A personalized individual semantics model for computing with linguistic intuitionistic fuzzy information and application in MCDM

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Research Article

Keywords: Multi-criteria decision-making, personalized individual semantics, linguistic intuitionistic fuzzy preference relations, additive consistency, Hamming distance

Posted Date: September 12th, 2022

DOI: https://doi.org/10.21203/rs.3.rs-1762844/v1

A personalized individual semantics model for computing with linguistic intuitionistic fuzzy information and application in MCDM

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Abstract This paper develops a personalized individual semantics (PISs) model for computing with linguistic intuitionistic fuzzy information and applies to evaluating different brands of mobile phones. First, considering that a linguistic term means different things to different decision-makers, a consistency-driven optimization model for checking the additive consistent linguistic intuitionistic fuzzy preference relations (LIFPRs) is constructed by considering the PISs of a decision maker. Besides, several optimization models is built to determine the PISs of linguistic terms in LIFPRs and obtain the acceptable additive consistent LIFPRs. Second, a new definition of Hamming distance between linguistic intuitionistic fuzzy numbers (LIFNs) is developed by considering the PISs of a decision maker, and several of its desirable properties are discussed. Then, the method of deriving the weight vectors of criteria is calculated based on the proposed distance measure. Subsequently, a framework of group decision making process with LIFPRs is offered, and the application of the proposed method is illustrated by using a multi-criteria decision-making problem about evaluating different brands of mobile phones. Finally, the comparative analysis is conducted to show the feasibility of proposed group decision-making method.

Keywords Multi-criteria decision-making, personalized individual semantics, linguistic intuitionistic fuzzy preference relations, additive consistency, Hamming distance

1. Introduction

Group decision-making is widespread in real-world decision-making problems. Many studies focused on all kinds of group decision-making problems and obtained fruitful results [1-3]. For group decision-making under fuzzy and uncertain environment, linguistic information is common way to express evaluation information. These group decision-making problems are named linguistic group decision-making problems. Traditionally, linguistic group decision-making utilizes single linguistic terms to express evaluation information. In order to enrich linguistic expressions for the decision-makers, a variety of linguistic formats are developed, such as interval linguistic [4], hesitant fuzzy linguistic term set [5], probabilistic linguistic term set [6], flexible linguistic expression [7], linguistic distribution [8], multi-granular linguistic distribution [9], hesitant intuitionistic linguistic distribution [10], etc. Wu et al. [8] provided a comprehensive review of the distributed linguistic representations in decision-making, including: Taxonomy, key elements and applications, and challenges in data science and explainable artificial intelligence. Herrera-Viedma et al. [11] provided a brief tour through the linguistic decision-making trends, studies, methodologies, and models developed in the last 50 years.

Motivated by intuitionistic fuzzy set and linguistic term set, Chen et al. [12] introduced the concept of linguistic intuitionistic fuzzy numbers (LIFNs). A LIFN expresses the decision-makers preferences with linguistic membership degree and linguistic non-membership, which the membership and non-membership are presented by linguistic terms. Since LIFNs can represent preferred and non-preferred qualitative judgments of decision-makers, the LIFNs are considered a flexible linguistic model for decision-makers to express their evaluation information. At present, LIFNs have been applied in many real-world decision-making problems, such as individual financial investment decision-making [13], three-way decision-making [14], scheme selection of design for disassembly [15], analysis of socioeconomic development [16], system failure probability evaluation [17], etc. And several group decision-making methods with LIFNs have been developed, such as outranking method [18], aggregation operator-based method [19], Preference Ranking Organization Method for Enrichment Evaluation method [20], and characteristic measure-based method [21].

In linguistic decision-making processes, computing with words is an important point to note about. Many studies focused on the method of computing with words and obtained fruitful results. The commonly used method is solving computing with words problems by means of mathematical functions between numerical values and linguistic terms.

Among them, the most notable ones are the following: (1) The method based on 2-tuples linguistic representation model. This method based on the canonical characteristic values of the linguistic terms to deal with linguistic term sets that are symmetrically distributed and not uniformly. Based on 2-tuples linguistic representation model, the numerical scales model is developed. (2) The method based on numerical scales model. This method establishes a one to one mapping between a numerical scale and the linguistic terms. Personalized individual semantics (PISs) model [22] is a common used method. (3) The method based on specific mathematical functions. This method assigns numerical values to linguistic terms to represent its corresponding semantic. Utilizing subscript values of the linguistic terms for computing with words is a common method [23]. (4) The method based on cloud model. Cloud model can characterize the uncertainties of linguistic information and identify the certainty degree of random variables using probability distributions. In order to make a comprehensive understanding to these computing with words methods, summary of these methods is listed in Table 1.

Table 1: Summary of some computing with words methods

Methods	Characteristic	Representative literatures
The method based on 2-tuples linguistic representation model	Uses the ordinary for representing linguistic information	Muhuri et al. [24], Wang et al. [25] and Giráldez-Cru et al. [26].
The method based on numerical scales model	Establishes a one to one mapping between the linguistic terms and a numerical scale	Dong et al. [27], Fan et al. [7] and Jiang et al. [28].
The method based on specific mathematical functions	Assigns numerical values to linguistic terms to represent its corresponding semantic	Fu et al. [23], Zhang et al. [29] and Mi et al. [30]
The method based on cloud model	Identify the certainty degree of random variables using probability distributions	Wang et al. [31], Xiao et al. [32] and Peng et al. [33].

Although there are several methods for computing with words, shortcomings still exist in some methods. Such as, the method of based on cloud model cannot guarantee the loss of information in the process of linguistic conversion, and the methods of based on 2-tuples linguistic representation model and specific mathematical functions are difficult to reflect words mean different things to different people, etc. In group decision-making, different decision-makers may have different understandings of words. There are mainly two methods to address this issue. The first method is utilizing multi-granularities linguistic term sets for decision-makers, and the second method is using a same linguistic term set for all decision-makers, while the decision-makers' PISs of the linguistic information are considered. For the latter one, constructing some optimization models to maximize the consistency of linguistic preference relations is a common used method [34-36], and it has been studied in several linguistic environments, such as incomplete linguistic [37], flexible linguistic expressions [28], comparative linguistic expressions [38, 39], probabilistic linguistic [40] , linguistic distribution [41, 42], etc.

Although the existing studies are effective for solving multi-criteria decision-making (MCDM) problems with LIFNs, there are still some limitations as follows. (1) Although several studies focused on solving the MCDM problems with LIFNs. Unfortunately, these studies did not take into account PISs. At present, some existing PISs model-based MCDM methods have been proposed for managing the PISs of linguistic term with comparative linguistic expressions, probabilistic linguistic, etc. But these methods are not suitable for managing the PISs in MCDM problems with LIFNs. (2) The existing studies assumed that the criteria weight vectors are the same for different decision-makers. However, the individuals may have their own criteria weight vectors due to the difference of preference, interest and background. PISs and criteria weight vectors have an important impact on group decision-making results. Therefore, it is necessary to develop a new method for managing PISs and criteria weight vectors in MCDM problems with LIFNs.

To eliminate above mention limitations, the consistency measures from the perspectives of additive consistent linguistic intuitionistic fuzzy preference relations (LIFPRs) are defined. And several optimization models is built to determine the PISs of linguistic terms in LIFPRs and obtain the acceptable additive consistent LIFPRs. The primary contributions of this study are summarized as follows.

(1) Considering that a linguistic term means different things to different decision-makers, a consistency-driven optimization model for checking the additive consistent LIFPRs is constructed by considering the PISs of a decision maker. Moreover, several optimization models is built to determine the PISs of linguistic terms in LIFPRs and obtain the acceptable additive consistent LIFPRs.

(2) A new definition of Hamming distance between LIFNs is developed by considering the PISs of a decision maker, and several of its desirable properties are discussed. Besides, based on the proposed distance measure, the method of deriving the weight vectors of criteria is introduced.

The remainder of the paper is organized as follows. In Section 2, basic concepts related to LIFNs, numerical scale model, and PISs model are reviewed. In Section 3, the concept of acceptable additive consistent LIFPRs based on PISs is introduced, and several programming models are developed for deriving the acceptable additive consistent LIFPRs. Besides, an algorithm is introduced to obtain complete or acceptable additive consistent LIFPRs. In Section 4, the MCDM problems with LIFPRs are introduced, and an optimization model is constructed for determining the weight vectors of criteria. Moreover, a framework of MCDM procedure with LIFPRs is introduced. In Section 5, the proposed method is illustrated by an example, and a comparative analysis is provided. Finally, conclusions are presented in Section 6.

2. Preliminaries

To carry out the following research, this part briefly reviews some basic concepts, including the concepts of LIFNs, numerical scale model, and PISs model.

2.1 LIFNs

Considering the fact that linguistic variables can only denote the qualitative preferred recognitions of the decision-makers, Chen et al. [12] introduced the concept of LIFNs, which can denote the qualitative preferred and non-preferred judgments of the decision-makers simultaneously.

Definition 1 [12]. A LIFN $\tilde{s}$ on the continuous linguistic term set $S_c = \{s_\alpha | \alpha \in [0, 2t]\}$ is expressed as $\tilde{s} = (s_\alpha, s_\beta)$, where s_α denoted the preferred qualitative degree and s_β denoted the non-preferred qualitative degree, such that $s_\alpha \oplus s_\beta \le s_{2t}$.

Obviously, LIFNs follow the principle of intuitionistic fuzzy numbers. The difference between them is LIFNs used qualitative degree to denote the preferred and non-preferred degrees, while intuitionistic fuzzy numbers used quantify. Afterwards, Meng et al. [43] proposed the concept of linguistic intuitionistic fuzzy preferences (LIFRs).

Definition 2 [43]. Let $x = \{x_1, x_2, \cdots, x_n\}$ be a finite object set, and $S_c = \{s_\alpha | \alpha \in [0, 2t]\}$ be a continuous linguistic term set. A LIFR $\tilde{R}$ is defined as $\tilde{R} = \left(\tilde{r}_{ij}\right)_{n\times n}$, where $\tilde{r}_{ij} = \left(s_{\mu_{ij}}, s_{v_{ij}}\right)$, $i, j = 1, 2, \cdots, n$, and $s_{\mu_{ij}}$ denoted the preferred qualitative degree and $s_{v_{ij}}$ denoted the non-preferred qualitative degree of the object x_i over x_j under the following conditions:

$$\begin{cases} s_{\mu_{ij}} = s_{v_{ji}}, s_{v_{ij}} = s_{\mu_{ji}} \\ s_{\mu_{ij}} \oplus s_{v_{ij}} \le s_{2t} \\ s_{\mu_{ii}} = s_{v_{ii}} = s_t \end{cases}. \tag{1}$$

To develop the concept of additive consistent LIFPRs, Meng et al. [43] first introduced the concept of reverse complementary LIFN (RCLIFN) $\tilde{s}^c = \left(s_{2t}s_\alpha, s_{2t}s_\beta\right)$, where $s_{2t}s_\alpha = s_{2t-\alpha}$ and $s_{2t}s_\beta = s_{2t-\beta}$, and then developed the concept of additive consistent LIFPRs.

Definition 3 [43]. Let $x = \{x_1, x_2, \ldots, x_n\}$ be a finite object set, $S_c = \{s_\alpha | \alpha \in [0, 2t]\}$ be a continuous linguistic term set, and $\tilde{R} = \left(\tilde{r}_{ij}\right)_{n\times n}$ be a LIFPR. Then, $\tilde{R}$ is additive consistency if and only if there is a set of the 0-1 indicator variables α_{ij}, $i, j = 1, 2, \ldots, n$, and $i < j$ such that:

$$\left(\alpha_{ij}\tilde{r}_{ij} \oplus \left(1-\alpha_{ij}\right)\tilde{r}_{ji}^c\right) \oplus \left(s_t, s_t\right) = \left(\alpha_{ik}\tilde{r}_{ik} \oplus \left(1-\alpha_{ik}\right)\tilde{r}_{ki}^c\right) \oplus \left(\alpha_{kj}\tilde{r}_{kj} \oplus \left(1-\alpha_{kj}\right)\tilde{r}_{jk}^c\right), \tag{2}$$

for each triple of (i, j, k), $i < k < j$, and $\tilde{r}_{ji}^c$ is a RCLIFN.

According to the relationship between interval linguistic fuzzy preference relations and LIFPRs, Meng et al. [43] provided the proof of Eq. (2) is equivalent to the following formulas:

$$\begin{cases} \left(\alpha_{ij}s_{u_{ij}} \oplus \left(1-\alpha_{ij}\right)\left(s_{2t}s_{v_{ij}}\right)\right) \oplus s_t = \left(\alpha_{ik}s_{u_{ik}} \oplus \left(1-\alpha_{ik}\right)\left(s_{2t}s_{v_{ik}}\right)\right) \oplus \left(\alpha_{kj}s_{u_{kj}} \oplus \left(1-\alpha_{kj}\right)\left(s_{2t}s_{v_{kj}}\right)\right) \\ \left(\alpha_{ij}s_{v_{ij}} \oplus \left(1-\alpha_{ij}\right)\left(s_{2t}s_{\mu_{ij}}\right)\right) \oplus s_t = \left(\alpha_{ik}s_{v_{ik}} \oplus \left(1-\alpha_{ik}\right)\left(s_{2t}s_{\mu_{ik}}\right)\right) \oplus \left(\alpha_{kj}s_{v_{kj}} \oplus \left(1-\alpha_{kj}\right)\left(s_{2t}s_{\mu_{kj}}\right)\right) \end{cases}. \tag{3}$$

2.2 Numerical scale model

As an extension of the linguistic 2-tuples, Dong et al. [44] developed the concept of numerical scale.

Definition 4 [44]. Let $S = \{s_0, s_1, \ldots, s_g\}$ and R respectively be a set of linguistic terms and real numbers. A function $NS : S \to R$ is called a numerical scale of S, and $NS(s_i)$ is the numerical index of s_i.

For a linguistic representation model with 2-tuple linguistic model (s_i, α), Dong et al. [44] introduced a numerical scale NS.

Definition 5 [44]. The numerical scale NS for (s_i, α) is defined as follows:

$$NS(s_i,\alpha)=\begin{cases} NS(s_i)+\alpha\times(NS(s_{i+1})-NS(s_i)), & \alpha\geq 0 \\ NS(s_i)+\alpha\times(NS(s_i)-NS(s_{i-1})), & \alpha<0 \end{cases}. \tag{4}$$

Obviously, the numerical scale defines a one to one mapping between a linguistic term set and a numerical scale. Afterwards, Dong et al. [45] proposed the inverse operator of numerical scale NS, which defines a one to one mapping between a numerical scale and a linguistic term set.

Definition 6 [45]**.** The inverse operator of numerical scale NS is defined as follows:

$$NS^{-1}(\gamma)=\begin{cases} \left(s_i,\dfrac{\gamma-NS(s_i)}{NS(s_{i+1})-NS(s_i)}\right), & NS(s_i)<\gamma<\dfrac{NS(s_i)+NS(s_{i+1})}{2} \\ \left(s_i,\dfrac{\gamma-NS(s_i)}{NS(s_i)-NS(s_{i-1})}\right), & \dfrac{NS(s_{i-1})+NS(s_i)}{2}\leq\gamma\leq NS(s_i) \end{cases}. \tag{5}$$

To demonstrate the validity of proposed numerical scale model, Dong et al. [46] provided a unified framework work of numerical scale connect to the 2-tuple linguistic model, the unbalanced linguistic model, and the proportional 2-tuple linguistic model.

2.3 Personalized individual semantics model

Since different decision-makers might have different understandings of the linguistic terms. In other words, words mean different things for different people. To address this issue, Li et al. [22] developed the concept of PISs. Owing to its practicability and effectiveness, the used of PISs has been developed in various linguistic forms. Such as, comparative linguistic expression [47], flexible linguistic expression [28], probabilistic linguistic term sets [40], and distribution linguistic term sets [42].

The PISs of linguistic terms have different formats, such as exact values [22], and unit intervals [38]. In the following section, the method of computing PISs of the linguistic terms in linguistic preference relations is viewed.

Let $A=\{a_1,a_2,\ldots,a_n\}$ and $E=\{e_1,e_2,\ldots,e_m\}$ respectively be the set of alternative and decision-makers. The decision-makers $e_1,e_2,\ldots,e_m$ provide the pairwise comparisons in the form of linguistic preference relations $P^k=\left(p_{ij}^k\right)_{n\times n}$, where $p_{ij}^k+p_{ji}^k=s_g$, $p_{ii}^k=s_{g/2}$, $p_{ij}^k\in S$, $i,j=1,2,\ldots,n$, $k=1,2,\ldots,m$. To derive the PISs of linguistic terms, Li et al. [48] developed the following model:

$$\max \quad CI\left(P^k\right)=1-\frac{4}{n(n-1)(n-2)}\sum_{i<j<z}\left|NS\left(p_{ij}^k\right)+NS\left(p_{jz}^k\right)-NS\left(p_{iz}^k\right)-0.5\right|$$

$$s.t.\begin{cases} NS\left(s_0\right)=0 \\ NS\left(s_g\right)=1 \\ NS\left(s_{g/2}\right)=0.5 \\ NS\left(s_i\right)\in\left[(i-1)/g,(i+1)/g\right], \quad i=1,2,\ldots,g-1, i\neq g/2 \\ NS\left(s_{i+1}\right)-NS\left(s_i\right)\geq\lambda, \quad i=0,1,\ldots,g-1 \end{cases}. \tag{6}$$

In Eq. (5), the objective function indicates the larger the value of $CI\left(P^k\right)$, the more consistent P^k is. The first to fourth constraints determined the range of numerical scales, and the fifth constraint assured the ordered of numerical scales. The constraint value $\lambda\in(0,1)$ is a small positive number, it can be set priori, such as $\lambda=0.01$.

3. Acceptable additive consistent LIFPRs

In this section, the concept of acceptable additive consistent LIFPRs based on PISs is introduced, and then several programming models are developed for deriving the acceptable additive consistent LIFPRs from unacceptable consistent ones. Finally, an algorithm is introduced to obtain complete or acceptable additive consistent LIFPRs.

3.1 The consistency of LIFPRs

Let s_α and s_β be any two linguistic numbers, λ be a nonnegative real number within interval $[0,1]$. Xu [49] developed the following operations: (1) $s_\alpha \oplus s_\beta = s_{\alpha+\beta}$; (2) $s_\alpha s_\beta = s_{\alpha-\beta}$; and (3) $\lambda s_\alpha = s_{\lambda\alpha}$. To further consider Eq. (3), according to the operations list above, it can be further simplified into:

$$\begin{cases} s_{\alpha_{ij}u_{ij}+\left(1-\alpha_{ij}\right)\left(2t-v_{ij}\right)+t} = s_{\alpha_{ik}u_{ik}+\left(1-\alpha_{ik}\right)\left(2t-v_{ik}\right)+\alpha_{kj}u_{kj}+\left(1-\alpha_{kj}\right)\left(2t-v_{kj}\right)} \\ s_{\alpha_{ij}v_{ij}+\left(1-\alpha_{ij}\right)\left(2t-\mu_{ij}\right)+t} = s_{\alpha_{ik}v_{ik}+\left(1-\alpha_{ik}\right)\left(2t-\mu_{ik}\right)+\alpha_{kj}v_{kj}+\left(1-\alpha_{kj}\right)\left(2t-\mu_{kj}\right)} \end{cases}. \tag{7}$$

To facilitate following discussion, some notations are used to denote the subscript values that is listed in Eq. (7), namely $p_{ij,1}=\alpha_{ij}u_{ij}+\left(1-\alpha_{ij}\right)\left(2t-v_{ij}\right)+t$, $p_{ij,2}^k=\alpha_{ik}u_{ik}+\left(1-\alpha_{ik}\right)\left(2t-v_{ik}\right)+\alpha_{kj}u_{kj}+\left(1-\alpha_{kj}\right)\left(2t-v_{kj}\right)$, $q_{ij,1}=\alpha_{ij}v_{ij}+\left(1-\alpha_{ij}\right)\left(2t-\mu_{ij}\right)+t$ and $q_{ij,2}^k=\alpha_{ik}v_{ik}+\left(1-\alpha_{ik}\right)\left(2t-\mu_{ik}\right)+\alpha_{kj}v_{kj}+\left(1-\alpha_{kj}\right)\left(2t-\mu_{kj}\right)$. In this way, Eq. (7) is equivalent to the following formulas:

$$\begin{cases} s_{p_{ij,1}} = s_{p_{ij,2}^k} \\ s_{q_{ij,1}} = s_{q_{ij,2}^k} \end{cases}. \tag{8}$$

Let $\tilde{R} = \left(\tilde{r}_{ij}\right)_{n\times n}$ be a LIFPR on the finite object set $x = \{x_1, x_2, \ldots, x_n\}$ for the continuous linguistic term set $S_c = \{s_\alpha \mid \alpha \in [0, 2t]\}$. If there is a set of the 0-1 indicator variables α_{ij}, $i, j = 1, 2, \ldots, n$, and $i < j$ makes Eq. (8) hold, then, $\tilde{R}$ is additive consistent LIFPR. Unfortunately, we cannot guarantee that Eq. (8) is always holds. In other words, when the LIFPR $\tilde{R}$ is inconsistent, Eq. (8) will not hold. Considering that the above mentioned equations do not constantly hold in general given a deviation between $s_{p_{ij,1}}$ and $s_{p_{ij,2}^k}$, $s_{q_{ij,1}}$ and $s_{q_{ij,2}^k}$ for a set of the 0-1 indicator variables α_{ij}, $i, j = 1, 2, \ldots, n$, and $i < j$. Moreover, the more deviation $s_{p_{ij,1}} \ominus s_{p_{ij,2}^k}$ and $s_{q_{ij,1}} \ominus s_{q_{ij,2}^k}$ approach to 0, the more the consistency is. Based on this fact, the following programming model is constructed based on PISs model:

$$\max \quad \delta$$
$$s.t. \begin{cases}
1 - \frac{4}{n(n-1)(n-2)} \sum_{i=1}^{k-1} \sum_{k=i+1}^{j-1} \sum_{j=k+1}^{n} \left| NS\left(s_{p_{ij,1}}\right) - NS\left(s_{p_{ij,2}^k}\right) \right| \geq \delta & (9-1) \\
1 - \frac{4}{n(n-1)(n-2)} \sum_{i=1}^{k-1} \sum_{k=i+1}^{j-1} \sum_{j=k+1}^{n} \left| NS\left(s_{q_{ij,1}}\right) - NS\left(s_{q_{ij,2}^k}\right) \right| \geq \delta & (9-2) \\
p_{ij,1} = \alpha_{ij} u_{ij} + \left(1 - \alpha_{ij}\right)\left(2t - v_{ij}\right) + t & (9-3) \\
p_{ij,2}^k = \alpha_{ik} u_{ik} + \left(1 - \alpha_{ik}\right)\left(2t - v_{ik}\right) + \alpha_{kj} u_{kj} + \left(1 - \alpha_{kj}\right)\left(2t - v_{kj}\right) & (9-4) \\
q_{ij,1} = \alpha_{ij} v_{ij} + \left(1 - \alpha_{ij}\right)\left(2t - \mu_{ij}\right) + t & (9-5) \\
q_{ij,2}^k = \alpha_{ik} v_{ik} + \left(1 - \alpha_{ik}\right)\left(2t - \mu_{ik}\right) + \alpha_{kj} v_{kj} + \left(1 - \alpha_{kj}\right)\left(2t - \mu_{kj}\right) & (9-6) \\
NS\left(s_0\right) = 0 & (9-7) \\
NS\left(s_{2t}\right) = 1 & (9-8) \\
NS\left(s_t\right) = 0.5 & (9-9) \\
NS\left(s_i\right) \in \left[(i-1-t)/2t, (i+1-t)/2t\right], \quad i = 1, 2, \ldots, 2t-1, i \neq t & (9-10) \\
NS\left(s_{i+1}\right) - NS\left(s_i\right) \geq \lambda, \quad i = 0, 1, \ldots, 2t-1 & (9-11) \\
\alpha_{ij} = 0 \vee 1, \quad i, j = 1, 2, \ldots, n, i < k < j & (9-12)
\end{cases}. \tag{9}$$

In Eq. (9), the symbol δ is the objective function. The larger the value δ, indicates the more deviation $s_{p_{ij,1}} \ominus s_{p_{ij,2}^k}$ and $s_{q_{ij,1}} \ominus s_{q_{ij,2}^k}$ approach to 0. The constraints (9-1) and (9-2) respectively denotes the distance between

$s_{p_{ij,1}}$ and $s_{p_{ij,2}^k}$, $s_{q_{ij,1}}$ and $s_{q_{ij,2}^k}$ based on PISs. The constraints from (9-3) to (9-6) respectively denotes the subscript values of linguistic variables $s_{p_{ij,1}}$, $s_{p_{ij,2}^k}$, $s_{q_{ij,1}}$ and $s_{q_{ij,2}^k}$. The constraints from (9-7) to (9-10) denotes the range of numerical scales, the constraint (9-11) assures the ordered of numerical scales. And constraint (9-12) is a set of the 0-1 indicator variables. Solving Eq. (9) by the software packages such as Lingo, Matlab. After solving Eq. (9), we obtain the objective function value δ and the personalized individual numerical scales for each linguistic term set $S_c = \{s_\alpha | \alpha \in [0,2t]\}$, that is, $NS(s_0)$, $NS(s_1)$, $\ldots$, $NS(s_{2t})$.

In Eq. (9), if the objective function value $\delta = 1$, indicates the relationships $\left|NS\left(s_{p_{ij,1}}\right) - NS\left(s_{p_{ij,2}^k}\right)\right| = 0$ and $\left|NS\left(s_{q_{ij,1}}\right) - NS\left(s_{q_{ij,2}^k}\right)\right| = 0$ hold. In other words, there is a set of the 0-1 indicator variables α_{ij}, $i, j = 1,2,\ldots,n$, and $i < k < j$ makes $s_{p_{ij,1}} = s_{p_{ij,2}^k}$ and $s_{q_{ij,1}} = s_{q_{ij,2}^k}$ hold simultaneously. In this case, $\tilde{R}$ is a complete additive consistent LIFPR. In contrary, If the objective function value $\delta \neq 1$, then $\tilde{R}$ is not a complete additive consistent LIFPR. Obviously, the objective function vale δ indicates the optimal consistency of LIFPR. For that, the consistency index of LIFPR can be defined as follows.

Definition 7. Let $\tilde{R} = \left(\tilde{r}_{ij}\right)_{n\times n}$ be a LIFPR on the finite object set $x = \{x_1, x_2, \ldots, x_n\}$, δ be the objective function vale derived from Eq. (9). Then the consistency index of $\tilde{R}$ is defined as $CI\left(\tilde{R}\right) = \delta$.

In the actual decision-making process, it is difficult to obtain complete consistent LIFPR and it's not necessary to obtain it sometimes. On the basis of above analysis, acceptable additive consistent LIFPRs are available. The concept of it is developed as follows.

Definition 8. Let $\tilde{R} = \left(\tilde{r}_{ij}\right)_{n\times n}$ be a LIFPR on the finite object set $x = \{x_1, x_2, \ldots, x_n\}$, δ_0 be the threshold value, and δ be the objective function vale derived from Eq. (9). If the relationship $\delta \geq \delta_0$ holds, then $\tilde{R}$ is considered as an acceptable additive consistent LIFPR.

From definition 8 can be seen, if $\delta < \delta_0$, indicatives that $\tilde{R}$ is not an acceptable additive consistent LIFPR, its consistency needs to be further improving, this will be conducted in the following section.

3.2 Consistency improving process from unacceptable ones

Consistency of preference relations is related to rationality. By comparison, inconsistent preference relations often lead to misleading solutions. Therefore, developing some approaches to obtain the expected consistency level is necessary. However, only few scholars focus on optimization-based method to obtain the expected consistent LIFPR at present. Therefore, in this section, several mathematical programming models are proposed to obtain acceptable additive consistent LIFPR which considering the PISs. There are two stages including, namely the first stage is derived the LIFPR with largest number of LIFNs in the upper triangular part, and the second stage is obtained the adjust LIFPR, which considering the minimum adjustment and PISs.

Stage 1. Derive the LIFPR with largest number of LIFNs

By solving Eq. (9), if we have $\delta \geq \delta_0$, then $\tilde{R}$ is an acceptable additive consistent LIFPR. In contrary, when $\delta < \delta_0$, $\tilde{R}$ is unacceptable consistency. Eq. (9) shows that there are maybe more than two LIFPRs have the same objective function value δ^*. To determine the unique LIFPR with the highest additive consistency level, the following programming model is developed:

$$\max \quad z = \sum_{i=1}^{j-1} \sum_{j=i+1}^{n} \alpha_{ij}$$
$$s.t. \begin{cases} 1 - \frac{4}{n(n-1)(n-2)} \sum_{i=1}^{k-1} \sum_{k=i+1}^{j-1} \sum_{j=k+1}^{n} \left| NS\left(s_{p_{ij,1}}\right) - NS\left(s_{p_{ij,2}^k}\right) \right| = \delta^* & (10-1) \\ 1 - \frac{4}{n(n-1)(n-2)} \sum_{i=1}^{k-1} \sum_{k=i+1}^{j-1} \sum_{j=k+1}^{n} \left| NS\left(s_{q_{ij,1}}\right) - NS\left(s_{q_{ij,2}^k}\right) \right| = \delta^* & (10-2) \\ p_{ij,1} = \alpha_{ij} u_{ij} + \left(1-\alpha_{ij}\right)\left(2t - v_{ij}\right) + t & (10-3) \\ p_{ij,2}^k = \alpha_{ik} u_{ik} + \left(1-\alpha_{ik}\right)\left(2t - v_{ik}\right) + \alpha_{kj} u_{kj} + \left(1-\alpha_{kj}\right)\left(2t - v_{kj}\right) & (10-4) \\ q_{ij,1} = \alpha_{ij} v_{ij} + \left(1-\alpha_{ij}\right)\left(2t - \mu_{ij}\right) + t & (10-5) \\ q_{ij,2}^k = \alpha_{ik} v_{ik} + \left(1-\alpha_{ik}\right)\left(2t - \mu_{ik}\right) + \alpha_{kj} v_{kj} + \left(1-\alpha_{kj}\right)\left(2t - \mu_{kj}\right) & (10-6) \\ NS\left(s_0\right) = 0 & (10-7) \\ NS\left(s_{2t}\right) = 1 & (10-8) \\ NS\left(s_t\right) = 0.5 & (10-9) \\ NS\left(s_i\right) \in \left[(i-1-t)/2t, (i+1-t)/2t\right], \quad i = 1,2,\ldots,2t-1, i \neq t & (10-10) \\ NS\left(s_{i+1}\right) - NS\left(s_i\right) \geq \lambda, \quad i = 0,1,\ldots,2t-1 & (10-11) \\ \alpha_{ij} = 0 \vee 1, \quad i,j = 1,2,\ldots,n, i < k < j & (10-12) \end{cases} . \quad (10)$$

In Eq. (10), the sum of the 0-1 indicator variables α_{ij} is the objective function. The larger the value z, indicates the more indicator variables set value 1, and the LIFPR with largest number of LIFNs is derived. The constraints (10-1) and (10-2) are based on the objective function value δ^* derived from Eq. (10). Others constraints are the same as those given in Eq. (9). After solving Eq. (10), a unique set of the 0-1 indicator variables α_{ij}^*, $i, j = 1, 2, \ldots, n$, and $i < j$ are obtained. If we return the indicator variables α_{ij}^* to Eq. (3), the associated LIFPR is derived, which has the largest number of LIFNs in the upper triangular part.

Stage 2. Obtain the adjust LIFPR

In this regard, when $\tilde{R}$ is unacceptable consistency, we need to adjust the original evaluation values provided by the decision-makers to ensure the ranking of objects reasonably. Meanwhile, in order to avoid the loss and distortion of evaluation information, the adjustment should be as small as possible to retain the decision-makers' original evaluation.

To do this, some notations are developed, let ϕ_{ij}^+ and ϕ_{ij}^-, $i, j = 1, 2, \ldots, n$, and $i < j$ respectively denote the adjustment positive deviation and adjustment negative deviation relate to the preferred qualitative degree $s_{\alpha_{ij}^* u_{ij} + (1-\alpha_{ij}^*)(2t - v_{ij}) + t}$, ϕ_{ij}^+, $\phi_{ij}^- \geq 0$ and $\phi_{ij}^+ \times \phi_{ij}^- = 0$. Similar, φ_{ij}^+ and φ_{ij}^-, $i, j = 1, 2, \ldots, n$, and $i < j$ respectively denote the adjustment positive deviation and adjustment negative deviation relate to the non-preferred qualitative degree $s_{\alpha_{ij}^* v_{ij} + (1-\alpha_{ij}^*)(2t - \mu_{ij}) + t}$, φ_{ij}^+, $\varphi_{ij}^- \geq 0$ and $\varphi_{ij}^+ \times \varphi_{ij}^- = 0$. Let $p_{ij,1}^* = \alpha_{ij}^* u_{ij} + (1-\alpha_{ij}^*)(2t - v_{ij}) + t + \phi_{ij}^- - \phi_{ij}^+$, $p_{ij,2}^{k*} = \alpha_{ik}^* u_{ik} + (1-\alpha_{ik}^*)(2t - v_{ik}) + \phi_{ik}^- - \phi_{ik}^+ + \alpha_{kj}^* u_{kj} + (1-\alpha_{kj}^*)(2t - v_{kj}) + \phi_{kj}^- - \phi_{kj}^+$, $q_{ij,1}^* = \alpha_{ij}^* v_{ij} + (1-\alpha_{ij}^*)(2t - \mu_{ij})$ $+ t + \varphi_{ij}^- - \varphi_{ij}^+$ and $q_{ij,2}^{k*} = \alpha_{ik}^* v_{ik} + (1-\alpha_{ik}^*)(2t - \mu_{ik}) + \varphi_{ik}^- - \varphi_{ik}^+ + \alpha_{kj}^* v_{kj} + (1-\alpha_{kj}^*)(2t - \mu_{kj}) + \varphi_{kj}^- - \varphi_{kj}^+$ be the subscript values of adjustment LIFPR. Then, the following programming model is developed:

$$\min \quad \sum_{i=1}^{j-1}\sum_{j=i+1}^{n}\left(\phi_{ij}^{+}+\phi_{ij}^{-}+\varphi_{ij}^{+}+\varphi_{ij}^{-}\right)$$

$$s.t.\begin{cases}
1-\dfrac{4}{n(n-1)(n-2)}\sum_{i=1}^{k-1}\sum_{k=i+1}^{j-1}\sum_{j=k+1}^{n}\left|NS\left(s_{p_{ij,1}^{*}}\right)-NS\left(s_{p_{ij,2}^{k*}}\right)\right|\geq\delta_0 & (11-1)\\
1-\dfrac{4}{n(n-1)(n-2)}\sum_{i=1}^{k-1}\sum_{k=i+1}^{j-1}\sum_{j=k+1}^{n}\left|NS\left(s_{q_{ij,1}^{*}}\right)-NS\left(s_{q_{ij,2}^{k*}}\right)\right|\geq\delta_0 & (11-2)\\
p_{ij,1}^{*}=\alpha_{ij}^{*}u_{ij}+\left(1-\alpha_{ij}^{*}\right)\left(2t-v_{ij}\right)+t+\phi_{ij}^{-}-\phi_{ij}^{+} & (11-3)\\
p_{ij,2}^{k*}=\alpha_{ik}^{*}u_{ik}+\left(1-\alpha_{ik}^{*}\right)\left(2t-v_{ik}\right)+\phi_{ik}^{-}-\phi_{ik}^{+}+\alpha_{kj}^{*}u_{kj}+\left(1-\alpha_{kj}^{*}\right)\left(2t-v_{kj}\right)+\phi_{kj}^{-}-\phi_{kj}^{+} & (11-4)\\
q_{ij,1}^{*}=\alpha_{ij}^{*}v_{ij}+\left(1-\alpha_{ij}^{*}\right)\left(2t-\mu_{ij}\right)+t+\varphi_{ij}^{-}-\varphi_{ij}^{+} & (11-5)\\
q_{ij,2}^{k*}=\alpha_{ik}^{*}v_{ik}+\left(1-\alpha_{ik}^{*}\right)\left(2t-\mu_{ik}\right)+\varphi_{ik}^{-}-\varphi_{ik}^{+}+\alpha_{kj}^{*}v_{kj}+\left(1-\alpha_{kj}^{*}\right)\left(2t-\mu_{kj}\right)+\varphi_{kj}^{-}-\varphi_{kj}^{+} & (11-6)\\
NS\left(s_0\right)=0 & (11-7)\\
NS\left(s_{2t}\right)=1 & (11-8)\\
NS\left(s_t\right)=0.5 & (11-9)\\
NS\left(s_i\right)\in\left[(i-1-t)/2t,(i+1-t)/2t\right],\quad i=1,2,\cdots,2t-1, i\neq t & (11-10)\\
NS\left(s_{i+1}\right)-NS\left(s_i\right)\geq\lambda,\quad i=0,1,\cdots,2t-1 & (11-11)\\
p_{ij,1}^{*},q_{ij,1}^{*}\leq 3t & (11-12)\\
p_{ij,1}^{*},q_{ij,1}^{*}\geq 0 & (11-13)\\
\phi_{ij}^{+}\times\phi_{ij}^{-}=0 & (11-14)\\
\varphi_{ij}^{+}\times\varphi_{ij}^{-}=0 & (11-15)\\
\phi_{ij}^{+},\phi_{ij}^{-},\varphi_{ij}^{+},\varphi_{ij}^{-}\geq 0,\quad i<k<j & (11-16)
\end{cases} \quad . \qquad (11)$$

In Eq. (11), the objective function ensures the adjustment as small as possible. The constraints (11-1) and (11-2) respectively denote the distance between $s_{p_{ij,1}^{*}}$ and $s_{p_{ij,2}^{k*}}$, $s_{q_{i,1}^{*}}$ and $s_{q_{ij,2}^{k*}}$ based on PISs, the threshold value δ_0 ensures the adjustment LIFPR meet the acceptable additive consistency. The constraints from (11-3) to (11-6) respectively denotes the subscript values of linguistic variables $s_{p_{ij,1}^{*}}$, $s_{p_{ij,2}^{k*}}$, $s_{q_{i,1}^{*}}$ and $s_{q_{ij,2}^{k*}}$. The constraints from (11-7) to (11-10) denotes the range of numerical scales, the constraint (11-11) assures the ordered of numerical scales. And constraints (11-12) and (11-13) ensure the adjustment subscript values within $[0,2t]$. The constraints from (11-14) to (11-16) denotes adjustment variables constraints. After solving Eq. (11), we obtain the adjustment variables values ϕ_{ij}^{+}, ϕ_{ij}^{-}, φ_{ij}^{+}, φ_{ij}^{-} and the personalized numerical scales for each linguistic term set $S_c=\left\{s_\alpha \middle| \alpha\in[0,2t]\right\}$, that is, $NS(s_0)$, $NS(s_1)$, $\cdots$, $NS(s_{2t})$.

According to the adjustment variables values, the adjustment LIFPR $\tilde{R}^* = \left(\tilde{r}_{ij}^*\right)_{n\times n}$ that meets the acceptable additive consistency is determined, where:

$$\tilde{r}_{ij}^* = \left(s_{\alpha_{ij}^* u_{ij} + \left(1-\alpha_{ij}^*\right)\left(2t - v_{ij}\right) + \phi_{ij}^- - \phi_{ij}^+}, s_{\alpha_{ij}^* v_{ij} + \left(1-\alpha_{ij}^*\right)\left(2t - \mu_{ij}\right) + \varphi_{ij}^- - \varphi_{ij}^+} \right) \tag{12}$$

3.3 An algorithm for obtaining complete or acceptable additive consistent LIFPRs

On the basis of above discussion, this subsection develops an algorithm for obtaining complete or acceptable additive consistent LIFPRs. The main steps are described in figure 1 and listed as follows.

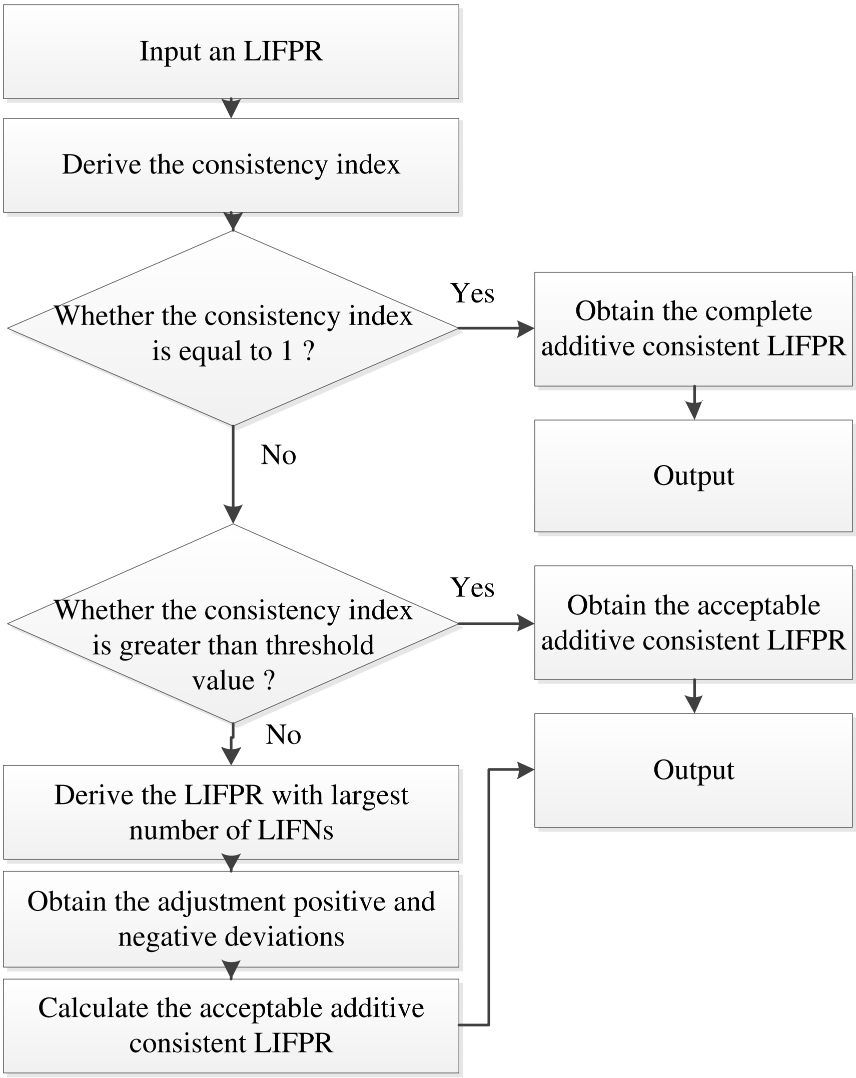

Fig.1. An algorithm for obtaining complete or acceptable additive consistent LIFPRs

Algorithm 1.

Input: An LIFPR $\tilde{R} = \left(\tilde{r}_{ij}\right)_{n\times n}$.

Output: The complete or acceptable additive consistent LIFPR $\tilde{R}^{*} = \left(\tilde{r}_{ij}^{*}\right)_{n\times n}$.

Step 1: Derive the consistency index of $\tilde{R}$.

According to Eq. (9), the objective function vale δ is obtained. On the basis of δ, there are three cases, namely:

Case 1: If $\delta = 1$, then output the complete additive consistent LIFPR.

Case 2: If $\delta_0 \le \delta < 1$, then output the acceptable additive consistent LIFPR.

Case 3: If $\delta < \delta_0$, the consistency of $\tilde{R}$ needs to be further improving, and conducts the following step.

Step 2: Derive the LIFPR with largest number of LIFNs.

According to Eq. (10), the 0-1 indicator variables α_{ij}^{*} ensure LIFPR with largest number of LIFNs is obtained, return the indicator variables α_{ij}^{*} to Eq. (3), the associated LIFPR is derived.

Step 3: Obtain the adjustment positive and negative deviations.

According to Eq. (11), deriving the adjustment positive deviation ϕ_{ij}^{+} and adjustment negative deviation ϕ_{ij}^{-} relate to the preferred qualitative degree, together with the adjustment positive deviation φ_{ij}^{+}, and adjustment negative deviation φ_{ij}^{-} relate to the non-preferred qualitative degree.

Step 4: Calculate the acceptable additive consistent LIFPR.

The acceptable additive consistent LIFPR is calculated based on Eq. (12).

To show the concrete application of the above algorithm for obtaining complete or acceptable additive consistent LIFPRs, an example from Meng et al. [43] is conducted as follows.

Example 1. Let $x = \{x_1, x_2, x_3, x_4\}$ be the given object set. The LIFPR $\tilde{R} = \left(\tilde{r}_{ij}\right)_{4\times 4}$ on x for the continuous linguistic term set $S_c = \{s_\alpha \mid \alpha \in [0,10]\}$ is defined as follows:

$$\tilde{R}=\begin{pmatrix} (s_5,s_5) & (s_3,s_6) & (s_7,s_2) & (s_6,s_2) \\ (s_6,s_3) & (s_5,s_5) & (s_4,s_5) & (s_6,s_3) \\ (s_2,s_7) & (s_5,s_4) & (s_5,s_5) & (s_2,s_4) \\ (s_2,s_6) & (s_3,s_6) & (s_4,s_2) & (s_5,s_5) \end{pmatrix}.$$

To judge the additive consistency of the LIFPR $\tilde{R}$, the following steps is conducted.

Step 1: Derive the consistency index of $\tilde{R}$.

Set $\lambda = 0.01$, according to Eq. (9), following model we have

$$\max \quad \delta$$

$$s.t.\begin{cases} 1-\frac{1}{6}\sum_{i=1}^{k-1}\sum_{k=i+1}^{j-1}\sum_{j=k+1}^{4}\left|NS\left(s_{p_{ij,1}}\right)-NS\left(s_{p_{ij,2}^{k}}\right)\right|\geq\delta \\ 1-\frac{1}{6}\sum_{i=1}^{k-1}\sum_{k=i+1}^{j-1}\sum_{j=k+1}^{4}\left|NS\left(s_{q_{ij,1}}\right)-NS\left(s_{q_{ij,2}^{k}}\right)\right|\geq\delta \\ p_{13,1}=7\alpha_{13}+(1-\alpha_{13})(10-2)+5 \\ p_{13,2}^{2}=3\alpha_{12}+(1-\alpha_{12})(10-6)+4\alpha_{23}+(1-\alpha_{23})(10-5) \\ q_{13,1}=2\alpha_{13}+(1-\alpha_{13})(10-7)+5 \\ q_{13,2}^{2}=6\alpha_{12}+(1-\alpha_{12})(10-3)+5\alpha_{23}+(1-\alpha_{23})(10-4) \\ p_{14,1}=6\alpha_{14}+(1-\alpha_{14})(10-2)+5 \\ p_{14,2}^{2}=3\alpha_{12}+(1-\alpha_{12})(10-6)+6\alpha_{24}+(1-\alpha_{24})(10-3) \\ q_{14,1}=2\alpha_{14}+(1-\alpha_{14})(10-6)+5 \\ q_{14,2}^{2}=6\alpha_{12}+(1-\alpha_{12})(10-3)+3\alpha_{24}+(1-\alpha_{24})(10-6) \\ p_{14,2}^{3}=7\alpha_{13}+(1-\alpha_{13})(10-2)+2\alpha_{34}+(1-\alpha_{34})(10-4) \\ q_{14,2}^{3}=2\alpha_{13}+(1-\alpha_{13})(10-7)+4\alpha_{34}+(1-\alpha_{34})(10-2) \\ p_{24,1}=6\alpha_{24}+(1-\alpha_{24})(10-3)+5 \\ p_{24,2}^{3}=4\alpha_{23}+(1-\alpha_{23})(10-5)+2\alpha_{34}+(1-\alpha_{34})(10-4) \\ q_{24,1}=3\alpha_{24}+(1-\alpha_{24})(10-6)+5 \\ q_{24,2}^{3}=5\alpha_{23}+(1-\alpha_{23})(10-4)+4\alpha_{34}+(1-\alpha_{34})(10-2) \\ NS(s_0)=0 \\ NS(s_{10})=1 \\ NS(s_5)=0.5 \\ NS(s_i)\in\left[(i-1-5)/10,(i+1-5)/10\right], \quad i=1,2,\cdots,9, i\neq 5 \\ NS(s_{i+1})-NS(s_i)\geq 0.01, \quad i=0,1,\cdots,9 \\ \alpha_{ij}=0\vee 1, \quad i,j=1,2,\cdots,4, i<j \end{cases}$$

By solving the above model, we have $\delta = 0.93$. If we set the threshold value $\delta_0 = 0.9$, since the relationship $\delta > \delta_0$ hold, $\tilde{R}$ is the acceptable additive consistent LIFPR..

4. A framework of MCDM procedure with LIFPRs

In this section, the MCDM problems with LIFPRs are firstly introduced, and then an optimization model is constructed for determining the weight vectors of criteria, which is taken into accounted PISs. Finally, a framework of MCDM procedure with LIFPRs is introduced.

4.1 The MCDM problems with LIFPRs

The MCDM problems involve n alternatives denoted as $A = \{a_1, a_2, \cdots, a_n\}$. Each alternative is assessed based on several feature criteria, denoted as $C = \{c_1, c_2, \cdots, c_m\}$ and $\omega = (\omega_1, \omega_2, \cdots, \omega_m)$ is the criteria' weight vector. We assume that the weights of criteria are completely unknown. The moderator provides the evaluation of alternative a_i, $i = 1, 2, \cdots, n$ under criterion c_j, $j = 1, 2, \cdots, m$, and denotes as $\tilde{R} = \left(\tilde{r}_{ij}\right)_{n \times m}$, which are LIFNs for the continuous linguistic term set $S_c = \{S_\alpha | \alpha \in [0, 2t]\}$. Besides, an expert term is invited to comment these n alternatives for each criterion using LIFPRs. Let $\tilde{R}^{c_l} = \left(\tilde{r}_{ij}^{c_l}\right)_{n \times n}$, $l = 1, 2, \cdots, m$ denote the LIFPRs on A for the continuous linguistic term set. The objective of the decision-making process is to find out the best choice, which utilizes the evaluation matrix $\tilde{R}$ and pairwise judgment matrices $\tilde{R}^{c_l}$, $l = 1, 2, \cdots, m$.

4.2 Calculate the weight vectors of the criteria

In the above-mentioned MCDM problems, the integrated pairwise judgment information is needed to derive best choice. For that, the development of the method to determine the weight vectors of the criteria is necessary. There are two stages including, namely the first stage is calculated the distance between any two LIFNs, and the second stage is obtained the weight vectors of the criteria, which based on the proposed distance measure.

Stage 1. Calculate the distance between any two LIFNs

The distance measure is a classical topic in the fuzzy set theory. As for LIFNs, Peng et al. [50] developed the Hamming distance utilized 2-tuples, and Zhang et al. [18] in view of linguistic scale functions introduced the generalized distance. Although some of distance measures have been proposed and successfully applied to the decision-making problems, it is found that there still exists some shortcoming where the PISs is seldom considered in the existing distance measures. Based on this fact, the new distance measure is developed as follows.

Definition 9. Let $\tilde{s}_1 = \left(s_{\alpha_1}, s_{\beta_1}\right)$ and $\tilde{s}_2 = \left(s_{\alpha_2}, s_{\beta_2}\right)$ be any two LIFNs. The Hamming distance between $\tilde{s}_1$ and $\tilde{s}_2$ is defined as follows:

$$D\left(\tilde{s}_1, \tilde{s}_2\right) = \frac{1}{2}\left(\left|NS\left(s_{\alpha_1}\right) - NS\left(s_{\alpha_2}\right)\right| + \left|NS\left(s_{\beta_1}\right) - NS\left(s_{\beta_2}\right)\right|\right), \tag{13}$$

where NS is the number scale function on $S_c = \left\{S_\alpha \middle| \alpha \in [0, 2t]\right\}$, and $NS(s_0)$, $NS(s_1)$, $\cdots$, $NS(s_{2t})$ is the PISs of the number scale function for a decision maker.

To derive the PISs of linguistic terms, the following model is developed:

$$\min \quad D\left(\tilde{s}_1, \tilde{s}_2\right) = \frac{1}{2}\left(\left|NS\left(s_{\alpha_1}\right) - NS\left(s_{\alpha_2}\right)\right| + \left|NS\left(s_{\beta_1}\right) - NS\left(s_{\beta_2}\right)\right|\right)$$

$$s.t. \begin{cases} NS(s_0) = 0 \\ NS(s_{2t}) = 1 \\ NS(s_t) = 0.5 \\ NS(s_i) \in \left[(i-1)/2t, (i+1)/2t\right], \quad i = 1, 2, \cdots, 2t-1, i \neq t \\ NS(s_{i+1}) - NS(s_i) \geq \lambda, \quad i = 0, 1, \cdots, 2t-1 \end{cases}. \tag{14}$$

To show the concrete application of the proposed distance measure, the following example is conducted.

Example 2. $\tilde{s}_1 = (s_3, s_6)$ and $\tilde{s}_2 = (s_2, s_4)$ be any two LIFNs the continuous linguistic term set $S_c = \left\{s_\alpha \middle| \alpha \in [0, 10]\right\}$. The Hamming distance between $\tilde{s}_1$ and $\tilde{s}_2$ is calculated as follows:

Set $\lambda = 0.01$, according to Eq. (14), following model we have

$$\min \quad D\left(\tilde{s}_1, \tilde{s}_2\right) = \frac{1}{2}\left(\left|NS(s_3) - NS(s_2)\right| + \left|NS(s_6) - NS(s_4)\right|\right)$$

$$s.t. \begin{cases} NS(s_0) = 0 \\ NS(s_{10}) = 1 \\ NS(s_5) = 0.5 \\ NS(s_2) \in [1/10, 3/10] \\ NS(s_3) \in [2/10, 4/10] \\ NS(s_4) \in [3/10, 5/10] \\ NS(s_6) \in [5/10, 7/10] \\ NS(s_6) - NS(s_5) \geq 0.01 \\ NS(s_5) - NS(s_4) \geq 0.01 \\ NS(s_4) - NS(s_3) \geq 0.01 \\ NS(s_3) - NS(s_2) \geq 0.01 \end{cases}$$

By solving the above model, we have $D\left(\tilde{s}_1, \tilde{s}_2\right) = 0.015$.

The proposed Hamming distance has the following property.

Property 1. Let $\tilde{s}_1 = \left(s_{\alpha_1}, s_{\beta_1}\right)$, $\tilde{s}_2 = \left(s_{\alpha_2}, s_{\beta_2}\right)$ and $\tilde{s}_3 = \left(s_{\alpha_3}, s_{\beta_3}\right)$ be any three LIFNs. The Hamming distance defined above has the following properties:

(1) $0 \leq D\left(\tilde{s}_1, \tilde{s}_2\right) \leq 1$;

(2) $D\left(\tilde{s}_1, \tilde{s}_2\right) = D\left(\tilde{s}_2, \tilde{s}_1\right)$;

(3) If $\tilde{s}_1 \leq \tilde{s}_2 \leq \tilde{s}_3$, then $D\left(\tilde{s}_1, \tilde{s}_3\right) \geq D\left(\tilde{s}_1, \tilde{s}_2\right)$ and $D\left(\tilde{s}_1, \tilde{s}_3\right) \geq D\left(\tilde{s}_2, \tilde{s}_3\right)$.

Proof: Obviously, the Hamming distance $D\left(\tilde{s}_1, \tilde{s}_2\right)$ satisfies (1) and (2) of Property 1, the proof of them are omitted, and the proof of (3) of is provided as follows.

If $\tilde{s}_1 \leq \tilde{s}_2 \leq \tilde{s}_3$, then $s_{\alpha_1} \leq s_{\alpha_2} \leq s_{\alpha_3}$ and $s_{\beta_1} \geq s_{\beta_2} \geq s_{\beta_3}$. Meanwhile NS is a strictly monotonically increasing and continuous function. Therefore, the following inequalities can be obtained:

$NS\left(s_{\alpha_1}\right) \leq NS\left(s_{\alpha_2}\right) \leq NS\left(s_{\alpha_3}\right)$ and $NS\left(s_{\beta_1}\right) \geq NS\left(s_{\beta_2}\right) \geq NS\left(s_{\beta_3}\right)$.

Then

$\left|NS\left(s_{\alpha_1}\right) - NS\left(s_{\alpha_3}\right)\right| \geq \left|NS\left(s_{\alpha_2}\right) - NS\left(s_{\alpha_3}\right)\right|$ and $\left|NS\left(s_{\beta_1}\right) - NS\left(s_{\beta_3}\right)\right| \geq \left|NS\left(s_{\beta_2}\right) - NS\left(s_{\beta_3}\right)\right|$.

Thus

$$\frac{1}{2}\left(\left|NS\left(s_{\alpha_1}\right)-NS\left(s_{\alpha_3}\right)\right|+\left|NS\left(s_{\beta_1}\right)-NS\left(s_{\beta_3}\right)\right|\right)\geq\frac{1}{2}\left(\left|NS\left(s_{\alpha_2}\right)-NS\left(s_{\alpha_3}\right)\right|+\left|NS\left(s_{\beta_2}\right)-NS\left(s_{\beta_3}\right)\right|\right).$$

Therefore, $D\left(\tilde{s}_1,\tilde{s}_3\right)\geq D\left(\tilde{s}_2,\tilde{s}_3\right)$ is true, and $D\left(\tilde{s}_1,\tilde{s}_3\right)\geq D\left(\tilde{s}_1,\tilde{s}_2\right)$ can be proven in a similar manner.

Stage 2. Obtain the weight vectors of the criteria

In the above-mentioned MCDM problems, the moderator provides the evaluation matrix $\tilde{R}=\left(\tilde{r}_{ij}\right)_{n\times m}$, where $\tilde{r}_{ij}$ denotes the evaluation of alternative a_i, $i=1,2,\ldots,n$ under criterion c_j, $j=1,2,\ldots,m$. In following section, motivated by the maximizing deviation method [51], a maximizing Hamming distance deviation is developed to obtain the criteria weight vectors under linguistic intuitionistic fuzzy environments.

First, the Hamming distance between the criteria c_j and other criteria c_g, $g=1,2,\ldots,m$, $g\neq j$ with respective to the alternative a_i is calculated as follows:

$$D_{ij}=\sum_{g=1,g\neq j}^{m}\frac{1}{2}\left(\left|NS\left(s_{\alpha_{ij}}\right)-NS\left(s_{\alpha_{ig}}\right)\right|+\left|NS\left(s_{\beta_{ij}}\right)-NS\left(s_{\beta_{ig}}\right)\right|\right),\tag{15}$$

where $i=1,2,\ldots,n$ and $j=1,2,\ldots,m$.

Second, the distance between alternative a_i and other alternatives a_j $j=1,2,\ldots,n$, with respective to the criteria c_j is derived as follows:

$$D_j=\sum_{i=1}^{n}\sum_{g=1,g\neq j}^{m}\frac{1}{2}\left(\left|NS\left(s_{\alpha_{ij}}\right)-NS\left(s_{\alpha_{ig}}\right)\right|+\left|NS\left(s_{\beta_{ij}}\right)-NS\left(s_{\beta_{ig}}\right)\right|\right),\tag{16}$$

where $j=1,2,\ldots,m$.

Third, the weighted distance function is then constructed:

$$D(w)=\sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{g=1,g\neq j}^{m}\frac{1}{2}\left(\left|NS\left(s_{\alpha_{ij}}\right)-NS\left(s_{\alpha_{ig}}\right)\right|+\left|NS\left(s_{\beta_{ij}}\right)-NS\left(s_{\beta_{ig}}\right)\right|\right)\omega_j,\tag{17}$$

where ω_j is the weight vectors of criteria.

Next, the following optimization model for computing the optimal weight vectors of criteria is constructed as follows:

$$
\begin{aligned}
&D(w)=\sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{g=1,g\neq j}^{m}\frac{1}{2}\left(\left|NS\left(s_{\alpha_{ij}}\right)-NS\left(s_{\alpha_{ig}}\right)\right|+\left|NS\left(s_{\beta_{ij}}\right)-NS\left(s_{\beta_{ig}}\right)\right|\right)\omega_j \\
&s.t.\quad \sum_{j=1}^{m}\omega_j=1,\quad \omega_j\geq 0,\quad j=1,2,\cdots,m
\end{aligned}
\quad . \tag{18}
$$

The Lagrange function is constructed to obtain the solution of Eq. (18) :

$$
L(w,\eta)=\sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{g=1,g\neq j}^{m}\frac{1}{2}\left(\left|NS\left(s_{\alpha_{ij}}\right)-NS\left(s_{\alpha_{ig}}\right)\right|+\left|NS\left(s_{\beta_{ij}}\right)-NS\left(s_{\beta_{ig}}\right)\right|\right)\omega_j+\eta\left(\sum_{j=1}^{m}\omega_j-1\right). \tag{19}
$$

The criteria weight vector is derived by solving the Lagrange function:

$$
\omega_j=\frac{\sum_{i=1}^{n}\sum_{g=1,g\neq j}^{m}\frac{1}{2}\left(\left|NS\left(s_{\alpha_{ij}}\right)-NS\left(s_{\alpha_{ig}}\right)\right|+\left|NS\left(s_{\beta_{ij}}\right)-NS\left(s_{\beta_{ig}}\right)\right|\right)}{\sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{g=1,g\neq j}^{m}\frac{1}{2}\left(\left|NS\left(s_{\alpha_{ij}}\right)-NS\left(s_{\alpha_{ig}}\right)\right|+\left|NS\left(s_{\beta_{ij}}\right)-NS\left(s_{\beta_{ig}}\right)\right|\right)}. \tag{20}
$$

where $j=1,2,\cdots,m$.

4.3 A framework of MCDM procedure with LIFPRs

The proposed decision-making procedure is summarized in the following steps.

Step 1: Form pairwise judgment matrices.

According to the determine criteria and alternatives, the expert team provides their comment on these n alternatives for each criterion in C using LIFPRs, and denotes as $\tilde{R}^{c_l}=\left(\tilde{r}_{ij}^{c_l}\right)_{n\times n}$, $l=1,2,\cdots,m$.

Step 2: Check and improve the consistency of LIFPRs.

With respective to algorithm 1, the complete or acceptable additive consistent LIFPRs are obtained, which is denoted by $\tilde{R}^{*,c_l}=\left(\tilde{r}_{ij}^{*,c_l}\right)_{n\times n}$, $l=1,2,\cdots,m$, and the corresponding PISs of number scale function $NS\left(s_{\tilde{r}_{ij}^{*,c_l}}\right)$, $l=1,2,\cdots,m$ is derived.

Step 3: Determine the weight vectors of criteria.

The weight vectors of criteria are determined according to Eq. (20).

Step 4: Compute the collective number scale function.

The collective number scale function is determined by the following formula:

$$\tilde{\vartheta}_i = \frac{1}{n}\sum_{j=1}^{n}\sum_{l=1}^{m}\omega^{c_l} NS\left(s_{\tilde{r}_{ij}^{*,c_l}}\right),\ \ i = 1,2,\cdots,n, \tag{21}$$

where ω^{c_l} is the weight vector of criteria, which is determined in Step 3.

Step 5: Calculate the score and accuracy functions of collective number scale function.

The score function of collective scale function is determined by the following formula:

$$LS\left(\tilde{\vartheta}_i\right) = NS\left(s_{\mu_i}\right) - NS\left(s_{v_i}\right),\ \ i = 1,2,\cdots,n. \tag{22}$$

The accuracy function of collective scale function is determined by the following formula:

$$LH\left(\tilde{\vartheta}_i\right) = NS\left(s_{\mu_i}\right) + NS\left(s_{v_i}\right),\ \ i = 1,2,\cdots,n. \tag{23}$$

Step 6: Rank the alternatives.

The ranking order of all alternatives is obtained by the value of score and accuracy functions of collective number scale function. The order relationship is defined as follows:

$$\begin{array}{l} \text{If } LS\left(\tilde{\vartheta}_i\right) > LS\left(\tilde{\vartheta}_j\right), \text{ then } \tilde{\vartheta}_i \succ \tilde{\vartheta}_j \\ \text{If } LS\left(\tilde{\vartheta}_i\right) = LS\left(\tilde{\vartheta}_j\right), \text{ then } \begin{cases} \text{If } LH\left(\tilde{\vartheta}_i\right) > LH\left(\tilde{\vartheta}_j\right), \text{ then } \tilde{\vartheta}_i \prec \tilde{\vartheta}_j \\ \text{If } LH\left(\tilde{\vartheta}_i\right) = LH\left(\tilde{\vartheta}_j\right), \text{ then } \tilde{\vartheta}_i \sim \tilde{\vartheta}_j \end{cases} \end{array}. \tag{24}$$

The proposed decision-making procedure is depicted in Fig. 2.

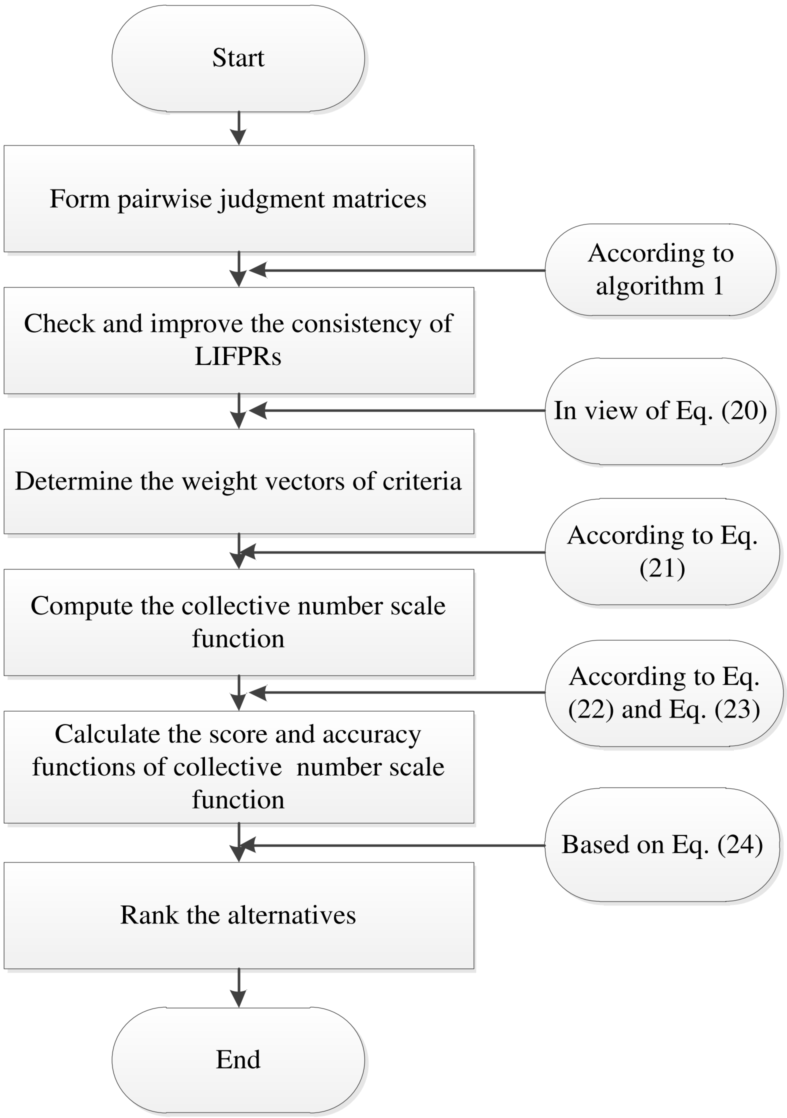

Fig.2. The proposed framework of decision-making process with LIFPRs

5. Illustrative example

In this section, evaluation of different brands of mobile phones (revised from) is provided to illustrate the application of the proposed method, and conjunction with comparative analysis is conducted.

Since the Motorola Corporation invented the first mobile phone in 1983. After decades of development, the function of mobile phones has undergone tremendous changes and the mobile phones have become one of the most important daily necessities. In the first half of 2016, the sales of mobile phones exceed 250 million in China. According to the sales, there are four major brands of mobile phones, including: (1) a_1 HUAWEI; (2) a_2 OPPO; (3) a_3 APPLE;

and (4) a_4 VIVO. In the above list brands of mobile phones, HUAWEI, OPPO and VIVO are made in China, while APPLE is made in America. There are different characteristics of each brand of mobile phones. However, several factors should be considered when we evaluate these four brands of mobile phones, such as: (1) c_1 appearance; (2) c_2 price; (3) c_3 performance; (4) c_4 quality, and (5) c_5 reputation. To purchase a number of cost-effective mobile phones, a business company invited an expert team to evaluate these four brands of mobile phones. To fully express the recognitions of the experts, they are permitted to apply linguistic variables in the predefined linguistic term set { s_0 : extremely bad; s_1 : very bad; s_2 : bad; s_3 : relatively bad; s_4 : a little bad; s_5 : fair; s_6 : a little good; s_7 : relatively good; s_8 : good; s_9 : very good; s_{10} : extremely good }. Furthermore, the experts are allowed to express the preferred and non-preferred opinions for each pair of brands of mobile phones. With respective to these four brands of mobile phones for each criterion, LIFPRs are listed in matrices $\tilde{R}^{c_l}$, $l = 1, 2, \ldots, 5$, and the moderator provides the evaluation of alternatives with LIFNs, which are listed in matrix $\tilde{R}$.

Take the evaluation values (s_3, s_6) from matrix $\tilde{R}^{c_1}$ for example. The expert term provides the preferred value s_3 and non-preferred value s_6 when assesses the brands of mobile phones HUAWEI to OPPO, and cannot determine exact numerals. In such case, the evaluation value can be modeled by a LIFPR (s_3, s_6). Other entries, that is, LIFPRs, in matrices $\tilde{R}^{c_l}$, $l = 1, 2, \ldots, 5$ are similarly explained.

$$\tilde{R}^{c_1} = \begin{pmatrix} (s_5,s_5) & (s_3,s_6) & (s_3,s_7) & (s_4,s_6) \\ (s_6,s_3) & (s_5,s_5) & (s_4,s_5) & (s_7,s_2) \\ (s_7,s_3) & (s_5,s_4) & (s_5,s_5) & (s_7,s_3) \\ (s_6,s_4) & (s_2,s_7) & (s_3,s_7) & (s_5,s_5) \end{pmatrix}, \quad \tilde{R}^{c_2} = \begin{pmatrix} (s_5,s_5) & (s_6,s_4) & (s_7,s_3) & (s_5,s_4) \\ (s_4,s_6) & (s_5,s_5) & (s_6,s_4) & (s_4,s_5) \\ (s_3,s_7) & (s_4,s_6) & (s_5,s_5) & (s_3,s_6) \\ (s_4,s_5) & (s_5,s_4) & (s_6,s_3) & (s_5,s_5) \end{pmatrix},$$

$$\tilde{R}^{c_3} = \begin{pmatrix} (s_5,s_5) & (s_4,s_5) & (s_3,s_6) & (s_6,s_4) \\ (s_5,s_4) & (s_5,s_5) & (s_4,s_5) & (s_6,s_3) \\ (s_6,s_3) & (s_5,s_4) & (s_5,s_5) & (s_7,s_2) \\ (s_4,s_6) & (s_3,s_6) & (s_2,s_7) & (s_5,s_5) \end{pmatrix}, \quad \tilde{R}^{c_4} = \begin{pmatrix} (s_5,s_5) & (s_2,s_8) & (s_3,s_6) & (s_3,s_5) \\ (s_8,s_2) & (s_5,s_5) & (s_5,s_5) & (s_6,s_3) \\ (s_6,s_3) & (s_5,s_5) & (s_5,s_5) & (s_7,s_2) \\ (s_5,s_3) & (s_3,s_6) & (s_2,s_7) & (s_5,s_5) \end{pmatrix}, \quad \text{and}$$

$$\tilde{R}^{c_5} = \begin{pmatrix} (s_5,s_5) & (s_3,s_6) & (s_2,s_8) & (s_4,s_5) \\ (s_6,s_3) & (s_5,s_5) & (s_4,s_6) & (s_6,s_4) \\ (s_8,s_2) & (s_6,s_4) & (s_5,s_5) & (s_7,s_2) \\ (s_5,s_4) & (s_4,s_6) & (s_2,s_7) & (s_5,s_5) \end{pmatrix}. \quad \tilde{R} = \begin{pmatrix} (s_8,s_2) & (s_7,s_3) & (s_5,s_4) & (s_8,s_2) & (s_6,s_3) \\ (s_6,s_3) & (s_6,s_3) & (s_7,s_2) & (s_7,s_3) & (s_8,s_2) \\ (s_7,s_2) & (s_6,s_4) & (s_8,s_2) & (s_6,s_2) & (s_4,s_5) \\ (s_6,s_3) & (s_8,s_2) & (s_5,s_4) & (s_7,s_2) & (s_5,s_5) \end{pmatrix}.$$

5.1 Illustration of the proposed method

The procedure for evaluating different brands of mobile phones using the proposed method is showed below.

Step 1: Form pairwise judgment matrices.

All pairwise judgment matrices for each criterion in C have been provided, as demonstrated in matrices 1-5.

Step 2: Check and improve the consistency of LIFPRs.

To check and improve the consistency of LIFPRs, algorithm 1 is developed in this subsection.

First, deriving the consistency index of $\tilde{R}^{c_l} = \left(\tilde{r}_{ij}^{c_l}\right)_{n\times n}$, $l = 1,2,\ldots,5$.

Suppose we set $\lambda = 0.01$ and $\delta_0 = 0.9$, for LIFPR $\tilde{R}^{c_1}$, by solving Eq. (9), we have: $\delta = 0.998$. Since $\delta > \delta_0$, then $\tilde{R}^{c_1}$ is acceptable additive consistent LIFPR, and the corresponding PISs of number scale function is obtained, which is listed in $NS\left(s_{\tilde{r}_{ij}^{*,c_1}}\right)$.

Similar, for LIFPR $\tilde{R}^{c_2}$, by solving Eq. (9), we have: $\delta = 0.983$. Since $\delta > \delta_0$, then $\tilde{R}^{c_2}$ is acceptable additive consistent LIFPR, and the corresponding PISs of number scale function is obtained, which is listed in $NS\left(s_{\tilde{r}_{ij}^{*,c_2}}\right)$.

In a same way, for LIFPR $\tilde{R}^{c_3}$, by solving Eq. (9), we have: $\delta = 1$. Since $\delta > \delta_0$, then $\tilde{R}^{c_3}$ is complete additive consistent LIFPR, and the corresponding PISs of number scale function is obtained, which is listed in $NS\left(s_{\tilde{r}_{ij}^{*,c_3}}\right)$.

Similar, for LIFPR $\tilde{R}^{c_4}$, by solving Eq. (9), we have: $\delta = 0.98$. Since $\delta > \delta_0$, then $\tilde{R}^{c_4}$ is acceptable additive consistent LIFPR, and the corresponding PISs of number scale function is obtained, which is listed in $NS\left(s_{\tilde{r}_{ij}^{*,c_4}}\right)$.

In a same way, for LIFPR $\underset{\sim}{R}^{c_5}$, by solving Eq. (9), we have: $\delta = 1$. Since $\delta > \delta_0$, then $\underset{\sim}{R}^{c_5}$ is complete additive consistent LIFPR, and the corresponding PISs of number scale function is obtained, which is listed in $NS\left(s_{\underset{\sim}{r}_{ij}^{*,c_5}}\right)$.

$$NS\left(s_{\underset{\sim}{r}_{ij}^{*,c_1}}\right)=\begin{pmatrix}(0.50,0.50) & (0.40,0.51) & (0.40,0.80) & (0.49,0.51)\\(0.51,0.40) & (0.50,0.50) & (0.49,0.50) & (0.79,0.30)\\(0.80,0.40) & (0.50,0.49) & (0.50,0.50) & (0.79,0.40)\\(0.51,0.49) & (0.30,0.79) & (0.40,0.79) & (0.50,0.50)\end{pmatrix},$$

$$NS\left(s_{\underset{\sim}{r}_{ij}^{*,c_2}}\right)=\begin{pmatrix}(0.50,0.50) & (0.60,0.41) & (0.80,0.40) & (0.50,0.41)\\(0.41,0.60) & (0.50,0.50) & (0.60,0.41) & (0.41,0.50)\\(0.40,0.80) & (0.41,0.60) & (0.50,0.50) & (0.40,0.60)\\(0.41,0.50) & (0.50,0.41) & (0.60,0.40) & (0.50,0.50)\end{pmatrix},$$

$$NS\left(s_{\underset{\sim}{r}_{ij}^{*,c_3}}\right)=\begin{pmatrix}(0.50,0.50) & (0.49,0.50) & (0.48,0.79) & (0.70,0.49)\\(0.50,0.49) & (0.50,0.50) & (0.49,0.50) & (0.72,0.30)\\(0.79,0.48) & (0.50,0.49) & (0.50,0.50) & (0.80,0.20)\\(0.49,0.70) & (0.30,0.72) & (0.20,0.80) & (0.50,0.50)\end{pmatrix},$$

$$NS\left(s_{\underset{\sim}{r}_{ij}^{*,c_4}}\right)=\begin{pmatrix}(0.50,0.50) & (0.20,0.90) & (0.30,0.79) & (0.40,0.50)\\(0.90,0.20) & (0.50,0.50) & (0.50,0.50) & (0.70,0.30)\\(0.79,0.30) & (0.50,0.50) & (0.50,0.50) & (0.80,0.20)\\(0.50,0.40) & (0.30,0.70) & (0.20,0.80) & (0.50,0.50)\end{pmatrix},$$ and

$$NS\left(s_{\underset{\sim}{r}_{i}^{*,c_5}}\right)=\begin{pmatrix}(0.50,0.50) & (0.30,0.70) & (0.29,0.76) & (0.40,0.50)\\(0.70,0.30) & (0.50,0.50) & (0.40,0.70) & (0.70,0.40)\\(0.76,0.29) & (0.70,0.40) & (0.50,0.50) & (0.71,0.29)\\(0.50,0.40) & (0.40,0.70) & (0.29,0.71) & (0.50,0.50)\end{pmatrix}.$$

Step 3: Determine the weight vectors of criteria.

The weight vectors of criteria are determined according to Eq. (20): $\omega^{c_1} = 0.15$, $\omega^{c_2} = 0.16$, $\omega^{c_3} = 0.23$, $\omega^{c_4} = 0.15$ and $\omega^{c_5} = 0.30$.

Step 4: Compute the collective number scale function.

The collective number scale function are determined according to Eq. (21): $\tilde{\vartheta}_1 = (0.45, 0.57)$, $\tilde{\vartheta}_2 = (0.56, 0.45)$, $\tilde{\vartheta}_3 = (0.61, 0.43)$ and $\tilde{\vartheta}_4 = (0.41, 0.59)$.

Step 5: Calculate the score and accuracy functions of collective number scale function.

According to Eq. (22), the score function of collective scale function is determined: $LS\left(\tilde{\vartheta}_1\right) = -0.12$, $LS\left(\tilde{\vartheta}_2\right) = 0.11$, $LS\left(\tilde{\vartheta}_3\right) = 0.18$ and $LS\left(\tilde{\vartheta}_4\right) = -0.18$.

According to Eq. (23), the accuracy function of collective scale function is determined: $LH\left(\tilde{\vartheta}_1\right) = 1.02$,

$LH\left(\tilde{\vartheta}_2\right) = 1.01$, $LH\left(\tilde{\vartheta}_3\right) = 1.04$ and $LH\left(\tilde{\vartheta}_4\right) = 1.00$.

Step 6: Rank the alternatives.

Since $LS\left(\tilde{\vartheta}_3\right) > LS\left(\tilde{\vartheta}_2\right) > LS\left(\tilde{\vartheta}_1\right) > LS\left(\tilde{\vartheta}_4\right)$, then $\tilde{\vartheta}_3 \succ \tilde{\vartheta}_2 \succ \tilde{\vartheta}_1 \succ \tilde{\vartheta}_4$ based on Eq. (24), by which we have $\text{APPLE} \succ \text{OPPO} \succ \text{HUAWEI} \succ \text{VIVO}$.

5.2 Comparative analysis and discussion

To validate the feasibility of the proposed method, we conducted a comparative study with other method based on the same illustrative example.

Meng et al. [43] first proposed the concept of LIFPR, and then introduced the concept of additive consistent LIFPR. Moreover, to obtain the complete consistent LIFPR, several goal programming models are developed based on additive consistency. Finally, these goal programming models have been extended to incomplete LIFPR. To better comparison, the results obtained by Meng et al. [43]'s method and the proposed method are summarized in Table 2. The detailed calculation process of Meng et al. [43]'s method can be found in [43].

Table 2: The ranking results derive from different methods

Methods	Ranking values				Ranking results
	$LS\left(\tilde{\vartheta}_1\right)$	$LS\left(\tilde{\vartheta}_2\right)$	$LS\left(\tilde{\vartheta}_3\right)$	$LS\left(\tilde{\vartheta}_4\right)$	
Meng et al. [43]'s method	-0.9489	1.5517	1.6875	-1.3501	$\tilde{\vartheta}_3 \succ \tilde{\vartheta}_2 \succ \tilde{\vartheta}_1 \succ \tilde{\vartheta}_4$
The proposed method	-0.12	0.11	0.18	-0.18	$\tilde{\vartheta}_3 \succ \tilde{\vartheta}_2 \succ \tilde{\vartheta}_1 \succ \tilde{\vartheta}_4$

As shown in Table 1, it can be easily found that the best alternative obtained from Meng et al. [43]'s method is the same as the proposed method, and the ranking results are also the same. This also confirms the effectiveness of the proposed method. Although these two methods both consider the additive consistency checking and improving processes, and taken into accounted the weight vectors of criteria. They are some difference. First, in linguistic decision-making, computing with words is an important point to note about. Meng et al. [43]'s method conducted it with subscript values, while the proposed method utilized PISs model. Based on the fact that words mean different things for different people, the proposed method utilized PISs model seems more reasonable. Second, consistency of preference relations is related to rationality. By comparison, inconsistent preference relations often lead to misleading solutions. Meng et al. [43]'s method only developed the complete consistency checking and improving process, while the proposed method not only considers the complete consistency, but also the acceptable consistency. In the actual decision-making process, it is difficult to obtain complete consistent LIFPR and it's not necessary to obtain it sometimes. On the basis of above analysis, acceptable additive consistent LIFPRs are available. For that, the proposed method has a wider background application. Moreover, the objective functions are different when considers the consistency checking process. In Meng et al. [43]'s method, the objective functions constructed based on minimizing the deviation from the target of the goal. However, the proposed method focuses on maximizing the parameter of satisfaction degree. The different perspectives for solving the problems lead to different decision-making results, but the proposed method takes the decision-makers' satisfaction degree into account, this is more suitable for solving decision-making problems in some backgrounds.

To verify the advantages of our approaches, we compare them with several representative models under the MCDM environment with LIFPRs. Table 3 presents the performances of these approaches regarding several indexes.

(1) Meng et al. [52]'s method: This method derived priority weights of alternatives based on two stages strategy. The first stage is estimating the missing elements in LIFPRs based on the properties of multiplicative consistent LIFPRs, and the second stage is deriving the priority weights based on complete LIFPRs. The method developed these processes in view of the subscript values of linguistic variables, and does not taken into accounted the psychological characteristics of decision-makers. Compared with Meng et al. [52]'s method, the proposed method utilized PISs model to computing with words and taken into accounted the psychological characteristics of decision-makers. The proposed method has advantage in representing the specific semantics of each individual.

(2) Wan et al. [53]'s method. This method studied the consensus reaching process with LIFNs. The process included two-stage consensus reaching method. The LIFNs evaluation values with high linguistic indeterminacy degrees are

modified in the first stage, and evaluation elements with high deviation elements are modified in the second stage. t The method developed these stages in view of the 2-tuple of linguistic variables, and does not taken into accounted the psychological characteristics of decision-makers. Compared with Wan et al. [53]'s method, the proposed method utilized PISs model to computing with words, and obtain the priority weight vectors by taking into account decision-makers' satisfaction degree. On account of these, the proposed method has advantage in avoiding the loss of individual information and considering the psychological characteristics of decision-makers.

(3) Jin et al. [54]'s method. This method developed a decision support model that simultaneously considered the individual consistency and group consensus for group decision-making with LIFPRs. The method defined the concept of multiplicative consistency of LIFPRs directly apply the concept of multiplicative consistency of linguistic fuzzy preference relations. As Meng et al. [55] noted, issues may exist as that for intuitionistic fuzzy preference relations. Compared with Jin et al. [54]'s method, the proposed method considered all the cases corresponds to the consistency LIFPRs. In view of these, the proposed method has advantage in avoiding the loss of information and the calculation process seems more reasonable.

Table 3: Comparisons of different MCDM approaches with LIFPRs

Method	Consistency type	Priority weights determination method	The method of computing with words	Whether consider the psychological characteristics
Meng et al. [52]'s method	multiplicative consistency	Based on complete multiplicative consistent LIFPR	Subscript values	No
Wan et al. [53]'s method	Not involved	Consensus reaching process	2-tuple	No
Jin et al. [54]'s method	multiplicative consistency	Based on complete or acceptable multiplicative consistent LIFPR and consensus reaching process	Subscript values	No
The proposed method	additive consistency	Based on complete or acceptable additive consistent LIFPR	PISs model	Yes

According to the comparison analysis, the method proposed in this study has the following advantages over other existing approaches.

(1) In linguistic decision-making, computing with words is an important point to note about. The proposed method utilized PISs model to computing with words, this ensures the proposed method has advantage in representing the specific semantics of each individual.

(2) The proposed method focuses on maximizing the parameter of satisfaction degree to construct objective functions when considers the consistency checking process. The decision-making process considers the psychological characteristics of decision-makers, and more suitable for decision-making problems in some backgrounds.

(3) The method of determining the weight vectors of criteria is developed. This method uses PISs model to computing with words.

6. Conclusion

This paper develops a PISs model for computing with linguistic intuitionistic fuzzy information and applies to evaluating different brands of mobile phones. First, a consistency-driven optimization model for checking the additive consistent LIFPRs is constructed by considering the PISs of a decision maker. Besides, several optimization models is built to determine the PISs of linguistic terms in LIFPRs and obtain the acceptable additive consistent LIFPRs. Second, a new definition of Hamming distance between LIFNs is introduced. Then, the method of deriving the weight vectors of criteria is developed based on the proposed distance measure. Subsequently, a framework of group decision-making process with LIFRs is offered, and the application of the proposed method is illustrated by evaluating different brands of mobile phones. Finally, the comparative analysis is presented to show the feasibility of the group decision-making method.

The present study provides several significant contributions for MCDM problems with LIFPRs. They are summarized as follows: (1) the proposed method utilized PISs model to computing with words, this ensures the proposed method has advantage in representing the specific semantics of each individual. (2) The proposed method focuses on maximizing the parameter of satisfaction degree to construct objective function, and more suitable for decision-making problems in some backgrounds. (3) A new definition of Hamming distance between LIFNs is introduced considering the PISs of a decision maker. In our future research, the framework of group decision-making process with LIFRs is designed by considering the consistency and consensus, and applied the proposed method to solve other practical MCDM problems.

Acknowledgement

The authors thank the anonymous reviewers and the editor for their insightful and constructive comments and suggestions that have led to an improved version of this paper. This work was supported by the National Natural Science Foundation of China (Nos.72061026), the Natural Science Foundation of Guangxi (Nos. 2020GXNSFAA297239), the Science and Technology Plan of Guangxi (gui ke AD20238006), and the National Natural Science Foundation of China (Nos.61866006).

Compliance with Ethical Standards

Conflict of Interest: Authors Jian Li, Li-li Niu, Qiongxia Chen, Feilong Li and Zhong-xing Wang declare that they have no conflict of interest.

Ethical approval: This article does not contain any studies with human participants performed by any of the authors.

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