Operation properties and (α,β)-equalities of complex intuitionistic fuzzy sets

Zengtai Gong (zt-gong@163.com)
Northwest Normal University https://orcid.org/0000-0001-5878-1506
Fangdi Wang
Northwest Normal University

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Operation properties and (α, β)-equalities of complex intuitionistic fuzzy sets *

Zengtai Gong[a†] Fangdi Wang[a,b]

[a]College of Mathematics and Statistics, Northwest Normal University, Lanzhou, Gansu, 730070, PR China

[b]Basic Subjects Department, Lanzhou Institute of Technology, Lanzhou, Gansu, 730050, PR China

Abstract: A complex intuitionistic fuzzy set is an innovative uncertainty set whose membership and non-membership functions take values in the unit circle in the complex plane. This paper investigates various operation properties and proposes a new distance measure for complex intuitionistic fuzzy sets. The distance of two complex intuitionistic fuzzy sets measures the difference between the grades of two complex intuitionistic fuzzy sets as well as that between the phases of the two complex intuitionistic fuzzy sets. This distance measure is then used to define (α, β)-equalities of complex intuitionistic fuzzy sets which coincide with those of intuitionistic fuzzy sets already defined in the literature if complex intuitionistic fuzzy sets reduce to traditional intuitionistic fuzzy sets. Two complex intuitionistic fuzzy sets are said to be (α, β)-equal if the distance between their membership degree is less than $1-\alpha$ and the distance between their non-membership degree is less than β. Meanwhile we shows how various operations between complex intuitionistic fuzzy sets affect given (α, β)-equalities of complex intuitionistic fuzzy sets. Finally, complex intuitionistic fuzzy relations are discussed and some examples are given to illuminate the results obtained in this paper.

Keywords: Complex intuitionistic fuzzy set; distance measure; (α, β)-equality; complex intuitionistic fuzzy relations; operation

1 Introduction

Atanassov [4, 5] introduced the concept of intuitionstic fuzzy set characterized by a membership function and a non-membership function, which is a generalization of fuzzy set [28]. The intuitionstic fuzzy sets overcome the restrictions of fuzzy sets in handling conflicting information concerning membership of objects and have numerous applications in modeling imprecision [17], pattern recognition [27], computational intelligence [13], decision making [25], medical diagnosis problems [26] and medical image segmentation [6, 15]. It is well known that the range of each membership function and non-membership function are limited to the interval $[0, 1]$, and their sum is also belongs to the interval $[0, 1]$, i.e., they all belong to the real numbers.

The question presented by Daniel Ramot and other researchers was that, what will be the result, if we change the co-domain in the fuzzy sets to complex numbers instead of real numbers? In 2002, Ramot et al. [23] proposed an important concept and called it complex fuzzy set, where the membership function $\mu(x)$ instead of being a real valued function with the rang of $[0, 1]$ is replaced by a complex-valued function of the form

$$\mu_S(x) = r_S(x) \cdot e^{i\omega_S(x)} \qquad (i = \sqrt{-1}),$$

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†Corresponding author: zt-gong@163.com, gongzt@nwnu.edu.cn (Z. Gong).

where $r_S(x)$ and $\omega_S(x)$ are both real valued function and $r_S(x) \in [0,1]$, $\mu_S(x)$ has a value in the range of complex unit circle. However, this concept is different from fuzzy complex number introduced and discussed by Buckley [7, 8, 9, 10] and Zhang [29]. Essentially as explained in [23] the complex fuzzy set still remains the characterization of the uncertainly through the amplitude of the grade of membership having a value in the range of $[0,1]$ whilst adding the membership phase $\omega_S(x)$. As explained in Ramot et al. [23], the key feature of complex fuzzy sets is the presence of phase and its membership. This gives those complex fuzzy sets wavelike properties which could result in constructive and destructive interference depending on the phase value. Thus property distinguishes these complex fuzzy sets from conventional fuzzy sets, fuzzy complex sets, and type 2 fuzzy sets [14, 16, 28, 29]. Several examples are given in [23] which demonstrate the utility of these complex fuzzy sets. They also define several important concepts such as the complement, union, intersection and complex fuzzy relations for such complex fuzzy sets. On the basis of Ramot's works, some researchers continued to study in the theory and applications of fuzzy complex analysis [18, 19, 30].

As the extension of the intuitionstic fuzzy sets, in 2012, Alkouri and Salleh [1] proposed a new innovative concept and called it complex intuitionistic fuzzy sets, where the membership function $\mu(x)$ and non-membership function $\nu(x)$ instead of being real valued functions with the rang of $[0,1]$ are replaced by complex-valued functions of the form

$$\mu_A(x) = r_A(x) \cdot e^{i\bar{\omega}_{\mu A}(x)} \qquad i = \sqrt{-1}$$

and

$$\nu_A(x) = s_A(x) \cdot e^{i\bar{\omega}_{\nu A}(x)} \qquad i = \sqrt{-1},$$

where $r_A(x)$ and $s_A(x)$ are real-valued functions and both belong to the interval $[0,1]$ such that $0 \leq r_A(x) + s_A(x) \leq 1$, also $\bar{\omega}_{\mu A}(x)$ and $\bar{\omega}_{\nu A}(x)$ are real-valued functions. They also discussed the basic operations on complex intuitionistic fuzzy sets, developed a formula for calculating distance among complex intuitionistic fuzzy sets and gave its application in a decision making problem [2, 3]. Rani and Garg [24] proposed a series of distance measures for complex intuitionistic fuzzy sets and applied them in pattern recognition and medical diagnosis problems.

On the other hand, with an attempt to show that "precise membership values should normally be of no practical significance", Pappis [22] introduced firstly the notion of "proximity measure". Hong and Hwang [21] then presented an important generalization. Further, Cai [11, 12] introduced and discussed δ-equalities of fuzzy sets and their properties. As the extension of the δ-equalities of fuzzy sets, the δ-equalities of complex fuzzy sets was discussed by Zhang et al [30]. Meanwhile, in 2013, Gong et al. [20] investigated the (α, β)-equalities of intuitionistic fuzzy sets by the dual triangle norms. In this paper, we build on the results obtained in Gong's paper by introducing some operations on complex intuitionistic fuzzy sets and their properties and then investigate the important concept of (α, β)-equalities which allows us to systematically develop measures of distance between, equality and similarity for complex intuitionistic fuzzy sets.

This paper is a continuing work of the papers of Alkouri and Salleh [1, 2, 3] and Gong et al. [20]. The rest of the paper is organized as follows: In Section 2, after reviewing the concept of complex intuitionistic fuzzy set, some operations of complex intuitionistic fuzzy sets are introduced, and their properties are discussed. Section 3 investigates (α, β)-equalities of complex intuitionistic fuzzy sets and discusses (α, β)-equalities for various implication operators. Complex intuitionistic fuzzy relations are discussed in Section 4 and some examples are given to illuminate the results obtained in this paper in Section 5. Conclusion is given in the Section 6.

2 Operations of complex intuitionistic fuzzy sets

Definition 2.1 ([1]) A complex intuitionistic fuzzy set A, defined on an universe of discourse U, is characterized by membership and non-membership functions $\mu_A(x)$ and $\nu_A(x)$, respectively, that assign any element $x \in U$ a complex-valued grade of both membership and non-membership in A.

By definition, the values of $\mu_A(x)$, $\nu_A(x)$, and their sum may receive all lying within the unit circle in the complex plane, and are on the form

$$\mu_A(x) = r_A(x) \cdot e^{i\bar{\omega}_{\mu A}(x)}$$

for membership function in A and

$$\nu_A(x) = s_A(x) \cdot e^{i\bar{\omega}_{\nu A}(x)}$$

for non-membership function in A, where $i = \sqrt{-1}$, each of $r_A(x)$ and $s_A(x)$ are real-valued functions and both belong to the interval $[0,1]$ such that $0 \leq r_A(x) + s_A(x) \leq 1$, also $e^{i\bar{\omega}_{\mu A}(x)}$ and $e^{i\bar{\omega}_{\nu A}(x)}$ are periodic function whose periodic law and principal period are, respectively, 2π and $0 < \omega_{\mu A}(x), \omega_{\nu A}(x) \leq 2\pi$, i.e., $\bar{\omega}_{\mu A}(x) = \omega_{\mu A}(x) + 2k\pi,\ \bar{\omega}_{\nu A}(x) = \omega_{\nu A}(x) + 2k\pi,\ k = 0, \pm 1, \pm 2, ...$, where $\omega_{\mu A}(x)$ and $\omega_{\nu A}(x)$ are the principal arguments. The principal arguments $\omega_{\mu A}(x)$ and $\omega_{\nu A}(x)$ will used in the following text.

Let $IF^\star(U)$ be the set of all complex intuitionistic fuzzy sets on U. The complex intuitionistic fuzzy set A may be represented as the set of ordered pairs

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\},$$

where $\mu_A(x) : U \to \{a | a \in C,\ |a| \leq 1\}$, $\nu_A(x) : U \to \{a^{'} | a^{'} \in C,\ |a^{'}| \leq 1\}$, and $|\mu_A(x) + \nu_A(x)| \leq 1$.

Definition 2.2 (1) A quasi-triangular norm T is a function $[0,1]^2 \times [0,1]^2 \to [0,1]^2$ that satisfies the following conditions:

(i) $T((1,1),(1,1)) = (1,1)$;

(ii) $T((a,a^{'}),(b,b^{'})) = T((b,b^{'}),(a,a^{'}))$;

(iii) $T((a,a^{'}),(b,b^{'})) \leq T((c,c^{'}),(d,d^{'}))$ whenever $a \leq c,\ a^{'} \leq c^{'}$ and $b \geq d,\ b^{'} \geq d^{'}$;

(iv) $T(T((a,a^{'}),(b,b^{'})),(c,c^{'})) = T((a,a^{'}),T((b,b^{'}),(c,c^{'})))$.

(2) A triangular norm T is a function $[0,1]^2 \times [0,1]^2 \to [0,1]^2$ that satisfies the conditions (i)-(iv) and the following condition:

(v) $T((0,0),(0,0)) = (0,0)$.

We said T is an s-norm, if a triangular norm T satisfies

(vi) $T((a,a^{'}),(0,0)) = (a,a^{'})$.

We said T is a t-norm, if a triangular norm T satisfies

(vii) $T((a,a^{'}),(1,1)) = (a,a^{'})$.

(3) We said a binary function $\bar{T}$:

$\bar{T} : IF^\star(U) \times IF^\star(U) \to IF^\star(U)$

$\bar{T}(A,B) \mapsto \langle \sup\limits_{x\in U} T_1(\mu_A(x), \mu_B(x)) \cdot e^{i \sup_{x\in U} T_2(\omega_{\mu A}(x), \omega_{\mu B}(x))}, \inf\limits_{x\in U} T_1(\nu_A(x), \nu_B(x)) \cdot e^{i \inf_{x\in U} T_2(\omega_{\nu A}(x), \omega_{\nu B}(x))}\rangle$

is a triangular norm if T_1 is a triangular norm and T_2 is a quasi-triangular norm; we said $\bar{T}$ is an s-norm if T_1 an s-norm; we said $\bar{T}$ is a t-norm if T_1 a t-norm.

Definition 2.3 (Complex Intuitionistic Fuzzy Union) Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in U\}$ be two complex intuitionistic fuzzy sets on U, $\mu_A(x) = r_A(x) \cdot e^{i\omega_{\mu A}(x)}$, $\mu_B(x) = r_B(x) \cdot e^{i\omega_{\mu B}(x)}$, $\nu_A(x) = s_A(x) \cdot e^{i\omega_{\nu A}(x)}$ and $\nu_B(x) = s_B(x) \cdot e^{i\omega_{\nu B}(x)}$ their membership and non-membership functions, respectively. The complex intuitionistic fuzzy union of A and B, denoted by $A \cup B = \{\langle x, \mu_{A\cup B}(x), \nu_{A\cup B}(x)\rangle : x \in U\}$, where

$$\mu_{A\cup B}(x) = r_{A\cup B}(x) \cdot e^{i\omega_{\mu(A\cup B)}(x)} = \max(r_A(x), r_B(x)) \cdot e^{i\max(\omega_{\mu A}(x), \omega_{\mu B}(x))} \tag{2.1}$$

and

$$\nu_{A\cup B}(x) = s_{A\cup B}(x) \cdot e^{i\omega_{\nu(A\cup B)}(x)} = \min(s_A(x), s_B(x)) \cdot e^{i\min(\omega_{\nu A}(x), \omega_{\nu B}(x))}. \tag{2.2}$$

Example 2.1 Let

$A = \frac{\langle 0.5\cdot e^{i1.2\pi}, 0.4\cdot e^{i0.8\pi}\rangle}{x} + \frac{\langle 0.4\cdot e^{i0.5\pi}, 0.6\cdot e^{i1.3\pi}\rangle}{y} + \frac{\langle 0.3\cdot e^{i2\pi}, 0.5\cdot e^{i1.5\pi}\rangle}{z}$,

$B = \frac{\langle 0.6\cdot e^{i0.2\pi}, 0.3\cdot e^{i1.8\pi}\rangle}{x} + \frac{\langle 0.2\cdot e^{i0.5\pi}, 0.6\cdot e^{i0.5\pi}\rangle}{y} + \frac{\langle 0.7\cdot e^{i\pi}, 0.1\cdot e^{i0.9\pi}\rangle}{z}$,

then $A \cup B = \frac{\langle 0.6\cdot e^{i1.2\pi}, 0.3\cdot e^{i0.8\pi}\rangle}{x} + \frac{\langle 0.4\cdot e^{i0.5\pi}, 0.6\cdot e^{i0.5\pi}\rangle}{y} + \frac{\langle 0.7\cdot e^{i2\pi}, 0.1\cdot e^{i0.9\pi}\rangle}{z}$. ♢

Theorem 2.1 The complex intuitionistic fuzzy union on $IF^\star(U)$ is an s-norm.

Proof Properties (i), (ii), (v) and (vi) can be easily verified from Definition 2.3. Here we only prove (iii) and (iv).

(iii) Let A, B, C and D be four complex intuitionistic fuzzy sets on U, $\mu_A(x) = r_A(x) \cdot e^{i\omega_{\mu A}(x)}$, $\mu_B(x) = r_B(x) \cdot e^{i\omega_{\mu B}(x)}$, $\mu_C(x) = r_C(x) \cdot e^{i\omega_{\mu C}(x)}$, $\mu_D(x) = r_D(x) \cdot e^{i\omega_{\mu D}(x)}$, $\nu_A(x) = s_A(x) \cdot e^{i\omega_{\nu A}(x)}$, $\nu_B(x) = s_B(x) \cdot e^{i\omega_{\nu B}(x)}$, $\nu_C(x) = s_C(x) \cdot e^{i\omega_{\nu C}(x)}$ and $\nu_D(x) = s_D(x) \cdot e^{i\omega_{\nu D}(x)}$ their membership and non-membership functions, respectively. Suppose $|\mu_A(x)| \leq |\mu_C(x)|$, $\omega_{\mu A}(x) \leq \omega_{\mu C}(x)$, $|\nu_A(x)| \geq |\nu_C(x)|$, $\omega_{\nu A}(x) \geq \omega_{\nu C}(x)$ and $|\mu_B(x)| \leq |\mu_D(x)|$, $\omega_{\mu B}(x) \leq \omega_{\mu D}(x)$, $|\nu_B(x)| \geq |\nu_D(x)|$, $\omega_{\nu B}(x) \geq \omega_{\nu D}(x)$, for any $x \in U$. We have

$$|\mu_{A\cup B}(x)| = \max(r_A(x), r_B(x)) \leq \max(r_C(x), r_D(x)) = |\mu_{C\cup D}(x)|,$$

$$\omega_{\mu(A\cup B)}(x) = \max(\omega_{\mu A}(x), \omega_{\mu B}(x)) \leq \max(\omega_{\mu C}(x), \omega_{\mu D}(x)) = \omega_{\mu(C\cup D)}(x),$$

and

$$|\nu_{A\cup B}(x)| = \min(s_A(x), s_B(x)) \geq \min(s_C(x), s_D(x)) = |\nu_{C\cup D}(x)|,$$

$$\omega_{\nu(A\cup B)}(x) = \min(\omega_{\nu A}(x), \omega_{\nu B}(x)) \geq \min(\omega_{\nu C}(x), \omega_{\nu D}(x)) = \omega_{\nu(C\cup D)}(x).$$

(iv) Suppose A, B and C be three complex intuitionistic fuzzy sets on U, $\mu_A(x) = r_A(x) \cdot e^{i\omega_{\mu A}(x)}$, $\mu_B(x) = r_B(x) \cdot e^{i\omega_{\mu B}(x)}$, $\mu_C(x) = r_C(x) \cdot e^{i\omega_{\mu C}(x)}$, $\nu_A(x) = s_A(x) \cdot e^{i\omega_{\nu A}(x)}$, $\nu_B(x) = s_B(x) \cdot e^{i\omega_{\nu B}(x)}$ and $\nu_C(x) = s_C(x) \cdot e^{i\omega_{\nu C}(x)}$ their membership and non-membership functions, respectively. Then

$$\begin{aligned}
\mu_{(A\cup B)\cup C}(x) &= r_{(A\cup B)\cup C}(x) \cdot e^{i\omega_{\mu((A\cup B)\cup C)}(x)} = \max(r_{A\cup B}(x), r_C(x)) \cdot e^{i\max(\omega_{\mu(A\cup B)}(x), \omega_{\mu C}(x))} \\
&= \max(\max(r_A(x), r_B(x)), r_C(x)) \cdot e^{i\max(\max(\omega_{\mu A}(x), \omega_{\mu B}(x)), \omega_{\mu C}(x))} \\
&= \max(r_A(x), \max(r_B(x), r_C(x)) \cdot e^{i\max(\omega_{\mu A}(x), \max(\omega_{\mu B}(x), \omega_{\mu C}(x))} \\
&= \mu_{A\cup(B\cup C)}(x). \\
\nu_{(A\cup B)\cup C}(x) &= s_{(A\cup B)\cup C}(x) \cdot e^{i\omega_{\nu((A\cup B)\cup C)}(x)} = \min(s_{A\cup B}(x), s_C(x)) \cdot e^{i\min(\omega_{\nu(A\cup B)}(x), \omega_{\nu C}(x))} \\
&= \min(\min(s_A(x), s_B(x)), s_C(x)) \cdot e^{i\min(\min(\omega_{\nu A}(x), \omega_{\nu B}(x)), \omega_{\nu C}(x))} \\
&= \min(s_A(x), \min(s_B(x), s_C(x)) \cdot e^{i\min(\omega_{\nu A}(x), \min(\omega_{\nu B}(x), \omega_{\nu C}(x))} \\
&= \nu_{A\cup(B\cup C)}(x).
\end{aligned}$$

□

Corollary 2.1 Let $C_\alpha \in IF^\star(U)$, $\alpha \in I$, $\mu_{C_\alpha}(x) = r_{C_\alpha}(x) \cdot e^{i\omega_{\mu C_\alpha}(x)}$ and $\nu_{C_\alpha}(x) = s_{C_\alpha}(x) \cdot e^{i\omega_{\nu C_\alpha}(x)}$ its membership and non-membership functions, respectively, where I is an arbitrary index set. Then $\bigcup_{\alpha\in I} C_\alpha \in IF^\star(U)$, and its membership and non-membership functions are

$$\mu_{\cup_{\alpha\in I} C_\alpha}(x) = \sup_{\alpha\in I} r_{C_\alpha}(x) \cdot e^{i\sup_{\alpha\in I}\omega_{\mu C_\alpha}(x)}$$

and

$$\nu_{\cup_{\alpha\in I} C_\alpha}(x) = \inf_{\alpha\in I} s_{C_\alpha}(x) \cdot e^{i\inf_{\alpha\in I}\omega_{\nu C_\alpha}(x)}.$$

Definition 2.4 (Complex Intuitionistic Fuzzy Intersection) Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in U\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in U\}$ be two complex intuitionistic fuzzy sets on U, $\mu_A(x) = r_A(x) \cdot e^{i\omega_{\mu A}(x)}$, $\mu_B(x) = r_B(x) \cdot e^{i\omega_{\mu B}(x)}$, $\nu_A(x) = s_A(x) \cdot e^{i\omega_{\nu A}(x)}$ and $\nu_B(x) = s_B(x) \cdot e^{i\omega_{\nu B}(x)}$ their membership and non-membership functions, respectively. The complex intuitionistic fuzzy intersection of A and B, denoted by $A \cap B = \{\langle x, \mu_{A\cap B}(x), \nu_{A\cap B}(x)\rangle : x \in U\}$, where

$$\mu_{A\cap B}(x) = r_{A\cap B}(x) \cdot e^{i\omega_{\mu(A\cap B)}(x)} = \min(r_A(x), r_B(x)) \cdot e^{i\min(\omega_{\mu A}(x), \omega_{\mu B}(x))} \tag{2.3}$$

and

$$\nu_{A\cap B}(x) = s_{A\cap B}(x) \cdot e^{i\omega_{\nu(A\cap B)}(x)} = \max(s_A(x), s_B(x)) \cdot e^{i\max(\omega_{\nu A}(x), \omega_{\nu B}(x))}. \tag{2.4}$$

Example 2.2 Let
$A = \frac{\langle 0.5\cdot e^{i1.2\pi}, 0.4\cdot e^{i0.8\pi}\rangle}{x} + \frac{\langle 0.4\cdot e^{i0.5\pi}, 0.6\cdot e^{i1.3\pi}\rangle}{y} + \frac{\langle 0.3\cdot e^{i2\pi}, 0.5\cdot e^{i1.5\pi}\rangle}{z}$,
$B = \frac{\langle 0.6\cdot e^{i0.2\pi}, 0.3\cdot e^{i1.8\pi}\rangle}{x} + \frac{\langle 0.2\cdot e^{i0.5\pi}, 0.6\cdot e^{i0.5\pi}\rangle}{y} + \frac{\langle 0.7\cdot e^{i\pi}, 0.1\cdot e^{i0.9\pi}\rangle}{z}$,
then $A \cap B = \frac{\langle 0.5\cdot e^{i0.2\pi}, 0.4\cdot e^{i1.8\pi}\rangle}{x} + \frac{\langle 0.2\cdot e^{i0.5\pi}, 0.6\cdot e^{i1.3\pi}\rangle}{y} + \frac{\langle 0.3\cdot e^{i\pi}, 0.5\cdot e^{i1.5\pi}\rangle}{z}$. ♢

Theorem 2.2 The complex intuitionistic fuzzy intersection on $IF^\star(U)$ is a t-norm.

Proof Properties (i), (ii), (v) and (vii) can be easily verified from Definition 2.3. Here we only prove (iii) and (iv).

(iii) Let A, B, C and D be four complex intuitionistic fuzzy sets on U, $\mu_A(x) = r_A(x) \cdot e^{i\omega_{\mu A}(x)}$, $\mu_B(x) = r_B(x) \cdot e^{i\omega_{\mu B}(x)}$, $\mu_C(x) = r_C(x) \cdot e^{i\omega_{\mu C}(x)}$, $\mu_D(x) = r_D(x) \cdot e^{i\omega_{\mu D}(x)}$, $\nu_A(x) = s_A(x) \cdot e^{i\omega_{\nu A}(x)}$, $\nu_B(x) = s_B(x) \cdot e^{i\omega_{\nu B}(x)}$, $\nu_C(x) = s_C(x) \cdot e^{i\omega_{\nu C}(x)}$ and $\nu_D(x) = s_D(x) \cdot e^{i\omega_{\nu D}(x)}$ their membership and non-membership functions, respectively. Suppose $|\mu_A(x)| \le |\mu_C(x)|$, $\omega_{\mu A}(x) \le \omega_{\mu C}(x)$, $|\nu_A(x)| \ge |\nu_C(x)|$, $\omega_{\nu A}(x) \ge \omega_{\nu C}(x)$ and $|\mu_B(x)| \le |\mu_D(x)|$, $\omega_{\mu B}(x) \le \omega_{\mu D}(x)$, $|\nu_B(x)| \ge |\nu_D(x)|$, $\omega_{\nu B}(x) \ge \omega_{\nu D}(x)$, for any $x \in U$. We have

$$|\mu_{A\cap B}(x)| = \min(r_A(x), r_B(x)) \le \min(r_C(x), r_D(x)) = |\mu_{C\cap D}(x)|,$$

$$\omega_{\mu(A\cap B)}(x) = \min(\omega_{\mu A}(x), \omega_{\mu B}(x)) \le \min(\omega_{\mu C}(x), \omega_{\mu D}(x)) = \omega_{\mu(C\cap D)}(x),$$

and

$$|\nu_{A\cap B}(x)| = \max(s_A(x), s_B(x)) \ge \max(s_C(x), s_D(x)) = |\nu_{C\cap D}(x)|,$$

$$\omega_{\nu(A\cap B)}(x) = \max(\omega_{\nu A}(x), \omega_{\nu B}(x)) \ge \max(\omega_{\nu C}(x), \omega_{\nu D}(x)) = \omega_{\nu(C\cap D)}(x).$$

(iv) Suppose A, B and C be three complex intuitionistic fuzzy sets on U, $\mu_A(x) = r_A(x) \cdot e^{i\omega_{\mu A}(x)}$, $\mu_B(x) = r_B(x) \cdot e^{i\omega_{\mu B}(x)}$, $\mu_C(x) = r_C(x) \cdot e^{i\omega_{\mu C}(x)}$, $\nu_A(x) = s_A(x) \cdot e^{i\omega_{\nu A}(x)}$, $\nu_B(x) = s_B(x) \cdot e^{i\omega_{\nu B}(x)}$ and $\nu_C(x) = s_C(x) \cdot e^{i\omega_{\nu C}(x)}$ their membership and non-membership functions, respectively. Then

$$\begin{aligned}
\mu_{(A\cap B)\cap C}(x) &= r_{(A\cap B)\cap C}(x) \cdot e^{i\omega_{\mu((A\cap B)\cap C)}(x)} = \min(r_{A\cap B}(x), r_C(x)) \cdot e^{i\min(\omega_{\mu(A\cap B)}(x), \omega_{\mu C}(x))} \\
&= \min(\min(r_A(x), r_B(x)), r_C(x)) \cdot e^{i\min(\min(\omega_{\mu A}(x), \omega_{\mu B}(x)), \omega_{\mu C}(x))} \\
&= \min(r_A(x), \min(r_B(x), r_C(x)) \cdot e^{i\min(\omega_{\mu A}(x), \min(\omega_{\mu B}(x), \omega_{\mu C}(x))} \\
&= \mu_{A\cap(B\cap C)}(x).
\end{aligned}$$

$$\begin{aligned}
\nu_{(A\cap B)\cap C}(x) &= s_{(A\cap B)\cap C}(x) \cdot e^{i\omega_{\nu((A\cap B)\cap C)}(x)} = \max(s_{A\cup B}(x), s_C(x)) \cdot e^{i\max(\omega_{\nu(A\cup B)}(x), \omega_{\nu C}(x))} \\
&= \max(\max(s_A(x), s_B(x)), s_C(x)) \cdot e^{i\max(\max(\omega_{\nu A}(x), \omega_{\nu B}(x)), \omega_{\nu C}(x))} \\
&= \max(s_A(x), \max(s_B(x), s_C(x)) \cdot e^{i\max(\omega_{\nu A}(x), \max(\omega_{\nu B}(x), \omega_{\nu C}(x))} \\
&= \nu_{A\cap(B\cap C)}(x).
\end{aligned}$$
□

Corollary 2.2 Let $C_\alpha \in IF^\star(U)$, $\alpha \in I$, $\mu_{C_\alpha}(x) = r_{C_\alpha}(x) \cdot e^{i\omega_{\mu C_\alpha}(x)}$ and $\nu_{C_\alpha}(x) = s_{C_\alpha}(x) \cdot e^{i\omega_{\nu C_\alpha}(x)}$ its membership and non-membership functions, where I is an arbitrary index set. Then $\bigcap_{\alpha\in I} C_\alpha \in IF^\star(U)$, and its membership and non-membership functions are

$$\mu_{\cap_{\alpha\in I} C_\alpha}(x) = \inf_{\alpha\in I} r_{C_\alpha}(x) \cdot e^{i\inf_{\alpha\in I}\omega_{\mu C_\alpha}(x)}$$

and

$$\nu_{\cap_{\alpha\in I}C_\alpha}(x)=\sup_{\alpha\in I}s_{C_\alpha}(x)\cdot e^{i\sup_{\alpha\in I}\omega_{\nu C_\alpha}(x)}.$$

Corollary 2.3 Let $C_{\alpha\beta}\in IF^\star(U),\ \alpha\in I_1,\ \beta\in I_2,\ \mu_{C_{\alpha\beta}}(x)=r_{C_{\alpha\beta}}(x)\cdot e^{i\omega_{\mu C_{\alpha\beta}}(x)}$ and $\nu_{C_{\alpha\beta}}(x)=s_{C_{\alpha\beta}}(x)\cdot e^{i\omega_{\nu C_{\alpha\beta}}(x)}$ its membership and non-membership functions, respectively, where I_1 and I_2 are two arbitrary index sets. Then $\bigcup_{\alpha\in I_1}\bigcap_{\beta\in I_2}C_{\alpha\beta},\ \bigcap_{\alpha\in I_1}\bigcup_{\beta\in I_2}C_{\alpha\beta}\in IF^\star(U)$, and their membership and non-membership functions are

$$\mu_{\cup_{\alpha\in I_1}\cap_{\beta\in I_2}C_{\alpha\beta}}(x)=\sup_{\alpha\in I_1}\inf_{\beta\in I_2}r_{C_{\alpha\beta}}(x)\cdot e^{i\sup_{\alpha\in I_1}\inf_{\beta\in I_2}\omega_{\mu C_{\alpha\beta}}(x)},$$

$$\mu_{\cap_{\alpha\in I_1}\cup_{\beta\in I_2}C_{\alpha\beta}}(x)=\inf_{\alpha\in I_1}\sup_{\beta\in I_2}r_{C_{\alpha\beta}}(x)\cdot e^{i\inf_{\alpha\in I_1}\sup_{\beta\in I_2}\omega_{\mu C_{\alpha\beta}}(x)},$$

and

$$\nu_{\cup_{\alpha\in I_1}\cap_{\beta\in I_2}C_{\alpha\beta}}(x)=\inf_{\alpha\in I_1}\sup_{\beta\in I_2}s_{C_{\alpha\beta}}(x)\cdot e^{i\inf_{\alpha\in I_1}\sup_{\beta\in I_2}\omega_{\nu C_{\alpha\beta}}(x)},$$

$$\nu_{\cap_{\alpha\in I_1}\cup_{\beta\in I_2}C_{\alpha\beta}}(x)=\sup_{\alpha\in I_1}\inf_{\beta\in I_2}s_{C_{\alpha\beta}}(x)\cdot e^{i\sup_{\alpha\in I_1}\inf_{\beta\in I_2}\omega_{\nu C_{\alpha\beta}}(x)}.$$

Corollary 2.4 Let $C_k\in IF^\star(U),\ k=1,2,...,\mu_{C_k}(x)=r_{C_k}(x)\cdot e^{i\omega_{\mu C_k}(x)}$ and $\nu_{C_k}(x)=s_{C_k}(x)\cdot e^{i\omega_{\nu C_k}(x)}$ its membership and non-membership functions, respectively. Then

$$\overline{\lim}_{n\to\infty}C_n=\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}C_k,\ \underline{\lim}_{n\to\infty}C_n=\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}C_k\in IF^\star(U),$$

and their membership and non-membership functions are

$$\mu_{\overline{\lim}_{n\to\infty}C_k}(x)=\inf_{n\geq1}\sup_{k\geq n}r_{C_k}(x)\cdot e^{i\inf_{n\geq1}\sup_{k\geq n}\omega_{\mu C_k}(x)},$$

$$\mu_{\underline{\lim}_{n\to\infty}C_k}(x)=\sup_{n\geq1}\inf_{k\geq n}r_{C_k}(x)\cdot e^{i\sup_{n\geq1}\inf_{k\geq n}\omega_{\mu C_k}(x)},$$

and

$$\nu_{\overline{\lim}_{n\to\infty}C_k}(x)=\sup_{n\geq1}\inf_{k\geq n}s_{C_k}(x)\cdot e^{i\sup_{n\geq1}\inf_{k\geq n}\omega_{\nu C_k}(x)},$$

$$\nu_{\underline{\lim}_{n\to\infty}C_k}(x)=\inf_{n\geq1}\sup_{k\geq n}s_{C_k}(x)\cdot e^{i\inf_{n\geq1}\sup_{k\geq n}\omega_{\nu C_k}(x)}.$$

Definition 2.5 (Complex Intuitionistic Fuzzy Complement) Let $A=\{\langle x,\mu_A(x),\nu_A(x)\rangle : x\in U\}$ be a complex intuitionistic fuzzy set on U, $\mu_A(x)=r_A(x)\cdot e^{i\omega_{\mu A}(x)}$ and $\nu_A(x)=s_A(x)\cdot e^{i\omega_{\nu A}(x)}$ its membership and non-membership functions, respectively. The complex intuitionistic fuzzy complement of A, denoted by $\overline{A}$ and defined by following ways:

(i) $\overline{A}=\{\langle x,\nu_A(x),\mu_A(x)\rangle\}$;

(ii) $\overline{A}=\{\langle x,\mu_{\bar A}(x),\nu_{\bar A}(x)\rangle\}$, where $\mu_{\bar A}(x)=r_{\bar A}(x)\cdot e^{i\omega_{\mu\bar A}(x)}$, $\nu_{\bar A}(x)=s_{\bar A}(x)\cdot e^{i\omega_{\nu\bar A}(x)}$ and $r_{\bar A}(x)=1-r_A(x),\ s_{\bar A}(x)=1-s_A(x), \omega_{\mu\bar A}(x)=\left\{\begin{array}{c}\omega_{\mu A}(x)\\ 2\pi-\omega_{\mu A}(x)=-\omega_{\mu A}(x),\\ \omega_{\mu A}(x)+\pi\end{array}\right.$ and $\omega_{\nu\bar A}(x)=\left\{\begin{array}{c}\omega_{\nu A}(x)\\ 2\pi-\omega_{\nu A}(x)=-\omega_{\nu A}(x).\\ \omega_{\nu A}(x)+\pi\end{array}\right.$

The following example use the first way of Definition 2.5 to calculate the complement of the complex intuitionistic fuzzy set A. Note that if the second way is used, the corresponding results also can be obtained.

Example 2.3 Let $A=\frac{\langle0.5\cdot e^{i1.2\pi},0.4\cdot e^{i0.8\pi}\rangle}{x}+\frac{\langle0.4\cdot e^{i0.5\pi},0.6\cdot e^{i1.3\pi}\rangle}{y}+\frac{\langle0.3\cdot e^{i2\pi},0.5\cdot e^{i1.5\pi}\rangle}{z}$,

then $\overline{A}=\frac{\langle0.4\cdot e^{i0.8\pi},0.5\cdot e^{i1.2\pi}\rangle}{x}+\frac{\langle0.6\cdot e^{i1.3\pi},0.4\cdot e^{i0.5\pi}\rangle}{y}+\frac{\langle0.5\cdot e^{i1.5\pi},0.3\cdot e^{i2\pi}\rangle}{z}$. ♢

Proposition 2.1 Let $A,\ B$ and C be any three complex intuitionistic fuzzy sets on U, then the following propositions hold:

(i) $A \cup A = A,\ A \cap A = A$;

(ii) $A \cup B = B \cup A,\ A \cap B = B \cap A$;

(iii) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C),\ (A \cap B) \cup C = (A \cup C) \cap (B \cup C)$;

(iv) $A \cap (B \cap C) = (A \cap B) \cap C,\ A \cup (B \cup C) = (A \cup B) \cup C$;

(v) $\overline{(A \cap B)} = \overline{A} \cup \overline{B},\ \overline{(A \cup B)} = \overline{A} \cap \overline{B}$;

(vi) $\overline{\overline{A}} = A$.

Proof Here we only prove (iii), (iv), (v) and (vi). Let $A,\ B$ and C be three complex intuitionistic fuzzy sets on U, $\mu_A(x) = r_A(x) \cdot e^{i\omega_{\mu A}(x)}$, $\mu_B(x) = r_B(x) \cdot e^{i\omega_{\mu B}(x)}$, $\mu_C(x) = r_C(x) \cdot e^{i\omega_{\mu C}(x)}$, $\nu_A(x) = s_A(x) \cdot e^{i\omega_{\nu A}(x)}$, $\nu_B(x) = s_B(x) \cdot e^{i\omega_{\nu B}(x)}$ and $\nu_C(x) = s_C(x) \cdot e^{i\omega_{\nu C}(x)}$ their membership and non-membership functions, respectively. The complement of A and B are $\overline{A} = \langle x, \nu_A(x), \mu_A(x)\rangle$ and $\overline{B} = \langle x, \nu_B(x), \mu_B(x)\rangle$, respectively. Then

(iii) First of all, we prove that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$, since

$$\begin{aligned}
\mu_{(A\cup B)\cap C}(x) &= r_{(A\cup B)\cap C}(x) \cdot e^{i\omega_{\mu((A\cup B)\cap C)}(x)} = \min(r_{A\cup B}(x), r_C(x)) \cdot e^{i\min(\omega_{\mu(A\cup B)}(x),\omega_{\mu C}(x))} \\
&= \min(\max(r_A(x), r_B(x)), r_C(x)) \cdot e^{i\min(\max(\omega_{\mu A}(x),\omega_{\mu B}(x)),\omega_{\mu C}(x))} \\
&= \max(\min(r_A(x), r_C(x)), \min(r_B(x), r_C(x))) \cdot e^{i\max(\min(\omega_{\mu A}(x),\omega_{\mu C}(x)),\min(\omega_{\mu B}(x),\omega_{\mu C}(x)))} \\
&= r_{(A\cap C)\cup(B\cap C)}(x) \cdot e^{i\omega_{\mu((A\cap C)\cup(B\cap C))}(x)} = \mu_{(A\cap C)\cup(B\cap C)}(x).
\end{aligned}$$

$$\begin{aligned}
\nu_{(A\cup B)\cap C}(x) &= s_{(A\cup B)\cap C}(x) \cdot e^{i\omega_{\nu((A\cup B)\cap C)}(x)} = \max(s_{A\cup B}(x), s_C(x)) \cdot e^{i\max(\omega_{\nu(A\cup B)}(x),\omega_{\nu C}(x))} \\
&= \max(\min(s_A(x), s_B(x)), s_C(x)) \cdot e^{i\max(\min(\omega_{\nu A}(x),\omega_{\nu B}(x)),\omega_{\nu C}(x))} \\
&= \min(\max(s_A(x), s_C(x)), \max(s_B(x), s_C(x))) \cdot e^{i\min(\max(\omega_{\nu A}(x),\omega_{\nu C}(x)),\max(\omega_{\nu B}(x),\omega_{\mu C}(x)))} \\
&= s_{(A\cap C)\cup(B\cap C)}(x) \cdot e^{i\omega_{\nu((A\cap C)\cup(B\cap C))}(x)} = \nu_{(A\cap C)\cup(B\cap C}(x).
\end{aligned}$$

It implies that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

Similarly, we can prove that $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

(iv) First of all, we prove that $A \cap (B \cap C) = (A \cap B) \cap C$, since

$$\begin{aligned}
\mu_{A\cap(B\cap C)}(x) &= r_{A\cap(B\cap C)}(x) \cdot e^{i\omega_{\mu(A\cap(B\cap C))}(x)} = \min(r_A(x), r_{B\cap C}(x)) \cdot e^{i\min(\omega_{\mu A}(x),\omega_{\mu(B\cap C)}(x))} \\
&= \min(r_A(x), \min(r_B(x), r_C(x))) \cdot e^{i\min(\omega_{\mu A}(x),\min(\omega_{\mu B}(x),\omega_{\mu C}(x)))} \\
&= \min(\min(r_A(x), r_B(x)), r_C(x)) \cdot e^{i\min(\min(\omega_{\mu A}(x),\omega_{\mu B}(x)),\omega_{\mu C}(x))} \\
&= r_{(A\cap B)\cap C}(x) \cdot e^{i\omega_{\mu((A\cap B)\cap C)}(x)} = \mu_{(A\cap B)\cap C}(x).
\end{aligned}$$

$$\begin{aligned}
\nu_{A\cap(B\cap C)}(x) &= s_{A\cap(B\cap C)}(x) \cdot e^{i\omega_{\nu(A\cap(B\cap C))}(x)} = \max(s_A(x), s_{B\cap C}(x)) \cdot e^{i\max(\omega_{\nu A}(x),\omega_{\nu(B\cap C)}(x))} \\
&= \max(s_A(x), \max(s_B(x), s_C(x))) \cdot e^{i\max(\omega_{\nu A}(x),\max(\omega_{\nu B}(x),\omega_{\nu C}(x)))} \\
&= \max(\max(s_A(x), s_B(x)), s_C(x)) \cdot e^{i\max(\max(\omega_{\nu A}(x),\omega_{\nu B}(x)),\omega_{\nu C}(x))} \\
&= s_{(A\cap B)\cap C}(x) \cdot e^{i\omega_{\nu((A\cap B)\cap C)}(x)} = \nu_{(A\cap B)\cap C}(x).
\end{aligned}$$

It implies that $A \cap (B \cap C) = (A \cap B) \cap C$.

Similarly, we can prove that $(A \cup B) \cup C = A \cup (B \cup C)$.

(v) First of all, we prove that $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$, since

$$\begin{aligned}
\mu_{\overline{(A\cap B)}}(x) &= \nu_{A\cap B}(x) = s_{A\cap B}(x) \cdot e^{i\omega_{\nu(A\cap B)}(x)} \\
&= \max(s_A(x), s_B(x)) \cdot e^{i\max(\omega_{\nu A}(x),\omega_{\mu B}(x))} \\
&= \mu_{\overline{A}\cup\overline{B}}(x). \\
\nu_{\overline{(A\cap B)}}(x) &= \mu_{A\cap B}(x) = r_{A\cap B}(x) \cdot e^{i\omega_{\mu(A\cap B)}(x)} \\
&= \min(r_A(x), r_B(x)) \cdot e^{i\min(\omega_{\mu A}(x),\omega_{\mu B}(x))} \\
&= \nu_{\overline{A}\cup\overline{B}}(x).
\end{aligned}$$

It implies that $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$.

Similarly, we can prove that $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$.

(vi) $\mu_{\overline{\overline{A}}}(x) = \nu_{\overline{A}}(x) = \mu_A(x)$ and $\nu_{\overline{\overline{A}}}(x) = \mu_{\overline{A}}(x) = \nu_A(x)$, i.e., $\overline{\overline{A}} = A$. $\square$

Definition 2.6 (Complex Intuitionistic Fuzzy Product) Let A and B be two complex intuitionistic fuzzy sets on U, $\mu_A(x) = r_A(x) \cdot e^{i\omega_{\mu A}(x)}$, $\nu_A(x) = s_A(x) \cdot e^{i\omega_{\nu A}(x)}$, $\mu_B(x) = r_B(x) \cdot e^{i\omega_{\mu B}(x)}$ and $\nu_B(x) = s_B(x) \cdot e^{i\omega_{\nu B}(x)}$ their membership and non-membership functions, respectively. The complex intuitionistic fuzzy product of A and B, denoted by $A \circ B = \{\langle x, \mu_{A\circ B}(x), \nu_{A\circ B}(x)\rangle : x \in U\}$, where

$$\mu_{A\circ B}(x) = r_{A\circ B}(x) \cdot e^{i\omega_{\mu(A\circ B)}(x)} = (r_A(x) \cdot r_B(x)) \cdot e^{i2\pi(\frac{\omega_{\mu A}(x)}{2\pi} \cdot \frac{\omega_{\mu B}(x)}{2\pi})} \tag{2.5}$$

and

$$\begin{aligned}\nu_{A\circ B}(x) &= s_{A\circ B}(x) \cdot e^{i\omega_{\nu(A\circ B)}(x)} \\ &= (s_A(x) + s_B(x) - s_A(x) \cdot s_B(x)) \cdot e^{i2\pi(\frac{\omega_{\nu A}(x)}{2\pi} + \frac{\omega_{\nu B}(x)}{2\pi} - \frac{\omega_{\nu A}(x)}{2\pi} \cdot \frac{\omega_{\nu B}(x)}{2\pi})}.\end{aligned} \tag{2.6}$$

Example 2.4 Let
$A = \frac{\langle 0.5\cdot e^{i1.2\pi}, 0.4\cdot e^{i0.8\pi}\rangle}{x} + \frac{\langle 0.4\cdot e^{i0.5\pi}, 0.6\cdot e^{i1.3\pi}\rangle}{y} + \frac{\langle 0.3\cdot e^{i2\pi}, 0.5\cdot e^{i1.5\pi}\rangle}{z}$,
$B = \frac{\langle 0.6\cdot e^{i0.2\pi}, 0.3\cdot e^{i1.8\pi}\rangle}{x} + \frac{\langle 0.2\cdot e^{i0.5\pi}, 0.6\cdot e^{i0.5\pi}\rangle}{y} + \frac{\langle 0.7\cdot e^{i\pi}, 0.1\cdot e^{i0.9\pi}\rangle}{z}$,
then $A \circ B = \frac{\langle 0.3\cdot e^{i0.12\pi}, 0.58\cdot e^{i1.88\pi}\rangle}{x} + \frac{\langle 0.08\cdot e^{i0.125\pi}, 0.84\cdot e^{i1.475\pi}\rangle}{y} + \frac{\langle 0.21\cdot e^{i\pi}, 0.55\cdot e^{i1.725\pi}\rangle}{z}$. ◊

Theorem 2.3 The complex intuitionistic fuzzy product on $IF^\star(U)$ is a t-norm.

Proof Properties (i), (ii), (v) and (vii) can be easily verified from Definition 2.3. Here we only prove (iii) and (iv).

(iii) Let A, B, C and D be four complex intuitionistic fuzzy sets on U, $\mu_A(x) = r_A(x) \cdot e^{i\omega_{\mu A}(x)}$, $\mu_B(x) = r_B(x) \cdot e^{i\omega_{\mu B}(x)}$, $\mu_C(x) = r_C(x) \cdot e^{i\omega_{\mu C}(x)}$, $\mu_D(x) = r_D(x) \cdot e^{i\omega_{\mu D}(x)}$, $\nu_A(x) = s_A(x) \cdot e^{i\omega_{\nu A}(x)}$, $\nu_B(x) = s_B(x) \cdot e^{i\omega_{\nu B}(x)}$, $\nu_C(x) = s_C(x) \cdot e^{i\omega_{\nu C}(x)}$ and $\nu_D(x) = s_D(x) \cdot e^{i\omega_{\nu D}(x)}$ their membership and non-membership functions, respectively. Suppose $|\mu_A(x)| \le |\mu_C(x)|$, $\omega_{\mu A}(x) \le \omega_{\mu C}(x)$, $|\nu_A(x)| \ge |\nu_C(x)|$, $\omega_{\nu A}(x) \ge \omega_{\nu C}(x)$ and $|\mu_B(x)| \le |\mu_D(x)|$, $\omega_{\mu B}(x) \le \omega_{\mu D}(x)$, $|\nu_B(x)| \ge |\nu_D(x)|$, $\omega_{\nu B}(x) \ge \omega_{\nu D}(x)$, for any $x \in U$. We have

$$|\mu_{A\circ B}(x)| = |r_A(x)| \cdot |r_B(x)| \le |r_C(x)| \cdot |r_D(x)| = |\mu_{C\circ D}(x)|,$$

$$\omega_{\mu(A\circ B)}(x) = 2\pi(\frac{\omega_{\mu A}(x)}{2\pi} \cdot \frac{\omega_{\mu B}(x)}{2\pi}) \le 2\pi(\frac{\omega_{\mu C}(x)}{2\pi} \cdot \frac{\omega_{\mu D}(x)}{2\pi}) = \omega_{\mu(C\circ D)}(x),$$

and

$$|\nu_{A\circ B}(x)| = |s_A(x)| + |s_B(x)| - |s_A(x)| \cdot |s_B(x)| \ge |s_C(x)| + |s_D(x)| - |s_C(x)| \cdot |s_D(x)| = |\nu_{C\circ D}(x)|,$$

$$\omega_{\nu(A\circ B)}(x) = 2\pi(\frac{\omega_{\nu A}(x)}{2\pi} + \frac{\omega_{\nu B}(x)}{2\pi} - \frac{\omega_{\nu A}(x)}{2\pi} \cdot \frac{\omega_{\nu B}(x)}{2\pi}) \ge 2\pi(\frac{\omega_{\nu C}(x)}{2\pi} + \frac{\omega_{\nu D}(x)}{2\pi} - \frac{\omega_{\nu C}(x)}{2\pi} \cdot \frac{\omega_{\nu D}(x)}{2\pi}) = \omega_{\nu(C\circ D)}(x).$$

(iv) Suppose A, B and C be three complex intuitionistic fuzzy sets on U, $\mu_A(x) = r_A(x) \cdot e^{i\omega_{\mu A}(x)}$, $\mu_B(x) = r_B(x) \cdot e^{i\omega_{\mu B}(x)}$, $\mu_C(x) = r_C(x) \cdot e^{i\omega_{\mu C}(x)}$, $\nu_A(x) = s_A(x) \cdot e^{i\omega_{\nu A}(x)}$, $\nu_B(x) = s_B(x) \cdot e^{i\omega_{\nu B}(x)}$ and $\nu_C(x) = s_C(x) \cdot e^{i\omega_{\nu C}(x)}$ their membership and non-membership functions, respectively. Then

$$\begin{aligned}\mu_{(A\circ B)\circ C}(x) &= r_{(A\circ B)\circ C}(x) \cdot e^{i\omega_{\mu((A\circ B)\circ C)}(x)} = (r_{A\circ B}(x) \cdot r_C(x)) \cdot e^{i2\pi(\frac{\omega_{\mu(A\circ B)}(x)}{2\pi} \cdot \frac{\omega_{\mu C}(x)}{2\pi})} \\ &= ((r_A(x) \cdot r_B(x)) \cdot r_C(x)) \cdot e^{i2\pi(\frac{2\pi(\frac{\omega_{\mu A}(x)}{2\pi} \cdot \frac{\omega_{\mu B}(x)}{2\pi})}{2\pi} \cdot \frac{\omega_{\mu C}(x)}{2\pi})} \\ &= (r_A(x) \cdot (r_B(x) \cdot r_C(x))) \cdot e^{i2\pi(\frac{\omega_{\mu A}(x)}{2\pi} \cdot \frac{2\pi(\frac{\omega_{\mu B}(x)}{2\pi} \cdot \frac{\omega_{\mu C}(x)}{2\pi})}{2\pi})} \\ &= \mu_{A\circ(B\circ C)}(x).\end{aligned}$$

$$\begin{aligned}\nu_{(A\circ B)\circ C}(x) &= s_{(A\circ B)\circ C}(x) \cdot e^{i\omega_{\nu((A\circ B)\circ C)}(x)} \\ &= (s_{A\circ B}(x) + s_C(x) - s_{A\circ B}(x) \cdot s_C(x)) \cdot e^{i2\pi(\frac{\omega_{\nu(A\circ B)}(x)}{2\pi} + \frac{\omega_{\nu C}(x)}{2\pi} - \frac{\omega_{\nu(A\circ B)}(x)}{2\pi} \cdot \frac{\omega_{\nu C}(x)}{2\pi})} \\ &= (s_A(x) + s_B(x) - s_A(x) \cdot s_B(x) + s_C(x) - (s_A(x) + s_B(x) - s_A(x) \cdot s_B(x)) \cdot s_C(x)) \cdot \\ &\quad e^{i2\pi(\frac{2\pi(\frac{\omega_{\nu A}(x)}{2\pi} + \frac{\omega_{\nu B}(x)}{2\pi} - \frac{\omega_{\nu A}(x)}{2\pi} \cdot \frac{\omega_{\nu B}(x)}{2\pi})}{2\pi} + \frac{\omega_{\nu C}(x)}{2\pi} - \frac{2\pi(\frac{\omega_{\nu A}(x)}{2\pi} + \frac{\omega_{\nu B}(x)}{2\pi} - \frac{\omega_{\nu A}(x)}{2\pi} \cdot \frac{\omega_{\nu B}(x)}{2\pi})}{2\pi} \cdot \frac{\omega_{\nu C}(x)}{2\pi})}\end{aligned}$$

$$
\begin{aligned}
&= (s_A(x) + s_B(x) + s_C(x) - s_B(x) \cdot s_C(x) - s_A(x) \cdot (s_B(x) + s_C(x) - s_B(x) \cdot s_C(x))) \cdot \\
&\quad e^{i2\pi(\frac{\omega_{\nu A}(x)}{2\pi} + \frac{2\pi(\frac{\omega_{\nu B}(x)}{2\pi} + \frac{\omega_{\nu C}(x)}{2\pi} - \frac{\omega_{\nu B}(x)}{2\pi} \cdot \frac{\omega_{\nu C}(x)}{2\pi})}{2\pi} - \frac{\omega_{\nu A}(x)}{2\pi} \cdot \frac{2\pi(\frac{\omega_{\nu B}(x)}{2\pi} + \frac{\omega_{\nu C}(x)}{2\pi} - \frac{\omega_{\nu B}(x)}{2\pi} \cdot \frac{\omega_{\nu C}(x)}{2\pi})}{2\pi})} \\
&= s_{A\circ(B\circ C)}(x) \cdot e^{i\omega_{\nu(A\circ(B\circ C))}(x)} \\
&= \nu_{A\circ(B\circ C)}(x). \qquad \square
\end{aligned}
$$

Corollary 2.5 Let $C_\alpha \in IF^\star(U),\ \alpha \in I,\ \mu_{C_\alpha}(x) = r_{C_\alpha}(x) \cdot e^{i\omega_{\mu C_\alpha}(x)}$ and $\nu_{C_\alpha}(x) = s_{C_\alpha}(x) \cdot e^{i\omega_{\nu C_\alpha}(x)}$ its membership and non-membership functions, where I is an arbitrary index set. Then $\prod_{\alpha \in I} C_\alpha = C_1 \circ C_2 \circ \cdots \circ C_\alpha \in IF^\star(U)$, and its membership and non-membership functions are

$$
\mu_{\prod_{\alpha\in I} C_\alpha}(x) = (r_{C_1}(x) \cdot r_{C_2}(x) \cdots r_{C_\alpha}(x)) \cdot e^{i2\pi(\frac{\omega_{\mu C_1}(x)}{2\pi} \cdot \frac{\omega_{\mu C_2}(x)}{2\pi} \cdots \frac{\omega_{\mu C_\alpha}(x)}{2\pi})}
$$

and

$$
\nu_{\prod_{\alpha\in I} C_\alpha}(x) = [s_{C_1}(x) + s_{C_2}(x) + \cdots + s_{C_\alpha}(x) - \cdots + (-1)^{\alpha-1}(s_{C_1}(x) \cdot s_{C_2}(x) \cdots s_{C_\alpha}(x))]
$$

$$
\cdot\, e^{i2\pi[(\frac{\omega_{\mu C_1}(x)}{2\pi} + \frac{\omega_{\mu C_2}(x)}{2\pi} + \cdots + \frac{\omega_{\mu C_\alpha}(x)}{2\pi}) - \cdots + \frac{(-1)^{\alpha-1}}{(2\pi)^2}(\frac{\omega_{\nu C_1}(x)}{2\pi} \cdot \frac{\omega_{\nu C_2}(x)}{2\pi} \cdots \frac{\omega_{\nu C_\alpha}(x)}{2\pi})]}.
$$

Definition 2.7 (Complex Intuitionistic Fuzzy Cartesian Product) Let $A_n,\ n = 1, 2, ..., N$ be N complex intuitionistic fuzzy sets on U, $\mu_{A_n}(x) = r_{A_n}(x) \cdot e^{i\omega_{\mu A_n}(x)}$, $\nu_{A_n}(x) = s_{A_n}(x) \cdot e^{i\omega_{\nu A_n}(x)}$ their membership and non-membership functions, respectively. The complex intuitionistic fuzzy Cartesian product of $A_n,\ n = 1, 2, ...N$, denoted by $A_1 \times A_2 \times \cdots \times A_N = \{\langle x, \mu_{A_1\times A_2\times\cdots\times A_N}(x), \nu_{A_1\times A_2\times\cdots\times A_N}(x)\rangle : x \in U\}$, where

$$
\begin{aligned}
\mu_{A_1\times A_2\times\cdots\times A_N}(x) &= r_{A_1\times A_2\times\cdots\times A_N}(x) \cdot e^{i\omega_{\mu(A_1\times A_2\times\cdots\times A_N)}(x)} \\
&= \min(r_{A_1}(x), r_{A_2}(x), \cdots, r_{A_N}(x)) \cdot e^{i\min(\omega_{\mu A_1}(x), \omega_{\mu A_2}(x), \cdots, \omega_{\mu A_N}(x))} \qquad (2.7)
\end{aligned}
$$

and

$$
\begin{aligned}
\nu_{A_1\times A_2\times\cdots\times A_N}(x) &= s_{A_1\times A_2\times\cdots\times A_N}(x) \cdot e^{i\omega_{\nu(A_1\times A_2\times\cdots\times A_N)}(x)} \\
&= \max(s_{A_1}(x), s_{A_2}(x), \cdots, s_{A_N}(x)) \cdot e^{i\max(\omega_{\nu A_1}(x), \omega_{\nu A_2}(x), \cdots, \omega_{\nu A_N}(x))}. \qquad (2.8)
\end{aligned}
$$

Example 2.5 Let

$$
\begin{aligned}
A &= \frac{\langle 0.5\cdot e^{i1.2\pi}, 0.4\cdot e^{i0.8\pi}\rangle}{x} + \frac{\langle 0.4\cdot e^{i0.5\pi}, 0.6\cdot e^{i1.3\pi}\rangle}{y} + \frac{\langle 0.3\cdot e^{i2\pi}, 0.5\cdot e^{i1.5\pi}\rangle}{z}, \\
B &= \frac{\langle 0.6\cdot e^{i0.2\pi}, 0.3\cdot e^{i1.8\pi}\rangle}{x} + \frac{\langle 0.2\cdot e^{i0.5\pi}, 0.6\cdot e^{i0.5\pi}\rangle}{y} + \frac{\langle 0.7\cdot e^{i\pi}, 0.1\cdot e^{i0.9\pi}\rangle}{z},
\end{aligned}
$$

$$
\begin{aligned}
\text{then } A \times B = &\frac{\langle 0.5\cdot e^{i0.2\pi}, 0.4\cdot e^{i1.8\pi}\rangle}{(x,x)} + \frac{\langle 0.2\cdot e^{i0.5\pi}, 0.6\cdot e^{i0.8\pi}\rangle}{(x,y)} + \frac{\langle 0.5\cdot e^{i\pi}, 0.4\cdot e^{i0.9\pi}\rangle}{(x,z)} \\
&+ \frac{\langle 0.4\cdot e^{i0.2\pi}, 0.6\cdot e^{i1.8\pi}\rangle}{(y,x)} + \frac{\langle 0.2\cdot e^{i0.5\pi}, 0.6\cdot e^{i1.3\pi}\rangle}{(y,y)} + \frac{\langle 0.4\cdot e^{i0.5\pi}, 0.6\cdot e^{i1.3\pi}\rangle}{(y,z)} \\
&+ \frac{\langle 0.3e^{i0.2\pi}, 0.5e^{i1.8\pi}\rangle}{(z,x)} + \frac{\langle 0.2e^{i0.5\pi}, 0.6e^{i1.5\pi}\rangle}{(z,y)} + \frac{\langle 0.3e^{i\pi}, 0.5e^{i1.5\pi}\rangle}{(z,z)}. \qquad \diamond
\end{aligned}
$$

Definition 2.8 (Complex Intuitionistic Fuzzy Probabilistic Sum) Let A and B be two complex intuitionistic fuzzy sets on U, $\mu_A(x) = r_A(x) \cdot e^{i\omega_{\mu A}(x)}$, $\nu_A(x) = s_A(x) \cdot e^{i\omega_{\nu A}(x)}$, $\mu_B(x) = r_B(x) \cdot e^{i\omega_{\mu B}(x)}$ and $\nu_B(x) = s_B(x) \cdot e^{i\omega_{\nu B}(x)}$ their membership and non-membership functions, respectively. The complex intuitionistic fuzzy probabilistic sum of A and B, denoted by $A \hat{+} B = \{\langle x, \mu_{A\hat{+}B}(x), \nu_{A\hat{+}B}(x)\rangle : x \in U\}$, where

$$
\begin{aligned}
\mu_{A\hat{+}B}(x) &= r_{A\hat{+}B}(x) \cdot e^{i\omega_{\mu(A\hat{+}B)}(x)} \\
&= (r_A(x) + r_B(x) - r_A(x) \cdot r_B(x)) \cdot e^{i2\pi(\frac{\omega_{\mu A}(x)}{2\pi} + \frac{\omega_{\mu B}(x)}{2\pi} - \frac{\omega_{\mu A}(x)}{2\pi} \cdot \frac{\omega_{\mu B}(x)}{2\pi})} \qquad (2.9)
\end{aligned}
$$

and

$$
\nu_{A\hat{+}B}(x) = s_{A\hat{+}B}(x) \cdot e^{i\omega_{\nu(A\hat{+}B)}(x)} = (s_A(x) \cdot s_B(x)) \cdot e^{i2\pi(\frac{\omega_{\nu A}(x)}{2\pi} \cdot \frac{\omega_{\nu B}(x)}{2\pi})}. \qquad (2.10)
$$

Example 2.6 Let

$A = \frac{\langle 0.5\cdot e^{i1.2\pi}, 0.4\cdot e^{i0.8\pi}\rangle}{x} + \frac{\langle 0.4\cdot e^{i0.5\pi}, 0.6\cdot e^{i1.3\pi}\rangle}{y} + \frac{\langle 0.3\cdot e^{i2\pi}, 0.5\cdot e^{i1.5\pi}\rangle}{z}$,

$B = \frac{\langle 0.6\cdot e^{i0.2\pi}, 0.3\cdot e^{i1.8\pi}\rangle}{x} + \frac{\langle 0.2\cdot e^{i0.5\pi}, 0.6\cdot e^{i0.5\pi}\rangle}{y} + \frac{\langle 0.7\cdot e^{i\pi}, 0.1\cdot e^{i0.9\pi}\rangle}{z}$,

then $A \widehat{+} B = \frac{\langle 0.8\cdot e^{i1.28\pi}, 0.12\cdot e^{i0.72\pi}\rangle}{x} + \frac{\langle 0.52\cdot e^{i0.875\pi}, 0.36\cdot e^{i0.325\pi}\rangle}{y} + \frac{\langle 0.79\cdot e^{i2\pi}, 0.05\cdot e^{i0.675\pi}\rangle}{z}$. ♢

Theorem 2.4 The complex intuitionistic fuzzy probabilistic sum on $IF^\star(U)$ is an s-norm.

Proof Properties (i), (ii), (v) and (vi) can be easily verified from Definition 2.3. Here we only prove (iii) and (iv).

(iii) Let A, B, C and D be four complex intuitionistic fuzzy sets on U, $\mu_A(x) = r_A(x)\cdot e^{i\omega_{\mu A}(x)}$, $\mu_B(x) = r_B(x)\cdot e^{i\omega_{\mu B}(x)}$, $\mu_C(x) = r_C(x)\cdot e^{i\omega_{\mu C}(x)}$, $\mu_D(x) = r_D(x)\cdot e^{i\omega_{\mu D}(x)}$, $\nu_A(x) = s_A(x)\cdot e^{i\omega_{\nu A}(x)}$, $\nu_B(x) = s_B(x)\cdot e^{i\omega_{\nu B}(x)}$, $\nu_C(x) = s_C(x)\cdot e^{i\omega_{\nu C}(x)}$ and $\nu_D(x) = s_D(x)\cdot e^{i\omega_{\nu D}(x)}$ their membership and non-membership functions, respectively. Suppose $|\mu_A(x)| \le |\mu_C(x)|$, $\omega_{\mu A}(x) \le \omega_{\mu C}(x)$, $|\nu_A(x)| \ge |\nu_C(x)|$, $\omega_{\nu A}(x) \ge \omega_{\nu C}(x)$ and $|\mu_B(x)| \le |\mu_D(x)|$, $\omega_{\mu B}(x) \le \omega_{\mu D}(x)$, $|\nu_B(x)| \ge |\nu_D(x)|$, $\omega_{\nu B}(x) \ge \omega_{\nu D}(x)$, for any $x \in U$. We have

$$|\mu_{A\widehat{+}B}(x)| = |r_A(x)| + |r_B(x)| - |r_A(x)|\cdot|r_B(x)| \le |r_C(x)| + |r_D(x)| - |r_C(x)|\cdot|r_D(x)| = |\mu_{C\widehat{+}D}(x)|,$$

$$\omega_{\mu(A\widehat{+}B)}(x) = 2\pi\left(\frac{\omega_{\mu A}(x)}{2\pi} + \frac{\omega_{\mu B}(x)}{2\pi} - \frac{\omega_{\mu A}(x)}{2\pi}\cdot\frac{\omega_{\mu B}(x)}{2\pi}\right) \le 2\pi\left(\frac{\omega_{\mu C}(x)}{2\pi} + \frac{\omega_{\mu D}(x)}{2\pi} - \frac{\omega_{\mu C}(x)}{2\pi}\cdot\frac{\omega_{\mu D}(x)}{2\pi}\right) = \omega_{\mu(C\widehat{+}D)}(x),$$

and

$$|\nu_{A\widehat{+}B}(x)| = |s_A(x)|\cdot|s_B(x)| \ge |s_C(x)|\cdot|s_D(x)| = |\nu_{C\widehat{+}D}(x)|,$$

$$\omega_{\nu(A\widehat{+}B)}(x) = 2\pi\left(\frac{\omega_{\nu A}(x)}{2\pi}\cdot\frac{\omega_{\nu B}(x)}{2\pi}\right) \ge 2\pi\left(\frac{\omega_{\nu C}(x)}{2\pi}\cdot\frac{\omega_{\nu D}(x)}{2\pi}\right) = \omega_{\nu(C\widehat{+}D)}(x).$$

(iv) Suppose A, B and C be three complex intuitionistic fuzzy sets on U, $\mu_A(x) = r_A(x)\cdot e^{i\omega_{\mu A}(x)}$, $\mu_B(x) = r_B(x)\cdot e^{i\omega_{\mu B}(x)}$, $\mu_C(x) = r_C(x)\cdot e^{i\omega_{\mu C}(x)}$, $\nu_A(x) = s_A(x)\cdot e^{i\omega_{\nu A}(x)}$, $\nu_B(x) = s_B(x)\cdot e^{i\omega_{\nu B}(x)}$ and $\nu_C(x) = s_C(x)\cdot e^{i\omega_{\nu C}(x)}$ their membership and non-membership functions, respectively. Then

$$\begin{aligned}
\mu_{(A\widehat{+}B)\widehat{+}C}(x) &= r_{(A\widehat{+}B)\widehat{+}C}(x)\cdot e^{i\omega_{\mu((A\widehat{+}B)\widehat{+}C)}(x)} \\
&= (r_{A\widehat{+}B}(x) + r_C(x) - r_{A\widehat{+}B}(x)\cdot r_C(x))\cdot e^{i2\pi\left(\frac{\omega_{\mu(A\widehat{+}B)}(x)}{2\pi} + \frac{\omega_{\mu C}(x)}{2\pi} - \frac{\omega_{\mu(A\widehat{+}B)}(x)}{2\pi}\cdot\frac{\omega_{\mu C}(x)}{2\pi}\right)} \\
&= (r_A(x) + r_B(x) - r_A(x)\cdot r_B(x) + r_C(x) - (r_A(x) + r_B(x) - r_A(x)\cdot r_B(x))\cdot r_C(x))\cdot \\
&\quad e^{i2\pi\left(\frac{2\pi\left(\frac{\omega_{\mu A}(x)}{2\pi} + \frac{\omega_{\mu B}(x)}{2\pi} - \frac{\omega_{\mu A}(x)}{2\pi}\cdot\frac{\omega_{\mu B}(x)}{2\pi}\right)}{2\pi} + \frac{\omega_{\mu C}(x)}{2\pi} - \frac{2\pi\left(\frac{\omega_{\mu A}(x)}{2\pi} + \frac{\omega_{\mu B}(x)}{2\pi} - \frac{\omega_{\mu A}(x)}{2\pi}\cdot\frac{\omega_{\mu B}(x)}{2\pi}\right)}{2\pi}\cdot\frac{\omega_{\mu C}(x)}{2\pi}\right)} \\
&= (r_A(x) + r_B(x) + r_C(x) - r_B(x)\cdot r_C(x) - r_A(x)\cdot(r_B(x) + r_C(x) - r_B(x)\cdot r_C(x)))\cdot \\
&\quad e^{i2\pi\left(\frac{\omega_{\mu A}(x)}{2\pi} + \frac{2\pi\left(\frac{\omega_{\mu B}(x)}{2\pi} + \frac{\omega_{\mu C}(x)}{2\pi} - \frac{\omega_{\mu B}(x)}{2\pi}\cdot\frac{\omega_{\mu C}(x)}{2\pi}\right)}{2\pi} - \frac{\omega_{\mu A}(x)}{2\pi}\cdot\frac{2\pi\left(\frac{\omega_{\mu B}(x)}{2\pi} + \frac{\omega_{\mu C}(x)}{2\pi} - \frac{\omega_{\mu B}(x)}{2\pi}\cdot\frac{\omega_{\mu C}(x)}{2\pi}\right)}{2\pi}\right)} \\
&= r_{A\widehat{+}(B\widehat{+}C)}(x)\cdot e^{i\omega_{\mu(A\widehat{+}(B\widehat{+}C))}(x)} \\
&= \mu_{A\widehat{+}(B\widehat{+}C)}(x).
\end{aligned}$$

$$\begin{aligned}
\nu_{(A\widehat{+}B)\widehat{+}C}(x) &= s_{(A\widehat{+}B)\widehat{+}C}(x)\cdot e^{i\omega_{\nu((A\widehat{+}B)\widehat{+}C)}(x)} = (s_{A\widehat{+}B}(x)\cdot s_C(x))\cdot e^{i2\pi\left(\frac{\omega_{\nu(A\widehat{+}B)}(x)}{2\pi}\cdot\frac{\omega_{\nu C}(x)}{2\pi}\right)} \\
&= ((s_A(x)\cdot s_B(x))\cdot s_C(x))\cdot e^{i2\pi\left(\frac{2\pi\left(\frac{\omega_{\nu A}(x)}{2\pi}\cdot\frac{\omega_{\nu B}(x)}{2\pi}\right)}{2\pi}\cdot\frac{\omega_{\nu C}(x)}{2\pi}\right)} \\
&= (s_A(x)\cdot(s_B(x)\cdot s_C(x)))\cdot e^{i2\pi\left(\frac{\omega_{\nu A}(x)}{2\pi}\cdot\frac{2\pi\left(\frac{\omega_{\nu B}(x)}{2\pi}\cdot\frac{\omega_{\nu C}(x)}{2\pi}\right)}{2\pi}\right)} \\
&= \nu_{A\widehat{+}(B\widehat{+}C)}(x).
\end{aligned}$$
□

Corollary 2.6 Let $C_\alpha \in IF^\star(U)$, $\alpha \in I$, $\mu_{C_\alpha}(x) = r_{C_\alpha}(x)\cdot e^{i\omega_{\mu C_\alpha}(x)}$ and $\nu_{C_\alpha}(x) = s_{C_\alpha}(x)\cdot e^{i\omega_{\nu C_\alpha}(x)}$ its membership and non-membership functions, where I is an arbitrary index set. Then $C_1\widehat{+}C_2\widehat{+}\cdots\widehat{+}C_\alpha \in IF^\star(U)$, and its membership and non-membership functions are

$$\mu_{C_1\widehat{+}C_2\widehat{+}\cdots\widehat{+}C_\alpha}(x) = [r_{C_1}(x) + r_{C_2}(x) + \cdots + r_{C_\alpha}(x) - \cdots + (-1)^{\alpha-1}(r_{C_1}(x)\cdot r_{C_2}(x)\cdots r_{C_\alpha}(x))]$$
$$\cdot\, e^{i2\pi\left[\left(\frac{\omega_{\mu C_1}(x)}{2\pi} + \frac{\omega_{\mu C_2}(x)}{2\pi} + \cdots + \frac{\omega_{\mu C_\alpha}(x)}{2\pi}\right) - \cdots + \frac{(-1)^{\alpha-1}}{(2\pi)^2}\left(\frac{\omega_{\mu C_1}(x)}{2\pi}\cdot\frac{\omega_{\mu C_2}(x)}{2\pi}\cdots\frac{\omega_{\mu C_\alpha}(x)}{2\pi}\right)\right]}$$

and

$$\nu_{C_1\hat{+}C_2\hat{+}\cdots\hat{+}C_\alpha}(x) = (s_{C_1}(x)\cdot s_{C_2}(x)\cdots s_{C_\alpha}(x))\cdot e^{i2\pi(\frac{\omega_{\nu C_1}(x)}{2\pi}\cdot\frac{\omega_{\nu C_2}(x)}{2\pi}\cdots\frac{\omega_{\nu C_\alpha}(x)}{2\pi})}.$$

Definition 2.9 (Complex Intuitionistic Fuzzy Bold Sum) Let A and B be two complex intuitionistic fuzzy sets on U, $\mu_A(x) = r_A(x)\cdot e^{i\omega_{\mu A}(x)}$, $\nu_A(x) = s_A(x)\cdot e^{i\omega_{\nu A}(x)}$, $\mu_B(x) = r_B(x)\cdot e^{i\omega_{\mu B}(x)}$ and $\nu_B(x) = s_B(x)\cdot e^{i\omega_{\nu B}(x)}$ their membership and non-membership functions, respectively. The complex intuitionistic fuzzy bold sum of A and B, denoted by $A\dot{\cup}B = \{\langle x, \mu_{A\dot{\cup}B}(x), \nu_{A\dot{\cup}B}(x)\rangle : x\in U\}$, where

$$\mu_{A\dot{\cup}B}(x) = r_{A\dot{\cup}B}(x)\cdot e^{i\omega_{\mu(A\dot{\cup}B)}(x)} = \min(1, r_A(x)+r_B(x))\cdot e^{i\min(2\pi,\omega_{\mu A}(x)+\omega_{\mu B}(x))} \tag{2.11}$$

and

$$\nu_{A\dot{\cup}B}(x) = s_{A\dot{\cup}B}(x)\cdot e^{i\omega_{\nu(A\dot{\cup}B)}(x)} = \max(0, r_A(x)+r_B(x)-1)\cdot e^{i\max(0,\omega_{\mu A}(x)+\omega_{\mu B}(x)-2\pi)}. \tag{2.12}$$

Example 2.7 Let

$A = \frac{\langle 0.5\cdot e^{i1.2\pi}, 0.4\cdot e^{i0.8\pi}\rangle}{x} + \frac{\langle 0.4\cdot e^{i0.5\pi}, 0.6\cdot e^{i1.3\pi}\rangle}{y} + \frac{\langle 0.3\cdot e^{i2\pi}, 0.5\cdot e^{i1.5\pi}\rangle}{z}$,

$B = \frac{\langle 0.6\cdot e^{i0.2\pi}, 0.3\cdot e^{i1.8\pi}\rangle}{x} + \frac{\langle 0.2\cdot e^{i0.5\pi}, 0.6\cdot e^{i0.5\pi}\rangle}{y} + \frac{\langle 0.7\cdot e^{i\pi}, 0.1\cdot e^{i0.9\pi}\rangle}{z}$,

then $A\dot{\cup}B = \frac{\langle 1\cdot e^{i1.4\pi}, 0\cdot e^{i0.6\pi}\rangle}{x} + \frac{\langle 0.6e^{i\pi}, 0.2\cdot e^{i0\pi}\rangle}{y} + \frac{\langle 1\cdot e^{i2\pi}, 0\cdot e^{i0.4\pi}\rangle}{z}$. ♢

Theorem 2.5 The complex intuitionistic fuzzy bold sum on $IF^\star(U)$ is an s-norm.

Proof Properties (i), (ii), (v) and (vi) can be easily verified from Definition 2.3. Here we only prove (iii) and (iv).

(iii) Let A, B, C and D be four complex intuitionistic fuzzy sets on U, $\mu_A(x) = r_A(x)\cdot e^{i\omega_{\mu A}(x)}$, $\mu_B(x) = r_B(x)\cdot e^{i\omega_{\mu B}(x)}$, $\mu_C(x) = r_C(x)\cdot e^{i\omega_{\mu C}(x)}$, $\mu_D(x) = r_D(x)\cdot e^{i\omega_{\mu D}(x)}$, $\nu_A(x) = s_A(x)\cdot e^{i\omega_{\nu A}(x)}$, $\nu_B(x) = s_B(x)\cdot e^{i\omega_{\nu B}(x)}$, $\nu_C(x) = s_C(x)\cdot e^{i\omega_{\nu C}(x)}$ and $\nu_D(x) = s_D(x)\cdot e^{i\omega_{\nu D}(x)}$ their membership and non-membership functions, respectively. Suppose $|\mu_A(x)| \le |\mu_C(x)|$, $\omega_{\mu A}(x)\le\omega_{\mu C}(x)$, $|\nu_A(x)|\ge|\nu_C(x)|$, $\omega_{\nu A}(x)\ge\omega_{\nu C}(x)$ and $|\mu_B(x)|\le|\mu_D(x)|$, $\omega_{\mu B}(x)\le\omega_{\mu D}(x)$, $|\nu_B(x)|\ge|\nu_D(x)|$, $\omega_{\nu B}(x)\ge\omega_{\nu D}(x)$, for any $x\in U$. We have

$$|\mu_{A\dot{\cup}B}(x)| = \min(1, r_A(x)+r_B(x)) \le \min(1, r_C(x)+r_D(x)) = |\mu_{C\dot{\cup}D}(x)|,$$

$$\omega_{\mu(A\dot{\cup}B)}(x) = \min(2\pi, \omega_{\mu A}(x)+\omega_{\mu B}(x)) \le \min(2\pi, \omega_{\mu C}(x)+\omega_{\mu D}(x)) = \omega_{\mu(C\dot{\cup}D)}(x),$$

and

$$|\nu_{A\dot{\cup}B}(x)| = \max(0, s_A(x)+s_B(x)-1) \ge \max(0, s_C(x)+s_D(x)-1) = |\nu_{C\dot{\cup}D}(x)|,$$

$$\omega_{\nu(A\dot{\cup}B)}(x) = \max(0, \omega_{\nu A}(x)+\omega_{\nu B}(x)-2\pi) \ge \max(0, \omega_{\nu C}(x)+\omega_{\nu D}(x)-2\pi) = \omega_{\nu(C\dot{\cup}D)}(x).$$

(iv) Suppose A, B and C be three complex intuitionistic fuzzy sets on U, $\mu_A(x) = r_A(x)\cdot e^{i\omega_{\mu A}(x)}$, $\mu_B(x) = r_B(x)\cdot e^{i\omega_{\mu B}(x)}$, $\mu_C(x) = r_C(x)\cdot e^{i\omega_{\mu C}(x)}$, $\nu_A(x) = s_A(x)\cdot e^{i\omega_{\nu A}(x)}$, $\nu_B(x) = s_B(x)\cdot e^{i\omega_{\nu B}(x)}$ and $\nu_C(x) = s_C(x)\cdot e^{i\omega_{\nu C}(x)}$ their membership and non-membership functions, respectively. Then

$$\begin{aligned}
\mu_{(A\dot{\cup}B)\dot{\cup}C}(x) &= r_{(A\dot{\cup}B)\dot{\cup}C}(x)\cdot e^{i\omega_{\mu((A\dot{\cup}B)\dot{\cup}C)}(x)} = \min(1, r_{A\dot{\cup}B}(x)+r_C(x))\cdot e^{i\min(2\pi,\omega_{\mu(A\dot{\cup}B)}(x)+\omega_{\mu C}(x))}\\
&= \min(1, \min(1, r_A(x)+r_B(x))+r_C(x))\cdot e^{i\min(2\pi,\min(2\pi,\omega_{\mu A}(x)+\omega_{\mu B}(x))+\omega_{\mu C}(x))}\\
&= \min(1, r_A(x)+\min(1, r_B(x)+r_C(x)))\cdot e^{i\min(2\pi,\omega_{\mu A}(x)+\min(2\pi,\omega_{\mu B}(x)+\omega_{\mu C}(x)))}\\
&= \min(1, r_A(x)+r_{B\dot{\cup}C}(x))\cdot e^{i\min(2\pi,\omega_{\mu A}(x)+\omega_{\mu(B\dot{\cup}C)}(x))}\\
&= \mu_{A\dot{\cup}(B\dot{\cup}C)}(x).\\
\nu_{(A\dot{\cup}B)\dot{\cup}C}(x) &= s_{(A\dot{\cup}B)\dot{\cup}C}(x)\cdot e^{i\omega_{\nu((A\dot{\cup}B)\dot{\cup}C)}(x)} = \max(0, s_{A\dot{\cup}B}(x)+s_C(x)-1)\cdot e^{i\max(0,\omega_{\nu(A\dot{\cup}B)}(x)+\omega_{\nu C}(x)-2\pi)}\\
&= \max(0, \max(0, s_A(x)+s_B(x)-1)+s_C(x)-1)\cdot e^{i\max(0,\max(0,\omega_{\nu A}(x)+\omega_{\nu B}(x)-2\pi)+\omega_{\nu C}(x)-2\pi)}\\
&= \max(0, s_A(x)+\max(0, s_B(x)+s_C(x)-1)-1)\cdot e^{i\max(0,\omega_{\nu A}(x)+\max(0,\omega_{\nu B}(x)+\omega_{\nu C}(x)-2\pi)-2\pi)}\\
&= \max(0, s_A(x)+s_{B\dot{\cup}C}(x)-1)\cdot e^{i\max(2\pi,\omega_{\nu A}(x)+\omega_{\nu(B\dot{\cup}C)}(x)-2\pi)}\\
&= \nu_{A\dot{\cup}(B\dot{\cup}C)}(x).
\end{aligned}$$

□

Corollary 2.7 Let $C_\alpha \in IF^\star(U)$, $\alpha \in I$, $\mu_{C_\alpha}(x) = r_{C_\alpha}(x) \cdot e^{i\omega_{\mu C_\alpha}(x)}$ and $\nu_{C_\alpha}(x) = s_{C_\alpha}(x) \cdot e^{i\omega_{\nu C_\alpha}(x)}$ its membership and non-membership functions, where I is an arbitrary index set. Then $C_1 \dot{\cup} C_2 \dot{\cup} \cdots \dot{\cup} C_\alpha \in IF^\star(U)$, and its membership and non-membership functions are

$$\mu_{C_1\dot{\cup}C_2\dot{\cup}\cdots\dot{\cup}C_\alpha}(x) = \min(1, r_{C_1}(x) + r_{C_2}(x) + \cdots + r_{C_\alpha}(x)) \cdot e^{i\min(2\pi, \omega_{\mu C_1}(x) + \omega_{\mu C_2}(x) + \cdots + \omega_{\mu C_\alpha}(x))}$$

and

$$\nu_{C_1\dot{\cup}C_2\dot{\cup}\cdots\dot{\cup}C_\alpha}(x) = \max(0, r_{C_1}(x) + r_{C_2}(x) + \cdots + r_{C_\alpha}(x) - 1) \cdot e^{i\max(0, \omega_{\nu C_1}(x) + \omega_{\nu C_2}(x) + \cdots + \omega_{\nu C_\alpha}(x) - 2\pi)}.$$

Definition 2.10 (Complex Intuitionistic Fuzzy Bold Intersection) Let A and B be two complex intuitionistic fuzzy sets on U, $\mu_A(x) = r_A(x) \cdot e^{i\omega_{\mu A}(x)}$, $\nu_A(x) = s_A(x) \cdot e^{i\omega_{\nu A}(x)}$, $\mu_B(x) = r_B(x) \cdot e^{i\omega_{\mu B}(x)}$ and $\nu_B(x) = s_B(x) \cdot e^{i\omega_{\nu B}(x)}$ their membership and non-membership functions, respectively. The complex intuitionistic fuzzy bold intersection of A and B, denoted by $A \dot{\cap} B = \{\langle x, \mu_{A\dot{\cap}B}(x), \nu_{A\dot{\cap}B}(x)\rangle : x \in U\}$, where

$$\mu_{A\dot{\cap}B}(x) = r_{A\dot{\cap}B}(x) \cdot e^{i\omega_{\mu(A\dot{\cap}B)}(x)} = \max(0, r_A(x) + r_B(x) - 1) \cdot e^{i\max(0, \omega_{\mu A}(x) + \omega_{\mu B}(x) - 2\pi)} \tag{2.13}$$

and

$$\nu_{A\dot{\cap}B}(x) = s_{A\dot{\cap}B}(x) \cdot e^{i\omega_{\nu(A\dot{\cap}B)}(x)} = \min(1, s_A(x) + s_B(x)) \cdot e^{i\min(2\pi, \omega_{\nu A}(x) + \omega_{\nu B}(x))}. \tag{2.14}$$

Example 2.8 Let

$A = \frac{\langle 0.5\cdot e^{i1.2\pi}, 0.4\cdot e^{i0.8\pi}\rangle}{x} + \frac{\langle 0.4\cdot e^{i0.5\pi}, 0.6\cdot e^{i1.3\pi}\rangle}{y} + \frac{\langle 0.3\cdot e^{i2\pi}, 0.5\cdot e^{i1.5\pi}\rangle}{z}$,

$B = \frac{\langle 0.6\cdot e^{i0.2\pi}, 0.3\cdot e^{i1.8\pi}\rangle}{x} + \frac{\langle 0.2\cdot e^{i0.5\pi}, 0.6\cdot e^{i0.5\pi}\rangle}{y} + \frac{\langle 0.7\cdot e^{i\pi}, 0.1\cdot e^{i0.9\pi}\rangle}{z}$,

then $A\dot{\cap}B = \frac{\langle 0.1\cdot e^{i0\pi}, 0.7\cdot e^{i2\pi}\rangle}{x} + \frac{\langle 0\cdot e^{i0\pi}, 1\cdot e^{i1.8\pi}\rangle}{y} + \frac{\langle 0\cdot e^{i\pi}, 0.6\cdot e^{i2\pi}\rangle}{z}$. ♢

Theorem 2.6 The complex intuitionistic fuzzy bold intersection on $IF^\star(U)$ is a t-norm.

Proof Properties (i), (ii), (v) and (vii) can be easily verified from Definition 2.3. Here we only prove (iii) and (iv).

(iii) Let A, B, C and D be four complex intuitionistic fuzzy sets on U, $\mu_A(x) = r_A(x) \cdot e^{i\omega_{\mu A}(x)}$, $\mu_B(x) = r_B(x) \cdot e^{i\omega_{\mu B}(x)}$, $\mu_C(x) = r_C(x) \cdot e^{i\omega_{\mu C}(x)}$, $\mu_D(x) = r_D(x) \cdot e^{i\omega_{\mu D}(x)}$, $\nu_A(x) = s_A(x) \cdot e^{i\omega_{\nu A}(x)}$, $\nu_B(x) = s_B(x) \cdot e^{i\omega_{\nu B}(x)}$, $\nu_C(x) = s_C(x) \cdot e^{i\omega_{\nu C}(x)}$ and $\nu_D(x) = s_D(x) \cdot e^{i\omega_{\nu D}(x)}$ their membership and non-membership functions, respectively. Suppose $|\mu_A(x)| \le |\mu_C(x)|$, $\omega_{\mu A}(x) \le \omega_{\mu C}(x)$, $|\nu_A(x)| \ge |\nu_C(x)|$, $\omega_{\nu A}(x) \ge \omega_{\nu C}(x)$ and $|\mu_B(x)| \le |\mu_D(x)|$, $\omega_{\mu B}(x) \le \omega_{\mu D}(x)$, $|\nu_B(x)| \ge |\nu_D(x)|$, $\omega_{\nu B}(x) \ge \omega_{\nu D}(x)$, for any $x \in U$. We have

$$|\mu_{A\dot{\cap}B}(x)| = \max(0, r_A(x) + r_B(x) - 1) \le \max(0, r_C(x) + r_D(x) - 1) = |\mu_{C\dot{\cap}D}(x)|,$$

$$\omega_{\mu(A\dot{\cap}B)}(x) = \max(0, \omega_{\mu A}(x) + \omega_{\mu B}(x) - 2\pi) \le \max(0, \omega_{\mu C}(x) + \omega_{\mu D}(x) - 2\pi) = \omega_{\mu(C\dot{\cap}D)}(x),$$

and

$$|\nu_{A\dot{\cap}B}(x)| = \min(1, s_A(x) + s_B(x)) \ge \min(1, s_C(x) + s_D(x)) = |\nu_{C\dot{\cap}D}(x)|,$$

$$\omega_{\nu(A\dot{\cap}B)}(x) = \min(2\pi, \omega_{\nu A}(x) + \omega_{\nu B}(x)) \ge \min(2\pi, \omega_{\nu C}(x) + \omega_{\nu D}(x)) = \omega_{\nu(C\dot{\cap}D)}(x).$$

(iv) Suppose A, B and C be three complex intuitionistic fuzzy sets on U, $\mu_A(x) = r_A(x) \cdot e^{i\omega_{\mu A}(x)}$, $\mu_B(x) = r_B(x) \cdot e^{i\omega_{\mu B}(x)}$, $\mu_C(x) = r_C(x) \cdot e^{i\omega_{\mu C}(x)}$, $\nu_A(x) = s_A(x) \cdot e^{i\omega_{\nu A}(x)}$, $\nu_B(x) = s_B(x) \cdot e^{i\omega_{\nu B}(x)}$ and $\nu_C(x) = s_C(x) \cdot e^{i\omega_{\nu C}(x)}$ their membership and non-membership functions, respectively. Then

$$\begin{aligned}\mu_{(A\dot{\cap}B)\dot{\cap}C}(x) &= r_{(A\dot{\cap}B)\dot{\cap}C}(x) \cdot e^{i\omega_{\mu((A\dot{\cap}B)\dot{\cap}C)}(x)} = \max(0, r_{A\dot{\cap}B}(x) + r_C(x) - 1) \cdot e^{i\max(0, \omega_{\mu(A\dot{\cap}B)}(x) + \omega_{\mu C}(x) - 2\pi)} \\ &= \max(0, \max(0, r_A(x) + r_B(x) - 1) + r_C(x) - 1) \cdot e^{i\max(0, \max(0, \omega_{\mu A}(x) + \omega_{\mu B}(x) - 2\pi) + \omega_{\mu C}(x) - 2\pi)} \\ &= \max(0, r_A(x) + \max(0, r_B(x) + r_C(x) - 1) - 1) \cdot e^{i\max(0, \omega_{\mu A}(x) + \max(0, \omega_{\mu B}(x) + \omega_{\mu C}(x) - 2\pi) - 2\pi)}\end{aligned}$$

$$
\begin{aligned}
&= \max(0, r_A(x) + r_{B\dot\cap C}(x) - 1) \cdot e^{i\max(2\pi, \omega_{\mu A}(x) + \omega_{\mu(B\dot\cap C)}(x) - 2\pi)} \\
&= \mu_{A\dot\cap(B\dot\cap C)}(x).
\end{aligned}
$$

$$
\begin{aligned}
\nu_{(A\dot\cap B)\dot\cap C}(x) &= s_{(A\dot\cap B)\dot\cap C}(x) \cdot e^{i\omega_{\nu((A\dot\cap B)\dot\cap C)}(x)} = \min(1, s_{A\dot\cap B}(x) + s_C(x)) \cdot e^{i\min(2\pi, \omega_{\nu(A\dot\cap B)}(x) + \omega_{\nu C}(x))} \\
&= \min(1, \min(1, s_A(x) + s_B(x)) + s_C(x)) \cdot e^{i\min(2\pi, \min(2\pi, \omega_{\nu A}(x) + \omega_{\nu B}(x)) + \omega_{\nu C}(x))} \\
&= \min(1, s_A(x) + \min(1, s_B(x) + s_C(x))) \cdot e^{i\min(2\pi, \omega_{\nu A}(x) + \min(2\pi, \omega_{\nu B}(x) + \omega_{\nu C}(x)))} \\
&= \min(1, s_A(x) + s_{B\dot\cap C}(x)) \cdot e^{i\min(2\pi, \omega_{\nu A}(x) + \omega_{\nu(B\dot\cap C)}(x))} \\
&= \nu_{A\dot\cap(B\dot\cap C)}(x).
\end{aligned}
$$

□

Corollary 2.8 Let $C_\alpha \in IF^\star(U)$, $\alpha \in I$, $\mu_{C_\alpha}(x) = r_{C_\alpha}(x) \cdot e^{i\omega_{\mu C_\alpha}(x)}$ and $\nu_{C_\alpha}(x) = s_{C_\alpha}(x) \cdot e^{i\omega_{\nu C_\alpha}(x)}$ its membership and non-membership functions, where I is an arbitrary index set. Then $C_1 \dot\cap C_2 \dot\cap \cdots \dot\cap C_\alpha \in IF^\star(U)$, and its membership and non-membership functions are

$$
\mu_{C_1\dot\cap C_2\dot\cap\cdots\dot\cap C_\alpha}(x) = \max\left(0, r_{C_1}(x) + r_{C_2}(x) + \cdots + r_{C_\alpha}(x) - 1\right) \cdot e^{i\max\left(0, \omega_{\mu C_1}(x) + \omega_{\mu C_2}(x) + \cdots + \omega_{\mu C_\alpha}(x) - 2\pi\right)}
$$

and

$$
\nu_{C_1\dot\cap C_2\dot\cap\cdots\dot\cap C_\alpha}(x) = \min\left(1, r_{C_1}(x) + r_{C_2}(x) + \cdots + r_{C_\alpha}(x)\right) \cdot e^{i\min\left(2\pi, \omega_{\mu C_1}(x) + \omega_{\mu C_2}(x) + \cdots + \omega_{\mu C_\alpha}(x)\right)}.
$$

Definition 2.11 (Complex Intuitionistic Fuzzy Bounded Difference) Let A and B be two complex intuitionistic fuzzy sets on U, $\mu_A(x) = r_A(x) \cdot e^{i\omega_{\mu A}(x)}$, $\nu_A(x) = s_A(x) \cdot e^{i\omega_{\nu A}(x)}$, $\mu_B(x) = r_B(x) \cdot e^{i\omega_{\mu B}(x)}$ and $\nu_B(x) = s_B(x) \cdot e^{i\omega_{\nu B}(x)}$ their membership and non-membership functions, respectively. The complex intuitionistic fuzzy bounded difference of A and B, denoted by $A|-|B = \{\langle x, \mu_{A|-|B}(x), \nu_{A|-|B}(x)\rangle : x \in U\}$, where

$$
\mu_{A|-|B}(x) = r_{A|-|B}(x) \cdot e^{i\omega_{\mu(A|-|B)}(x)} = \max\left(0, r_A(x) - r_B(x)\right) \cdot e^{i\max\left(0, \omega_{\mu A}(x) - \omega_{\mu B}(x)\right)} \tag{2.15}
$$

and

$$
\nu_{A|-|B}(x) = s_{A|-|B}(x) \cdot e^{i\omega_{\nu(A|-|B)}(x)} = \min\left(1, 1 - s_A(x) + s_B(x)\right) \cdot e^{i\min\left(2\pi, 2\pi - \omega_{\nu A}(x) + \omega_{\nu B}(x)\right)}. \tag{2.16}
$$

Example 2.9 Let

$A = \frac{\langle 0.5\cdot e^{i1.2\pi}, 0.4\cdot e^{i0.8\pi}\rangle}{x} + \frac{\langle 0.4\cdot e^{i0.5\pi}, 0.6\cdot e^{i1.3\pi}\rangle}{y} + \frac{\langle 0.3\cdot e^{i2\pi}, 0.5\cdot e^{i1.5\pi}\rangle}{z}$,

$B = \frac{\langle 0.6\cdot e^{i0.2\pi}, 0.3\cdot e^{i1.8\pi}\rangle}{x} + \frac{\langle 0.2\cdot e^{i0.5\pi}, 0.6\cdot e^{i0.5\pi}\rangle}{y} + \frac{\langle 0.7\cdot e^{i\pi}, 0.1\cdot e^{i0.9\pi}\rangle}{z}$,

then $A|-|B = \frac{\langle 0\cdot e^{i\pi}, 0.9\cdot e^{i2\pi}\rangle}{x} + \frac{\langle 0.2\cdot e^{i0\pi}, 1\cdot e^{i1.2\pi}\rangle}{y} + \frac{\langle 0\cdot e^{i\pi}, 0.6\cdot e^{i1.4\pi}\rangle}{z}$. ♢

Definition 2.12 (Complex Intuitionistic Fuzzy Symmetrical Difference) Let A and B be two complex intuitionistic fuzzy sets on U, $\mu_A(x) = r_A(x) \cdot e^{i\omega_{\mu A}(x)}$, $\nu_A(x) = s_A(x) \cdot e^{i\omega_{\nu A}(x)}$, $\mu_B(x) = r_B(x) \cdot e^{i\omega_{\mu B}(x)}$ and $\nu_B(x) = s_B(x) \cdot e^{i\omega_{\nu B}(x)}$ their membership and non-membership functions, respectively. The complex intuitionistic fuzzy symmetrical difference of A and B, denoted by $A\nabla B = \{\langle x, \mu_{A\nabla B}(x), \nu_{A\nabla B}(x)\rangle : x \in U\}$, where

$$
\mu_{A\nabla B}(x) = r_{A\nabla B}(x) \cdot e^{i\omega_{\mu(A\nabla B)}(x)} = |r_A(x) - r_B(x)| \cdot e^{i|\omega_{\mu A}(x) - \omega_{\mu B}(x)|} \tag{2.17}
$$

and

$$
\nu_{A\nabla B}(x) = s_{A\nabla B}(x) \cdot e^{i\omega_{\nu(A\nabla B)}(x)} = |1 - s_B(x) - s_A(x)| \cdot e^{i|2\pi - \omega_{\nu B}(x) - \omega_{\nu A}(x)|}. \tag{2.18}
$$

Example 2.10 Let

$A = \frac{\langle 0.5\cdot e^{i1.2\pi}, 0.4\cdot e^{i0.8\pi}\rangle}{x} + \frac{\langle 0.4\cdot e^{i0.5\pi}, 0.6\cdot e^{i1.3\pi}\rangle}{y} + \frac{\langle 0.3\cdot e^{i2\pi}, 0.5\cdot e^{i1.5\pi}\rangle}{z}$,

$B = \frac{\langle 0.6\cdot e^{i0.2\pi}, 0.3\cdot e^{i1.8\pi}\rangle}{x} + \frac{\langle 0.2\cdot e^{i0.5\pi}, 0.6\cdot e^{i0.5\pi}\rangle}{y} + \frac{\langle 0.7\cdot e^{i\pi}, 0.1\cdot e^{i0.9\pi}\rangle}{z}$,

then $A\nabla B = \frac{\langle 0.1\cdot e^{i\pi}, 0.3\cdot e^{i0.6\pi}\rangle}{x} + \frac{\langle 0.2\cdot e^{i0\pi}, 0.2\cdot e^{i0.2\pi}\rangle}{y} + \frac{\langle 0.4\cdot e^{i\pi}, 0.4\cdot e^{i0.4\pi}\rangle}{z}$. ♢

Definition 2.13 (Complex Intuitionistic Fuzzy Convex Linear Sum) Let A and B be two complex intuitionistic fuzzy sets on U, $\mu_{_A}(x) = r_{_A}(x) \cdot e^{i\omega_{\mu A}(x)}$, $\nu_{_A}(x) = s_{_A}(x) \cdot e^{i\omega_{\nu A}(x)}$, $\mu_{_B}(x) = r_{_B}(x) \cdot e^{i\omega_{\mu B}(x)}$ and $\nu_{_B}(x) = s_{_B}(x) \cdot e^{i\omega_{\nu B}(x)}$ their membership and non-membership functions, respectively. The complex intuitionistic fuzzy convex linear sum of min and max of A and B, denoted by $A||_\lambda B = \{\langle x, \mu_{_{A||_\lambda B}}(x), \nu_{_{A||_\lambda B}}(x)\rangle : x \in U\}$, $\lambda \in [0,1]$, where

$$\mu_{_{A||_\lambda B}}(x) = r_{_{A||_\lambda B}}(x) \cdot e^{i\omega_{\mu(A||_\lambda B)}(x)} = [\lambda \min(r_{_A}(x), r_{_B}(x)) + (1-\lambda)\max(r_{_A}(x), r_{_B}(x))] \cdot e^{i[\lambda \min(\omega_{\mu A}(x), \omega_{\mu B}(x)) + (1-\lambda)\max(\omega_{\mu A}(x), \omega_{\mu B}(x))]}, \tag{2.19}$$

and

$$\nu_{_{A||_\lambda B}}(x) = s_{_{A||_\lambda B}}(x) \cdot e^{i\omega_{\nu(A||_\lambda B)}(x)} = [\lambda \max(s_{_A}(x), s_{_B}(x)) + (1-\lambda)\min(s_{_A}(x), s_{_B}(x))] \cdot e^{i[\lambda \max(\omega_{\nu A}(x), \omega_{\nu B}(x)) + (1-\lambda)\min(\omega_{\nu A}(x), \omega_{\nu B}(x))]}. \tag{2.20}$$

Example 2.11 Let

$A = \frac{\langle 0.5 \cdot e^{i1.2\pi}, 0.4 \cdot e^{i0.8\pi}\rangle}{x} + \frac{\langle 0.4 \cdot e^{i0.5\pi}, 0.6 \cdot e^{i1.3\pi}\rangle}{y} + \frac{\langle 0.3 \cdot e^{i2\pi}, 0.5 \cdot e^{i1.5\pi}\rangle}{z}$,

$B = \frac{\langle 0.6 \cdot e^{i0.2\pi}, 0.3 \cdot e^{i1.8\pi}\rangle}{x} + \frac{\langle 0.2 \cdot e^{i0.5\pi}, 0.6 \cdot e^{i0.5\pi}\rangle}{y} + \frac{\langle 0.7 \cdot e^{i\pi}, 0.1 \cdot e^{i0.9\pi}\rangle}{z}$,

then $A||_\lambda B = \frac{\langle 0.57 \cdot e^{i0.9\pi}, 0.33 \cdot e^{i1.1\pi}\rangle}{x} + \frac{\langle 0.34 \cdot e^{i0.5\pi}, 0.6 \cdot e^{i0.74\pi}\rangle}{y} + \frac{\langle 0.58 \cdot e^{i1.7\pi}, 0.22 \cdot e^{i1.08\pi}\rangle}{z}$ when $\lambda = 0.3$. $\diamondsuit$

3 (α, β)-Equalities of complex intuitionistic fuzzy sets

In this section, we define a new distance measure for complex intuitionistic fuzzy sets. The distance of two complex intuitionistic fuzzy sets measures the difference between the grades of two complex intuitionistic fuzzy sets as well as that between the phases of the two complex intuitionistic fuzzy sets. This distance measure is then used to define (α, β)-equalities of complex intuitionistic fuzzy sets which coincide with those of intuitionistic fuzzy sets already defined in the literature if complex intuitionistic fuzzy sets reduce to traditional intuitionistic fuzzy sets.

Definition 3.1 A distance between two complex intuitionistic fuzzy sets is a function $d : (IF^\star(U), IF^\star(U)) \to [0,1]$, for any $A, B, C \in IF^\star(U)$, satisfying the following properties:

(i) $0 \le d(A,B) \le 1$, $d(A,B) = 0$ if and only if $A = B$;

(ii) $d(A,B) = d(B,A)$; and

(iii) $d(A,B) \le d(A,C) + d(C,B)$.

In the following, we introduce two functions $\rho(\mu_{_A}, \mu_{_B})$ and $\rho(\nu_{_A}, \nu_{_B})$ which play an important role in the remainder of this paper.

Let A and B be two complex intuitionistic fuzzy sets on U, $\mu_{_A}(x) = r_{_A}(x) \cdot e^{i\omega_{\mu A}(x)}$, $\mu_{_B}(x) = r_{_B}(x) \cdot e^{i\omega_{\mu B}(x)}$, $\nu_{_A}(x) = s_{_A}(x) \cdot e^{i\omega_{\nu A}(x)}$ and $\nu_{_B}(x) = s_{_B}(x) \cdot e^{i\omega_{\nu B}(x)}$ their membership and non-membership functions, respectively. We define

Definition 3.2

$$\rho(\mu_{_A}, \mu_{_B}) = \max(\sup_{x \in U} |r_{_A}(x) - r_{_B}(x)|, \frac{1}{2\pi} \sup_{x \in U} |\omega_{\mu A}(x) - \omega_{\mu B}(x)|) \tag{3.1}$$

and

$$\rho(\nu_{_A}, \nu_{_B}) = \max(\sup_{x \in U} |s_{_A}(x) - s_{_B}(x)|, \frac{1}{2\pi} \sup_{x \in U} |\omega_{\nu A}(x) - \omega_{\nu B}(x)|), \tag{3.2}$$

then

$$d(A,B) = \frac{1}{2}(\rho(\mu_{_A}, \mu_{_B}) + \rho(\nu_{_A}, \nu_{_B})). \tag{3.3}$$

Theorem 3.1 $d(A,B)$ defined by the equality (3.3) is a distance function of complex intuitionistic fuzzy sets on U.

Proof (i) and (ii) can be easily verified from Definition 3.2. Here we only prove (iii).

Let $A,\ B$ and C be complex intuitionistic fuzzy sets on U, $\mu_A(x) = r_A(x)\cdot e^{i\omega_{\mu A}(x)}$, $\mu_B(x) = r_B(x)\cdot e^{i\omega_{\mu B}(x)}$, $\mu_C(x) = r_C(x)\cdot e^{i\omega_{\mu C}(x)}$, $\nu_A(x) = s_A(x)\cdot e^{i\omega_{\nu A}(x)}$, $\nu_B(x) = s_B(x)\cdot e^{i\omega_{\nu B}(x)}$ and $\nu_C(x) = s_C(x)\cdot e^{i\omega_{\nu C}(x)}$ their membership and non-membership functions, respectively.

(iii) $d(A,B) = \frac{1}{2}(\rho(\mu_A,\mu_B)+\rho(\nu_A,\nu_B))$
$= \frac{1}{2}(\max(\sup_{x\in U}|r_A(x)-r_B(x)|, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A}(x)-\omega_{\mu B}(x)|)+\max(\sup_{x\in U}|s_A(x)-s_B(x)|, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A}(x)-\omega_{\nu B}(x)|))$
$\le \frac{1}{2}(\max(\sup_{x\in U}(|r_A(x)-r_C(x)|+|r_C(x)-r_B(x)|), \frac{1}{2\pi}\sup_{x\in U}(|\omega_{\mu A}(x)-\omega_{\mu C}(x)|+|\omega_{\mu C}(x)-\omega_{\mu B}(x)|))+\max(\sup_{x\in U}(|s_A(x)-s_C(x)|+|s_C(x)-s_B(x)|), \frac{1}{2\pi}\sup_{x\in U}(|\omega_{\nu A}(x)-\omega_{\nu C}(x)|+|\omega_{\nu C}(x)-\omega_{\nu B}(x)|)))$
$= \frac{1}{2}(\max(\sup_{x\in U}|r_A(x)-r_C(x)|, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A}(x)-\omega_{\mu C}(x)|)+\max(\sup_{x\in U}|s_A(x)-s_C(x)|, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A}(x)-\omega_{\nu C}(x)|)+\frac{1}{2}(\max(\sup_{x\in U}|r_C(x)-r_B(x)|, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu C}(x)-\omega_{\mu B}(x)|)+\max(\sup_{x\in U}|s_C(x)-s_B(x)|, \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu C}(x)-\omega_{\nu B}(x)|))$
$= \frac{1}{2}(\rho(\mu_A,\mu_C)+\rho(\nu_A,\nu_C))+\frac{1}{2}(\rho(\mu_B,\mu_C)+\rho(\nu_B,\nu_C))$
$= d(A,C)+d(C,B).$ □

Example 3.1 Let
$A = \frac{\langle 0.5\cdot e^{i1.2\pi}, 0.4\cdot e^{i0.8\pi}\rangle}{x}+\frac{\langle 0.4\cdot e^{i0.5\pi}, 0.6\cdot e^{i1.3\pi}\rangle}{y}+\frac{\langle 0.3\cdot e^{i2\pi}, 0.5\cdot e^{i1.5\pi}\rangle}{z}$,
$B = \frac{\langle 0.6\cdot e^{i0.2\pi}, 0.3\cdot e^{i1.8\pi}\rangle}{x}+\frac{\langle 0.2\cdot e^{i0.5\pi}, 0.6\cdot e^{i0.5\pi}\rangle}{y}+\frac{\langle 0.7\cdot e^{i\pi}, 0.1\cdot e^{i0.9\pi}\rangle}{z}$.

Since $\sup_{x\in U}|r_A(x)-r_B(x)| = 0.4$, $\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A}(x)-\omega_{\mu B}(x)| = 0.5$, $\sup_{x\in U}|s_A(x)-s_B(x)| = 0.4$, and $\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A}(x)-\omega_{\nu B}(x)| = 0.5$, therefore $\rho(\mu_A,\mu_B) = 0.5$ and $\rho(\nu_A,\nu_B) = 0.5$, so $d(A,B) = \frac{1}{2}(\rho(\mu_A,\mu_B)+\rho(\nu_A,\nu_B)) = 0.5$. ♢

Note 3.1 It is easy to see that, if A and B are two intuitionistic fuzzy sets on U, then

$$\rho(\mu_A,\mu_B) = \sup_{x\in U}|\mu_A(x)-\mu_B(x)|,\ \rho(\nu_A,\nu_B) = \sup_{x\in U}|\nu_A(x)-\nu_B(x)|$$

and

$$d(A,B) = \frac{1}{2}(\rho(\mu_A,\mu_B)+\rho(\nu_A,\nu_B)).$$

Definition 3.3 ([20]) Let U be an universe of discourse, A and B be two intuitionistic fuzzy sets on U, $\mu_A(x),\ \mu_B(x),\ \nu_A(x)$ and $\nu_B(x)$ their membership and non-membership functions, respectively. Then A and B are said to be (α,β)-equal, if and only if

$$\sup_{x\in U}|\mu_A(x)-\mu_B(x)| \le 1-\alpha,\ \sup_{x\in U}|\nu_A(x)-\nu_B(x)| \le \beta,$$

where $0\le\alpha\le 1,\ 0\le\beta\le 1$, and $\alpha+\beta\le 1$. Symbolically, we denote $A = (\alpha,\beta)B$. In this way we say A and B construct a (α,β)-equality.

Lemma 3.1 Let

$$\alpha_1 * \alpha_2 = \max(0, \alpha_1+\alpha_2-1) \tag{3.4}$$

and

$$\beta_1 * \beta_2 = \min(1, \beta_1+\beta_2), \tag{3.5}$$

where $0\le\alpha_1,\alpha_2\le 1,\ 0\le\beta_1,\beta_2\le 1$ and $\alpha_1+\beta_1\le 1,\ \alpha_2+\beta_2\le 1$. Then

(i) $0*\alpha_1 = 0,\ 0*\beta_1 = \beta_1,\ \forall\ \alpha_1\in[0,1],\ \beta_1\in[0,1]$;

(ii) $1 * \alpha_1 = \alpha_1,\ 1 * \beta_1 = 1,\ \forall\ \alpha_1 \in [0,1],\ \beta_1 \in [0,1]$;

(iii) $0 \le \alpha_1 * \alpha_2 \le 1,\ 0 \le \beta_1 * \beta_2 \le 1,\ \forall\ \alpha_1,\ \alpha_2 \in [0,1],\ \beta_1,\ \beta_2 \in [0,1]$;

(iv) $\alpha_1 \le \alpha \Rightarrow \alpha_1 * \alpha_2 \le \alpha * \alpha_2,\ \beta_1 \le \beta \Rightarrow \beta_1 * \beta_2 \le \beta * \beta_2,\ \forall\ \alpha_1,\ \alpha,\ \alpha_2 \in [0,1],\ \beta_1,\ \beta,\ \beta_2 \in [0,1]$;

(v) $\alpha_1 * \alpha_2 = \alpha_2 * \alpha_1,\ \beta_1 * \beta_2 = \beta_2 * \beta_1,\ \forall\ \alpha_1,\ \alpha_2 \in [0,1],\ \beta_1,\ \beta_2 \in [0,1]$;

(vi) $(\alpha_1 * \alpha_2) * \alpha_3 = \alpha_1 * (\alpha_2 * \alpha_3),\ (\beta_1 * \beta_2) * \beta_3 = \beta_1 * (\beta_2 * \beta_3),\ \forall\ \alpha_1,\ \alpha_2,\ \alpha_3 \in [0,1],\ \beta_1,\ \beta_2,\ \beta_3 \in [0,1]$.

Definition 3.4 Let A and B be two complex intuitionistic fuzzy sets on U, $\mu_A(x) = r_A(x) \cdot e^{i\omega_{\mu A}(x)}$, $\mu_B(x) = r_B(x) \cdot e^{i\omega_{\mu B}(x)}$, $\nu_A(x) = s_A(x) \cdot e^{i\omega_{\nu A}(x)}$ and $\nu_B(x) = s_B(x) \cdot e^{i\omega_{\nu B}(x)}$ their membership and non-membership functions, respectively. Then A and B are said to be (α, β)-equal, if and only if

$$\rho(\mu_A, \mu_B) \le 1 - \alpha,\ \rho(\nu_A, \nu_B) \le \beta, \tag{3.6}$$

where $0 \le \alpha \le 1,\ 0 \le \beta \le 1$, and $\alpha + \beta \le 1$. Symbolically, we denote $A = (\alpha, \beta)B$. In this way we say A and B construct a (α, β)-equality.

Note 3.2 Two complex intuitionistic fuzzy sets A and B are said to build a (α, β)-equality if $\rho(\mu_A, \mu_B) \le 1 - \alpha$ and $\rho(\nu_A, \nu_B) \le \beta$. An advantage of using $1 - \alpha$ rather than δ is that the interpretation of α can comply with common sense. That is, the greater α is, the more equal the two complex intuitionistic fuzzy sets are; the smaller β is, the more equal the two complex intuitionistic fuzzy sets are; and if $\alpha = 1$ or $\beta = 0$, then the two complex intuitionistic fuzzy sets are strictly equal.

Theorem 3.2 Let A and B be two complex intuitionistic fuzzy sets on U. Then

(i) $A = (0,1)B$;

(ii) $A = (1,0)B \Leftrightarrow A = B$;

(iii) $A = (\alpha, \beta)B \Leftrightarrow B = (\alpha, \beta)A$;

(iv) $A = (\alpha_1, \beta_1)B$ and $\alpha_1 \ge \alpha_2,\ \beta_1 \le \beta_2 \Rightarrow A = (\alpha_2, \beta_2)B$;

(v) If $\forall\ i \in I,\ A = (\alpha_i, \beta_i)B$, where I is an index set and $\sup_{i\in I} \alpha_i + \sup_{i\in I} \beta_i \le 1$, then $A = (\sup_{i\in I} \alpha_i, \sup_{i\in I} \beta_i)B$;

(vi) Let $A = (\alpha_1, \beta_1)B$. If there exists an unique α and β, such $A = (\alpha, \beta)B$ for any A and B, then $\alpha \le \alpha_1,\ \beta \ge \beta_1$.

Proof Properties (i)-(iv) can be easily proved. Here we only prove properties (v) and (vi).

(v) Since $A = (\alpha_i, \beta_i)B$ for any $i \in I$, we have

$$\rho(\mu_A, \mu_B) = \max(\sup_{x\in U} |r_A(x) - r_B(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu A}(x) - \omega_{\mu B}(x)|) \le 1 - \alpha_i,$$

and

$$\rho(\nu_A, \nu_B) = \max(\sup_{x\in U} |s_A(x) - s_B(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu A}(x) - \omega_{\nu B}(x)|) \le \beta_i,$$

therefore

$$\sup_{x\in U} |r_A(x) - r_B(x)| \le 1 - \sup_{i\in I} \alpha_i,\ \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu A}(x) - \omega_{\mu B}(x)| \le 1 - \sup_{i\in I} \alpha_i,$$

and

$$\sup_{x\in U} |s_A(x) - s_B(x)| \le \sup_{i\in I} \beta_i,\ \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu A}(x) - \omega_{\nu B}(x)| \le \sup_{i\in I} \beta_i,$$

hence

$$\rho(\mu_A, \mu_B) = \max(\sup_{x\in U} |r_A(x) - r_B(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu A}(x) - \omega_{\mu B}(x)|) \le 1 - \sup_{i\in I} \alpha_i,$$

and

$$\rho(\nu_A, \nu_B) = \max(\sup_{x\in U} |s_A(x) - s_B(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu A}(x) - \omega_{\nu B}(x)|) \le \sup_{i\in I} \beta_i.$$

It implies that $A = (\sup_{i\in I} \alpha_i, \sup_{i\in I} \beta_i)B$.

(vi) Let $\alpha_1 = 1 - \rho(\mu_A, \mu_B),\ \beta_1 = \rho(\nu_A, \nu_B)$. Then $A = (\alpha_1, \beta_1)B$. Obviously, if $A = (\alpha, \beta)B$, we have $1 - \alpha_1 = \rho(\mu_A, \mu_B) \le 1 - \alpha$ and $\beta_1 = \rho(\mu_A, \mu_B) \le \beta$. There must be $\alpha \le \alpha_1,\ \beta \ge \beta_1$. □

Theorem 3.3 If $A = (\alpha_1, \beta_1)B$ and $B = (\alpha_2, \beta_2)C$, then $A = (\alpha, \beta)C$, where $\alpha = \alpha_1 * \alpha_2,\ \beta = 1 - \alpha_1 * \alpha_2$.

Proof Since $A = (\alpha_1, \beta_1)B$ and $B = (\alpha_2, \beta_2)C$, we have

$$\rho(\mu_{_A}, \mu_{_B}) = \max(\sup_{x\in U} |r_{_A}(x) - r_{_B}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu A}(x) - \omega_{\mu B}(x)|) \leq 1 - \alpha_1,$$

$$\rho(\nu_{_A}, \nu_{_B}) = \max(\sup_{x\in U} |s_{_A}(x) - s_{_B}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu A}(x) - \omega_{\nu B}(x)|) \leq \beta_1,$$

and

$$\rho(\mu_{_B}, \mu_{_C}) = \max(\sup_{x\in U} |r_{_B}(x) - r_{_C}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu B}(x) - \omega_{\mu C}(x)|) \leq 1 - \alpha_2,$$

$$\rho(\nu_{_B}, \nu_{_C}) = \max(\sup_{x\in U} |s_{_B}(x) - s_{_C}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu B}(x) - \omega_{\nu C}(x)|) \leq \beta_2,$$

therefore

$$\sup_{x\in U} |r_{_A}(x) - r_{_B}(x)| \leq 1 - \alpha_1,\ \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu A}(x) - \omega_{\mu B}(x)| \leq 1 - \alpha_1,$$

$$\sup_{x\in U} |s_{_A}(x) - s_{_B}(x)| \leq \beta_1,\ \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu A}(x) - \omega_{\nu B}(x)| \leq \beta_1,$$

and

$$\sup_{x\in U} |r_{_B}(x) - r_{_C}(x)| \leq 1 - \alpha_2,\ \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu B}(x) - \omega_{\mu C}(x)| \leq 1 - \alpha_2,$$

$$\sup_{x\in U} |s_{_B}(x) - s_{_C}(x)| \leq \beta_2,\ \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu B}(x) - \omega_{\nu C}(x)| \leq \beta_2,$$

consequently, we have

$\rho(\mu_{_A}, \mu_{_C}) = \max(\sup_{x\in U} |r_{_A}(x) - r_{_C}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu A}(x) - \omega_{\mu C}(x)|)$
$\leq \max(\sup_{x\in U} |r_{_A}(x) - r_{_B}(x)| + \sup_{x\in U} |r_{_B}(x) - r_{_C}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu A}(x) - \omega_{\mu B}(x)| + \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu B}(x) - \omega_{\mu C}(x)|)$
$\leq \max((1 - \alpha_1) + (1 - \alpha_2), (1 - \alpha_1) + (1 - \alpha_2)) = (1 - \alpha_1) + (1 - \alpha_2) = 1 - (\alpha_1 + \alpha_2 - 1),$
furthermore, note that $\rho(\mu_{_A}, \mu_{_C}) \leq 1$. Hence

$$\rho(\mu_{_A}, \mu_{_C}) \leq 1 - \max(0, \alpha_1 + \alpha_2 - 1) = 1 - \alpha_1 * \alpha_2 = 1 - \alpha.$$

$\rho(\nu_{_A}, \nu_{_C}) = \max(\sup_{x\in U} |s_{_A}(x) - s_{_C}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu A}(x) - \omega_{\nu C}(x)|)$
$\leq \max(\sup_{x\in U} |s_{_A}(x) - s_{_B}(x)| + \sup_{x\in U} |s_{_B}(x) - s_{_C}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu A}(x) - \omega_{\nu B}(x)| + \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu B}(x) - \omega_{\nu C}(x)|)$
$\leq \max(\beta_1 + \beta_2, \beta_1 + \beta_2) = \beta_1 + \beta_2.$

That is to say

$$\rho(\nu_{_A}, \nu_{_C}) \leq \beta_1 + \beta_2 \leq 1 - \alpha_1 + 1 - \alpha_2 = 1 - (\alpha_1 + \alpha_2 - 1)$$
$$= 1 - \max(0, \alpha_1 + \alpha_2 - 1) = 1 - \alpha_1 * \alpha_2 = \beta.$$

It implies that $A = (\alpha, \beta)C$. □

Theorem 3.4 If $A_1 = (\alpha_1, \beta_1)B_1$ and $A_2 = (\alpha_2, \beta_2)B_2$, then $A_1 \cup A_2 = (\min(\alpha_1, \alpha_2), \max(\beta_1, \beta_2))(B_1 \cup B_2)$.

Proof Since $A_1 = (\alpha_1, \beta_1)B_1$ and $A_2 = (\alpha_2, \beta_2)B_2$, we have

$$\rho(\mu_{_{A_1}}, \mu_{_{B_1}}) = \max(\sup_{x\in U} |r_{_{A_1}}(x) - r_{_{B_1}}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu A_1}(x) - \omega_{\mu B_1}(x)|) \leq 1 - \alpha_1,$$

$$\rho(\nu_{_{A_1}}, \nu_{_{B_1}}) = \max(\sup_{x\in U} |s_{_{A_1}}(x) - s_{_{B_1}}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu A_1}(x) - \omega_{\nu B_1}(x)|) \leq \beta_1,$$

and

$$\rho(\mu_{A_2}, \mu_{B_2}) = \max(\sup_{x\in U} |r_{A_2}(x) - r_{B_2}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu A_2}(x) - \omega_{\mu B_2}(x)|) \leq 1 - \alpha_2,$$

$$\rho(\nu_{A_2}, \nu_{B_2}) = \max(\sup_{x\in U} |s_{A_2}(x) - s_{B_2}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu A_2}(x) - \omega_{\nu B_2}(x)|) \leq \beta_2,$$

therefore

$$\sup_{x\in U} |r_{A_1}(x) - r_{B_1}(x)| \leq 1 - \alpha_1, \ \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu A_1}(x) - \omega_{\mu B_1}(x)| \leq 1 - \alpha_1,$$

$$\sup_{x\in U} |s_{A_1}(x) - s_{B_1}(x)| \leq \beta_1, \ \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu A_1}(x) - \omega_{\nu B_1}(x)| \leq \beta_1,$$

and

$$\sup_{x\in U} |r_{A_2}(x) - r_{B_2}(x)| \leq 1 - \alpha_2, \ \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu A_2}(x) - \omega_{\mu B_2}(x)| \leq 1 - \alpha_2,$$

$$\sup_{x\in U} |s_{A_2}(x) - s_{B_2}(x)| \leq \beta_2, \ \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu A_2}(x) - \omega_{\nu B_2}(x)| \leq \beta_2,$$

consequently, we have

$$\rho(\mu_{A_1\cup A_2}, \mu_{B_1\cup B_2}) = \max(\sup_{x\in U} |r_{A_1\cup A_2}(x) - r_{B_1\cup B_2}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu(A_1\cup A_2)}(x) - \omega_{\mu(B_1\cup B_2)}(x)|),$$

$$\rho(\nu_{A_1\cup A_2}, \nu_{B_1\cup B_2}) = \max(\sup_{x\in U} |s_{A_1\cup A_2}(x) - s_{B_1\cup B_2}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu(A_1\cup A_2)}(x) - \omega_{\nu(B_1\cup B_2)}(x)|),$$

where

$$\begin{aligned}
\sup_{x\in U} |r_{A_1\cup A_2}(x) - r_{B_1\cup B_2}(x)| &= \sup_{x\in U} |\max(r_{A_1}(x), r_{A_2}(x)) - \max(r_{B_1}(x), r_{B_2}(x))| \\
&\leq \sup_{x\in U} \max(|r_{A_1}(x) - r_{B_1}(x)|, |r_{A_2}(x) - r_{B_2}(x)|) \\
&\leq \sup_{x\in U} \max(1-\alpha_1, 1-\alpha_2) \leq 1 - \min(\alpha_1, \alpha_2). \\
\tfrac{1}{2\pi} \sup_{x\in U} |\omega_{\mu(A_1\cup A_2)}(x) - \omega_{\mu(B_1\cup B_2)}(x)| &= \tfrac{1}{2\pi} \sup_{x\in U} |\max(\omega_{\mu A_1}(x), \omega_{\mu A_2}(x)) - \max(\omega_{\mu B_1}(x), \omega_{\mu B_2}(x))| \\
&\leq \tfrac{1}{2\pi} \sup_{x\in U} \max(|\omega_{\mu A_1}(x) - \omega_{\mu B_1}(x)|, |\omega_{\mu A_2}(x) - \omega_{\mu B_2}(x)|) \\
&\leq \tfrac{1}{2\pi} \sup_{x\in U} \max(1-\alpha_1, 1-\alpha_2) \leq 1 - \min(\alpha_1, \alpha_2). \\
\sup_{x\in U} |s_{A_1\cup A_2}(x) - s_{B_1\cup B_2}(x)| &= \sup_{x\in U} |\min(s_{A_1}(x), s_{A_2}(x)) - \min(s_{B_1}(x), s_{B_2}(x))| \\
&\leq \sup_{x\in U} \min(|s_{A_1}(x) - s_{B_1}(x)|, |s_{A_2}(x) - s_{B_2}(x)|) \\
&\leq \sup_{x\in U} \min(\beta_1, \beta_2) \leq \max(\beta_1, \beta_2). \\
\tfrac{1}{2\pi} \sup_{x\in U} |\omega_{\nu(A_1\cup A_2)}(x) - \omega_{\nu(B_1\cup B_2)}(x)| &= \tfrac{1}{2\pi} \sup_{x\in U} |\min(\omega_{\nu A_1}(x), \omega_{\nu A_2}(x)) - \min(\omega_{\nu B_1}(x), \omega_{\nu B_2}(x))| \\
&\leq \tfrac{1}{2\pi} \sup_{x\in U} \min(|\omega_{\nu A_1}(x) - \omega_{\nu B_1}(x)|, |\omega_{\nu A_2}(x) - \omega_{\nu B_2}(x)|) \\
&\leq \tfrac{1}{2\pi} \sup_{x\in U} \min(\beta_1, \beta_2) \leq \max(\beta_1, \beta_2).
\end{aligned}$$

It implies that

$$A_1 \cup A_2 = (\min(\alpha_1, \alpha_2), \max(\beta_1, \beta_2))(B_1 \cup B_2).$$

□

Corollary 3.1 If $A_i = (\alpha_i, \beta_i)B_i, \ i \in I$, where I is an index set, then $\cup_{i\in I} A_i = (\inf_{i\in I} \alpha_i, \sup_{i\in I} \beta_i) \cup_{i\in I} B_i$.

Theorem 3.5 If $A_1 = (\alpha_1, \beta_1)B_1$ and $A_2 = (\alpha_2, \beta_2)B_2$, then $A_1 \cap A_2 = (\min(\alpha_1, \alpha_2), \max(\beta_1, \beta_2))(B_1 \cap B_2)$.

Proof Since $A_1 = (\alpha_1, \beta_1)B_1$ and $A_2 = (\alpha_2, \beta_2)B_2$, we have

$$\rho(\mu_{A_1}, \mu_{B_1}) = \max(\sup_{x\in U} |r_{A_1}(x) - r_{B_1}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu A_1}(x) - \omega_{\mu B_1}(x)|) \leq 1 - \alpha_1,$$

$$\rho(\nu_{A_1}, \nu_{B_1}) = \max(\sup_{x\in U} |s_{A_1}(x) - s_{B_1}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu A_1}(x) - \omega_{\nu B_1}(x)|) \leq \beta_1,$$

and

$$\rho(\mu_{A_2}, \mu_{B_2}) = \max(\sup_{x\in U} |r_{A_2}(x) - r_{B_2}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu A_2}(x) - \omega_{\mu B_2}(x)|) \leq 1 - \alpha_2,$$

$$\rho(\nu_{A_2}, \nu_{B_2}) = \max(\sup_{x\in U} |s_{A_2}(x) - s_{B_2}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu A_2}(x) - \omega_{\nu B_2}(x)|) \leq \beta_2,$$

therefore

$$\sup_{x\in U} |r_{A_1}(x) - r_{B_1}(x)| \leq 1 - \alpha_1,\ \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu A_1}(x) - \omega_{\mu B_1}(x)| \leq 1 - \alpha_1,$$

$$\sup_{x\in U} |s_{A_1}(x) - s_{B_1}(x)| \leq \beta_1,\ \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu A_1}(x) - \omega_{\nu B_1}(x)| \leq \beta_1,$$

and

$$\sup_{x\in U} |r_{A_2}(x) - r_{B_2}(x)| \leq 1 - \alpha_2,\ \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu A_2}(x) - \omega_{\mu B_2}(x)| \leq 1 - \alpha_2,$$

$$\sup_{x\in U} |s_{A_2}(x) - s_{B_2}(x)| \leq \beta_2,\ \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu A_2}(x) - \omega_{\nu B_2}(x)| \leq \beta_2,$$

consequently, we have

$$\rho(\mu_{A_1\cap A_2}, \mu_{B_1\cap B_2}) = \max(\sup_{x\in U} |r_{A_1\cap A_2}(x) - r_{B_1\cap B_2}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu(A_1\cap A_2)}(x) - \omega_{\mu(B_1\cap B_2)}(x)|),$$

$$\rho(\nu_{A_1\cap A_2}, \nu_{B_1\cap B_2}) = \max(\sup_{x\in U} |s_{A_1\cap A_2}(x) - s_{B_1\cap B_2}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu(A_1\cap A_2)}(x) - \omega_{\nu(B_1\cap B_2)}(x)|),$$

where

$$\begin{aligned}
\sup_{x\in U} |r_{A_1\cap A_2}(x) - r_{B_1\cap B_2}(x)| &= \sup_{x\in U} |\min(r_{A_1}(x), r_{A_2}(x)) - \min(r_{B_1}(x), r_{B_2}(x))| \\
&\leq \sup_{x\in U} \max(|r_{A_1}(x) - r_{B_1}(x)|, |r_{A_2}(x) - r_{B_2}(x)|) \\
&\leq \sup_{x\in U} \max(1-\alpha_1, 1-\alpha_2) \leq 1 - \min(\alpha_1, \alpha_2). \\
\tfrac{1}{2\pi} \sup_{x\in U} |\omega_{\mu(A_1\cap A_2)}(x) - \omega_{\mu(B_1\cap B_2)}(x)| &= \tfrac{1}{2\pi} \sup_{x\in U} |\min(\omega_{\mu A_1}(x), \omega_{\mu A_2}(x)) - \min(\omega_{\mu B_1}(x), \omega_{\mu B_2}(x))| \\
&\leq \tfrac{1}{2\pi} \sup_{x\in U} \max(|\omega_{\mu A_1}(x) - \omega_{\mu B_1}(x)|, |\omega_{\mu A_2}(x) - \omega_{\mu B_2}(x)|) \\
&\leq \tfrac{1}{2\pi} \sup_{x\in U} \max(1-\alpha_1, 1-\alpha_2) \leq 1 - \min(\alpha_1, \alpha_2). \\
\sup_{x\in U} |s_{A_1\cap A_2}(x) - s_{B_1\cap B_2}(x)| &= \sup_{x\in U} |\max(s_{A_1}(x), s_{A_2}(x)) - \max(s_{B_1}(x), s_{B_2}(x))| \\
&\leq \sup_{x\in U} \max(|s_{A_1}(x) - s_{B_1}(x)|, |s_{A_2}(x) - s_{B_2}(x)|) \\
&\leq \sup_{x\in U} \max(\beta_1, \beta_2) \leq \max(\beta_1, \beta_2). \\
\tfrac{1}{2\pi} \sup_{x\in U} |\omega_{\nu(A_1\cap A_2)}(x) - \omega_{\nu(B_1\cap B_2)}(x)| &= \tfrac{1}{2\pi} \sup_{x\in U} |\max(\omega_{\nu A_1}(x), \omega_{\nu A_2}(x)) - \max(\omega_{\nu B_1}(x), \omega_{\nu B_2}(x))| \\
&\leq \tfrac{1}{2\pi} \sup_{x\in U} \max(|\omega_{\nu A_1}(x) - \omega_{\nu B_1}(x)|, |\omega_{\nu A_2}(x) - \omega_{\nu B_2}(x)|) \\
&\leq \tfrac{1}{2\pi} \sup_{x\in U} \max(\beta_1, \beta_2) \leq \max(\beta_1, \beta_2).
\end{aligned}$$

It implies that

$$A_1 \cap A_2 = (\min(\alpha_1, \alpha_2), \max(\beta_1, \beta_2))(B_1 \cap B_2).$$

□

Corollary 3.2 If $A_i = (\alpha_i, \beta_i)B_i,\ i \in I$, where I is an index set, then $\cap_{i\in I} A_i = (\inf_{i\in I} \alpha_i, \sup_{i\in I} \beta_i) \cap_{i\in I} B_i$.

Theorem 3.6 If $A = (\alpha, \beta)B$, then $\overline{A} = (\alpha, \beta)\overline{B}$.

Proof Since $A = (\alpha, \beta)B$, we have

$$\rho(\mu_A, \mu_B) = \max(\sup_{x\in U} |r_A(x) - r_B(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu A}(x) - \omega_{\mu B}(x)|) \leq 1 - \alpha,$$

$$\rho(\nu_A, \nu_B) = \max(\sup_{x\in U} |s_A(x) - s_B(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu A}(x) - \omega_{\nu B}(x)|) \leq \beta,$$

therefore

$\rho(\mu_{\bar{A}}, \mu_{\bar{B}})$
$= \max(\sup_{x\in U} |r_{\bar{A}}(x) - r_{\bar{B}}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu\bar{A}}(x) - \omega_{\mu\bar{B}}(x)|)$
$= \max(\sup_{x\in U} |(1 - r_A(x)) - (1 - r_B(x))|, \frac{1}{2\pi} \sup_{x\in U} |(2\pi - \omega_{\mu A}(x)) - (2\pi - \omega_{\mu B}(x))|)$
$= \max(\sup_{x\in U} |r_A(x) - r_B(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu A}(x) - \omega_{\mu B}(x)|) = \rho(\mu_A, \mu_B) \leq 1 - \alpha.$
$\rho(\nu_{\bar{A}}, \nu_{\bar{B}})$
$= \max(\sup_{x\in U} |s_{\bar{A}}(x) - s_{\bar{B}}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu\bar{A}}(x) - \omega_{\nu\bar{B}}(x)|)$
$= \max(\sup_{x\in U} |(1 - s_A(x)) - (1 - s_B(x))|, \frac{1}{2\pi} \sup_{x\in U} |(2\pi - \omega_{\nu A}(x)) - (2\pi - \omega_{\nu B}(x))|)$
$= \max(\sup_{x\in U} |s_A(x) - s_B(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu A}(x) - \omega_{\nu B}(x)|) = \rho(\nu_A, \nu_B) \leq \beta.$
It implies that $\overline{A} = (\alpha, \beta)\overline{B}$. □

Corollary 3.3 If $A_{ij} = (\alpha_{ij}, \beta_{ij})B_{ij},\ i \in I_1,\ j \in I_2$, where I_1 and I_2 are two index sets, then

$$\cup_{i\in I_1} \cap_{j\in I_2} A_{ij} = (\inf_{i\in I_1} \inf_{j\in I_2} \alpha_{ij}, \sup_{i\in I_1} \sup_{j\in I_2} \beta_{ij}) \cup_{i\in I_1} \cap_{j\in I_2} B_{ij}$$

and

$$\cap_{i\in I_1} \cup_{j\in I_2} A_{ij} = (\inf_{i\in I_1} \inf_{j\in I_2} \alpha_{ij}, \sup_{i\in I_1} \sup_{j\in I_2} \beta_{ij}) \cap_{i\in I_1} \cup_{j\in I_2} B_{ij}.$$

Corollary 3.4 Suppose $A_i = (\alpha_i, \beta_i)B_i,\ i = 1, 2, \cdots$. Let

$$\lim_{n\to\infty} \sup A_i = \cap_{n=1}^{\infty} \cup_{i=n}^{\infty} A_i, \lim_{n\to\infty} \inf A_i = \cup_{n=1}^{\infty} \cap_{i=n}^{\infty} A_i$$

and

$$\lim_{n\to\infty} \sup B_i = \cap_{n=1}^{\infty} \cup_{i=n}^{\infty} B_i, \lim_{n\to\infty} \inf B_i = \cup_{n=1}^{\infty} \cap_{i=n}^{\infty} B_i,$$

then

$$\lim_{n\to\infty} \sup A_n = (\inf_{n\geq 1} \alpha_n, \sup_{n\geq 1} \beta_n) \lim_{n\to\infty} \sup B_n$$

and

$$\lim_{n\to\infty} \inf A_n = (\inf_{n\geq 1} \alpha_n, \sup_{n\geq 1} \beta_n) \lim_{n\to\infty} \inf B_n.$$

Theorem 3.7 If $A_1 = (\alpha_1, \beta_1)B_1$ and $A_2 = (\alpha_2, \beta_2)B_2$, then $A_1 \circ A_2 = (\alpha_1 * \alpha_2, 1 - \alpha_1 * \alpha_2)(B_1 \circ B_2)$.

Proof Since $A_1 = (\alpha_1, \beta_1)B_1$ and $A_2 = (\alpha_2, \beta_2)B_2$, we have

$$\rho(\mu_{A_1}, \mu_{B_1}) = \max(\sup_{x\in U} |r_{A_1}(x) - r_{B_1}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu A_1}(x) - \omega_{\mu B_1}(x)|) \leq 1 - \alpha_1,$$

$$\rho(\nu_{A_1}, \nu_{B_1}) = \max(\sup_{x\in U} |s_{A_1}(x) - s_{B_1}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu A_1}(x) - \omega_{\nu B_1}(x)|) \leq \beta_1,$$

and

$$\rho(\mu_{A_2}, \mu_{B_2}) = \max(\sup_{x\in U} |r_{A_2}(x) - r_{B_2}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu A_2}(x) - \omega_{\mu B_2}(x)|) \leq 1 - \alpha_2,$$

$$\rho(\nu_{A_2}, \nu_{B_2}) = \max(\sup_{x\in U} |s_{A_2}(x) - s_{B_2}(x)|, \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu A_2}(x) - \omega_{\nu B_2}(x)|) \leq \beta_2,$$

therefore

$$\sup_{x\in U} |r_{A_1}(x) - r_{B_1}(x)| \leq 1 - \alpha_1,\ \ \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu A_1}(x) - \omega_{\mu B_1}(x)| \leq 1 - \alpha_1,$$

$$\sup_{x\in U} |s_{A_1}(x) - s_{B_1}(x)| \leq \beta_1,\ \ \frac{1}{2\pi} \sup_{x\in U} |\omega_{\nu A_1}(x) - \omega_{\nu B_1}(x)| \leq \beta_1,$$

and

$$\sup_{x\in U} |r_{A_2}(x) - r_{B_2}(x)| \leq 1 - \alpha_2,\ \ \frac{1}{2\pi} \sup_{x\in U} |\omega_{\mu A_2}(x) - \omega_{\mu B_2}(x)| \leq 1 - \alpha_2,$$

$$\sup_{x\in U}|s_{A_2}(x)-s_{B_2}(x)|\leq\beta_2,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A_2}(x)-\omega_{\nu B_2}(x)|\leq\beta_2,$$

consequently, we have

$$\rho(\mu_{A_1\circ A_2},\mu_{B_1\circ B_2})=\max(\sup_{x\in U}|r_{A_1\circ A_2}(x)-r_{B_1\circ B_2}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\circ A_2)}(x)-\omega_{\mu(B_1\circ B_2)}(x)|),$$

$$\rho(\nu_{A_1\circ A_2},\nu_{B_1\circ B_2})=\max(\sup_{x\in U}|s_{A_1\circ A_2}(x)-s_{B_1\circ B_2}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\circ A_2)}(x)-\omega_{\nu(B_1\circ B_2)}(x)|),$$

where

$\sup_{x\in U}|r_{A_1\circ A_2}(x)-r_{B_1\circ B_2}(x)|$
$=\sup_{x\in U}|r_{A_1}(x)\cdot r_{A_2}(x)-r_{B_1}(x)\cdot r_{B_2}(x)|$
$=\sup_{x\in U}|r_{A_1}(x)\cdot r_{A_2}(x)-r_{A_2}(x)\cdot r_{B_1}(x)+r_{A_2}(x)\cdot r_{B_1}(x)-r_{B_1}(x)\cdot r_{B_2}(x)|$
$=\sup_{x\in U}|r_{A_2}(x)(r_{A_1}(x)-r_{B_1}(x))+r_{B_1}(x)(r_{A_2}(x)-r_{B_2}(x))|$
$\leq\sup_{x\in U}|r_{A_1}(x)-r_{B_1}(x)|+\sup_{x\in U}|r_{A_2}(x)-r_{B_2}(x)|$
$\leq 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1)$.

We note that

$$\sup_{x\in U}|r_{A_1\circ A_2}(x)-r_{B_1\circ B_2}(x)|\leq 1,$$

so we have

$$\sup_{x\in U}|r_{A_1\circ A_2}(x)-r_{B_1\circ B_2}(x)|\leq 1-(\alpha_1+\alpha_2-1)=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2.$$

$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\circ A_2)}(x)-\omega_{\mu(B_1\circ B_2)}(x)|$
$=\frac{1}{2\pi}\sup_{x\in U}|2\pi(\frac{\omega_{\mu A_1}(x)}{2\pi}\cdot\frac{\omega_{\mu A_2}(x)}{2\pi})-2\pi(\frac{\omega_{\mu B_1}(x)}{2\pi}\cdot\frac{\omega_{\mu B_2}(x)}{2\pi})|$
$=\frac{1}{2\pi}\sup_{x\in U}|\frac{\omega_{\mu A_1}(x)\cdot\omega_{\mu A_2}(x)}{2\pi}-\frac{\omega_{\mu A_2}(x)\cdot\omega_{\mu B_1}(x)}{2\pi}+\frac{\omega_{\mu A_2}(x)\cdot\omega_{\mu B_1}(x)}{2\pi}-\frac{\omega_{\mu B_1}(x)\cdot\omega_{\mu B_2}(x)}{2\pi}|$
$=\frac{1}{2\pi}\sup_{x\in U}|\frac{\omega_{\mu A_2}(x)(\omega_{\mu A_1}(x)-\omega_{\mu B_1}(x))}{2\pi}+\frac{\omega_{\mu B_1}(x)(\omega_{\mu A_2}(x)-\omega_{\mu B_2}(x))}{2\pi}|$
$\leq\frac{1}{2\pi}(\sup_{x\in U}|\omega_{\mu A_1}(x)-\omega_{\mu B_1}(x)|+\sup_{x\in U}|\omega_{\mu A_2}(x)-\omega_{\mu B_2}(x)|)$
$\leq 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1)$.

We note that

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\circ A_2)}(x)-\omega_{\mu(B_1\circ B_2)}(x)|\leq 1,$$

so we have

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\circ A_2)}(x)-\omega_{\mu(B_1\circ B_2)}(x)|\leq 1-(\alpha_1+\alpha_2-1)=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2.$$

$\sup_{x\in U}|s_{A_1\circ A_2}(x)-s_{B_1\circ B_2}(x)|$
$=\sup_{x\in U}|s_{A_1}(x)+s_{A_2}(x)-s_{A_1}(x)\cdot s_{A_2}(x)-(s_{B_1}(x)+s_{B_2}(x)-s_{B_1}(x)\cdot s_{B_2}(x))|$
$=\sup_{x\in U}|(1-s_{B_2}(x))(s_{A_1}(x)-s_{B_1}(x))+(1-s_{A_1}(x))(s_{A_2}(x)-s_{B_2}(x))|$
$\leq\sup_{x\in U}|s_{A_1}(x)-s_{B_1}(x)|+\sup_{x\in U}|s_{A_2}(x)-s_{B_2}(x)|$
$\leq\beta_1+\beta_2$.

We note that

$$\sup_{x\in U}|s_{A_1\circ A_2}(x)-s_{B_1\circ B_2}(x)|\leq 1,$$

so we have

$$\sup_{x\in U}|s_{A_1\circ A_2}(x)-s_{B_1\circ B_2}(x)|\leq\beta_1+\beta_2\leq 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1)$$
$$=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2.$$

$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\circ A_2)}(x)-\omega_{\nu(B_1\circ B_2)}(x)|$

$= \frac{1}{2\pi}\sup_{x\in U}|2\pi(\frac{\omega_{\nu A_1}(x)}{2\pi}+\frac{\omega_{\nu A_2}(x)}{2\pi}-\frac{\omega_{\nu A_1}(x)}{2\pi}\cdot\frac{\omega_{\nu A_2}(x)}{2\pi})-2\pi(\frac{\omega_{\nu B_1}(x)}{2\pi}+\frac{\omega_{\nu B_2}(x)}{2\pi}-\frac{\omega_{\nu B_1}(x)}{2\pi}\cdot\frac{\omega_{\nu B_2}(x)}{2\pi})|$
$= \sup_{x\in U}|(1-\frac{\omega_{\nu A_2}(x)}{2\pi})(\frac{\omega_{\nu A_1}(x)}{2\pi}-\frac{\omega_{\nu B_1}(x)}{2\pi})+(1-\frac{\omega_{\nu B_1}(x)}{2\pi})(\frac{\omega_{\nu A_2}(x)}{2\pi}-\frac{\omega_{\nu B_2}(x)}{2\pi})|$
$\leq \sup_{x\in U}(|1-\frac{\omega_{\nu A_2}(x)}{2\pi}||\frac{\omega_{\nu A_1}(x)}{2\pi}-\frac{\omega_{\nu B_1}(x)}{2\pi}|+|1-\frac{\omega_{\nu B_1}(x)}{2\pi}||\frac{\omega_{\nu A_2}(x)}{2\pi}-\frac{\omega_{\nu B_2}(x)}{2\pi}|)$
$\leq \frac{1}{2\pi}(\sup_{x\in U}|\omega_{\nu A_1}(x)-\omega_{\nu B_1}(x)|+|\omega_{\nu A_2}(x)-\omega_{\nu B_2}(x)|)$
$\leq \beta_1+\beta_2.$
We note that

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\circ A_2)}(x)-\omega_{\nu(B_1\circ B_2)}(x)|\leq 1,$$

so we have

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\circ A_2)}(x)-\omega_{\nu(B_1\circ B_2)}(x)|\leq\beta_1+\beta_2\leq 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1)$$

$$=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2.$$

It implies that

$$A_1\circ A_2=(\alpha_1*\alpha_2,1-\alpha_1*\alpha_2)(B_1\circ B_2).$$

□

Corollary 3.5 Suppose $A_i=(\alpha_i,\beta_i)B_i,\ i\in I$, where I is an index set, then $A_1\circ A_2\circ\cdots\circ A_i=(\alpha_1*\alpha_2*\cdots*\alpha_i,1-\alpha_1*\alpha_2*\cdots*\alpha_i)(B_1\circ B_2\circ\cdots\circ B_i)$.

Theorem 3.8 If $A_n=(\alpha_n,\beta_n)B_n,\ n=1,2,...,N$, then $A_1\times A_2\times\cdots\times A_n=(\inf_{1\leq n\leq N}\alpha_n,\sup_{1\leq n\leq N}\beta_n)(B_1\times B_2\times\cdots\times B_n)$.

Proof Trivial from Definition 2.7 and Definition 3.4. □

Theorem 3.9 If $A_1=(\alpha_1,\beta_1)B_1$ and $A_2=(\alpha_2,\beta_2)B_2$, then $A_1\hat{+}A_2=(\alpha_1*\alpha_2,1-\alpha_1*\alpha_2)(B_1\hat{+}B_2)$.

Proof Since $A_1=(\alpha_1,\beta_1)B_1$ and $A_2=(\alpha_2,\beta_2)B_2$, we have

$$\rho(\mu_{A_1},\mu_{B_1})=\max(\sup_{x\in U}|r_{A_1}(x)-r_{B_1}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A_1}(x)-\omega_{\mu B_1}(x)|)\leq 1-\alpha_1,$$

$$\rho(\nu_{A_1},\nu_{B_1})=\max(\sup_{x\in U}|s_{A_1}(x)-s_{B_1}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A_1}(x)-\omega_{\nu B_1}(x)|)\leq\beta_1,$$

and

$$\rho(\mu_{A_2},\mu_{B_2})=\max(\sup_{x\in U}|r_{A_2}(x)-r_{B_2}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A_2}(x)-\omega_{\mu B_2}(x)|)\leq 1-\alpha_2,$$

$$\rho(\nu_{A_2},\nu_{B_2})=\max(\sup_{x\in U}|s_{A_2}(x)-s_{B_2}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A_2}(x)-\omega_{\nu B_2}(x)|)\leq\beta_2,$$

therefore

$$\sup_{x\in U}|r_{A_1}(x)-r_{B_1}(x)|\leq 1-\alpha_1,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A_1}(x)-\omega_{\mu B_1}(x)|\leq 1-\alpha_1,$$

$$\sup_{x\in U}|s_{A_1}(x)-s_{B_1}(x)|\leq\beta_1,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A_1}(x)-\omega_{\nu B_1}(x)|\leq\beta_1,$$

and

$$\sup_{x\in U}|r_{A_2}(x)-r_{B_2}(x)|\leq 1-\alpha_2,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A_2}(x)-\omega_{\mu B_2}(x)|\leq 1-\alpha_2,$$

$$\sup_{x\in U}|s_{A_2}(x)-s_{B_2}(x)|\leq\beta_2,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A_2}(x)-\omega_{\nu B_2}(x)|\leq\beta_2,$$

consequently, we have

$$\rho(\mu_{A_1\hat{+}A_2},\mu_{B_1\hat{+}B_2})=\max(\sup_{x\in U}|r_{A_1\hat{+}A_2}(x)-r_{B_1\hat{+}B_2}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\hat{+}A_2)}(x)-\omega_{\mu(B_1\hat{+}B_2)}(x)|),$$

$$\rho(\nu_{A_1\hat{+}A_2},\nu_{B_1\hat{+}B_2})=\max(\sup_{x\in U}|s_{A_1\hat{+}A_2}(x)-s_{B_1\hat{+}B_2}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\hat{+}A_2)}(x)-\omega_{\nu(B_1\hat{+}B_2)}(x)|),$$

where

$\sup_{x\in U}|r_{A_1\hat{+}A_2}(x)-r_{B_1\hat{+}B_2}(x)|$
$=\sup_{x\in U}|r_{A_1}(x)+r_{A_2}(x)-r_{A_1}(x)\cdot r_{A_2}(x)-(r_{B_1}(x)+r_{B_2}(x)-r_{B_1}(x)\cdot r_{B_2}(x))|$
$=\sup_{x\in U}|(1-r_{B_2}(x))(r_{A_1}(x)-r_{B_1}(x))+(1-r_{A_1}(x))(r_{A_2}(x)-r_{B_2}(x))|$
$\leq\sup_{x\in U}|r_{A_1}(x)-r_{B_1}(x)|+\sup_{x\in U}|r_{A_2}(x)-r_{B_2}(x)|$
$\leq 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1).$

We note that

$$\sup_{x\in U}|r_{A_1\hat{+}A_2}(x)-r_{B_1\hat{+}B_2}(x)|\leq 1,$$

so we have

$$\sup_{x\in U}|r_{A_1\hat{+}A_2}(x)-r_{B_1\hat{+}B_2}(x)|\leq 1-(\alpha_1+\alpha_2-1)=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2.$$

$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\hat{+}A_2)}(x)-\omega_{\mu(B_1\hat{+}B_2)}(x)|$
$=\frac{1}{2\pi}\sup_{x\in U}|2\pi(\frac{\omega_{\mu A_1}(x)}{2\pi}+\frac{\omega_{\mu A_2}(x)}{2\pi}-\frac{\omega_{\mu A_1}(x)}{2\pi}\cdot\frac{\omega_{\mu A_2}(x)}{2\pi})-2\pi(\frac{\omega_{\mu B_1}(x)}{2\pi}+\frac{\omega_{\mu B_2}(x)}{2\pi}-\frac{\omega_{\mu B_1}(x)}{2\pi}\cdot\frac{\omega_{\mu B_2}(x)}{2\pi})|$
$=\sup_{x\in U}|(1-\frac{\omega_{\mu A_2}(x)}{2\pi})(\frac{\omega_{\mu A_1}(x)}{2\pi}-\frac{\omega_{\mu B_1}(x)}{2\pi})+(1-\frac{\omega_{\mu B_1}(x)}{2\pi})(\frac{\omega_{\mu A_2}(x)}{2\pi}-\frac{\omega_{\mu B_2}(x)}{2\pi})|$
$\leq\sup_{x\in U}(|1-\frac{\omega_{\mu A_2}(x)}{2\pi}||\frac{\omega_{\mu A_1}(x)}{2\pi}-\frac{\omega_{\mu B_1}(x)}{2\pi}|+|1-\frac{\omega_{\mu B_1}(x)}{2\pi}||\frac{\omega_{\mu A_2}(x)}{2\pi}-\frac{\omega_{\mu B_2}(x)}{2\pi}|)$
$\leq\frac{1}{2\pi}(\sup_{x\in U}|\omega_{\mu A_1}(x)-\omega_{\mu B_1}(x)|+|\omega_{\mu A_2}(x)-\omega_{\mu B_2}(x)|)$
$\leq 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1).$

We note that

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\hat{+}A_2)}(x)-\omega_{\nu(B_1\hat{+}B_2)}(x)|\leq 1,$$

so we have

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\hat{+}A_2)}(x)-\omega_{\nu(B_1\hat{+}B_2)}(x)|\leq 1-(\alpha_1+\alpha_2-1)=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2.$$

$\sup_{x\in U}|s_{A_1\hat{+}A_2}(x)-s_{B_1\hat{+}B_2}(x)|$
$=\sup_{x\in U}|s_{A_1}(x)\cdot s_{A_2}(x)-s_{B_1}(x)\cdot s_{B_2}(x)|$
$=\sup_{x\in U}|s_{A_1}(x)\cdot s_{A_2}(x)-s_{A_2}(x)\cdot s_{B_1}(x)+s_{A_2}(x)\cdot s_{B_1}(x)-s_{B_1}(x)\cdot s_{B_2}(x)|$
$=\sup_{x\in U}|s_{A_2}(x)(s_{A_1}(x)-s_{B_1}(x))+s_{B_1}(x)(s_{A_2}(x)-s_{B_2}(x))|$
$\leq\sup_{x\in U}|(s_{A_1}(x)-s_{B_1}(x))|+\sup_{x\in U}|(s_{A_2}(x)-s_{B_2}(x))|$
$\leq\beta_1+\beta_2.$

We note that

$$\sup_{x\in U}|s_{\nu_{(A_1\hat{+}A_2)}}(x)-s_{\nu_{(B_1\hat{+}B_2)}}(x)|\leq 1,$$

so we have

$$\sup_{x\in U}|s_{\nu_{(A_1\hat{+}A_2)}}(x)-s_{\nu_{(B_1\hat{+}B_2)}}(x)|\leq\beta_1+\beta_2\leq 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1)$$

$$=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2.$$

$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\hat{+}A_2)}(x)-\omega_{\nu(B_1\hat{+}B_2)}(x)|$
$=\frac{1}{2\pi}\sup_{x\in U}|2\pi(\frac{\omega_{\nu A_1}(x)}{2\pi}\cdot\frac{\omega_{\nu A_2}(x)}{2\pi})-2\pi(\frac{\omega_{\nu B_1}(x)}{2\pi}\cdot\frac{\omega_{\nu B_2}(x)}{2\pi})|$
$=\frac{1}{2\pi}\sup_{x\in U}|\frac{\omega_{\nu A_1}(x)\cdot\omega_{\nu A_2}(x)}{2\pi}-\frac{\omega_{\nu A_2}(x)\cdot\omega_{\nu B_1}(x)}{2\pi}+\frac{\omega_{\nu A_2}(x)\cdot\omega_{\nu B_1}(x)}{2\pi}-\frac{\omega_{\nu B_1}(x)\cdot\omega_{\nu B_2}(x)}{2\pi}|$
$=\frac{1}{2\pi}\sup_{x\in U}|\frac{\omega_{\nu A_2}(x)(\omega_{\nu A_1}(x)-\omega_{\nu B_1}(x))}{2\pi}+\frac{\omega_{\nu B_1}(x)(\omega_{\nu A_2}(x)-\omega_{\nu B_2}(x))}{2\pi}|$
$\leq\frac{1}{2\pi}(\sup_{x\in U}|\omega_{\nu A_1}(x)-\omega_{\nu B_1}(x)|+\sup_{x\in U}|\omega_{\nu A_2}(x)-\omega_{\nu B_2}(x)|)$
$\leq\beta_1+\beta_2.$

We note that

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\hat{+}A_2)}(x)-\omega_{\nu(B_1\hat{+}B_2)}(x)|\leq 1,$$

so we have

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\hat{+}A_2)}(x)-\omega_{\nu(B_1\hat{+}B_2)}(x)|\leq \beta_1+\beta_2\leq 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1)$$

$$=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2.$$

It implies that

$$A_1\hat{+}A_2=(\alpha_1*\alpha_2,1-\alpha_1*\alpha_2)(B_1\hat{+}B_2).$$

□

Corollary 3.6 Suppose $A_i=(\alpha_i,\beta_i)B_i,\ i\in I$, where I is an index set, then $A_1\hat{+}A_2\hat{+}\cdots\hat{+}A_i=(\alpha_1*\alpha_2*\cdots*\alpha_i,1-\alpha_1*\alpha_2*\cdots*\alpha_i)(B_1\hat{+}B_2\hat{+}\cdots\hat{+}B_i)$.

Theorem 3.10 If $A_1=(\alpha_1,\beta_1)B_1$ and $A_2=(\alpha_2,\beta_2)B_2$, then $A_1\dot{\cup}A_2=(\alpha_1*\alpha_2,1-\alpha_1*\alpha_2)(B_1\dot{\cup}B_2)$.

Proof Since $A_1=(\alpha_1,\beta_1)B_1$ and $A_2=(\alpha_2,\beta_2)B_2$, we have

$$\rho(\mu_{A_1},\mu_{B_1})=\max(\sup_{x\in U}|r_{A_1}(x)-r_{B_1}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A_1}(x)-\omega_{\mu B_1}(x)|)\leq 1-\alpha_1,$$

$$\rho(\nu_{A_1},\nu_{B_1})=\max(\sup_{x\in U}|s_{A_1}(x)-s_{B_1}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A_1}(x)-\omega_{\nu B_1}(x)|)\leq \beta_1,$$

and

$$\rho(\mu_{A_2},\mu_{B_2})=\max(\sup_{x\in U}|r_{A_2}(x)-r_{\mu_{B_2}}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A_2}(x)-\omega_{\mu B_2}(x)|)\leq 1-\alpha_2,$$

$$\rho(\nu_{A_2},\nu_{B_2})=\max(\sup_{x\in U}|s_{A_2}(x)-s_{\nu_{B_2}}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A_2}(x)-\omega_{\nu B_2}(x)|)\leq \beta_2,$$

therefore

$$\sup_{x\in U}|r_{A_1}(x)-r_{B_1}(x)|\leq 1-\alpha_1,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A_1}(x)-\omega_{\mu B_1}(x)|\leq 1-\alpha_1,$$

$$\sup_{x\in U}|s_{A_1}(x)-s_{B_1}(x)|\leq \beta_1,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A_1}(x)-\omega_{\nu B_1}(x)|\leq \beta_1,$$

and

$$\sup_{x\in U}|r_{A_2}(x)-r_{B_2}(x)|\leq 1-\alpha_2,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A_2}(x)-\omega_{\mu B_2}(x)|\leq 1-\alpha_2,$$

$$\sup_{x\in U}|s_{A_2}(x)-s_{B_2}(x)|\leq \beta_2,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A_2}(x)-\omega_{\nu B_2}(x)|\leq \beta_2,$$

consequently, we have

$$\rho(\mu_{A_1\dot{\cup}A_2},\mu_{B_1\dot{\cup}B_2})=\max(\sup_{x\in U}|r_{A_1\dot{\cup}A_2}(x)-r_{B_1\dot{\cup}B_2}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\dot{\cup}A_2)}(x)-\omega_{\mu(B_1\dot{\cup}B_2)}(x)|),$$

$$\rho(\nu_{A_1\dot{\cup}A_2},\nu_{B_1\dot{\cup}B_2})=\max(\sup_{x\in U}|s_{A_1\dot{\cup}A_2}(x)-s_{B_1\dot{\cup}B_2}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\dot{\cup}A_2)}(x)-\omega_{\nu(B_1\dot{\cup}B_2)}(x)|),$$

where

$\sup_{x\in U}|r_{A_1\dot{\cup}A_2}(x)-r_{B_1\dot{\cup}B_2}(x)|$
$=\sup_{x\in U}|\min(1,r_{A_1}(x)+r_{A_2}(x))-\min(1,r_{B_1}(x)+r_{B_2}(x))|$
$\leq\sup_{x\in U}|r_{A_1}(x)+r_{A_2}(x)-r_{B_1}(x)-r_{B_2}(x)|$
$\leq\sup_{x\in U}|r_{A_1}(x)-r_{B_1}(x)|+\sup_{x\in U}|r_{A_2}(x)-r_{B_2}(x)|$
$\leq 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1)$.

We note that

$$\sup_{x\in U}|r_{A_1\dot{\cup}A_2}(x)-r_{B_1\dot{\cup}B_2}(x)|\le 1,$$

so we have

$$\sup_{x\in U}|r_{A_1\dot{\cup}A_2}(x)-r_{B_1\dot{\cup}B_2}(x)|\le 1-(\alpha_1+\alpha_2-1)=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2.$$

$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\dot{\cup}A_2)}(x)-\omega_{\mu(B_1\dot{\cup}B_2)}(x)|$
$=\frac{1}{2\pi}\sup_{x\in U}|\min(2\pi,\omega_{\mu A_1}(x)+\omega_{\mu A_2}(x))-\min(2\pi,\omega_{\mu B_1}(x)+\omega_{\mu B_2}(x))|$
$\le\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A_1}(x)+\omega_{\mu A_2}(x)-\omega_{\mu B_1}(x)-\omega_{\mu B_2}(x)|$
$\le\frac{1}{2\pi}(\sup_{x\in U}|\omega_{\mu A_1}(x)-\omega_{\mu B_1}(x)|+\sup_{x\in U}|\omega_{\mu A_2}(x)-\omega_{\mu B_2}(x)|)$
$\le 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1).$
We note that

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\dot{\cup}A_2)}(x)-\omega_{\mu(B_1\dot{\cup}B_2)}(x)|\le 1,$$

so we have

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\dot{\cup}A_2)}(x)-\omega_{\mu(B_1\dot{\cup}B_2)}(x)|\le 1-(\alpha_1+\alpha_2-1)=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2.$$

$\sup_{x\in U}|s_{A_1\dot{\cup}A_2}(x)-s_{B_1\dot{\cup}B_2}(x)|$
$=\sup_{x\in U}|\max(0,s_{A_1}(x)+s_{A_2}(x)-1)-\max(0,s_{B_1}(x)+s_{B_2}(x)-1)|$
$\le\sup_{x\in U}|s_{A_1}(x)+s_{A_2}(x)-s_{B_1}(x)-s_{B_2}(x)|$
$\le\sup_{x\in U}|s_{A_1}(x)-s_{B_1}(x)|+\sup_{x\in U}|s_{A_2}(x)-s_{B_2}(x)|$
$\le\beta_1+\beta_2.$
We note that

$$\sup_{x\in U}|s_{A_1\dot{\cup}A_2}(x)-s_{B_1\dot{\cup}B_2}(x)|\le 1,$$

so we have

$$\sup_{x\in U}|s_{A_1\dot{\cup}A_2}(x)-s_{B_1\dot{\cup}B_2}(x)|\le\beta_1+\beta_2\le 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1)$$

$$=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2.$$

$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\dot{\cup}A_2)}(x)-\omega_{\nu(B_1\dot{\cup}B_2)}(x)|$
$=\frac{1}{2\pi}\sup_{x\in U}|\max(0,\omega_{\nu A_1}(x)+\omega_{\nu A_2}(x)-2\pi)-\max(0,\omega_{\nu B_1}(x)+\omega_{\nu B_2}(x)-2\pi)|$
$\le\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A_1}(x)+\omega_{\nu A_2}(x)-\omega_{\nu B_1}(x)-\omega_{\nu B_2}(x)|$
$\le\frac{1}{2\pi}(\sup_{x\in U}|\omega_{\nu A_1}(x)-\omega_{\nu B_1}(x)|+\sup_{x\in U}|\omega_{\nu A_2}(x)-\omega_{\nu B_2}(x)|)$
$\le\beta_1+\beta_2.$
We note that

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\dot{\cup}A_2)}(x)-\omega_{\nu(B_1\dot{\cup}B_2)}(x)|\le 1,$$

so we have

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\dot{\cup}A_2)}(x)-\omega_{\nu(B_1\dot{\cup}B_2)}(x)|\le\beta_1+\beta_2\le 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1)$$

$$=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2.$$

It implies that

$$A_1\dot{\cup}A_2=(\alpha_1*\alpha_2,1-\alpha_1*\alpha_2)(B_1\dot{\cup}B_2).$$

□

Corollary 3.7 Suppose $A_i = (\alpha_i, \beta_i)B_i$, $i \in I$, where I is an index set, then $A_1 \dot{\cup} A_2 \dot{\cup} \cdots \dot{\cup} A_i = (\alpha_1 * \alpha_2 * \cdots * \alpha_i, 1 - \alpha_1 * \alpha_2 * \cdots * \alpha_i)(B_1 \dot{\cup} B_2 \dot{\cup} \cdots \dot{\cup} B_i)$.

Theorem 3.11 If $A_1 = (\alpha_1, \beta_1)B_1$ and $A_2 = (\alpha_2, \beta_2)B_2$, then $A_1 \dot{\cap} A_2 = (\alpha_1 * \alpha_2, 1 - \alpha_1 * \alpha_2)(B_1 \dot{\cap} B_2)$.

Proof Since $A_1 = (\alpha_1, \beta_1)B_1$ and $A_2 = (\alpha_2, \beta_2)B_2$, we have

$$\rho(\mu_{A_1}, \mu_{B_1}) = \max(\sup_{x \in U} |r_{A_1}(x) - r_{B_1}(x)|, \frac{1}{2\pi} \sup_{x \in U} |\omega_{\mu A_1}(x) - \omega_{\mu B_1}(x)|) \leq 1 - \alpha_1,$$

$$\rho(\nu_{A_1}, \nu_{B_1}) = \max(\sup_{x \in U} |s_{A_1}(x) - s_{B_1}(x)|, \frac{1}{2\pi} \sup_{x \in U} |\omega_{\nu A_1}(x) - \omega_{\nu B_1}(x)|) \leq \beta_1,$$

and

$$\rho(\mu_{A_2}, \mu_{B_2}) = \max(\sup_{x \in U} |r_{A_2}(x) - r_{B_2}(x)|, \frac{1}{2\pi} \sup_{x \in U} |\omega_{\mu A_2}(x) - \omega_{\mu B_2}(x)|) \leq 1 - \alpha_2,$$

$$\rho(\nu_{A_2}, \nu_{B_2}) = \max(\sup_{x \in U} |s_{A_2}(x) - s_{B_2}(x)|, \frac{1}{2\pi} \sup_{x \in U} |\omega_{\nu A_2}(x) - \omega_{\nu B_2}(x)|) \leq \beta_2,$$

therefore

$$\sup_{x \in U} |r_{A_1}(x) - r_{B_1}(x)| \leq 1 - \alpha_1, \ \frac{1}{2\pi} \sup_{x \in U} |\omega_{\mu A_1}(x) - \omega_{\mu B_1}(x)| \leq 1 - \alpha_1,$$

$$\sup_{x \in U} |s_{A_1}(x) - s_{B_1}(x)| \leq \beta_1, \ \frac{1}{2\pi} \sup_{x \in U} |\omega_{\nu A_1}(x) - \omega_{\nu B_1}(x)| \leq \beta_1,$$

and

$$\sup_{x \in U} |r_{A_2}(x) - r_{B_2}(x)| \leq 1 - \alpha_2, \ \frac{1}{2\pi} \sup_{x \in U} |\omega_{\mu A_2}(x) - \omega_{\mu B_2}(x)| \leq 1 - \alpha_2,$$

$$\sup_{x \in U} |s_{A_2}(x) - s_{B_2}(x)| \leq \beta_2, \ \frac{1}{2\pi} \sup_{x \in U} |\omega_{\nu A_2}(x) - \omega_{\nu B_2}(x)| \leq \beta_2,$$

consequently, we have

$$\rho(\mu_{A_1 \dot{\cap} A_2}, \mu_{B_1 \dot{\cap} B_2}) = \max(\sup_{x \in U} |r_{A_1 \dot{\cap} A_2}(x) - r_{B_1 \dot{\cap} B_2}(x)|, \frac{1}{2\pi} \sup_{x \in U} |\omega_{\mu(A_1 \dot{\cap} A_2)}(x) - \omega_{\mu(B_1 \dot{\cap} B_2)}(x)|),$$

$$\rho(\nu_{A_1 \dot{\cap} A_2}, \nu_{B_1 \dot{\cap} B_2}) = \max(\sup_{x \in U} |s_{A_1 \dot{\cap} A_2}(x) - s_{B_1 \dot{\cap} B_2}(x)|, \frac{1}{2\pi} \sup_{x \in U} |\omega_{\nu(A_1 \dot{\cap} A_2)}(x) - \omega_{\nu(B_1 \dot{\cap} B_2)}(x)|),$$

where

$\sup_{x \in U} |r_{A_1 \dot{\cap} A_2}(x) - r_{B_1 \dot{\cap} B_2}(x)|$
$= \sup_{x \in U} |\max(0, r_{A_1}(x) + r_{A_2}(x) - 1) - \max(0, r_{B_1}(x) + r_{B_2}(x) - 1)|$
$\leq \sup_{x \in U} |r_{A_1}(x) + r_{A_2}(x) - r_{B_1}(x) - r_{B_2}(x)|$
$\leq \sup_{x \in U} |r_{A_1}(x) - r_{B_1}(x)| + \sup_{x \in U} |r_{A_2}(x) - r_{B_2}(x)|$
$\leq 1 - \alpha_1 + 1 - \alpha_2 = 1 - (\alpha_1 + \alpha_2 - 1)$.

We note that

$$\sup_{x \in U} |r_{A_1 \dot{\cap} A_2}(x) - r_{B_1 \dot{\cap} B_2}(x)| \leq 1,$$

so we have

$$\sup_{x \in U} |r_{A_1 \dot{\cap} A_2}(x) - r_{B_1 \dot{\cap} B_2}(x)| \leq 1 - (\alpha_1 + \alpha_2 - 1) = 1 - \max(0, \alpha_1 + \alpha_2 - 1) = 1 - \alpha_1 * \alpha_2.$$

$\frac{1}{2\pi} \sup_{x \in U} |\omega_{\mu(A_1 \dot{\cap} A_2)}(x) - \omega_{\mu(B_1 \dot{\cap} B_2)}(x)|$
$= \frac{1}{2\pi} \sup_{x \in U} |\max(0, \omega_{\mu A_1}(x) + \omega_{\mu A_2}(x) - 2\pi) - \max(0, \omega_{\mu B_1}(x) + \omega_{\mu B_2}(x) - 2\pi)|$
$\leq \frac{1}{2\pi} \sup_{x \in U} |\omega_{\mu A_1}(x) + \omega_{\mu A_2}(x) - \omega_{\mu B_1}(x) - \omega_{\mu B_2}(x)|$
$\leq \frac{1}{2\pi} (\sup_{x \in U} |\omega_{\mu A_1}(x) - \omega_{\mu B_1}(x)| + \sup_{x \in U} |\omega_{\mu A_2}(x) - \omega_{\mu B_2}(x)|)$
$\leq 1 - \alpha_1 + 1 - \alpha_2 = 1 - (\alpha_1 + \alpha_2 - 1) = 1 - \alpha_1 * \alpha_2$.

We note that

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\dot{\cap}A_2)}(x)-\omega_{\mu(B_1\dot{\cap}B_2)}(x)|\le 1,$$

so we have

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\dot{\cap}A_2)}(x)-\omega_{\mu(B_1\dot{\cap}B_2)}(x)|\le 1-(\alpha_1+\alpha_2-1)=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2.$$

$\sup_{x\in U}|s_{A_1\dot{\cap}A_2}(x)-s_{B_1\dot{\cap}B_2}(x)|$
$=\sup_{x\in U}|\min(1,s_{A_1}(x)+s_{A_2}(x))-\min(1,s_{B_1}(x)+s_{B_2}(x))|$
$\le\sup_{x\in U}|s_{A_1}(x)+s_{A_2}(x)-s_{B_1}(x)-s_{B_2}(x)|$
$\le\sup_{x\in U}|s_{A_1}(x)-s_{B_1}(x)|+\sup_{x\in U}|s_{A_2}(x)-s_{B_2}(x)|$
$\le\beta_1+\beta_2.$

We note that

$$\sup_{x\in U}|s_{A_1\dot{\cap}A_2}(x)-s_{B_1\dot{\cap}B_2}(x)|\le 1,$$

so we have

$$\sup_{x\in U}|s_{A_1\dot{\cap}A_2}(x)-s_{B_1\dot{\cap}B_2}(x)|\le\beta_1+\beta_2\le 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1)$$
$$=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2.$$

$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\dot{\cap}A_2)}(x)-\omega_{\nu(B_1\dot{\cap}B_2)}(x)|$
$=\frac{1}{2\pi}\sup_{x\in U}|\min(2\pi,\omega_{\nu A_1}(x)+\omega_{\nu A_2}(x))-\min(2\pi,\omega_{\nu B_1}(x)+\omega_{\nu B_2}(x))|$
$\le\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A_1}(x)+\omega_{\nu A_2}(x)-\omega_{\nu B_1}(x)-\omega_{\nu B_2}(x)|$
$\le\frac{1}{2\pi}(\sup_{x\in U}|\omega_{\nu A_1}(x)-\omega_{\nu B_1}(x)|+\sup_{x\in U}|\omega_{\nu A_2}(x)-\omega_{\nu B_2}(x)|)$
$\le\beta_1+\beta_2.$

We note that

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\dot{\cap}A_2)}(x)-\omega_{\nu(B_1\dot{\cap}B_2)}(x)|\le 1,$$

so we have

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\dot{\cap}A_2)}(x)-\omega_{\nu(B_1\dot{\cap}B_2)}(x)|\le\beta_1+\beta_2\le 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1)$$
$$=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2.$$

It implies that

$$A_1\dot{\cap}A_2=(\alpha_1*\alpha_2,1-\alpha_1*\alpha_2)(B_1\dot{\cap}B_2).$$

□

Corollary 3.8 Suppose $A_i=(\alpha_i,\beta_i)B_i,\ i\in I$, where I is an index set, then $A_1\dot{\cap}A_2\dot{\cap}\cdots\dot{\cap}A_i=(\alpha_1*\alpha_2*\cdots*\alpha_i,1-\alpha_1*\alpha_2*\cdots*\alpha_i)(B_1\dot{\cap}B_2\dot{\cap}\cdots\dot{\cap}B_i)$.

Theorem 3.12 If $A_1=(\alpha_1,\beta_1)B_1$ and $A_2=(\alpha_2,\beta_2)B_2$, then $A_1|-|A_2=(\alpha_1*\alpha_2,1-\alpha_1*\alpha_2)(B_1|-|B_2)$.

Proof Since $A_1=(\alpha_1,\beta_1)B_1$ and $A_2=(\alpha_2,\beta_2)B_2$, we have

$$\rho(\mu_{A_1},\mu_{B_1})=\max(\sup_{x\in U}|r_{A_1}(x)-r_{B_1}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A_1}(x)-\omega_{\mu B_1}(x)|)\le 1-\alpha_1,$$

$$\rho(\nu_{A_1},\nu_{B_1})=\max(\sup_{x\in U}|s_{A_1}(x)-s_{B_1}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A_1}(x)-\omega_{\nu B_1}(x)|)\le\beta_1,$$

and

$$\rho(\mu_{A_2},\mu_{B_2})=\max(\sup_{x\in U}|r_{A_2}(x)-r_{B_2}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A_2}(x)-\omega_{\mu B_2}(x)|)\le 1-\alpha_2,$$

$$\rho(\nu_{A_2},\nu_{B_2})=\max(\sup_{x\in U}|s_{A_2}(x)-s_{B_2}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A_2}(x)-\omega_{\nu B_2}(x)|)\leq\beta_2,$$

therefore

$$\sup_{x\in U}|r_{A_1}(x)-r_{B_1}(x)|\leq 1-\alpha_1,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A_1}(x)-\omega_{\mu B_1}(x)|\leq 1-\alpha_1,$$

$$\sup_{x\in U}|s_{A_1}(x)-s_{B_1}(x)|\leq\beta_1,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A_1}(x)-\omega_{\nu B_1}(x)|\leq\beta_1,$$

and

$$\sup_{x\in U}|r_{A_2}(x)-r_{B_2}(x)|\leq 1-\alpha_2,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A_2}(x)-\omega_{\mu B_2}(x)|\leq 1-\alpha_2,$$

$$\sup_{x\in U}|s_{A_2}(x)-s_{B_2}(x)|\leq\beta_2,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A_2}(x)-\omega_{\nu B_2}(x)|\leq\beta_2,$$

consequently, we have

$$\rho(\mu_{A_1|-|A_2},\mu_{B_1|-|B_2})=\max(\sup_{x\in U}|r_{A_1|-|A_2}(x)-r_{B_1|-|B_2}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1|-|A_2)}(x)-\omega_{\mu(B_1|-|B_2)}(x)|),$$

$$\rho(\nu_{A_1|-|A_2},\nu_{B_1|-|B_2})=\max(\sup_{x\in U}|s_{A_1|-|A_2}(x)-s_{B_1|-|B_2}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1|-|A_2)}(x)-\omega_{\nu(B_1|-|B_2)}(x)|),$$

where

$\sup_{x\in U}|r_{A_1|-|A_2}(x)-r_{B_1|-|B_2}(x)|$
$=\sup_{x\in U}|\max(0,r_{A_1}(x)-r_{A_2}(x))-\max(0,r_{B_1}(x)-r_{B_2}(x))|$
$\leq\sup_{x\in U}|r_{A_1}(x)-r_{A_2}(x)-r_{B_1}(x)+r_{B_2}(x)|$
$\leq\sup_{x\in U}|r_{A_1}(x)-r_{B_1}(x)|+\sup_{x\in U}|r_{A_2}(x)-r_{B_2}(x)|$
$\leq 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1).$

We note that

$$\sup_{x\in U}|r_{A_1|-|A_2}(x)-r_{B_1|-|B_2}(x)|\leq 1,$$

so we have

$$\sup_{x\in U}|r_{A_1|-|A_2}(x)-r_{B_1|-|B_2}(x)|\leq 1-(\alpha_1+\alpha_2-1)=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2.$$

$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1|-|A_2)}(x)-\omega_{\mu(B_1|-|B_2)}(x)|$
$=\frac{1}{2\pi}\sup_{x\in U}|\max(0,\omega_{\mu A_1}(x)-\omega_{\mu A_2}(x))-\max(0,\omega_{\mu B_1}(x)-\omega_{\mu B_2}(x))|$
$\leq\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A_1}(x)-\omega_{\mu A_2}(x)-\omega_{\mu B_1}(x)+\omega_{\mu B_2}(x)|$
$\leq\frac{1}{2\pi}(\sup_{x\in U}|\omega_{\mu A_1}(x)-\omega_{\mu B_1}(x)|+\sup_{x\in U}|\omega_{\mu A_2}(x)-\omega_{\mu B_2}(x)|)$
$\leq 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1).$

We note that

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1|-|A_2)}(x)-\omega_{\mu(B_1|-|B_2)}(x)|\leq 1,$$

so we have

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1|-|A_2)}(x)-\omega_{\mu(B_1|-|B_2)}(x)|\leq 1-(\alpha_1+\alpha_2-1)=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2.$$

$\sup_{x\in U}|s_{A_1|-|A_2}(x)-s_{B_1|-|B_2}(x)|$
$=\sup_{x\in U}|\min(1,1-s_{A_1}(x)+s_{A_2}(x))-\min(1,1-s_{B_1}(x)+s_{B_2}(x))|$
$\leq\sup_{x\in U}|-s_{A_1}(x)+s_{A_2}(x)+r_{B_1}(x)-r_{B_2}(x)|$
$\leq\sup_{x\in U}|s_{A_1}(x)-s_{B_1}(x)|+\sup_{x\in U}|s_{A_2}(x)-s_{B_2}(x)|$
$\leq\beta_1+\beta_2.$

We note that

$$\sup_{x\in U}|s_{A_1|-|A_2}(x)-s_{B_1|-|B_2}(x)|\leq 1,$$

so we have

$$\sup_{x\in U}|s_{A_1|-|A_2}(x)-s_{B_1|-|B_2}(x)|\le\beta_1+\beta_2\le 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1)$$

$$=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2.$$

$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1|-|A_2)}(x)-\omega_{\nu(B_1|-|B_2)}(x)|$
$=\frac{1}{2\pi}\sup_{x\in U}|\min(2\pi,2\pi-\omega_{\nu A_1}(x)+\omega_{\nu A_2}(x))-\min(2\pi,2\pi-\omega_{\nu B_1}(x)+\omega_{\nu B_2}(x))|$
$\le\frac{1}{2\pi}\sup_{x\in U}|-\omega_{\nu A_1}(x)+\omega_{\nu A_2}(x)+\omega_{\nu B_1}(x)-\omega_{\nu B_2}(x)|$
$\le\frac{1}{2\pi}(\sup_{x\in U}|\omega_{\nu A_1}(x)-\omega_{\nu B_1}(x)|+\sup_{x\in U}|\omega_{\nu A_2}(x)-\omega_{\nu B_2}(x)|)$
$\le\beta_1+\beta_2$.

We note that

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1|-|A_2)}(x)-\omega_{\nu(B_1|-|B_2)}(x)|\le 1,$$

so we have

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1|-|A_2)}(x)-\omega_{\nu(B_1|-|B_2)}(x)|\le\beta_1+\beta_2\le 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1)$$

$$=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2.$$

It implies that

$$A_1|-|A_2=(\alpha_1*\alpha_2,1-\alpha_1*\alpha_2)(B_1|-|B_2).$$

□

Corollary 3.9 Suppose $A_i=(\alpha_i,\beta_i)B_i,\ i\in I$, where I is an index set, then $A_1|-|A_2|-|\cdots|-|A_i=(\alpha_1*\alpha_2*\cdots*\alpha_i,1-\alpha_1*\alpha_2*\cdots*\alpha_i)(B_1|-|B_2|-|\cdots|-|B_i)$.

Theorem 3.13 If $A_1=(\alpha_1,\beta_1)B_1$ and $A_2=(\alpha_2,\beta_2)B_2$, then $A_1\nabla A_2=(\alpha_1*\alpha_2,1-\alpha_1*\alpha_2)(B_1\nabla B_2)$.

Proof Since $A_1=(\alpha_1,\beta_1)B_1$ and $A_2=(\alpha_2,\beta_2)B_2$, we have

$$\rho(\mu_{A_1},\mu_{B_1})=\max(\sup_{x\in U}|r_{A_1}(x)-r_{B_1}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A_1}(x)-\omega_{\mu B_1}(x)|)\le 1-\alpha_1,$$

$$\rho(\nu_{A_1},\nu_{B_1})=\max(\sup_{x\in U}|s_{A_1}(x)-s_{B_1}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A_1}(x)-\omega_{\nu B_1}(x)|)\le\beta_1,$$

and

$$\rho(\mu_{A_2},\mu_{B_2})=\max(\sup_{x\in U}|r_{A_2}(x)-r_{B_2}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A_2}(x)-\omega_{\mu B_2}(x)|)\le 1-\alpha_2,$$

$$\rho(\nu_{A_2},\nu_{B_2})=\max(\sup_{x\in U}|s_{A_2}(x)-s_{B_2}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A_2}(x)-\omega_{\nu B_2}(x)|)\le\beta_2,$$

therefore

$$\sup_{x\in U}|r_{A_1}(x)-r_{B_1}(x)|\le 1-\alpha_1,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A_1}(x)-\omega_{\mu B_1}(x)|\le 1-\alpha_1,$$

$$\sup_{x\in U}|s_{A_1}(x)-s_{B_1}(x)|\le\beta_1,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A_1}(x)-\omega_{\nu B_1}(x)|\le\beta_1,$$

and

$$\sup_{x\in U}|r_{A_2}(x)-r_{B_2}(x)|\le 1-\alpha_2,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A_2}(x)-\omega_{\mu B_2}(x)|\le 1-\alpha_2,$$

$$\sup_{x\in U}|s_{A_2}(x)-s_{B_2}(x)|\le\beta_2,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A_2}(x)-\omega_{\nu B_2}(x)|\le\beta_2,$$

consequently, we have

$$\rho(\mu_{A_1\nabla A_2},\mu_{B_1\nabla B_2})=\max(\sup_{x\in U}|r_{A_1\nabla A_2}(x)-r_{B_1\nabla B_2}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\nabla A_2)}(x)-\omega_{\mu(B_1\nabla B_2)}(x)|),$$

$$\rho(\nu_{A_1\nabla A_2},\nu_{B_1\nabla B_2})=\max(\sup_{x\in U}|s_{A_1\nabla A_2}(x)-s_{B_1\nabla B_2}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\nabla A_2)}(x)-\omega_{\nu(B_1\nabla B_2)}(x)|),$$

where

$\sup_{x\in U}|r_{A_1\nabla A_2}(x)-r_{B_1\nabla B_2}(x)|$
$=\sup_{x\in U}||r_{A_1}(x)-r_{A_2}(x)|-|r_{B_1}(x)-r_{B_2}(x)||$
$=\sup_{x\in U}|\max(r_{A_1}(x)-r_{A_2}(x),r_{A_2}(x)-r_{A_1}(x))-\max(r_{B_1}(x)-r_{B_2}(x),r_{B_2}(x)-r_{B_1}(x))|$
$\leq\sup_{x\in U}|r_{A_1}(x)-r_{B_1}(x)|+\sup_{x\in U}|r_{A_2}(x)-r_{B_2}(x)|$
$\leq 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1).$

We note that

$$\sup_{x\in U}|r_{A_1\nabla A_2}(x)-r_{B_1\nabla B_2}(x)|\leq 1,$$

so we have

$$\sup_{x\in U}|r_{A_1\nabla A_2}(x)-r_{B_1\nabla B_2}(x)|\leq 1-(\alpha_1+\alpha_2-1)=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2.$$

$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\nabla A_2)}(x)-\omega_{\mu(B_1\nabla B_2)}(x)|$
$=\frac{1}{2\pi}\sup_{x\in U}||\omega_{\mu A_1}(x)-\omega_{\mu A_2}(x)|-|\omega_{\mu B_1}(x)-\omega_{\mu B_2}(x)||$
$=\frac{1}{2\pi}\sup_{x\in U}|\max(\omega_{\mu A_1}(x)-\omega_{\mu A_2}(x),\omega_{\mu A_2}(x)-\omega_{\mu A_1}(x))-\max(\omega_{\mu B_1}(x)-\omega_{\mu B_2}(x),\omega_{\mu B_2}(x)-\omega_{\mu B_1}(x))|$
$\leq\frac{1}{2\pi}(\sup_{x\in U}|\omega_{\mu A_1}(x)-\omega_{\mu B_1}(x)|+\sup_{x\in U}|\omega_{\mu A_2}(x)-\omega_{\mu B_2}(x)|)$
$\leq 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1).$

We note that

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\nabla A_2)}(x)-\omega_{\mu(B_1\nabla B_2)}(x)|\leq 1,$$

so we have

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1\nabla A_2)}(x)-\omega_{\mu(B_1\nabla B_2)}(x)|\leq 1-(\alpha_1+\alpha_2-1)=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2.$$

$\sup_{x\in U}|s_{A_1\nabla A_2}(x)-s_{B_1\nabla B_2}(x)|$
$=\sup_{x\in U}||1-s_{A_2}(x)-s_{A_1}(x)|-|1-s_{B_2}(x)-s_{B_1}(x)||$
$\leq\sup_{x\in U}|-s_{A_1}(x)-s_{A_2}(x)+s_{B_1}(x)+s_{B_2}(x)|$
$\leq\sup_{x\in U}|s_{A_1}(x)-s_{B_1}(x)|+\sup_{x\in U}|s_{A_2}(x)-s_{B_2}(x)|$
$\leq\beta_1+\beta_2.$

We note that

$$\sup_{x\in U}|s_{A_1\nabla A_2}(x)-s_{B_1\nabla B_2}(x)|\leq 1,$$

so we have

$$\sup_{x\in U}|s_{A_1\nabla A_2}(x)-s_{B_1\nabla B_2}(x)|\leq\beta_1+\beta_2\leq 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1)$$

$$=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2.$$

$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\nabla A_2)}(x)-\omega_{\nu(B_1\nabla B_2)}(x)|$
$=\frac{1}{2\pi}\sup_{x\in U}||2\pi-\omega_{\nu A_2}(x)-\omega_{\nu A_1}(x)|-|2\pi-\omega_{\nu B_2}(x)+\omega_{\nu B_1}(x)||$
$\leq\frac{1}{2\pi}\sup_{x\in U}|-\omega_{\nu A_1}(x)-\omega_{\nu A_2}(x)+\omega_{\nu B_1}(x)+\omega_{\nu B_2}(x)|$
$\leq\frac{1}{2\pi}(\sup_{x\in U}|\omega_{\nu A_1}(x)-\omega_{\nu B_1}(x)|+\sup_{x\in U}|\omega_{\nu A_2}(x)-\omega_{\nu B_2}(x)|)$
$\leq\beta_1+\beta_2.$

We note that

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\nabla A_2)}(x)-\omega_{\nu(B_1\nabla B_2)}(x)|\leq 1,$$

so we have

$$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1\nabla A_2)}(x)-\omega_{\nu(B_1\nabla B_2)}(x)|\leq \beta_1+\beta_2\leq 1-\alpha_1+1-\alpha_2=1-(\alpha_1+\alpha_2-1)$$

$$=1-\max(0,\alpha_1+\alpha_2-1)=1-\alpha_1*\alpha_2.$$

It implies that

$$A_1\nabla A_2=(\alpha_1*\alpha_2,1-\alpha_1*\alpha_2)(B_1\nabla B_2).$$

□

Corollary 3.10 Suppose $A_i=(\alpha_i,\beta_i)B_i,\ i\in I$, where I is an index set, then $A_1\nabla A_2\nabla\cdots\nabla A_i=(\alpha_1*\alpha_2*\cdots*\alpha_i,1-\alpha_1*\alpha_2*\cdots*\alpha_i)(B_1\nabla B_2\nabla\cdots\nabla B_i)$.

Theorem 3.14 If $A_1=(\alpha_1,\beta_1)B_1$ and $A_2=(\alpha_2,\beta_2)B_2$, then $A_1||_\lambda A_2=(\min(\alpha_1,\alpha_2),\max(\beta_1,\beta_2))(B_1||_\lambda B_2)$.

Proof Since $A_1=(\alpha_1,\beta_1)B_1$ and $A_2=(\alpha_2,\beta_2)B_2$, we have

$$\rho(\mu_{A_1},\mu_{B_1})=\max(\sup_{x\in U}|r_{A_1}(x)-r_{B_1}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A_1}(x)-\omega_{\mu B_1}(x)|)\leq 1-\alpha_1,$$

$$\rho(\nu_{A_1},\nu_{B_1})=\max(\sup_{x\in U}|s_{A_1}(x)-s_{B_1}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A_1}(x)-\omega_{\nu B_1}(x)|)\leq \beta_1,$$

and

$$\rho(\mu_{A_2},\mu_{B_2})=\max(\sup_{x\in U}|r_{A_2}(x)-r_{B_2}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A_2}(x)-\omega_{\mu B_2}(x)|)\leq 1-\alpha_2,$$

$$\rho(\nu_{A_2},\nu_{B_2})=\max(\sup_{x\in U}|s_{A_2}(x)-s_{B_2}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A_2}(x)-\omega_{\nu B_2}(x)|)\leq \beta_2,$$

therefore

$$\sup_{x\in U}|r_{A_1}(x)-r_{B_1}(x)|\leq 1-\alpha_1,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A_1}(x)-\omega_{\mu B_1}(x)|\leq 1-\alpha_1,$$

$$\sup_{x\in U}|s_{A_1}(x)-s_{B_1}(x)|\leq \beta_1,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A_1}(x)-\omega_{\nu B_1}(x)|\leq \beta_1,$$

and

$$\sup_{x\in U}|r_{A_2}(x)-r_{B_2}(x)|\leq 1-\alpha_2,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu A_2}(x)-\omega_{\mu B_2}(x)|\leq 1-\alpha_2,$$

$$\sup_{x\in U}|s_{A_2}(x)-s_{B_2}(x)|\leq \beta_2,\ \frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu A_2}(x)-\omega_{\nu B_2}(x)|\leq \beta_2,$$

consequently, we have

$$\rho(\mu_{A_1||_\lambda A_2},\mu_{B_1||_\lambda B_2})=\max(\sup_{x\in U}|r_{A_1||_\lambda A_2}(x)-r_{B_1||_\lambda B_2}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1||_\lambda A_2)}(x)-\omega_{\mu(B_1||_\lambda B_2)}(x)|),$$

$$\rho(\nu_{A_1||_\lambda A_2},\nu_{B_1||_\lambda B_2})=\max(\sup_{x\in U}|s_{A_1||_\lambda A_2}(x)-s_{B_1||_\lambda B_2}(x)|,\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1||_\lambda A_2)}(x)-\omega_{\nu(B_1||_\lambda B_2)}(x)|),$$

where

$\sup_{x\in U}|r_{A_1||_\lambda A_2}(x)-r_{B_1||_\lambda B_2}(x)|$
$=\sup_{x\in U}[\lambda|\min(r_{A_1}(x),r_{A_2}(x))-\min(r_{B_1}(x),r_{B_2}(x))|+(1-\lambda)|\max(r_{A_1}(x),r_{A_2}(x))-\max(r_{B_1}(x),r_{B_2}(x))|]$
$\leq\sup_{x\in U}[\lambda\max(|r_{A_1}(x)-r_{A_2}(x)|,|r_{B_1}(x)-r_{B_2}(x)|)+(1-\lambda)\max(|r_{A_1}(x)-r_{A_2}(x)|,|r_{B_1}(x)-r_{B_2}(x)|)]$
$\leq\max(1-\alpha_1,1-\alpha_2)=1-\min(\alpha_1,\alpha_2).$
$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\mu(A_1||_\lambda A_2)}(x)-\omega_{\mu(B_1||_\lambda B_2)}(x)|$

$= \frac{1}{2\pi}\sup_{x\in U}[\lambda|\min(\omega_{\mu_{A_1}}(x),\omega_{\mu_{A_2}}(x)) - \min(\omega_{\mu_{B_1}}(x),\omega_{\mu_{B_2}}(x))| + (1-\lambda)|\max(\omega_{\mu_{A_1}}(x),\omega_{\mu_{A_2}}(x)) - \max(\omega_{\mu_{B_1}}(x),\omega_{\mu_{B_2}}(x))|]$

$\leq \frac{1}{2\pi}\sup_{x\in U}[\lambda\max(|\omega_{\mu_{A_1}}(x)-\omega_{\mu_{A_2}}(x)|,|\omega_{\mu_{B_1}}(x)-\omega_{\mu_{B_2}}(x)|)+(1-\lambda)\max(|\omega_{\mu_{A_1}}(x)-\omega_{\mu_{A_2}}(x)|,|\omega_{\mu_{B_1}}(x)-\omega_{\mu_{B_2}}(x)|)]$

$\leq \max(1-\alpha_1, 1-\alpha_2) = 1-\min(\alpha_1,\alpha_2)$.

$\sup_{x\in U}|s_{A_1||_\lambda A_2}(x) - s_{B_1||_\lambda B_2}(x)|$

$= \sup_{x\in U}[\lambda|\max(s_{A_1}(x),s_{A_2}(x))-\max(s_{B_1}(x),s_{B_2}(x))|+(1-\lambda)|\min(s_{A_1}(x),s_{A_2}(x))-\min(s_{B_1}(x),s_{B_2}(x))|]$

$\leq \sup_{x\in U}[\lambda\max(|s_{A_1}(x)-s_{A_2}(x)|,|s_{B_1}(x)-s_{B_2}(x)|)+(1-\lambda)\max(|s_{A_1}(x)-s_{A_2}(x)|,|s_{B_1}(x)-s_{B_2}(x)|]$

$\leq \max(\beta_1,\beta_2)$.

$\frac{1}{2\pi}\sup_{x\in U}|\omega_{\nu(A_1||_\lambda A_2)}(x) - \omega_{\nu(B_1||_\lambda B_2)}(x)|$

$= \frac{1}{2\pi}\sup_{x\in U}[\lambda|\max(\omega_{\nu_{A_1}}(x),\omega_{\nu_{A_2}}(x)) - \max(\omega_{\nu_{B_1}}(x),\omega_{\nu_{B_2}}(x))| + (1-\lambda)|\min(\omega_{\nu_{A_1}}(x),\omega_{\nu_{A_2}}(x)) - \min(\omega_{\nu_{B_1}}(x),\omega_{\nu_{B_2}}(x))|]$

$\leq \frac{1}{2\pi}\sup_{x\in U}[\lambda\max(|\omega_{\nu_{A_1}}(x)-\omega_{\nu_{A_2}}(x)|,|\omega_{\nu_{B_1}}(x)-\omega_{\nu_{B_2}}(x)|)+(1-\lambda)\max(|\omega_{\nu_{A_1}}(x)-\omega_{\nu_{A_2}}(x)|,|\omega_{\nu_{B_1}}(x)-\omega_{\nu_{B_2}}(x)|)]$

$\leq \max(\beta_1,\beta_2)$.

It implies that

$$A_1||_\lambda A_2 = (\min(\alpha_1,\alpha_2),\max(\beta_1,\beta_2))(B_1||_\lambda B_2).$$

□

Corollary 3.11 Suppose $A_i = (\alpha_i,\beta_i)B_i,\ i\in I$, where I is an index set, then $A_1||_\lambda A_2||_\lambda\cdots||_\lambda A_i = (\inf(\alpha_1,\alpha_2,\cdots,\alpha_i),\sup(\beta_1,\beta_2,\cdots,\beta_i))(B_1||_\lambda B_2||_\lambda\cdots||_\lambda B_i)$.

4 Complex intuitionistic fuzzy relations

In this section, complex intuitionistic fuzzy relations are discussed.

Definition 4.1 ([4, 5]) Let U and W be two arbitrary finite non-empty sets. An intuitionistic fuzzy relation $R(U,W)$ is an intuitionistic fuzzy subset of the product space $U\times W$. The relation $R(U,W)$ is characterized by the membership function $\mu_R(x,y): U\times W\to[0,1]$ and the non-membership function $\nu_R(x,y): U\times W\to[0,1]$ with the condition

$$0\leq \mu_R(x,y)+\nu_R(x,y)\leq 1$$

for all $x\in U$ and $y\in W$.

Like any intuitionistic fuzzy set, $R(U,W)$ may be represented as the set of ordered pairs

$$R(U,W) = \{((x,y),\mu_R(x,y),\nu_R(x,y)) \mid (x,y)\in U\times W\}.$$

Definition 4.2 Let U and W be two arbitrary finite non-empty sets. A complex intuitionistic fuzzy relation $R(U,W)$ is a complex intuitionistic fuzzy subset of the product space $U\times W$. The relation $R(U,W)$ is characterized by the membership function $\mu_R(x,y): U\times W\to\{a|a\in C,|a|\leq 1\}$ and the non-membership function $\nu_R(x,y): U\times W\to\{a^{'}|a^{'}\in C,|a^{'}|\leq 1\}$ with the condition

$$|\mu_R(x,y)+\nu_R(x,y)|\leq 1$$

where $x\in U$ and $y\in W$, $\mu_R(x,y)$ and $\nu_R(x,y)$ assign each pair (x,y) a complex-valued grade of membership and a complex-valued grade of non-membership to the set $R(U,W)$.

Like any complex intuitionistic fuzzy set, $R(U,W)$ may be represented as the set of ordered pairs

$$R(U,W) = \{((x,y),\mu_R(x,y),\nu_R(x,y)) \mid (x,y)\in U\times W\}.$$

The value $\mu_R(x,y)$ and $\nu_R(x,y)$ may receive lie within the unit circle in the complex plane, and are on the form $\mu_R(x,y) = r_R(x) \cdot e^{i\bar{\omega}_{\mu R}(x)}$ and $\nu_R(x,y) = s_R(x) \cdot e^{i\bar{\omega}_{\nu R}(x)}$, where $i = \sqrt{-1}$, each of $r_R(x)$ and $s_R(x)$ are real-valued and both belong to the interval $[0,1]$ such that $0 \le r_R(x) + s_R(x) \le 1$, also $\bar{\omega}_{\mu R}(x)$ and $\bar{\omega}_{\nu R}(x)$ are periodic function whose periodic law and principal period are, respectively, 2π and $0 < \omega_{\mu R}(x), \omega_{\nu R}(x) \le 2\pi$.

The complex membership function $\mu_R(x,y)$ and the complex non-membership function $\nu_R(x,y)$ are to be interpreted in the following manner:

(i) $r_R(x)$ represents a degree of interaction or interconnectedness between the elements of U and W; Correspondingly $s_R(x)$ represents a degree of no connection or no interaction between the elements of U and W;

(ii) $\bar{\omega}_{\mu R}(x)$ represents the phase of association, interaction, or interconnectedness between the elements of U and W; Correspondingly $\bar{\omega}_{\nu R}(x)$ represents the phase of no connection or no interaction between the elements of U and W.

Note 4.1 Without the phase terms $\bar{\omega}_{\mu R}(x)$ and $\bar{\omega}_{\nu R}(x)$, a complex intuitionistic fuzzy relation $R(U,W)$ reduces to a traditional intuitionistic fuzzy relation $R(U,W)$.

5 Examples

As is well-known, in the practice of financial work, we can make accurate evaluation and judgment on the advantages and disadvantages of the economic benefits of enterprises by dissecting and analyzing the financial situation and operating results of enterprises. The selection and application of financial indicators as evaluation and judgment standards is particularly important. In this section, we consider financial indicators selection and application between two companies below which involves the significance of the phase terms of a complex intuitionistic fuzzy relation and the application of operation of complex intuitionistic fuzzy set.

Example 5.1 Let U be the set of financial indicators or indexes of the **A** company. Possible elements of this set are return on equity, total asserts turnover, current asserts turnover, asset-liability rate, quick rate, capital accumulation rate, etc. Let W be the set of financial indicators of the **B** company. Let the complex intuitionistic fuzzy relation $R(U,W)$ represent the relation of influence of **A** company indicators on **B** financial indicators, i.e., y is influenced by x, where $x \in U$ and $y \in W$.

The membership function of the complex intuitionistic fuzzy relation $R(U,W)$, $\mu_R(x,y)$, is a complex valued, with an amplitude term and a phase term. The amplitude term indicates the degree of influence of an **A** company indicator on a **B** company indicator. An amplitude term with a value close to 0 implies a small degree of influence, while a value close to 1 suggests a large degree of influence. The phase term indicates the "*phase*" of influence, or time lag that characterizes the influence of an **A** company indicator on a **B** company indicator.

The non-membership function of the complex intuitionistic fuzzy relation $R(U,W)$, $\nu_R(x,y)$, is also a complex valued, with an amplitude term and a phase term. The amplitude term indicates the degree of uninfluence of an **A** company indicator on a **B** company indicator. An amplitude term with a value close to 0 implies a small degree of uninfluence, while a value close to 1 suggests a large degree of uninfluence. The phase term indicates the "*phase*" of uninfluence, or time lag that characterizes the uninfluence of an **A** company indicator on a **B** company indicator.

Consider, for example, let x =asset-liability rate, y=capital accumulation rate. Then $\mu_R(x,y)$ and $\nu_R(x,y)$ are the grade of membership and non-membership associated with the statement **A** company asset-liability rate influence **B** company capital accumulation rate. The value of $\mu_R(x,y)$ and $\nu_R(x,y)$ may be obtained from an expert.

Suppose an expert was to state that "**A** company asset-liability rate has a great influence on **B** company capital accumulation rate, and the effect of a decline or increase in **A** company asset-liability rate is evident in **B** company capital accumulation rate in two-four months. While the degrees to which **A** company asset-liability rate has no influence on **B** company capital accumulation rate is small, and the no effect of a decline or increase in **A** company asset-liability rate is evident in **B** company capital accumulation rate in two-four months." If $R(U, W)$ is a traditional intuitionistic fuzzy relation, the degree of membership $\mu_R(x, y) = 0.85$, the grade of non-membership $\nu_R(x, y) = 0.1$ and all information regrading the time frame of the interaction and no interaction between these two economic indexes would be lost. However, $R(U, W)$ is a complex intuitionistic fuzzy relation, thus $\mu_R(x, y)$ and $\nu_R(x, y)$ can be assigned two complex value which include all of the information provided by the expert.

Assume $R(U, W)$ measures interactions between **A** company and **B** company financial indexes in the limited time frame of 12 months. Then

$$\mu_R(x, y) = 0.85 \cdot e^{i2\pi \frac{3}{12}}, \tag{5.1}$$

and

$$\nu_R(x, y) = 0.1 \cdot e^{i2\pi \frac{3}{12}}, \tag{5.2}$$

thus

$$R(x, y) = (0.85 \cdot e^{i2\pi \frac{3}{12}}, 0.1 \cdot e^{i2\pi \frac{3}{12}}). \tag{5.3}$$

$\diamondsuit$

Note that the amplitude of μ_R(asset-liability rate, capital accumulation rate) and ν_R(asset-liability rate, capital accumulation rate) were selected to be 0.85 and 0.1, similar to the degree of membership and the degree of non-membership of a intuitionistic fuzzy set. The phase term was chosen to be $2\pi(\frac{3}{12})$ as an average of "two-four months," normalized by 12 months-the maximum time frame the relation was designed to take into account.

Example 5.2 Based on the Example 5.1, let V be the set of development indicators or indexes of the city, such as consumer price index, producer price index, etc. Now, consider the following two complex intuitionistic fuzzy relations.

(1) The relation $R(U, W)$, discussed in detail in the Example 5.1, representing the relation of influence of **A** company financial indexes on **B** company financial indexes.

(2) The relation $R(W, V)$, representing the relation of influence of **B** company financial indexes on city development indicators.

Let x=return on equity, y=total asserts turnover, z=producer price index, where $x \in U, y \in W$ and $z \in V$.

Suppose the following information is available from an expert.

(1) The influence of **A** company return on equity on **B** company total asserts turnover is medium, and its effect is evident in four-six months, while no-influence of **A** company of return on equity on **B** company total asserts turnover is medium, and its effect is not evident in four-six months. According to Definition 4.2, we have

$$R(x, y) = (0.55 \cdot e^{i2\pi \frac{5}{12}}, 0.4 \cdot e^{i2\pi \frac{5}{12}}). \tag{5.4}$$

(2) The influence of **B** company total asserts turnover on city development producer price index is verge large, and its effect is evident in nine-ten months, while no-influence of **B** company of total asserts turnover on city development producer price index is very small, and its effect is not evident in nine-ten months. According to Definition 4.2, we have

$$R(y, z) = (0.9 \cdot e^{i2\pi \frac{8}{12}}, 0.05 \cdot e^{i2\pi \frac{8}{12}})). \tag{5.5}$$

The two relations defined above may be combined in order to produce a third relation, $R(U,V)$, the relation of influence of **A** company return on equity on city development producer price index. The relation $R(U,V)$ is obtained through the composition of relation $R(x,y)$ and $R(y,z)$. It is possible to provide a general and rigorous definition for the composition of complex intuitionistic fuzzy relation. In this example, we consider the composition of the two degree of membership and the two degree of non-membership derived above: $\mu_R(x,y)$, $\mu_R(y,z)$, $\nu_R(x,y)$, and $\nu_R(y,z)$.

The result of this composition is the degree of membership and non-membership $\mu_R(x,z)$ and $\nu_R(x,z)$. From intuitive consideration, we suggest that the value of $\mu_R(x,z)$ and $\nu_R(x,z)$ should equal the product of $\mu_R(x,y)$ and $\mu_R(y,z)$ and the product of $\nu_R(x,y)$ and $\nu_R(y,z)$, i.e., $R(x,z)$ equal the product of $R(x,y)$ and $R(y,z)$. According to Definition 2.6,

$$R(x,z) = R(x,y) \circ R(y,z), \tag{5.6}$$

where

$$\mu_R(x,z) = \mu_R(x,y) \circ \mu_R(y,z) = 0.55 \cdot e^{i2\pi\frac{5}{12}} \circ 0.9 \cdot e^{i2\pi\frac{8}{12}} = 0.495 \cdot e^{i2\pi\frac{3.3}{12}} \tag{5.7}$$

and

$$\nu_R(x,z) = \nu_R(x,y) \circ \nu_R(y,z) = 0.4 \cdot e^{i2\pi\frac{5}{12}} \circ 0.05 \cdot e^{i2\pi\frac{8}{12}} = 0.38 \cdot e^{i2\pi\frac{9.7}{12}} \tag{5.8}$$

thus

$$R(x,z) = (0.495 \cdot e^{i2\pi\frac{3.3}{12}}, 0.43 \cdot e^{i2\pi\frac{9.7}{12}}). \tag{5.9}$$

$\diamondsuit$

Note that for the membership function, the amplitude term of $\mu_R(x,z)$ is derived by intersecting the amplitudes of $\mu_R(x,y)$ and $\mu_R(y,z)$, with product used as the intersection function of choice. The phase term of $\mu_R(x,z)$ is also derived by intersecting the amplitudes of $\mu_R(x,y)$ and $\mu_R(y,z)$, with product used as the intersection function of choice. While for the non-membership function, the amplitude term of $\nu_R(x,z)$ is derived by intersecting the amplitudes of $\nu_R(x,y)$ and $\nu_R(y,z)$, with probabilistic sum used as the union function of choice. The phase term of $\nu_R(x,z)$ is also derived by intersecting the amplitudes of $\nu_R(x,y)$ and $\nu_R(y,z)$, probabilistic sum used as the union function of choice.

Hence, the use of multiplication in this example makes good intuitive sense. Note that the product operation emphasizes a unique property of complex intuitionistic fuzzy setsthe complex algebra of its grades of membership and non-membership. It is a feature of complex intuitionistic fuzzy sets that is difficult to reproduce using traditional intuitionistic fuzzy sets.

6 Conclusion

In this paper, we have investigated the properties of various operations on complex intuitionistic fuzzy sets and introduced a new distance measure for complex intuitionistic fuzzy sets. This distance measure was then used to defined (α,β)-equalities of complex intuitionistic fuzzy sets which subsumed (α,β)-equalities of intuitionistic fuzzy sets defined in references [20]. Two complex intuitionistic fuzzy sets are said to be (α,β)-equal if $\rho(\mu_A,\mu_B) \leq 1-\alpha$ and $\rho(\nu_A,\nu_B) \leq \beta$. The importance of the work presented in this paper can be justified in theory as well as in practice. For example, in section 5 we consider financial indicators selection and application between two companies below which involves the significance of the phase terms of a complex intuitionistic fuzzy relation and the application of operation of complex intuitionistic fuzzy set.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with animals performed by any of the authors.

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