2\mu_{R_*}(\widetilde{B}_{i2}, \widetilde{B}_2^-)) + \cdots + (W_t\mu_{R_*}(\widetilde{B}_{it}, \widetilde{B}_t^-))$, $i = 1, 2, \ldots, p$.

Then, we get D_i^+ and D_i^- as $2n + 1$ fuzzy numbers and are denoted by

$$D_i^+ = (d_{i1}^+, d_{i2}^+, \ldots, d_{i(2n+1)}^+) \text{ and}$$
$$D_i^- = (d_{i1}^-, d_{i2}^-, \ldots, d_{i(2n+1)}^-), i = 1, 2, \ldots, p.$$

Then,

$$\overline{D}^+ = (\overline{d}_1^+, \overline{d}_2^+, \ldots, \overline{d}_{2n+1}^+) \text{ and}$$
$$\overline{D}^- = (\overline{d}_1^-, \overline{d}_2^-, \ldots, \overline{d}_{2n+1}^-),$$

where

$$\overline{d}_1^+ = \frac{1}{p}\sum_{i=1}^{p} d_{i1}^+$$

Table 8 Ideal and anti-ideal solutions of internet service providers

	A^+	A^-
C_1	(0.6111, 0.7222, 0.8333, 0.9444, 1)	(0.3333, 0.4444, 0.5556, 0.6667, 0.7778)
C_2	(0.5714, 0.6857, 0.8, 0.9143, 1)	(0.3429, 0.4571, 0.5714, 0.6857, 0.8)
C_3	(0.5714, 0.6857, 0.8, 0.9143, 1)	(0.3429, 0.4571, 0.5714, 0.6857, 0.8)
C_4	(0.5714, 0.6857, 0.8, 0.9143, 1)	(0.4, 0.5143, 0.6286, 0.7429, 0.8571)
C_5	(1,1,1,1,1)	(0.6986, 0.6986, 0.6986, 0.6986, 0.6986)

Table 9 Relative preference degree of ideal solution over internet providers on criteria

$\mu_{R*}(\widetilde{G}_j^+, \widetilde{G}_{ij})$	A_1	A_2	A_3	A_4
C_1	0.5	0.6230	0.8095	0.6825
C_2	0.9072	0.5	0.7480	0.5630
C_3	0.7480	0.5	0.6221	0.5590
C_4	0.6850	0.5	0.5590	0.5630
C_5	0.5	0.7250	1	0.5319

Table 10 Relative preference degree of internet service providers over anti-ideal solution on criteria

$\mu_{R*}(\widetilde{G}_{ij}, \widetilde{G}_j^-)$	A_1	A_2	A_3	A_4
C_1	0.8095	0.6865	0.5	0.6270
C_2	0.6260	0.7480	0.5	0.6851
C_3	0.5	0.7480	0.6260	0.6851
C_4	0.5	0.6850	0.6260	0.6260
C_5	1	0.7750	0.5	0.9681

$$\overline{d}_2^+ = \frac{1}{p}\sum_{i=1}^{p} d_{i2}^+$$

$$\overline{d}_3^+ = \frac{1}{p}\sum_{i=1}^{p} d_{i3}^+$$

$$\vdots$$

$$\overline{d}_{2n+1}^+ = \frac{1}{p}\sum_{i=1}^{p} d_{i(2n+1)}^+$$

and

$$\overline{d}_1^- = \frac{1}{p}\sum_{i=1}^{p} d_{i1}^-$$

$$\overline{d}_2^- = \frac{1}{p}\sum_{i=1}^{p} d_{i2}^-$$

$$\vdots$$

$$\overline{d}_{2n+1}^- = \frac{1}{p}\sum_{i=1}^{p} d_{i(2n+1)}^-.$$

Now, the relative closeness coefficient D_i of the alternative A_i is defined as

$$D_i = \frac{\mu_{R_*}(D_i^-, \overline{D}^-)}{\mu_{R_*}(D_i^-, \overline{D}^-) + \mu_{R_*}(D_i^+, \overline{D}^+)},\ i = 1, 2, ...p,$$

where $\mu_{R_*}(D_i^-, \overline{D}^-)$ is the relative preference degree of D_i^- over $\overline{D}^-$ and $\mu_{R_*}(D_i^+, \overline{D}^+)$ is the relative preference degree of D_i^+ over D^+.

Clearly, $D_i,\ i = 1, 2, \ldots, p$ is in the interval [0, 1]. The bigger the value $D_i,\ i = 1, 2, \ldots, p$ is closer the ideal solution is. Alternatively, smaller the value $D_i,\ i = 1, 2, \ldots, p$ is closer the anti-ideal solution is. Thus, the P alternatives are ranked according to their relative closeness coefficients $D_1, D_2, \ldots, D_p$.

Analysis of computational complexity

The computational complexity of the proposed algorithm is $O(mn)$, where m is the number of alternatives with n criteria. Like in the other FMCDM methods, it is assumed that the number of experts is a constant for the purpose of computational complexity. If $m = O(n)$, then the algorithm takes quadratic ($O(n^2)$) time. In general, FMCDM methods have $O(mn)$ computational complexity. This coincides with the complexity of the algorithm given in this paper.

Example 5.1 In real-world conditions, selection of an internet service provider is often based on various criteria such as Monthly cost (C_5) as Cost criteria and Volume of data (C_1), Speed of internet (C_2), Subscription to OTT platforms (C_3), dependable customer service (C_4) as benefit criteria. If the available internet service providers (A_1, A_2, A_3, A_4) are assessed on this criterion by four different experts (E_1, E_2, E_3, E_4) with each expert considering one of the benefit criteria to be more important than others, then there will be a dilemma in the selection as each service provider may have at least one parameter in which they are superior to other internet service providers. In such situations, we provide a solution to arrive at a decision as follows.

Table 1 represents the consolidated information about the performance of various internet service providers (A_1, A_2, A_3, A_4) with respect to various selection criteria (C_1, C_2, C_3, C_4, C_5) by various experts (E_1, E_2, E_3, E_4) wherein the feedback on benefit criteria is expressed as Very Good (VG),

Table 11 Weighted preference degree of ideal solution over internet service providers on criteria

$W_j \mu_{R*}(\widetilde{G}_j^+, \widetilde{G}_{ij})$	A_1	A_2	A_3	A_4
C_1	(0.25, 0.3, 0.35, 0.4, 0.4375)	(0.3115, 0.3738, 0.4361,0.4984,0.5451)	(0.4048, 0.4857,0.5667,0.6476,0.7083)	(0.3413,0.4095,0.4778,0.5460, 0.5972)
C_2	(0.2722, 0.3629,0.4536,0.5443,0.6350)	(0.15,0.20,0.25,0.30,0.35)	(0.2244, 0.2992, 0.3740,0.4488,0.5236)	(0.1689, 0.2252, 0.2815,0.3378,0.3941)
C_3	(0.3740, 0.4488, 0.5236,0.5984,0.6545)	(0.25, 0.3, 0.35, 0.4, 0.4375)	(0.3111, 0.3733,0.4355,0.4977,0.5443)	(0.2795, 0.3354, 0.3913,0.4472,0.4891)
C_4	(0.2740, 0.3425,0.4110,0.4795,0.5480)	(0.20, 0.25, 0.3, 0.35, 0.4)	(0.2236,0.2795,0.3354,0.3913,0.4472)	(0.2252,0.2815,0.3378,0.3941,0.4504)
C_5	(0.20, 0.25, 0.3, 0.35, 0.4)	(0.2900, 0.3625, 0.4350, 0.5075, 0.5800)	(0.4, 0.5,0.6,0.7,0.8)	(0.2128, 0.2660, 0.3191, 0.3723,0.4255)

Table 12 Weighted preference degree of internet service providers over anti-ideal solution on criteria

$W_j \mu_{R*}(\widetilde{G}_{ij}, \widetilde{G}_j^-)$	A_1	A_2	A_3	A_4
C_1	(0.4048, 0.4857, 0.5667, 0.6476, 0.7083)	(0.3443, 0.4119, 0.4806,0.5492,0.6007)	(0.25, 0.3, 0.35, 0.4, 0.4375)	(0.3135,0.3762,0.4389,0.5016, 0.5486)
C_2	(0.1878, 0.2504,0.3130,0.3756,0.4382)	(0.2244, 0.2992, 0.3740,0.4488,0.5236)	(0.15, 0.20, 0.25, 0.3, 0.35)	(0.2055, 0.2740, 0.3426,0.4111,0.4796)
C_3	(0.25, 0.3, 0.35, 0.4, 0.4375)	(0.3740, 0.4488, 0.5236,0.5984,0.6545)	(0.3130, 0.3756,0.4382,0.5008,0.5978)	(0.3426, 0.4111, 0.4796,0.5481,0.5995)
C_4	(0.20, 0.25, 0.3, 0.35, 0.4)	(0.2740, 0.3425,0.4110,0.4795,0.5480)	(0.2504,0.3130,0.3756,0.4382,0.5008)	(0.2504,0.3130,0.3756,0.4382,0.5008)
C_5	(0.4, 0.5,0.6,0.7,0.8)	(0.3100, 0.3875, 0.4650, 0.5425, 0.6200)	(0.20, 0.25, 0.3, 0.35, 0.4)	(0.3872, 0.4841, 0.5809, 0.6777,0.7745)

Good (G), Average (A), Poor (P) or Very Poor (VP). The feedback on cost criteria, on the other hand, is expressed as numerical values (for example, here it represents price of the service).

Linguistic terms used for expressing the feedback on benefit criteria are assigned with fuzzy numbers as shown in Table 2.

Table 3 represents weights of criteria and their corresponding fuzzy numbers.

Ideal and anti-ideal solutions of 4 internet service providers are presented in Table 8. Subsequent calculations are shown in Tables 9, 10, 11, 12, 13, 14, 15, 16.

From Table 17, the relative closeness coefficients of 4 internet service providers are A_1 : 0.4993, A_2 : 0.5767, A_3 : 0.3607 and A_4 : 0.5624 and hence the rank order of internet service providers is $A_2 > A_4 > A_1 > A_3$. This indicates that the internet service provider A_2 is better compared to other internet service providers.

6 Validation with existing methods

Wang (2014) proposed an algorithm to give decision in a triangular FMCDM using relative preference relation along with TOPSIS method. In the present study, we extended Wang's method to arrive at a decision in FMCDM problems involving $2n+1$ fuzzy numbers. Similar extension is possible for FMCDM problems involving $2n$ fuzzy numbers.

The suitability of the proposed method was verified by comparing it with the popularly used multi-criteria decision-making methods such as VIKOR, MOORA and ELECTRE. Prior to the application of these methods, alternative values of various criteria given by experts and weights of criteria from the above example are converted to crisp values by centroid defuzzification method.

1. **VIKOR:** Following table of values is obtained by applying VIKOR method (Opricovic (1998, 2002)).

	A_1	A_2	A_3	A_4
S	1.5384	0.3488	2.3193	0.5928
R	0.6950	0.2182	0.6950	0.4504
Q	0.8019	0	1	0.3054

From the above table, rank order of the alternatives is $A_2 > A_4 > A_1 > A_3$.

2. **Multi-objective Optimization by Ratio analysis (MOORA):** By applying MOORA reference point method (Brauers and Zavadskas (2006, 2010)), deviations from the reference points and ranking of the alternatives are presented in the following table.

	C_1	C_2	C_3	C_4	C_5	max. value	Ranking
A_1	0	0.0386	0.11	0.0682	0	0.11	3
A_2	0.0527	0	0	0	0.0404	0.0527	1
A_3	0.1329	0.0793	0.0535	0.0211	0.1111	0.1329	4
A_4	0.0775	0.0203	0.0253	0.0235	0.005	0.0775	2

3. **ELECTRE:** By using ELECTRE method (Roy (1991)), following global matrix is obtained.

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{pmatrix} - & 0 & 0 & 0 \\ 0 & - & 1 & 0 \\ 1 & 0 & - & 0 \\ 1 & 0 & 1 & - \end{pmatrix} \end{array}.$$

From the global matrix, it is clear that $A_2 > A_3 > A_1$ and $A_4 > A_3 > A_1$ which can also be expressed as $A_2 \geq A_4 > A_3 > A_1$.

The rank order of alternatives obtained from all three methods indicates the proposed method's suitability in FMCDM problems. The proposed method involves less operational complexity, it is easy to compute, and it minimizes the loss of information.

7 Conclusions

We have defined $2n + 1$ and $2n$ fuzzy numbers as generalizations of triangular and trapezoidal fuzzy numbers, respectively. Fuzzy preference relation and relative preference relation of Wang (2015b) are extended to rank $2n$ and $2n + 1$ fuzzy numbers and the results are compared with Wang and Lee (2008) method. Wang (2014)'s method was extended to arrive at a decision in FMCDM problems when the given data is in terms of $2n + 1$ fuzzy numbers. An illustrative example was provided to explain the suitability of the proposed method, and the results were validated using VIKOR, MOORA and ELECTRE methods. The proposed method can also be extended to $2n$ fuzzy numbers.

Table 13 Weighted preference degrees of ideal solution over internet service providers and average

	Weighted preference degrees
D_1^+	(1.3702, 1.7042, 2.0382, 2.3722, 2.6750)
D_2^+	(1.2015, 1.4863, 1.7711, 2.0559, 2.3126)
D_3^+	(1.5639, 1.9377, 2.3116, 2.6854, 3.0234)
D_4^+	(1.2277, 1.5176, 1.8079, 2.0974, 2.3563)
Average $\overline{D}^+$	(1.3408, 1.6615, 1.9822, 2.3027, 2.5918)

Table 14 Weighted preference degrees of internet service providers over anti-ideal solution and average

	Weighted preference degrees
D_1^-	(1.4426, 1.7861, 2.1297, 2.4732, 2.7840)
D_2^-	(1.5257, 1.8899, 2.2542, 2.6184, 2.9468)
D_3^-	(1.1634, 1.4386, 1.7138, 1.9890, 2.2361)
D_4^-	(1.4992, 1.8584, 2.2176, 2.5767, 2.9030)
Average $\overline{D}^-$	(1.4077, 1.7433, 2.0788, 2.4143, 2.7175)

Table 15 Relative preference degrees $\mu_P^*(D_i^+, \overline{D}^+)$

$\mu_{R*}(D_1^+, \overline{D}^+)$	0.5218
$\mu_P^*(D_2^+, \overline{D}^+)$	0.4183
$\mu_P^*(D_3^+, \overline{D}^+)$	0.6277
$\mu_P^*(D_4^+, \overline{D}^+)$	0.4323

Table 16 Relative preference degrees $\mu_P^*(D_i^-, \overline{D}^-)$

$\mu_{R*}(D_1^-, \overline{D}^-)$	0.5203
$\mu_P^*(D_2^-, \overline{D}^-)$	0.5700
$\mu_P^*(D_3^-, \overline{D}^-)$	0.3542
$\mu_P^*(D_4^-, \overline{D}^-)$	0.5555

Table 17 Relative closeness coefficients of internet service providers

D_1	0.4993
D_2	0.5767
D_3	0.3607
D_4	0.5624

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Declarations

Conflicts of interest The authors declare that they have no conflict of interest.

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