Multispace and Multilevel BDDC

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Abstract

BDDC method is the most advanced method from the Balancing family of iterative substructuring methods for the solution of large systems of linear algebraic equations arising from discretization of elliptic boundary value problems. In the case of many substructures, solving the coarse problem exactly becomes a bottleneck. Since the coarse problem in BDDC has the same structure as the original problem, it is straightforward to apply the BDDC method recursively to solve the coarse problem only approximately. In this paper, we formulate a new family of abstract Multispace BDDC methods and give condition number bounds from the abstract additive Schwarz preconditioning theory. The Multilevel BDDC is then treated as a special case of the Multispace BDDC and abstract multilevel condition number bounds are given. The abstract bounds yield polylogarithmic condition number bounds for an arbitrary fixed number of levels and scalar elliptic problems discretized by finite elements in two and three spatial dimensions. Numerical experiments confirm the theory.

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1   Introduction

The BDDC (Balancing Domain Decomposition by Constraints) method by Dohrmann [4] is the most advanced method from the BDD family introduced by Mandel [12]. It is a Neumann-Neumann iterative substructuring method of Schwarz type [5] that iterates on the system of primal variables reduced to the interfaces between the substructures. The BDDC method is closely related to the FETI-DP method (Finite Element Tearing and Interconnecting - Dual, Primal) by Farhat et al. [6, 7]. FETI-DP is a dual method that iterates on a system for Lagrange multipliers that enforce continuity on the interfaces, with some "coarse" variables treated as primal, and it is a further development of the FETI method by Farhat and Roux [8]. Polylogarithmic condition number estimates for BDDC were obtained in [13, 14] and a proof that the eigenvalues of BDDC and FETI-DP are actually the same except for eigenvalue equal to one was given in Mandel et

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al. [14]. Simpler proofs of the equality of eigenvalues were obtained by Li and Widlund [10], and also by Brenner and Sung [3], who also gave an example when BDDC has an eigenvalue equal to one but FETI-DP does not. In the case of many substructures, solving the coarse problem exactly becomes a bottleneck. However, since the coarse problem in BDDC has the same form as the original problem, the BDDC method can be applied recursively to solve the coarse problem only approximately. This leads to a multilevel form of BDDC in a straightforward manner, see Dohrmann [4]. Polylogarithmic condition number bounds for three-level BDDC (BDDC with two coarse levels) were proved in two and three spatial dimensions by Tu [20, 19].

In this paper, we present a new abstract Multispace BDDC method. The method extends a simple variational setting of BDDC from Mandel and Sousedík [15], which could be understood as an abstract version of BDDC by partial subassembly in Li and Widlund [11]. However, we do not adopt their change of variables, which does not seem to be suitable in our abstract setting. We provide a condition number estimate for the abstract Multispace BDDC method, which generalizes the estimate for a single space from [15]. The proof is based on the abstract additive Schwarz theory by Dryja and Widlund [5]. Many BDDC formulations (with an explicit treatment of substructure interiors, after reduction to substructure interfaces, with two levels, and multilevel) are then viewed as abstract Multispace BDDC with a suitable choice of spaces and operators, and abstract condition number estimates for those BDDC methods follow. This result in turn gives a polylogarithmic condition number bound for Multilevel BDDC applied to a second-order scalar elliptic model problems, with an arbitrary number of levels. A brief presentation of the main results of the paper, without proofs and with a simplified formulation of the Multispace BDDC estimate, is contained in the conference paper [16].

The paper is organized as follows. In Sec. 2 we introduce the abstract problem setting. In Sec. 3 we formulate an abstract Multispace BDDC as an additive Schwarz preconditioner. In Sec. 4 we introduce the settings of a model problem using finite element discretization. In Sec. 5 we recall the algorithm of the original (two-level) BDDC method and formulate it as a Multispace BDDC. In Sec. 6 we generalize the algorithm to obtain Multilevel BDDC and we also give an abstract condition number bound. In Sec. 7 we derive the condition number bound for the model problem. Finally, in Sec. 8, we report on numerical results.

2 Abstract Problem Setting

We wish to solve an abstract linear problem

$$u \in X : a(u,v) = \langle f_X, v \rangle, \quad \forall v \in X, \tag{1}$$

where X is a finite dimensional linear space, $a(\cdot, \cdot)$ is a symmetric positive definite bilinear form defined on X, $f_X \in X'$ is the right-hand side with X' denoting the dual space of X, and $\langle \cdot, \cdot \rangle$ is the duality pairing. The form $a(\cdot, \cdot)$ is also called the energy inner product, the value of the quadratic form $a(u,u)$ is called the energy of u, and the norm $\|u\|_a = a(u,u)^{1/2}$ is called the energy norm. The operator $A_X : X \mapsto X'$ associated with a is defined by

$$a(u,v) = \langle A_X u, v \rangle, \quad \forall u, v \in X.$$

A preconditioner is a mapping $B : X' \to X$ and we will look for preconditioners such that $\langle r, Br \rangle$ is also symmetric and positive definite on X'. It is well known that then $BA_X : X \to X$ has only real positive eigenvalues, and convergence of the preconditioned conjugate gradients method is bounded using the condition number

$$\kappa = \frac{\lambda_{\max}(BA_X)}{\lambda_{\min}(BA_X)},$$

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which we wish to bound above.

All abstract spaces in this paper are finite dimensional linear spaces and we make no distinction between a linear operator and its matrix.

3 Abstract Multispace BDDC

To introduce abstract Multispace BDDC preconditioner, suppose that the bilinear form a is defined and symmetric positive semidefinite on some larger space $W \supset X$. The preconditioner is derived from the abstract additive Schwarz theory, however we decompose some space between X and W rather than X as it would be done in the additive Schwarz method: In the design of the preconditioner, we choose spaces V_k, $k = 1, \ldots, M$, such that

$$X \subset \sum_{k=1}^{M} V_k \subset W. \tag{2}$$

Assumption 1 *The form $a(\cdot, \cdot)$ is positive definite on each V_k separately.*

Algorithm 2 (Abstract Multispace BDDC) *Given spaces V_k and linear operators Q_k, $k = 1, \ldots, M$ such that $a(\cdot, \cdot)$ is positive definite on each space V_k, and*

$$X \subset \sum_{k=1}^{M} V_k, \quad Q_k : V_k \to X,$$

define the preconditioner $B : r \in X' \longmapsto u \in X$ by

$$B : r \mapsto u = \sum_{k=1}^{M} Q_k v_k, \quad v_k \in V_k : \quad a(v_k, z_k) = \langle r, Q_k z_k \rangle, \quad \forall z_k \in V_k. \tag{3}$$

We formulate the condition number bound first in the full strength allowed by the proof. The bound used in the rest of this paper will be a corollary.

Theorem 3 *Define for all $k = 1, \ldots, M$ the spaces $V_k^{\mathcal{M}}$ by*

$$V_k^{\mathcal{M}} = \left\{ v_k \in V_k : \forall z_k \in V_k : Q_k v_k = Q_k z_k \implies \|v_k\|_a^2 \le \|z_k\|_a^2 \right\}.$$

If there exist constants C_0, ω, and a symmetric matrix $\mathcal{E} = (e_{ij})_{i,j=1}^{M}$, such that

$$\forall u \in X \quad \exists v_k \in V_k, \ k = 1, \ldots, M : u = \sum_{k=1}^{M} Q_k v_k, \quad \sum_{k=1}^{M} \|v_k\|_a^2 \le C_0 \|u\|_a^2 \tag{4}$$

$$\forall k = 1, \ldots, M \quad \forall v_k \in V_k^{\mathcal{M}} : \|Q_k v_k\|_a^2 \le \omega \|v_k\|_a^2 \tag{5}$$

$$\forall z_k \in Q_k V_k, \ k = 1, \ldots, M : a(z_i, z_j) \le e_{ij} \|z_i\|_a \|z_j\|_a, \tag{6}$$

then the preconditioner from Algorithm 2 satisfies

$$\kappa = \frac{\lambda_{\max}(BA_X)}{\lambda_{\min}(BA_X)} \le C_0 \omega \rho(\mathcal{E}).$$

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Proof. We interpret the Multispace BDDC preconditioner as an abstract additive Schwarz method. An abstract additive Schwarz method is specified by a decomposition of the space X into subspaces,

$$X = X_1 + \ldots + X_M, \tag{7}$$

and by symmetric positive definite bilinear forms b_i on X_i. The preconditioner is a linear operator

$$B : X' \to X, \qquad B : r \mapsto u,$$

defined by solving the following variational problems on the subspaces and adding the results,

$$B : r \mapsto u = \sum_{k=1}^{M} u_k, \quad u_k \in X_k : \quad b_k(u_k, y_k) = \langle r, y_k \rangle, \quad \forall y_k \in X_k. \tag{8}$$

Dryja and Widlund [5] proved that if there exist constants C_0, ω, and a symmetric matrix $\mathcal{E} = (e_{ij})_{i,j=1}^{M}$, such that

$$\forall u \in X \; \exists u_k \in X_k, \; k = 1, \ldots, M : u = \sum_{k=1}^{M} u_k, \; \sum_{k=1}^{M} \|u_k\|_{b_k}^2 \leq C_0 \|u\|_a^2 \tag{9}$$

$$\forall k = 1, \ldots, M \; \forall u_k \in X_k : \|u_k\|_a^2 \leq \omega \|u_k\|_{b_k}^2 \tag{10}$$

$$\forall u_k \in X_k, \; k = 1, \ldots, M : a(u_i, u_j) \leq e_{ij} \|u_i\|_a \|u_j\|_a \tag{11}$$

then

$$\kappa = \frac{\lambda_{\max}(BA_X)}{\lambda_{\min}(BA_X)} \leq C_0 \omega \rho(\mathcal{E}),$$

where ρ is the spectral radius.

Now the idea of the proof is essentially to map the assumptions of the abstract additive Schwarz estimate from the decomposition (7) of the space X to the decomposition (2). Define the spaces

$$X_k = Q_k V_k.$$

We will show that the preconditioner (3) satisfies (8), where b_k is defined by

$$b_k(u_k, y_k) = a(G_k x, G_k z), \quad x, z \in X, \quad u_k = Q_k G_k x, \quad y_k = Q_k G_k z. \tag{12}$$

with the operators $G_k : X \to V_k^{\mathcal{M}}$ defined by

$$G_k : u \mapsto v_k, \quad \frac{1}{2} a(v_k, v_k) \to \min, \text{ s.t. } v_k \in V_k^{\mathcal{M}}, \; u = \sum_{k=1}^{M} Q_k v_k, \tag{13}$$

First, from the definition of operators G_k, spaces X_k, and because a is positive definite on V_k by Assumption (1), it follows that $G_k x$ and $G_k z$ in (12) exist and are unique, so b_k is defined correctly. To prove (8), let v_k be as in (3) and note that v_k is the solution of

$$\frac{1}{2} a(v_k, v_k) - \langle r, Q_k v_k \rangle \to \min, \quad v_k \in V_k.$$

Consequently, the preconditioner (3) is an abstract additive Schwarz method and we only need to verify the inequalities (9)–(11). To prove (9), let $u \in X$. Then, with v_k from the assumption (4) and with $u_k = Q_k G_k v_k$ as in (12), it follows that

$$u = \sum_{k=1}^{M} u_k, \quad \sum_{k=1}^{M} \|u_k\|_{b_k}^2 = \sum_{k=1}^{M} \|v_k\|_a^2 \leq C_0 \|u\|_a^2.$$

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Next, let $u_k \in X_k$. From the definitions of X_k and $V_k^{\mathcal{M}}$, it follows that there exist unique $v_k \in V_k^{\mathcal{M}}$ such that $u_k = Q_k v_k$. Using the assumption (5) and the definition of b_k in (12), we get

$$\|u_k\|_a^2 = \|Q_k v_k\|_a^2 \leq \omega \|v_k\|_a^2 = \omega \|u_k\|_{b_k}^2,$$

which gives (10). Finally, (6) is the same as (11). ∎

The next Corollary was given without proof in [16, Lemma 1]. This is the special case of Theorem 3 that will be actually used. In the case when $M = 1$, this result was proved in [15].

Corollary 4 *Assume that the subspaces V_k are energy orthogonal, the operators Q_k are projections, $a(\cdot, \cdot)$ is positive definite on each space V_k, and*

$$\forall u \in X : \left[u = \sum_{k=1}^{M} v_k, \ v_k \in V_k \right] \implies u = \sum_{k=1}^{M} Q_k v_k. \tag{14}$$

Then the abstract Multispace BDDC preconditioner from Algorithm 2 satisfies

$$\kappa = \frac{\lambda_{\max}(BA_X)}{\lambda_{\min}(BA_X)} \leq \omega = \max_k \sup_{v_k \in V_k} \frac{\|Q_k v_k\|_a^2}{\|v_k\|_a^2} . \tag{15}$$

Proof. We only need to verify the assumptions of Theorem 3. Let $u \in X$ and choose v_k as the energy orthogonal projections of u on V_k. First, since the spaces V_k are energy orthogonal, $u = \sum v_k$, Q_k are projections, and from (14) $u = \sum Q_k v_k$, we get that $\|u\|_a^2 = \sum \|v_k\|_a^2$ which proves (4) with $C_0 = 1$. Next, the assumption (5) becomes the definition of ω in (15). Finally, (6) with $\mathcal{E} = I$ follows from the orthogonality of subspaces V_k. ∎

Remark 5 *The assumption (14) can be written as*

$$\left. \sum_{k=1}^{M} Q_k P_k \right|_X = I,$$

where P_k is the a-orthogonal projection from $\bigoplus_{j=1}^{M} V_j$ onto V_k. Hence, the property (14) is a type of decomposition of unity.

In the case when $M = 1$, (14) means that the projection Q_1 is onto X.

4 Finite Element Problem Setting

Let Ω be a bounded domain in $\mathbb{R}^d$, $d = 2$ or 3, decomposed into N nonoverlapping subdomains Ω^s, $s = 1, ..., N$, which form a conforming triangulation of the domain Ω. Subdomains will be also called substructures. Each substructure is a union of Lagrangian $P1$ or $Q1$ finite elements with characteristic mesh size h, and the nodes of the finite elements between substructures coincide. The nodes contained in the intersection of at least two substructures are called boundary nodes. The union of all boundary nodes is called the interface Γ. The interface Γ is a union of three different types of open sets: *faces*, *edges*, and *vertices*. The substructure vertices will be also called corners. For the case of regular substructures, such as cubes or tetrahedrons, we can use standard geometric definition of faces, edges, and vertices; cf., e.g., [9] for a more general definition.

In this paper, we find it more convenient to use the notation of abstract linear spaces and linear operators between them instead of the space $\mathbb{R}^n$ and matrices. The results can be easily converted

to the matrix language by choosing a finite element basis. The space of the finite element functions on Ω will be denoted as U. Let W^s be the space of finite element functions on substructure Ω^s, such that all of their degrees of freedom on $\partial \Omega^s \cap \partial \Omega$ are zero. Let

$$W = W^1 \times \cdots \times W^N,$$

and consider a bilinear form arising from the second-order scalar elliptic problem as

$$a\,(u,v) = \sum_{s=1}^{N} \int_{\Omega^S} \nabla u \nabla v\, dx, \quad u,v \in W. \tag{16}$$

Now $U \subset W$ is the subspace of all functions from W that are continuous across the substructure interfaces. We are interested in the solution of the problem (1) with $X = U$,

$$u \in U : a(u,v) = \langle f,v \rangle, \quad \forall v \in U, \tag{17}$$

where the bilinear form a is associated on the space U with the system operator A, defined by

$$A : U \mapsto U', \quad a(u,v) = \langle Au, v \rangle \text{ for all } u,v \in U, \tag{18}$$

and $f \in U'$ is the right-hand side. Hence, (17) is equivalent to

$$Au = f. \tag{19}$$

Define $U_I \subset U$ as the subspace of functions that are zero on the interface Γ, i.e., the "interior" functions. Denote by P the energy orthogonal projection from W onto U_I,

$$P : w \in W \longmapsto v_I \in U_I : a\,(v_I, z_I) = a\,(w, z_I), \quad \forall z_I \in U_I.$$

Functions from $(I - P)\,W$, i.e., from the nullspace of P, are called discrete harmonic; these functions are a-orthogonal to U_I and energy minimal with respect to increments in U_I. Next, let $\widehat{W}$ be the space of all discrete harmonic functions that are continuous across substructure boundaries, that is

$$\widehat{W} = (I - P)\,U. \tag{20}$$

In particular,
$$U = U_I \oplus \widehat{W}, \quad U_I \perp_a \widehat{W}. \tag{21}$$

A common approach in substructuring is to reduce the problem to the interface. The problem (17) is equivalent to two independent problems on the energy orthogonal subspaces U_I and $\widehat{W}$, and the solution u satisfies $u = u_I + \widehat{u}$, where

$$u \in U_I : a(u_I, v_I) = \langle f, v_I \rangle, \quad \forall v_I \in U_I, \tag{22}$$
$$u \in \widehat{W} : a(\widehat{u}, \widehat{v}) = \langle f, \widehat{v} \rangle, \quad \forall \widehat{v} \in \widehat{W}. \tag{23}$$

The solution of the interior problem (22) decomposes into independent problems, one per each substructure. The reduced problem (23) is then solved by preconditioned conjugate gradients. The reduced problem (23) is usually written equivalently as

$$u \in \widehat{W} : s(\widehat{u}, \widehat{v}) = \langle g, \widehat{v} \rangle, \quad \forall \widehat{v} \in \widehat{W},$$

where s is the form a restricted on the subspace $\widehat{W}$, and g is the reduced right hand side, i.e., the functional f restricted to the space $\widehat{W}$. The reduced right-hand side g is usually written as

$$\langle g, \widehat{v} \rangle = \langle f, \widehat{v} \rangle - a(u_I, \widehat{v}), \quad \forall \widehat{v} \in \widehat{W}, \tag{24}$$

because $a(u_I, \widehat{v}) = 0$ by (21). In the implementation, the process of passing to the reduced problem becomes the elimination of the internal degrees of freedom of the substructures, also known as static condensation. The matrix of the reduced bilinear form s in the basis defined by interface degrees of freedom becomes the Schur complement, and (24) becomes the reduced right-hand side. For details on the matrix formulation, see, e.g., [17, Sec. 4.6] or [18, Sec. 4.3].

The BDDC method is a two-level preconditioner characterized by the selection of certain *coarse degrees of freedom*, such as values at the corners and averages over edges or faces of substructures. Define $\widetilde{W} \subset W$ as the subspace of all functions such that the values of any coarse degrees of freedom have a common value for all relevant substructures and vanish on $\partial\Omega$, and $\widetilde{W}_\Delta \subset W$ as the subspace of all function such that their coarse degrees of freedom vanish. Next, define $\widetilde{W}_\Pi$ as the subspace of all functions such that their coarse degrees of freedom between adjacent substructures coincide, and such that their energy is minimal. Clearly, functions in $\widetilde{W}_\Pi$ are uniquely determined by the values of their coarse degrees of freedom, and

$$\widetilde{W}_\Delta \perp_a \widetilde{W}_\Pi, \quad \text{and} \quad \widetilde{W} = \widetilde{W}_\Delta \oplus \widetilde{W}_\Pi. \tag{25}$$

We assume that

$$a \text{ is positive definite on } \widetilde{W}. \tag{26}$$

That is the case when a is positive definite on the space U, where the problem (1) is posed, and there are sufficiently many coarse degrees of freedom. We further assume that the coarse degrees of freedom are zero on all functions from U_I, that is,

$$U_I \subset \widetilde{W}_\Delta. \tag{27}$$

In other words, the coarse degrees of freedom depend on the values on substructure boundaries only. From (25) and (27), it follows that the functions in $\widetilde{W}_\Pi$ are discrete harmonic, that is,

$$\widetilde{W}_\Pi = (I - P)\, \widetilde{W}_\Pi. \tag{28}$$

Next, let E be a projection from $\widetilde{W}$ onto U, defined by taking some weighted average on substructure interfaces. That is, we assume that

$$E : \widetilde{W} \to U, \quad EU = U, \quad E^2 = E. \tag{29}$$

Since a projection is the identity on its range, it follows that E does not change the interior degrees of freedom,

$$EU_I = U_I, \tag{30}$$

since $U_I \subset U$. Finally, we show that the operator $(I - P)\, E$ is a projection. From (30) it follows that E does not change interior degrees of freedom, so $EP = P$. Then, using the fact that $I - P$ and E are projections, we get

$$\begin{aligned}
[(I - P)\, E]^2 &= (I - P)\, E\, (I - P)\, E \\
&= (I - P)\, (E - P)\, E \\
&= (I - P)\, (I - P)\, E = (I - P)\, E.
\end{aligned} \tag{31}$$

Remark 6 *In [14, 15], the whole analysis was done in spaces of discrete harmonic functions after eliminating U_I, and the space $\widehat{W}$ was the solution space. In particular, $\widehat{W}$ consisted of discrete harmonic functions only, while the same space here would be $(I - P)\widehat{W}$. The decomposition of this space used in [14, 15] would be in our context written as*

$$(I - P)\widetilde{W} = (I - P)\widetilde{W}_\Delta \oplus \widetilde{W}_\Pi, \quad (I - P)\widetilde{W}_\Delta \perp_a \widetilde{W}_\Pi. \tag{32}$$

In the next section, the space X will be either U or $\widehat{W}$.

5 Two-level BDDC as Multispace BDDC

We show several different ways the original, two-level, BDDC algorithm can be interpreted as multispace BDDC. We consider first BDDC applied to the reduced problem (23), that is, (1) with $X = \widehat{W}$. This was the formulation considered in [14]. Define the space of discrete harmonic functions with coarse degrees of freedom continuous across the interface

$$\widetilde{W}_\Gamma = (I - P)\,\widetilde{W}.$$

Because we work in the space of discrete harmonic functions and the output of the averaging operator E is not discrete harmonic, denote

$$E_\Gamma = (I - P)\,E. \tag{33}$$

In an implementation, discrete harmonic functions are represented by the values of their degrees of freedom on substructure interfaces, cf., e.g. [18]; hence, the definition (33) serves formal purposes only, so that everything can be written in terms of discrete harmonic functions without passing to the matrix formulation.

Algorithm 7 ([15], BDDC on the reduced problem) *Define the preconditioner $r \in \widehat{W}' \longmapsto u \in \widehat{W}$ by*

$$u = E_\Gamma w_\Gamma, \quad w_\Gamma \in \widetilde{W}_\Gamma : a\,(w_\Gamma, z_\Gamma) = \langle r, E_\Gamma z_\Gamma \rangle, \quad \forall z_\Gamma \in \widetilde{W}_\Gamma. \tag{34}$$

Proposition 8 ([15]) *The BDDC preconditioner on the reduced problem in Algorithm 7 is the abstract Multispace BDDC from Algorithm 2 with $M = 1$ and the space and operator given by*

$$X = \widehat{W}, \quad V_1 = \widetilde{W}_\Gamma, \quad Q_1 = E_\Gamma. \tag{35}$$

Also, the assumptions of Corollary 4 are satisfied.

Proof. We only need to note that the bilinear form $a(\cdot, \cdot)$ is positive definite on $\widetilde{W}_\Gamma \subset \widetilde{W}$ by (26), and the operator E_Γ defined by (33) is a projection by (31). The projection E_Γ is onto $\widehat{W}$ because E is onto U by (29), and $I - P$ maps U onto $\widehat{W}$ by the definition of $\widehat{W}$ in (20). ∎

Using the decomposition (32), we can split the solution in the space $\widetilde{W}_\Gamma$ into the independent solution of two subproblems: mutually independent problems on substructures as the solution in the space $\widetilde{W}_{\Gamma\Delta} = (I - P)\widetilde{W}_\Delta$, and a solution of global coarse problem in the space $\widetilde{W}_\Pi$. The space $\widetilde{W}_\Gamma$ has a decomposition

$$\widetilde{W}_\Gamma = \widetilde{W}_{\Gamma\Delta} \oplus \widetilde{W}_\Pi, \quad \text{and} \quad \widetilde{W}_{\Gamma\Delta} \perp_a \widetilde{W}_\Pi, \tag{36}$$

the same as the decomposition (32), and Algorithm 7 can be rewritten as follows.

Algorithm 9 ([14], BDDC on the reduced problem) *Define the preconditioner* $r \in \widehat{W}' \longmapsto u \in \widehat{W}$ *by* $u = E_\Gamma (w_{\Gamma\Delta} + w_\Pi)$, *where*

$$w_{\Gamma\Delta} \in \widetilde{W}_{\Gamma\Delta} : a(w_{\Gamma\Delta}, z_{\Gamma\Delta}) = \langle r, E_\Gamma z_{\Gamma\Delta} \rangle, \quad \forall z_{\Gamma\Delta} \in \widetilde{W}_{\Gamma\Delta}, \tag{37}$$

$$w_\Pi \in \widetilde{W}_\Pi : a(w_\Pi, z_{\Gamma\Pi}) = \langle r, E_\Gamma z_{\Gamma\Pi} \rangle, \quad \forall z_{\Gamma\Pi} \in \widetilde{W}_\Pi. \tag{38}$$

Proposition 10 *The BDDC preconditioner on the reduced problem in Algorithm 9 is the abstract Multispace BDDC from Algorithm 2 with $M = 2$ and the spaces and operators given by*

$$X = \widehat{W}, \quad V_1 = \widetilde{W}_{\Gamma\Delta}, \quad V_2 = \widetilde{W}_\Pi, \quad Q_1 = Q_2 = E_\Gamma. \tag{39}$$

Also, the assumptions of Corollary 4 are satisfied.

Proof. Let $r \in \widehat{W}'$. Define the vectors v_i, $i = 1, 2$ in Multispace BDDC by (3) with V_i and Q_i given by (39). Let u, $w_{\Gamma\Delta}$, w_Π be the quantities in Algorithm 9, defined by (37)-(38). Using the decomposition (36), any $w_\Gamma \in \widetilde{W}_\Gamma$ can be written uniquely as $w_\Gamma = w_{\Gamma\Delta} + w_\Pi$ for some $w_{\Gamma\Delta}$ and w_Π corresponding to (3) as $v_1 = w_{\Gamma\Delta}$ and $v_2 = w_\Pi$, and $u = E_\Gamma (w_{\Gamma\Delta} + w_\Pi)$.

To verify the assumptions of Corollary 4, note that the decomposition (36) is a−orthogonal, $a(\cdot, \cdot)$ is positive definite on both $\widetilde{W}_{\Gamma\Delta}$ and $\widetilde{W}_\Pi$ as subspaces of $\widetilde{W}_\Gamma$ by (26), and E_Γ is a projection by (31). ∎

Next, we present a BDDC formulation on the space U with explicit treatment of interior functions in the space U_I as in [4, 13], i.e., in the way the BDDC algorithm was originally formulated.

Algorithm 11 ([4, 13], original BDDC) *Define the preconditioner* $r \in U' \longmapsto u \in U$ *as follows. Compute the interior pre-correction*

$$u_I \in U_I : a(u_I, z_I) = \langle r, z_I \rangle, \quad \forall z_I \in U_I. \tag{40}$$

Set up the updated residual

$$r_B \in U', \quad \langle r_B, v \rangle = \langle r, v \rangle - a(u_I, v), \quad \forall v \in U. \tag{41}$$

Compute the substructure correction

$$u_\Delta = E w_\Delta, \quad w_\Delta \in \widetilde{W}_\Delta : a(w_\Delta, z_\Delta) = \langle r_B, E z_\Delta \rangle, \quad \forall z_\Delta \in \widetilde{W}_\Delta. \tag{42}$$

Compute the coarse correction

$$u_\Pi = E w_\Pi, \quad w_\Pi \in \widetilde{W}_\Pi : a(w_\Pi, z_\Pi) = \langle r_B, E z_\Pi \rangle, \quad \forall z_\Pi \in \widetilde{W}_\Pi. \tag{43}$$

Add the corrections

$$u_B = u_\Delta + u_\Pi.$$

Compute the interior post-correction

$$v_I \in U_I : a(v_I, z_I) = a(u_B, z_I), \quad \forall z_I \in U_I. \tag{44}$$

Apply the combined corrections

$$u = u_B - v_I + u_I. \tag{45}$$

The interior corrections (40) and (44) decompose into independent Dirichlet problems, one for each substructure. The substructure correction (42) decomposes into independent constrained Neumann problems, one for each substructure. Thus, the evaluation of the preconditioner requires three problems to be solved in each substructure, plus solution of the coarse problem (43). In addition, the substructure corrections can be solved in parallel with the coarse problem.

Remark 12 *As it is well known [4], the first interior correction (40) can be omitted in the implementation by starting the iterations from an initial solution such that the residual in the interior of the substructures is zero,*

$$a\left(u, z_I\right) - \langle f_X, z_I \rangle = 0, \quad \forall z_I \in U_I,$$

i.e., such that the error is discrete harmonic. Then the output of the preconditioner is discrete harmonic and thus the errors in all the CG iterations (which are linear combinations of the original error and outputs from the preconditioner) are also discrete harmonic by induction.

The following proposition will be the starting point for the multilevel case.

Proposition 13 *The original BDDC preconditioner in Algorithm 11 is the abstract Multispace BDDC from Algorithm 2 with $M = 3$ and the spaces and operators given by*

$$X = U, \quad V_1 = U_I, \quad V_2 = (I - P)\widetilde{W}_\Delta, \quad V_3 = \widetilde{W}_\Pi, \tag{46}$$

$$Q_1 = I, \quad Q_2 = Q_3 = (I - P)E, \tag{47}$$

and the assumptions of Corollary 4 are satisfied.

Proof. Let $r \in U'$. Define the vectors v_i, $i = 1, 2, 3$, in Multispace BDDC by (3) with the spaces V_i given by (46) and with the operators Q_i given by (47). Let u_I, r_B, w_Δ, w_Π, u_B, v_I, and u be the quantities in Algorithm 11, defined by (40)-(45).

First, with $V_1 = U_I$, the definition of v_1 in (3) with $k = 1$ is identical to the definition of u_I in (40), so $u_I = v_1$.

Next, consider $w_\Delta \in \widetilde{W}_\Delta$ defined in (42). We show that w_Δ satisfies (3) with $k = 2$, i.e., $v_2 = w_\Delta$. So, let $z_\Delta \in \widetilde{W}_\Delta$ be arbitrary. From (42) and (41),

$$a\left(w_\Delta, z_\Delta\right) = \langle r_B, E z_\Delta \rangle = \langle r, E z_\Delta \rangle - a\left(u_I, E z_\Delta\right). \tag{48}$$

Now from the definition of u_I by (40) and the fact that $P E z_\Delta \in U_I$, we get

$$\langle r, P E z_\Delta \rangle - a\left(u_I, P E z_\Delta\right) = 0, \tag{49}$$

and subtracting (49) from (48) gives

$$a\left(w_\Delta, z_\Delta\right) = \langle r, (I - P) E z_\Delta \rangle - a\left(u_I, (I - P) E z_\Delta\right)$$
$$= \langle r, (I - P) E z_\Delta \rangle,$$

because $a\left(u_I, (I - P) E z_\Delta\right) = 0$ by orthogonality. To verify (3), it is enough to show that $P w_\Delta = 0$; then $w_\Delta \in (I - P)\widetilde{W}_\Delta = V_2$. Since P is an a-orthogonal projection, it holds that

$$a\left(P w_\Delta, P w_\Delta\right) = a\left(w_\Delta, P w_\Delta\right) = \langle r_B, E P w_\Delta \rangle = 0, \tag{50}$$

where we have used $EU_I \subset U_I$ following the assumption (30) and the equality

$$\langle r_B, z_I \rangle = \langle r, z_I \rangle - a\,(u_I, z_I) = 0$$

for any $z_I \in U_I$, which follows from (41) and (40). Since a is positive definite on $\widetilde{W} \supset U_I$ by assumption (26), it follows from (50) that $Pw_\Delta = 0$.

In exactly the same way, from (43) – (45), we get that if $w_\Pi \in \widetilde{W}_\Pi$ is defined by (43), then $v_3 = w_\Pi$ satisfies (3) with $k = 3$. (The proof that $Pw_\Pi = 0$ can be simplified but there is nothing wrong with proceeding exactly as for w_Δ.)

Finally, from (44), $v_I = P\,(Ew_\Delta + Ew_\Pi)$, so

$$\begin{aligned}
u &= u_I + (u_B - v_I) \\
&= u_I + (I - P)\,Ew_\Delta + (I - P)\,Ew_\Pi \\
&= Q_1 v_1 + Q_2 v_2 + Q_3 v_3.
\end{aligned}$$

It remains to verify the assumptions of Corollary 4.

First, the spaces $\widetilde{W}_\Pi$ and $\widetilde{W}_\Delta$ are a-orthogonal by (25) and, from (27),

$$(I - P)\,\widetilde{W}_\Delta \subset \widetilde{W}_\Delta,$$

thus $(I - P)\,\widetilde{W}_\Delta \perp_a \widetilde{W}_\Pi$. Clearly, $(I - P)\,\widetilde{W}_\Delta \perp_a U_I$. Since $\widetilde{W}_\Pi$ consists of discrete harmonic functions from (28), so $\widetilde{W}_\Pi \perp_a U_I$, it follows that the spaces V_i, $i = 1, 2, 3$, given by (46), are a-orthogonal.

Next, $(I - P)\,E$ is by (31) a projection, and so are the operators Q_i from (47).

It remains to prove the decomposition of unity (14). Let

$$u' = u_I + w_\Delta + w_\Pi \in U, \quad u_I \in U_I, \quad w_\Delta \in (I - P)\,\widetilde{W}_\Delta, \quad w_\Pi \in \widetilde{W}_\Pi, \qquad (51)$$

and let

$$v = u_I + (I - P)\,Ew_\Delta + (I - P)\,Ew_\Pi.$$

From (51), $w_\Delta + w_\Pi \in U$ since $u' \in U$ and $u_I \in U_I \subset U$. Then $E\,(w_\Delta + w_\Pi) = w_\Delta + w_\Pi$ by (29), so

$$\begin{aligned}
v &= u_I + (I - P)\,E\,(w_\Delta + w_\Pi) \\
&= u_I + (I - P)\,(w_\Delta + w_\Pi) \\
&= u_I + w_\Delta + w_\Pi = u',
\end{aligned}$$

because both w_Δ and w_Π are discrete harmonic. ∎

The next Theorem shows an equivalence of the three Algorithms introduced above.

Theorem 14 *The eigenvalues of the preconditioned operators from Algorithm 7, and Algorithm 9 are exactly the same. They are also the same as the eigenvalues from Algorithm 11, except possibly for multiplicity of eigenvalue equal to one.*

Proof. From the decomposition (36), we can write any $w \in \widetilde{W}_\Gamma$ uniquely as $w = w_\Delta + w_\Pi$ for some $w_\Delta \in \widetilde{W}_{\Gamma\Delta}$ and $w_\Pi \in \widetilde{W}_\Pi$, so the preconditioned operators from Algorithms 7 and 9 are spectrally equivalent and we need only to show their spectral equivalence to the preconditioned operator from

Algorithm 11. First, we note that the operator $A : U \mapsto U'$ defined by (18), and given in the block form as

$$A = \left[\begin{array}{cc} \mathcal{A}_{II} & \mathcal{A}_{I\Gamma} \\ \mathcal{A}_{\Gamma I} & \mathcal{A}_{\Gamma\Gamma} \end{array} \right],$$

with blocks

$$\mathcal{A}_{II} : U_I \to U_I', \quad \mathcal{A}_{I\Gamma} : U_I \to \widehat{W}',$$
$$\mathcal{A}_{\Gamma I} : \widehat{W} \to U_I', \quad \mathcal{A}_{\Gamma\Gamma} : \widehat{W} \to \widehat{W}',$$

is block diagonal and $\mathcal{A}_{\Gamma I} = \mathcal{A}_{I\Gamma} = 0$ for any $u \in U$, written as $u = u_I + \widehat{w}$, because $U_I \perp_a \widehat{W}$. Next, we note that the block $\mathcal{A}_{\Gamma\Gamma} : \widehat{W}' \to \widehat{W}$ is the Schur complement operator corresponding to the form s. Finally, since the block $\mathcal{A}_{II}$ is used only in the preprocessing step, the preconditioned operator from Algorithms 7 and 9 is simply $M_{\Gamma\Gamma}\mathcal{A}_{\Gamma\Gamma} : r \in \widehat{W}' \to u \in \widehat{W}$.

Let us now turn to Algorithm 11. Let the residual $r \in U$ be written as $r = r_I + r_\Gamma$, where $r_I \in U_I'$ and $r_\Gamma \in \widehat{W}'$. Taking $r_\Gamma = 0$, we get $r = r_I$, and it follows that $r_B = u_B = v_I = 0$, so $u = u_I$. On the other hand, taking $r = r_\Gamma$ gives $u_I = 0$, $r_B = r_\Gamma$, $v_I = Pu_B$ and finally $u = (I - P) E(w_\Delta + w_\Pi)$, so $u \in \widehat{W}$. This shows that the off-diagonal blocks of the preconditioner M are zero, and therefore it is block diagonal

$$M = \left[\begin{array}{cc} M_{II} & 0 \\ 0 & M_{\Gamma\Gamma} \end{array} \right].$$

Next, let us take $u = u_I$, and consider $r_\Gamma = 0$. The algorithm returns $r_B = u_B = v_I = 0$, and finally $u = u_I$. This means that $M_{II}\mathcal{A}_{II}u_I = u_I$, so $M_{II} = \mathcal{A}_{II}^{-1}$. The operator $A : U \to U'$, and the block preconditioned operator $MA : r \in U' \to u \in U$ from Algorithm 11 can be written, respectively, as

$$A = \left[\begin{array}{cc} \mathcal{A}_{II} & 0 \\ 0 & \mathcal{A}_{\Gamma\Gamma} \end{array} \right], \quad MA = \left[\begin{array}{cc} I & 0 \\ 0 & M_{\Gamma\Gamma}\mathcal{A}_{\Gamma\Gamma} \end{array} \right],$$

where the right lower block $M_{\Gamma\Gamma}\mathcal{A}_{\Gamma\Gamma} : r \in \widehat{W}' \to u \in \widehat{W}$ is exactly the same as the preconditioned operator from Algorithms 7 and 9. ∎

The BDDC condition number estimate is well known from [13]. Following Theorem 14 and Corollary 4, we only need to estimate $\|(I - P) Ew\|_a$ on $\widehat{W}$.

Theorem 15 ([13]) *The condition number of the original BDDC algorithm satisfies $\kappa \leq \omega$, where*

$$\omega = \max \left\{ \sup_{w \in \widetilde{W}} \frac{\|(I - P) Ew\|_a^2}{\|w\|_a^2}, 1 \right\} \leq C \left(1 + \log \frac{H}{h} \right)^2. \tag{52}$$

Remark 16 *In [13], the theorem was formulated by taking the supremum over the space of discrete harmonic functions $(I - P)\widetilde{W}$. However, the supremum remains the same by taking the larger space $\widetilde{W} \supset (I - P)\widetilde{W}$, since*

$$\frac{\|(I - P) Ew\|_a^2}{\|w\|_a^2} \leq \frac{\|(I - P) E (I - P) w\|_a^2}{\|(I - P) w\|_a^2}$$

from $E (I - P) = E$, which follows from (30), and from $\|w\|_a \geq \|(I - P) w\|_a$, which follows from the a-orthogonality of the projection P.

Before proceeding into the Multilevel BDDC section, let us write concisely the spaces and operators involved in the two-level preconditioner as

$$U_I \overset{P}{\underset{\subset}{\in}} U \overset{E}{\underset{\subset}{\in}} \widetilde{W}_\Delta \oplus \widetilde{W}_\Pi = \widetilde{W} \subset W.$$

We are now ready to extend this decomposition into the multilevel case.

6 Multilevel BDDC and an Abstract Bound

In this section, we generalize the two-level BDDC preconditioner to multiple levels, using the abstract Multispace BDDC framework from Algorithm 2. The substructuring components from Section 5 will be denoted by an additional subscript $_1$, as Ω_1^s, $s = 1, \dots N_1$, etc., and called level 1. The level 1 coarse problem (43) will be called the level 2 problem. It has the same finite element structure as the original problem (1) on level 1, so we put $U_2 = \widetilde{W}_{\Pi 1}$. Level 1 substructures are level 2 elements and level 1 coarse degrees of freedom are level 2 degrees of freedom. Repeating this process recursively, level $i - 1$ substructures become level i elements, and the level i substructures are agglomerates of level i elements. Level i substructures are denoted by Ω_i^s, $s = 1, \dots, N_i$, and they are assumed to form a conforming triangulation with a characteristic substructure size H_i. For convenience, we denote by Ω_0^s the original finite elements and put $H_0 = h$. The interface Γ_i on level i is defined as the union of all level i boundary nodes, i.e., nodes shared by at least two level i substructures, and we note that $\Gamma_i \subset \Gamma_{i-1}$. Level $i - 1$ coarse degrees of freedom become level i degrees of freedom. The shape functions on level i are determined by minimization of energy with respect to level $i - 1$ shape functions, subject to the value of exactly one level i degree of freedom being one and others level i degrees of freedom being zero. The minimization is done on each level i element (level $i - 1$ substructure) separately, so the values of level $i - 1$ degrees of freedom are in general discontinuous between level $i - 1$ substructures, and only the values of level i degrees of freedom between neighboring level i elements coincide.

The development of the spaces on level i now parallels the finite element setting in Section 4. Denote $U_i = \widetilde{W}_{\Pi i-1}$. Let W_i^s be the space of functions on the substructure Ω_i^s, such that all of their degrees of freedom on $\partial \Omega_i^s \cap \partial \Omega$ are zero, and let

$$W_i = W_i^1 \times \cdots \times W_i^{N_i}.$$

Then $U_i \subset W_i$ is the subspace of all functions from W that are continuous across the interfaces Γ_i. Define $U_{Ii} \subset U_i$ as the subspace of functions that are zero on Γ_i, i.e., the functions "interior" to the level i substructures. Denote by P_i the energy orthogonal projection from W_i onto U_{Ii},

$$P_i : w_i \in W_i \longmapsto v_{Ii} \in U_{Ii} : a\left(v_{Ii}, z_{Ii}\right) = a\left(w_i, z_{Ii}\right), \quad \forall z_{Ii} \in U_{Ii}.$$

Functions from $(I - P_i) W_i$, i.e., from the nullspace of P_i, are called discrete harmonic on level i; these functions are a-orthogonal to U_{Ii} and energy minimal with respect to increments in U_{Ii}. Denote by $\widehat{W}_i \subset U_i$ the subspace of discrete harmonic functions on level i, that is

$$\widehat{W}_i = (I - P_i) U_i. \tag{53}$$

In particular, $U_{Ii} \perp_a \widehat{W}_i$. Define $\widetilde{W}_i \subset W_i$ as the subspace of all functions such that the values of any coarse degrees of freedom on level i have a common value for all relevant level i substructures and vanish on $\partial \Omega_i^s \cap \partial \Omega$, and $\widetilde{W}_{\Delta i} \subset W_i$ as the subspace of all functions such that their level i coarse degrees of freedom vanish. Define $\widetilde{W}_{\Pi i}$ as the subspace of all functions such that their level i coarse degrees of freedom between adjacent substructures coincide, and such that their energy is minimal. Clearly, functions in $\widetilde{W}_{\Pi i}$ are uniquely determined by the values of their level i coarse degrees of freedom, and

$$\widetilde{W}_{\Delta i} \perp_a \widetilde{W}_{\Pi i}, \quad \widetilde{W}_i = \widetilde{W}_{\Delta i} \oplus \widetilde{W}_{\Pi i}. \tag{54}$$

We assume that the level i coarse degrees of freedom are zero on all functions from U_{Ii}, that is,

$$U_{Ii} \subset \widetilde{W}_{\Delta i}. \tag{55}$$

$$
\begin{array}{ccccccccccc}
 & & U & = & \widetilde{W}_{\Pi 0} & & & & & & \\
 & & \shortparallel & & & & & & & & \\
U_{I1} & \overset{P_1}{\Subset} & U_1 & \overset{E_1}{\Subset} & \widetilde{W}_{\Pi 1} & \oplus & \widetilde{W}_{\Delta 1} & = & \widetilde{W}_1 & \subset & W_1 \\
 & & & & \shortparallel & & & & & & \\
 & & U_{I2} & \overset{P_2}{\Subset} & U_2 & \overset{E_2}{\Subset} & \widetilde{W}_{\Pi 2} & \oplus & \widetilde{W}_{\Delta 2} & = & \widetilde{W}_2 \subset W_2 \\
 & & & & \downarrow I_2 & & \shortparallel & & & & \\
 & & & & \widetilde{U}_2 & & \vdots & & & & \\
 & & & & & & \shortparallel & & & & \\
 & & U_{I,L-1} & \overset{P_{L-1}}{\Subset} & U_{L-1} & \overset{E_{L-1}}{\Subset} & \widetilde{W}_{\Pi L-1} & \oplus & \widetilde{W}_{\Delta L-1} & = & \widetilde{W}_{L-1} \subset W_{L-1} \\
 & & & & \downarrow I_{L-1} & & \shortparallel & & & & \\
 & & & & \widetilde{U}_{L-1} & & U_L & & & & \\
 & & & & & & \downarrow I_L & & & & \\
 & & & & & & \widetilde{U}_L & & & &
\end{array}
$$

Figure 1: Space decompositions, embeddings and projections in the Multilevel BDDC

In other words, level i coarse degrees of freedom depend on the values on level i substructure boundaries only. From (54) and (55), it follows that the functions in $\widetilde{W}_{\Pi i}$ are discrete harmonic on level i, that is

$$\widetilde{W}_{\Pi i} = (I - P_i)\widetilde{W}_{\Pi i}. \tag{56}$$

Let E be a projection from $\widetilde{W}_i$ onto U_i, defined by taking some weighted average on Γ_i

$$E_i : \widetilde{W}_i \to U_i, \quad E_i U_{Ii} = U_{Ii}, \quad E_i^2 = E_i.$$

Since projection is the identity on its range, E_i does not change the level i interior degrees of freedom, in particular

$$E_i U_{Ii} = U_{Ii}. \tag{57}$$

Finally, we introduce an interpolation $I_i : U_i \to \widetilde{U}_i$ from level i degrees of freedom to functions in some classical finite element space $\widetilde{U}_i$ with the same degrees of freedom as U_i. The space $\widetilde{U}_i$ will be used for comparison purposes, to invoke known inequalities for finite elements. A more detailed description of the properties of I_i and the spaces $\widetilde{U}_i$ is postponed to the next section.

The hierarchy of spaces and operators is shown concisely in Figure 1. The Multilevel BDDC method is defined recursively [4, 16] by solving the coarse problem on level i only approximately, by one application of the preconditioner on level $i - 1$. Eventually, at level, $L - 1$, the coarse problem, which is the level L problem, is solved exactly. We need a more formal description of the method here, which is provided by the following algorithm.

Algorithm 17 (Multilevel BDDC) *Define the preconditioner $r_1 \in U_1' \longmapsto u_1 \in U_1$ as follows:*
   for $i = 1, \ldots, L - 1$,

Compute interior pre-correction on level i,

$$u_{Ii} \in U_{Ii} : a(u_{Ii}, z_{Ii}) = \langle r_i, z_{Ii} \rangle, \quad \forall z_{Ii} \in U_{Ii}. \tag{58}$$

Get updated residual on level i,

$$r_{Bi} \in U_i, \quad \langle r_{Bi}, v_i \rangle = \langle r_i, v_i \rangle - a\left(u_{Ii}, v_i\right), \quad \forall v_i \in U_i. \tag{59}$$

Find the substructure correction on level i:

$$w_{\Delta i} \in W_{\Delta i} : a\left(w_{\Delta i}, z_{\Delta i}\right) = \langle r_{Bi}, E_i z_{\Delta i} \rangle, \quad \forall z_{\Delta i} \in W_{\Delta i}. \tag{60}$$

Formulate the coarse problem on level i,

$$w_{\Pi i} \in W_{\Pi i} : a\left(w_{\Pi i}, z_{\Pi i}\right) = \langle r_{Bi}, E_i z_{\Pi i} \rangle, \quad \forall z_{\Pi i} \in W_{\Pi i}, \tag{61}$$

*If $i = L - 1$, solve the coarse problem directly and set $u_L = w_{\Pi L-1}$,
    otherwise set up the right-hand side for level $i + 1$,*

$$r_{i+1} \in \widetilde{W}'_{\Pi i}, \quad \langle r_{i+1}, z_{i+1} \rangle = \langle r_{Bi}, E_i z_{i+1} \rangle, \quad \forall z_{i+1} \in \widetilde{W}_{\Pi i} = U_{i+1}, \tag{62}$$

end.
for $i = L - 1, \ldots, 1$,

Average the approximate corrections on substructure interfaces on level i,

$$u_{Bi} = E_i\left(w_{\Delta i} + u_{i+1}\right). \tag{63}$$

Compute the interior post-correction on level i,

$$v_{Ii} \in U_{Ii} : a\left(v_{Ii}, z_{Ii}\right) = a\left(u_{Bi}, z_{Ii}\right), \quad \forall z_{Ii} \in U_{Ii}. \tag{64}$$

Apply the combined corrections,

$$u_i = u_{Ii} + u_{Bi} - v_{Ii}. \tag{65}$$

end.

We can now show that the Multilevel BDDC can be cast as the Multispace BDDC on energy orthogonal spaces, using the hierarchy of spaces from Figure 1.

Lemma 18 *The Multilevel BDDC preconditioner in Algorithm 17 is the abstract Multispace BDDC preconditioner from Algorithm 2 with $M = 2L - 1$, and the spaces and operators*

$$
\begin{aligned}
X &= U_1, \quad V_1 = U_{I1}, \quad V_2 = (I - P_1)\widetilde{W}_{\Delta 1}, \quad V_3 = U_{I2}, \\
V_4 &= (I - P_2)\widetilde{W}_{\Delta 2}, \quad V_5 = U_{I3}, \quad \ldots \\
V_{2L-4} &= (I - P_{L-2})\widetilde{W}_{\Delta L-2}, \quad V_{2L-3} = U_{IL-1}, \\
V_{2L-2} &= (I - P_{L-1})\widetilde{W}_{\Delta L-1}, \quad V_{2L-1} = \widetilde{W}_{\Pi L-1},
\end{aligned}
\tag{66}
$$

$$
\begin{aligned}
Q_1 &= I, \quad Q_2 = Q_3 = (I - P_1)E_1, \\
Q_4 &= Q_5 = (I - P_1)E_1(I - P_2)E_2, \quad \ldots \\
Q_{2L-4} &= Q_{2L-3} = (I - P_1)E_1 \cdots (I - P_{L-2})E_{L-2}, \\
Q_{2L-2} &= Q_{2L-1} = (I - P_1)E_1 \cdots (I - P_{L-1})E_{L-1},
\end{aligned}
\tag{67}
$$

and the assumptions of Corollary 4 are satisfied.

Proof. Let $r_1 \in U_1'$. Define the vectors v_k, $k = 1, \ldots, 2L - 1$ by (3) with the spaces and operators given by (66)-(67), and let u_{Ii}, r_{Bi}, $w_{\Delta i}$, $w_{\Pi i}$, r_{i+1}, u_{Bi}, v_{Ii}, and u_i be the quantities in Algorithm 17, defined by (58)-(65).

First, with $V_1 = U_{I1}$, the definition of v_1 in (3) is (58) with $i = 1$ and $u_{I1} = v_1$. We show that in general, for level $i = 1, \ldots, L - 1$, and space $k = 2i - 1$, we get (3) with $V_k = U_{Ii}$, so that $v_k = u_{Ii}$ and in particular $v_{2L-3} = u_{IL-1}$. So, let $z_{Ii} \in U_{Ii}$, $i = 2, \ldots, L - 1$, be arbitrary. From (58) using (62) and (59),

$$a(u_{Ii}, z_{Ii}) = \langle r_i, z_{Ii} \rangle = \langle r_{Bi-1}, E_{i-1}z_{Ii} \rangle = \tag{68}$$
$$= \langle r_{i-1}, E_{i-1}z_{Ii} \rangle - a(u_{Ii-1}, E_{i-1}z_{Ii}).$$

Since from (58) using the fact that $P_{i-1}E_{i-1}z_{Ii} \in U_{Ii-1}$ it follows that

$$\langle r_{i-1}, P_{i-1}E_{i-1}z_{Ii} \rangle - a(u_{Ii-1}, P_{i-1}E_{i-1}z_{Ii}) = 0,$$

we get from (68),

$$a(u_{Ii}, z_{Ii}) = \langle r_{i-1}, (I - P_{i-1})E_{i-1}z_{Ii} \rangle - a(u_{Ii-1}, (I - P_{i-1})E_{i-1}z_{Ii}),$$

and because $a(u_{Ii-1}, (I - P_{i-1})E_{i-1}z_{Ii}) = 0$ by orthogonality, we get

$$a(u_{Ii}, z_{Ii}) = \langle r_{i-1}, (I - P_{i-1})E_{i-1}z_{Ii} \rangle.$$

Repeating this process recursively using (68), we finally get

$$a(u_{Ii}, z_{Ii}) = \langle r_{i-1}, (I - P_{i-1})E_{i-1}z_{Ii} \rangle = \quad \ldots$$
$$= \langle r_1, (I - P_1)E_1 \cdots (I - P_{i-1})E_{i-1}z_{Ii} \rangle.$$

Next, consider $w_{\Delta i} \in \widetilde{W}_{\Delta i}$ defined by (60). We show that for $i = 1, \ldots, L - 1$, and $k = 2i$, we get (3) with $V_k = \widetilde{W}_{\Delta i}$, so that $v_k = w_{\Delta i}$ and in particular $v_{2L-2} = w_{\Delta L-1}$. So, let $z_{\Delta i} \in \widetilde{W}_{\Delta i}$ be arbitrary. From (60) using (59),

$$a(w_{\Delta i}, z_{\Delta i}) = \langle r_{Bi}, E_i z_{\Delta i} \rangle = \langle r_i, E_i z_{\Delta i} \rangle - a(u_{Ii}, E_i z_{\Delta i}). \tag{69}$$

From the definition of u_{Ii} by (58) and since $P_i E_i z_{\Delta i} \in U_{Ii}$ it follows that

$$\langle r_i, P_i E_i z_{\Delta i} \rangle - a(u_{Ii}, P_i E_i z_{\Delta i}) = 0,$$

so (69) gives

$$a(w_{\Delta i}, z_{\Delta i}) = \langle r_i, (I - P_i)E_i z_{\Delta i} \rangle - a(u_{Ii}, (I - P_i)E_i z_{\Delta i}).$$

Next, because $a(u_{Ii}, (I - P_i)E_i z_{\Delta i}) = 0$ by orthogonality, and using (62),

$$a(w_{\Delta i}, z_{\Delta i}) = \langle r_i, (I - P_i)E_i z_{\Delta i} \rangle = \langle r_{Bi-1}, E_{i-1}(I - P_i)E_i z_{\Delta i} \rangle.$$

Repeating this process recursively, we finally get

$$a(w_{\Delta i}, z_{\Delta i}) = \langle r_i, (I - P_i)E_i z_{\Delta i} \rangle = \quad \ldots$$
$$= \langle r_1, (I - P_1)E_1 \cdots (I - P_i)E_i z_{\Delta i} \rangle.$$

To verify (3), it remains to show that $P_i w_{\Delta i} = 0$; then $w_{\Delta i} \in (I - P_i)\widetilde{W}_{\Delta i} = V_k$. Since P_i is an a-orthogonal projection, it holds that

$$a(P_i w_{\Delta i}, P_i w_{\Delta i}) = a(w_{\Delta i}, P_i w_{\Delta i}) = \langle r_{Bi}, E_i P_i w_{\Delta i} \rangle = 0,$$

where we have used $E_i U_{Ii} \subset U_{Ii}$ following the assumption (57) and the equality

$$\langle r_{Bi}, z_{Ii} \rangle = \langle r_i, z_{Ii} \rangle - a\left(u_{Ii}, z_{Ii}\right) = 0$$

for any $z_{Ii} \in U_{Ii}$, which follows from (58) and (59).

In exactly the same way, we get that if $w_{\Pi L-1} \in \widetilde{W}_{\Pi L-1}$ is defined by (61), then $v_{2L-1} = w_{\Pi L-1}$ satisfies (3) with $k = 2L - 1$.

Finally, from (63)-(65) for any $i = L - 2, \ldots, 1$, we get

$$
\begin{aligned}
u_i &= u_{Ii} + u_{Bi} - v_{Ii} \\
&= u_{Ii} + (I - P_i) E_i \left(w_{\Delta i} + u_{i+1}\right) \\
&= u_{Ii} + (I - P_i) E_i \left[w_{\Delta i} + u_{Ii+1} + (I - P_{i+1}) E_{i+1} \left(w_{\Delta i+1} + u_{i+2}\right)\right] \\
&= u_{Ii} + \\
&\quad + (I - P_i) E_i \left[w_{\Delta i} + \ldots + (I - P_{L-1}) E_{L-1} \left(w_{\Delta L-1} + u_{\Pi L-1}\right)\right],
\end{aligned}
$$

and, in particular for u_1,

$$
\begin{aligned}
u_1 &= u_{I1} + \\
&\quad + (I - P_1) E_1 \left[w_{\Delta 1} + \ldots + (I - P_{L-1}) E_{L-1} \left(w_{\Delta L-1} + u_{\Pi L-1}\right)\right] \\
&= Q_1 v_1 + Q_2 v_2 + \ldots + Q_{2L-2} v_{2L-2} + Q_{2L-1} v_{2L-1}.
\end{aligned}
$$

It remains to verify the assumptions of Corollary 4.

The spaces $\widetilde{W}_{\Pi i}$ and $\widetilde{W}_{\Delta i}$, for all $i = 1, \ldots, L - 1$, are a-orthogonal by (54) and from (55),

$$(I - P_i) \widetilde{W}_{\Delta i} \subset \widetilde{W}_{\Delta i},$$

thus $(I - P_i) \widetilde{W}_{\Delta i}$ is a-orthogonal to $\widetilde{W}_{\Pi i}$. Since $\widetilde{W}_{\Pi i} = U_{i+1}$ consists of discrete harmonic functions on level i from (56), and $U_{Ii+1} \subset U_{i+1}$, it follows by induction that the spaces V_k, given by (66), are a-orthogonal.

We now show that the operators Q_k defined by (67) are projections. From our definitions, coarse degrees of freedom on substructuring level i (from which we construct the level $i+1$ problem) depend only on the values of degrees of freedom on the interface Γ_i and $\Gamma_j \subset \Gamma_i$ for $j \geq i$. Then,

$$(I - P_j) E_j (I - P_i) E_i (I - P_j) E_j = (I - P_i) E_i (I - P_j) E_j. \tag{70}$$

Using (70) and since $(I - P_1) E_1$ is a projection by (31), we get

$$
\begin{aligned}
\left[(I - P_1) E_1 \cdots (I - P_i) E_i\right]^2 &= (I - P_1) E_1 (I - P_1) E_1 \cdots (I - P_i) E_i \\
&= (I - P_1) E_1 \cdots (I - P_i) E_i,
\end{aligned}
$$

so the operators Q_k from (67) are projections.

It remains to prove the decomposition of unity (14). Let $u_i \in U_i$, such that

$$u_i' = u_{Ii} + w_{\Delta i} + u_{i+1}, \tag{71}$$

$$u_{Ii} \in U_{Ii}, \quad w_{\Delta i} \in (I - P_i) \widetilde{W}_{\Delta i}, \quad u_{i+1} \in U_{i+1} \tag{72}$$

and

$$v_i = u_{Ii} + (I - P_i) E_i w_{\Delta i} + (I - P_i) E_i u_{i+1}. \tag{73}$$

From (71), $w_{\Delta i} + u_{i+1} \in U_i$ since $u_i \in U_i$ and $u_{Ii} \in U_{Ii} \subset U_i$. Then $E_i [w_{\Delta i} + u_{i+1}] = w_{\Delta i} + u_{i+1}$ by (57), so

$$v_i = u_{Ii} + (I - P_i)E_i [w_{\Delta i} + u_{i+1}] = u_{Ii} + (I - P_i) [w_{\Delta i} + u_{i+1}] =$$
$$= u_{Ii} + w_{\Delta i} + u_{i+1} = u_{Ii} + w_{\Delta i} + u_{i+1} = u'_i,$$

because $w_{\Delta i}$ and u_{i+1} are discrete harmonic on level i. The fact that u_{i+1} in (71) and (73) are the same on arbitrary level i can be proved in exactly the same way using induction and putting u_{i+1} in (71) as

$$u_{i+1} = u_{Ii+1} + \ldots + w_{\Delta L-1} + w_{\Pi L-1},$$
$$u_{Ii+1} \in U_{Ii+1}, \quad w_{\Delta L-1} \in (I - P_{L-1})\widetilde{W}_{\Delta L-1}, \quad w_{\Pi L-1} \in \widetilde{W}_{\Pi L-1},$$

and in (73) as

$$u_{i+1} = u_{Ii+1} + \ldots + (I - P_{i+1}) E_{i+1} \cdots (I - P_{L-1}) E_{L-1} (w_{\Delta L-1} + w_{\Pi L-1}).$$

∎

The following bound follows from writing of the Multilevel BDDC as Multispace BDDC in Lemma 18 and the estimate for Multispace BDDC in Corollary 4.

Lemma 19 *If for some $\omega \geq 1$,*

$$\|(I - P_1)E_1 w_{\Delta 1}\|_a^2 \leq \omega \|w_{\Delta 1}\|_a^2 \quad \forall w_{\Delta 1} \in (I - P_1)\widetilde{W}_{\Delta 1},$$
$$\|(I - P_1)E_1 u_{I2}\|_a^2 \leq \omega \|u_{I2}\|_a^2 \quad \forall u_{I2} \in U_{I2},$$
$$\ldots \tag{74}$$
$$\|(I - P_1)E_1 \cdots (I - P_{L-1})E_{L-1} w_{\Pi L-1}\|_a^2 \leq \omega \|w_{\Pi L-1}\|_a^2 \quad \forall w_{\Pi L-1} \in \widetilde{W}_{\Pi L-1},$$

then the Multilevel BDDC preconditioner (Algorithm 17) satisfies $\kappa \leq \omega$.

Proof. Choose the spaces and operators as in (66)-(67) so that $u_{I1} = v_1 \in V_1 = U_{I1}$, $w_{\Delta 1} = v_2 \in V_2 = (I - P_1)\widetilde{W}_{\Delta 1}$, ..., $w_{\Pi L-1} = v_{2L-1} \in V_{2L-1} = \widetilde{W}_{\Pi L-1}$. The bound now follows from Corollary 4. ∎

Lemma 20 *If for some $\omega_i \geq 1$,*

$$\|(I - P_i)E_i w_i\|_a^2 \leq \omega_i \|w_i\|_a^2, \quad \forall w_i \in \widetilde{W}_i, \quad i = 1, \ldots, L - 1, \tag{75}$$

then the Multilevel BDDC preconditioner (Algorithm 17) satisfies $\kappa \leq \prod_{i=1}^{L-1}\omega_i$.

Proof. Note from Lemma 19 that $(I - P_1)\widetilde{W}_{\Delta 1} \subset \widetilde{W}_{\Delta 1} \subset \widetilde{W}_1$, $U_{I2} \subset \widetilde{W}_{\Pi 1} \subset \widetilde{W}_1$, and generally $(I - P_i)\widetilde{W}_{\Delta i} \subset \widetilde{W}_{\Delta i} \subset \widetilde{W}_i$, $U_{Ii+1} \subset \widetilde{W}_{\Pi i} \subset \widetilde{W}_i$. ∎

7 Condition Number Bound for the Model Problem

Let $|w|_{a(\Omega_i^s)}$ be the energy norm of a function $w \in \widetilde{W}_{\Pi i}$, $i = 1, \ldots, L - 1$, restricted to subdomain Ω_i^s, $s = 1, \ldots N_i$, i.e., $|w|_{a(\Omega_i^s)}^2 = \int_{\Omega_i^s} \nabla w \nabla w \, dx$, and let $\|w\|_a$ be the norm obtained by piecewise integration over each Ω_i^s. To apply Lemma 20 to the model problem presented in Section 5, we need

to generalize the estimate from Theorem 15 to coarse levels. To this end, let $I_{i+1} : \widetilde{W}_{\Pi i} \to \widetilde{U}_{i+1}$ be an interpolation from the level i coarse degrees of freedom (i.e., level $i + 1$ degrees of freedom) to functions in another space $\widetilde{U}_{i+1}$ and assume that, for all $i = 1, \ldots, L - 1$, and $s = 1, \ldots, N_i$, the interpolation satisfies for all $w \in \widetilde{W}_{\Pi i}$ and for all Ω_{i+1}^s the equivalence

$$c_{i,1} \left| I_{i+1} w \right|_{a(\Omega_{i+1}^s)}^2 \leq \left| I_i w \right|_{a(\Omega_{i+1}^s)}^2 \leq c_{i,2} \left| I_{i+1} w \right|_{a(\Omega_{i+1}^s)}^2, \tag{76}$$

which implies by Lemma 22 also the equivalence

$$c_{i,1} \left| I_{i+1} w \right|_{H^{1/2}(\partial\Omega_{i+1}^s)}^2 \leq \left| I_i w \right|_{H^{1/2}(\partial\Omega_{i+1}^s)}^2 \leq c_{i,2} \left| I_{i+1} w \right|_{H^{1/2}(\partial\Omega_{i+1}^s)}^2, \tag{77}$$

with $c_{i,2}/c_{i,1} \leq \text{const}$ bounded independently of $H_0, \ldots, H_{i+1}$.

Remark 21 *Since $I_1 = I$, the two norms are the same on $\widetilde{W}_{\Pi 0} = \widetilde{U}_1 = U_1$.*

For the three-level BDDC in two dimensions, the result of Tu [20, Lemma 4.2], which is based on the lower bound estimates by Brenner and Sung [2], can be written in our settings for all $w \in \widetilde{W}_{\Pi 1}$ and for all Ω_2^s as

$$c_{1,1} \left| I_2 w \right|_{a(\Omega_2^s)}^2 \leq \left| w \right|_{a(\Omega_2^s)}^2 \leq c_{1,2} \left| I_2 w \right|_{a(\Omega_2^s)}^2, \tag{78}$$

where I_2 is a piecewise (bi)linear interpolation given by values at corners of level 1 substructures, and $c_{1,2}/c_{1,1} \leq \text{const}$ independently of H/h.

For the three-level BDDC in three dimensions, the result of Tu [19, Lemma 4.5], which is based on the lower bound estimates by Brenner and He [1], can be written in our settings for all $w \in \widetilde{W}_{\Pi 1}$ and for all Ω_2^s as

$$c_{1,1} \left| I_2 w \right|_{H^{1/2}(\partial\Omega_2^s)}^2 \leq \left| w \right|_{H^{1/2}(\partial\Omega_2^s)}^2 \leq c_{1,2} \left| I_2 w \right|_{H^{1/2}(\partial\Omega_2^s)}^2, \tag{79}$$

where I_2 is an interpolation from the coarse degrees of freedom given by the averages over substructure edges, and $c_{1,2}/c_{1,1} \leq \text{const}$ independently of H/h.

We note that the level 2 substructures are called subregions in [20, 19]. Since $I_1 = I$, with $i = 1$ the equivalence (78) corresponds to (76), and (79) to (77).

The next Lemma establishes the equivalence of seminorms on a factor space from the equivalence of norms on the original space. Let $V \subset U$ be finite dimensional spaces and $\|\cdot\|_A$ a norm on U and define

$$\left| u \right|_{U/V,A} = \min_{v \in V} \|u - v\|_A. \tag{80}$$

We will be using (80) for the norm on the space of discrete harmonic functions $(I - P_i)\widetilde{W}_i$ with V as the space of interior functions U_{Ii}, and also with V as the space $\widetilde{W}_{\Delta i}$. In particular, since $\widetilde{W}_{\Pi i} \subset (I - P_i)\widetilde{W}_i$, we have

$$w \in \widetilde{W}_{\Pi i}, \quad \|w\|_a = \min_{w_\Delta \in \widetilde{W}_{\Delta i}} \|w - w_\Delta\|_a \tag{81}$$

Lemma 22 *Let $\|\cdot\|_A$, $\|\cdot\|_B$ be norms on U, and*

$$c_1 \|u\|_B^2 \leq \|u\|_A^2 \leq c_2 \|u\|_B^2, \quad \forall u \in U. \tag{82}$$

Then for any subspace $V \subset U$,

$$c_1 \left| u \right|_{U/V,B}^2 \leq \left| u \right|_{U/V,A}^2 \leq c_2 \left| u \right|_{U/V,B}^2,$$

resp.,

$$c_1 \min_{v \in V} \|u - v\|_B^2 \leq \min_{v \in V} \|u - v\|_A^2 \leq c_2 \min_{v \in V} \|u - v\|_B^2. \tag{83}$$

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Proof. From the definition (80) of the norm on a factor space, we get

$$|u|_{U/V,A} = \min_{v \in V} \|u - v\|_A = \|u - v_A\|_A .$$

for some v_A. Let v_B be defined similarly. Then

$$|u|^2_{U/V,A} = \min_{v \in V} \|u - v\|^2_A = \|u - v_A\|^2_A \leq \|u - v_B\|^2_A \leq$$

$$\leq c_2 \|u - v_B\|^2_B = c_2 \min_{v \in V} \|u - v\|^2_B = c_2 |u|^2_{U/V,B} ,$$

which is the right hand side inequality in (83). The left hand side inequality follows by switching the notation for $\|\cdot\|_A$ and $\|\cdot\|_B$. ∎

Lemma 23 *For all $i = 0, \ldots, L - 1$, and $s = 1, \ldots, N_i$,*

$$c_{i,1} \, |I_{i+1}w|^2_{a(\Omega^s_{i+1})} \leq |w|^2_{a(\Omega^s_{i+1})} \leq c_{k,2} \, |I_{i+1}w|^2_{a(\Omega^s_{i+1})} , \quad \forall w \in \widetilde{W}_{\Pi i}, \; \forall \Omega^s_{i+1}, \tag{84}$$

with $c_{i,2}/c_{i,1} \leq C_i$, independently of $H_0, \ldots, H_{i+1}$.

Proof. The proof follows by induction. For $i = 0$, (84) holds because $I_1 = I$. Suppose that (84) holds for some $i < L - 2$ and let $w \in \widetilde{W}_{\Pi i+1}$. From the definition of $\widetilde{W}_{\Pi i+1}$ by energy minimization,

$$|w|_{a(\Omega^s_{i+1})} = \min_{w_\Delta \in \widetilde{W}_{\Delta i+1}} |w - w_\Delta|_{a(\Omega^s_{i+1})} . \tag{85}$$

From (85), the induction assumption, and Lemma 22 eq. (83), it follows that

$$c_{i,1} \min_{w_\Delta \in \widetilde{W}_{\Delta i+1}} |I_{i+1}w - I_{i+1}w_\Delta|^2_{a(\Omega^s_{i+2})} \tag{86}$$

$$\leq \min_{w_\Delta \in \widetilde{W}_{\Delta i+1}} |w - w_\Delta|^2_{a(\Omega^s_{i+2})} \leq c_{i,2} \min_{w_\Delta \in \widetilde{W}_{\Delta i+1}} |I_{i+1}w - I_{i+1}w_\Delta|^2_{a(\Omega^s_{i+2})}$$

From the assumption (76), applied to the functions of the form $I_{i+1}w$ on Ω^s_{i+2},

$$c_1 \, |I_{i+2}w|^2_{a(\Omega^s_{i+2})} \leq \min_{w_\Delta \in \widetilde{W}_{\Delta i+1}} |I_{i+1}w - I_{i+1}w_\Delta|^2_{a(\Omega^s_{i+2})} \leq c_2 \, |I_{i+2}w|^2_{a(\Omega^s_{i+2})} \tag{87}$$

with c_2/c_1, bounded independently of $H_0, \ldots, H_{i+1}$. Then (85), (86) and (87) imply (84) with $C_i = C_{i-1} c_2/c_1$. ∎

Next, we generalize the estimate from Theorem 15 to coarse levels.

Lemma 24 *For all substructuring levels $i = 1, \ldots, L - 1$,*

$$\|(I - P_i)E_i w_i\|^2_a \leq C_i \left(1 + \log \frac{H_i}{H_{i-1}} \right)^2 \|w_i\|^2_a , \quad \forall w_i \in U_i. \tag{88}$$

Proof. From (84), summation over substructures on level i gives

$$c_{i,1} \|I_i w\|^2_a \leq \|w\|^2_a \leq c_{i,2} \|I_i w\|^2_a , \quad \forall w \in U_i. \tag{89}$$

Next, in our context, using the definition of P_i and (83), we get

$$\|I_i(I - P_i)E_i w_i\|^2_a = \min_{u_{Ii} \in U_{Ii}} \|I_i E_i w_i - I_i u_{Ii}\|^2_a ,$$

so from (52) for some $\overline{C}_i$ and all $i = 1, \ldots, L-1$,

$$\min_{u_{Ii} \in U_{Ii}} \|I_i E_i w_i - I_i u_{Ii}\|_a^2 \leq \overline{C}_i \left(1 + \log \frac{H_i}{H_{i-1}}\right)^2 \|I_i w_i\|_a^2, \quad \forall w_i \in U_i. \tag{90}$$

Similarly, from (89) and (90) it follows that

$$\begin{aligned}
\|(I - P_i)E_i w_i\|_a^2 &= \min_{u_{Ii} \in U_{Ii}} \|E_i w_i - u_{Ii}\|_a^2 \\
&\leq c_{i,2} \min_{u_{Ii} \in U_{Ii}} \|I_i E_i w_i - I_i u_{Ii}\|_a^2 \\
&\leq c_{i,2} \overline{C}_i \left(1 + \log \frac{H_i}{H_{i-1}}\right)^2 \|I_i w_i\|_a^2 \\
&\leq \frac{c_{i,2} \overline{C}_i}{c_{i,1}} \left(1 + \log \frac{H_i}{H_{i-1}}\right)^2 \|w_i\|_a^2,
\end{aligned}$$

which is (88) with

$$C_i = \frac{c_{i,2} \overline{C}_i}{c_{i,1}},$$

and $c_{i,2}/c_{i,1}$ from Lemma 23. ∎

Theorem 25 *The Multilevel BDDC for the model problem and corner coarse function in 2D and edge coarse functions in 3D satisfies the condition number estimate*

$$\kappa \leq \prod_{i=1}^{L-1} C_i \left(1 + \log \frac{H_i}{H_{i-1}}\right)^2.$$

Proof. The proof follows from Lemmas 20 and 24, with $\omega_i = C_i \left(1 + \log \frac{H_i}{H_{i-1}}\right)^2$. ∎

Remark 26 *For $L = 3$ in two and three dimensions we recover the estimates by Tu [20, 19], respectively.*

Remark 27 *While for standard (two-level) BDDC it is immediate that increasing the coarse space and thus decreasing the space $\widetilde{W}$ cannot increase the condition number bound, this is an open problem for the multilevel method. In fact, the 3D numerical results in the next section suggest that this may not be the case.*

Corollary 28 *In the case of uniform coarsening, i.e. with $H_i/H_{i-1} = H/h$ and the same geometry of decomposition on all levels $i = 1, \ldots L - 1$, we get*

$$\kappa \leq C^{L-1} \left(1 + \log H/h\right)^{2(L-1)}. \tag{91}$$

8   Numerical Examples

Numerical examples are presented in this section for the Poisson equation in two and three dimensions. The problem domain in 2D (3D) is the unit square (cube), and standard bilinear (trilinear) finite elements are used for the discretization. The substructures at each level are squares or cubes, and periodic essential boundary conditions are applied to the boundary of the domain.

Figure 2: Plot of data in Tables 1 and 2 for $H_i/H_{i-1} = 3$ at all levels.

Figure 3: Plot of data in Tables 1 and 2 for uniform coarsening.

Table 1: Two dimensional (2D) results. The letters C and E designate the use of corners and edges, respectively, in the coarse space.

L	C		C+E		n	n_Γ
	iter	cond	iter	cond		
$H_i/H_{i-1} = 3$ at all levels.						
2	8	1.92	5	1.08	144	80
3	13	3.10	7	1.34	1296	720
4	17	5.31	9	1.60	11,664	6480
5	23	9.22	10	1.85	104,976	58,320
6	31	16.07	11	2.12	944,748	524,880
7	42	28.02	13	2.45	8,503,056	4,723,920
$H_i/H_{i-1} = 4$ at all levels.						
2	9	2.20	6	1.14	256	112
3	15	4.02	8	1.51	4096	1792
4	21	7.77	10	1.88	65,536	28,672
5	30	15.2	12	2.24	1,048,576	458,752
6	42	29.7	13	2.64	16,777,216	7,340,032
$H_i/H_{i-1} = 8$ at all levels.						
2	10	2.99	7	1.33	1024	240
3	19	7.30	11	2.03	65,536	15,360
4	31	18.6	13	2.72	4,194,304	983,040
5	50	47.38	15	3.40	268,435,456	62,914,560
$H_i/H_{i-1} = 12$ at all levels.						
2	11	3.52	8	1.46	2304	368
3	21	10.12	12	2.39	331,776	52,992
4	39	29.93	15	3.32	47,775,744	7,630,848
$H_i/H_{i-1} = 16$ at all levels.						
2	11	3.94	8	1.56	4096	496
3	23	12.62	13	2.67	1,048,576	126,976
4	43	41.43	16	3.78	268,435,456	32,505,856

This choice of boundary conditions allows us to solve very large problems on a single processor since all substructure matrices are identical for a given level.

The preconditioned conjugate gradient algorithm is used to solve the associated linear systems to a relative residual tolerance of 10^{-8} for random right-hand-sides with zero mean value. The zero mean condition is required since, for periodic boundary conditions, the null space of the coefficient matrix is the unit vector. The coarse problem always has 4^2 (4^3) subdomains at the coarsest level for 2D (3D) problems.

The number of levels (L), the number of iterations (iter), and condition number estimates (cond) obtained from the conjugate gradient iterations are reported in Tables 1 and 2. The letters C, E, and F designate the use of corners, edges, or faces in the coarse space. For example, C+E means that both corners and edges are used in the coarse space. For 2D and 3D problems, the theory is applicable to coarse spaces C and E, respectively. Also shown in the tables are the total number of unknowns (n) and the number of unknowns (n_Γ) on subdomain boundaries at the finest level.

The results in Tables 1 and 2 are displayed in Figure 2 for a fixed value of $H_i/H_{i-1} = 3$. In two dimensions we observe very different behavior depending on the particular form of the coarse space. If only corners are used in 2D, then there is very rapid growth of the condition number with increasing numbers of levels as predicted by the theory. In contrast, if both corners and edges are used in the 2D coarse space, then the condition number appears to vary linearly with L for the the number of levels considered. Our explanation is that a bound similar to Theorem 25 still applies to

Table 2: Three dimensional (3D) results. The letters C, E, and F designate the use of corners, edges, and faces, respectively, in the coarse space.

L	E		C+E		C+E+F		n	n_Γ
	iter	cond	iter	cond	iter	cond		
$H_i/H_{i-1} = 3$ at all levels.								
2	10	1.85	8	1.47	5	1.08	1728	1216
3	14	3.02	12	2.34	8	1.50	46,656	32,832
4	18	4.74	18	5.21	11	2.20	1,259,712	886,464
5	23	7.40	26	14.0	16	3.98	34,012,224	23,934,528
$H_i/H_{i-1} = 4$ at all levels.								
2	10	1.94	9	1.66	6	1.16	4096	2368
3	15	3.51	14	3.24	10	1.93	262,144	151,552
4	20	6.09	22	9.95	14	3.05	16,777,216	9,699,328
$H_i/H_{i-1} = 8$ at all levels.								
2	12	2.37	11	2.24	8	1.50	32,768	10,816
3	19	5.48	20	7.59	14	3.32	16,777,216	5,537,792
$H_i/H_{i-1} = 10$ at all levels.								
2	12	2.56	12	2.47	9	1.69	64,000	17,344
3	20	6.39	22	10.1	16	3.85	64,000,000	17,344,000

the favorable 2D case, though possibly with (much) smaller constants, so the exponential growth of the condition number is no longer apparent. The results in Tables 1 and 2 are also displayed in Figure 3 for fixed numbers of levels. The observed growth of condition numbers for the case of uniform coarsening is consistent with the estimate in (91).

Similar trends are present in 3D, but the beneficial effects of using more enriched coarse spaces are much less pronounced. Interestingly, when comparing the use of edges only (E) with corners and edges (C+E) in the coarse space, the latter does not always lead to smaller numbers of iterations or condition numbers for more than two levels. The fully enriched coarse space (C+E+F), however, does give the best results in terms of iterations and condition numbers. It should be noted that the present 3D theory in Theorem 25 covers only the use of the edges only, and the present theory does not guarantee that the condition number (or even its bound) decrease with increasing the coarse space (Remark 27).

In summary, the numerical examples suggest that better performance, especially in 2D, can be obtained when using a fully enriched coarse space. Doing so does not incur a large computational expense since there is never the need to solve a large coarse problem exactly with the multilevel approach. Finally, we note that a large number of levels is not required to solve very large problems. For example, the number of unknowns in 3D for a 4-level method with a coarsening ratio of $H_i/H_{i-1} = 10$ at all levels is $(10^4)^3 = 10^{12}$.

References

[1] S. C. BRENNER AND Q. HE, *Lower bounds for three-dimensional nonoverlapping domain decomposition algorithms*, Numer. Math., 93 (2003), pp. 445–470.

[2] S. C. BRENNER AND L.-Y. SUNG, *Lower bounds for nonoverlapping domain decomposition preconditioners in two dimensions*, Math. Comp., 69 (2000), pp. 1319–1339.

[3] ——, *BDDC and FETI-DP without matrices or vectors*, Comput. Methods Appl. Mech. Engrg., 196 (2007), pp. 1429–1435.

[4] C. R. Dohrmann, *A preconditioner for substructuring based on constrained energy minimization*, SIAM J. Sci. Comput., 25 (2003), pp. 246–258.

[5] M. Dryja and O. B. Widlund, *Schwarz methods of Neumann-Neumann type for three-dimensional elliptic finite element problems*, Comm. Pure Appl. Math., 48 (1995), pp. 121–155.

[6] C. Farhat, M. Lesoinne, P. LeTallec, K. Pierson, and D. Rixen, *FETI-DP: a dual-primal unified FETI method. I. A faster alternative to the two-level FETI method*, Internat. J. Numer. Methods Engrg., 50 (2001), pp. 1523–1544.

[7] C. Farhat, M. Lesoinne, and K. Pierson, *A scalable dual-primal domain decomposition method*, Numer. Linear Algebra Appl., 7 (2000), pp. 687–714. Preconditioning techniques for large sparse matrix problems in industrial applications (Minneapolis, MN, 1999).

[8] C. Farhat and F.-X. Roux, *A method of finite element tearing and interconnecting and its parallel solution algorithm*, Internat. J. Numer. Methods Engrg., 32 (1991), pp. 1205–1227.

[9] A. Klawonn and O. B. Widlund, *Dual-primal FETI methods for linear elasticity*, Comm. Pure Appl. Math., 59 (2006), pp. 1523–1572.

[10] J. Li and O. B. Widlund, *FETI-DP, BDDC, and block Cholesky methods*, Internat. J. Numer. Methods Engrg., 66 (2006), pp. 250–271.

[11] ——, *On the use of inexact subdomain solvers for BDDC algorithms*, Comput. Methods Appl. Mech. Engrg., 196 (2007), pp. 1415–1428.

[12] J. Mandel, *Balancing domain decomposition*, Comm. Numer. Methods Engrg., 9 (1993), pp. 233–241.

[13] J. Mandel and C. R. Dohrmann, *Convergence of a balancing domain decomposition by constraints and energy minimization*, Numer. Linear Algebra Appl., 10 (2003), pp. 639–659. Dedicated to the 70th birthday of Ivo Marek.

[14] J. Mandel, C. R. Dohrmann, and R. Tezaur, *An algebraic theory for primal and dual substructuring methods by constraints*, Appl. Numer. Math., 54 (2005), pp. 167–193.

[15] J. Mandel and B. Sousedík, *Adaptive selection of face coarse degrees of freedom in the BDDC and the FETI-DP iterative substructuring methods*, Comput. Methods Appl. Mech. Engrg., 196 (2007), pp. 1389–1399.

[16] J. Mandel, B. Sousedík, and C. R. Dohrmann, *On multilevel BDDC*, Lecture Notes in Computational Science and Engineering, 60 (2007), pp. 287–294. Domain Decomposition Methods in Science and Engineering XVII.

[17] B. F. Smith, P. E. Bjørstad, and W. D. Gropp, *Domain decomposition*, Cambridge University Press, Cambridge, 1996. Parallel multilevel methods for elliptic partial differential equations.

[18] A. Toselli and O. Widlund, *Domain decomposition methods—algorithms and theory*, vol. 34 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 2005.

[19] X. Tu, *Three-level BDDC in three dimensions*, SIAM J. Sci. Comput., 29 (2007), pp. 1759–1780.

[20] ———, *Three-level BDDC in two dimensions*, Internat. J. Numer. Methods Engrg., 69 (2007), pp. 33–59.