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„Extraktion quantifizierbarer Information aus komplexen Systemen"

Dual Pricing of Multi-Exercise Options under Volume Constraints

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Preprint 13

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Dual pricing of multi-exercise options under volume constraints

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Abstract

In this paper we study the pricing problem of multi-exercise options under volume constraints. The volume constraint is modeled by an adapted process with values in the positive integers, which describes the maximal number of rights to be exercised at a given time. We derive a representation of the marginal value of an additional nth right as a standard single stopping problem with a modified cash-flow process. This representation then leads to a dual pricing formula, which generalizes a result by Meinshausen and Hambly (2004) from the standard multi-exercise option (with at most one right per time step) to general constraints. We also state an explicit Monte Carlo algorithm for computing confidence intervals for the price of multi-exercise options under volume constraints and present numerical results for the pricing of a swing contract in an electricity market.

Keywords: duality, option pricing, Monte Carlo simulation, multi-exercise options, swing options.

AMS classification: 91B28, 60G40, 62L15, 65C05.

1 Introduction

Motivated by the pricing problem of swing contracts in energy markets, several numerical algorithms for solving multiple stopping problems by simulation have been developed during the last years. One class of algorithms tries to solve the corresponding backward dynamic program approximatively. Carmona and Touzi (2008) estimate the conditional expectations within the dynamic program by a Malliavin calculus approach, which originates in Lions and Regnier (2001) and was further developed in Bouchard et al. (2004). Meinshausen and Hambly (2004) approximate the dynamic program by the least-squares Monte-Carlo method, which was applied for American options by Longstaff and Schwartz (2001) and Tsitsiklis and Van Roy (2001) and

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can be traced back to Carrière (1996). The paper of Meinshausen and Hambly (2004) contains, however, as its main result a generalization of the dual pricing approach to multi-exercise options. While the approximate dynamic programming approach naturally leads to close-to-optimal exercise rules and, thus, to lower price bounds, the dual approach can be applied to calculate upper price bounds. The basic ideas of this dual approach for a single stopping problem (and hence for the pricing of American options) can already be found in a paper by Davis and Karatzas (1994). The value of these ideas for numerical option pricing was however first discovered independently by Rogers (2002) and Haugh and Kogan (2004). Finally, we mention the simulation approach by Ibáñez (2004), who approximates the early exercise frontier for some particular pay-off functions by simulation, and the policy improvement method by Bender and Schoenmakers (2006). For options on a low-dimensional underlying one can also apply trinomial forests to solve the backward dynamic program, see e.g. the paper by Jaillet et al. (2004).

A main drawback of the algorithms, mentioned so far, is that they are mostly designed for options which can be exercised several times throughout the lifetime of the option, but at most once per time point (e.g. per day). However, in practice, for many swing options the number of exercise rights, which is admitted per day, is not constant. For instance, in an off-peak swing option the holder can buy electricity in off-peak hours only, which yields twice as many rights on holidays or weekends than on business days.

The scope of the present paper is twofold. Firstly, we study the multiple stopping problem in discrete time under general volume constraints. Precisely, the volume constraint can be any non-anticipative stochastic process taking values in thepositive integers. We derive a representation of the marginal value for an additional nth right as a single stopping problem with a modified cash-flow. The representation implies a recursive description of the multiple stopping problem under volume constraints in terms of optimal stopping times. To the best of our knowledge, this formulation is novel even in the case of a standard multiple stopping problem with at most one right per day. In particular, it is different from the standard recursion for multiple stopping problems in terms of the value functions, which dates back to Haggstrom (1967), i.e.: The holder decides between a) exercising a right and entering a contract with one less right tomorrow and b) entering a contract with the same number of rights tomorrow. Having the representation of the marginal value of the multi-exercise option under volume constraints as a modified single stopping problem at hand, we can apply the duality theory for single stopping problems. In this way we obtain a dual pricing formula for multi-exercise options. In the standard setting with at most one right per day, we show that the formula coincides with the one derived by Meinshausen and Hambly (2004). We believe, however, that our approach sheds new light on the connection between single and multiple stopping problems even in this situation. For general volume constraints the formula appears to be new. Secondly, we explain how the theoretical results lead to a Monte Carlo algorithm for computing confidence intervals on the price of multi-exercise options under general volume constraints. The algorithm can, again, be interpreted as a generalization of the one by Meinshausen and Hambly (2004). We also present some numerical experiments for swing options in the context of an electricity market.

The paper is organized as follows: In Section 2 we discuss the theoretical main results of the paper. Section 3 is devoted to the conception of a Monte Carlo pricing algorithm, while Section 4 contains the numerical experiments. The proofs of the main results are postponed to Sections 5–7.

2 Discussion of the main results

In this section we state and discuss the main results of the present paper. To this end we first fix some notations. Let $(\Omega, \mathcal{F}, (\mathcal{F}_i)_{i=0,\ldots,I}, P)$ denote a filtered probability space in discrete time. From a financial point of view we assume that P is already a fixed pricing measure for some given discounted market of tradable securities. We denote by $Z(i)$ an adapted stochastic process, which we consider a cash-flow. We assume that $E[|Z(i)|] < \infty$ for all i and that $Z(I)$ is bounded from below, and denote a strict lower bound by $b_{\min}$, i.e.

$$Z(I) > b_{\min}, \quad P\text{-a.s.} \tag{1}$$

The holder of the multi-exercise option with n rights is entitled to exercise the cash-flow n times. She may exercise several rights at the same time, but is subjected to some volume constraint. The constraint is modeled by an adapted, positive, integer-valued stochastic process $c(i)$, which describes the number of exercises which are allowed at time i.

We fix a positive integer N and presume that we are only interested in contracts with $n \leq N$ total exercise rights. Moreover we suppose that $c(I) = N$, i.e. one is allowed to exercise all rights at the final time. In fact, if we set $Z(I) = 0$, then we may think of the time I as an additional 'virtual' time point and 'exercising' a right at time I then means that the investor decides not to exercise this right. Similarly, negative values at $Z(I)$ can be imposed in order to penalize not-exercising some rights. We shall say that c is the *unit volume constraint*, if $c(i) = 1$ for all $i = 1, \ldots, I-1$. This situation corresponds to the standard multiple stopping problem in discrete time as considered e.g. in Bender and Schoenmakers (2006) and Meinshausen and Hambly (2004).

We exemplify this setup by swing options which are important derivatives in energy markets.

Example 2.1. We consider the following stylized version of a *swing option*. The holder of the swing option can buy between $N_{\min}$ and $N_{\max}$ packages of energy for a strike price K per package over a time period $i = 0, \ldots, \mathcal{I}$. She can choose the number of packages, she buys, within the above limits and the time points, at which she buys the packages. At each time i the holder is allowed to buy at most $c(i)$ packages. The constraint $c(i)$ is fixed in the contract. A typical example is that one is only entitled to exercise a right in off-peak hours. In a somewhat simplified framework this means that twice as many rights can be exercised on weekends or holidays than on business days, which leads to a time-dependent but deterministic choice of c. This contract can be cast into our framework as follows: Suppose that an adapted stochastic process $S(i) > 0$ models the average off-peak price for one package of energy at day i, whence the cash-flow is given by $Z(i) = S(i) - K$. We consider the multi-exercise option with up to $N = N_{\max}$ exercise rights and extend the time grid by two points, i.e. we set $I = \mathcal{J}+2$. Moreover we define

$$\begin{aligned} Z(\mathcal{J}+1) &= 0, \quad c(\mathcal{J}+1) = N_{\max} - N_{\min} \\ Z(\mathcal{J}+2) &= -K, \quad c(\mathcal{J}+1) = N_{\max}. \end{aligned}$$

This means that up to $N_{\max} - N_{\min}$ rights can be left not-exercised without any cost. If more than $N_{\max} - N_{\min}$ are not exercised the holder must pay a penalty of K per additional not-exercised right. If we presume that the holder behaves rationally, then she will never leave more than $N_{\max} - N_{\min}$ not-exercised, because exercising at any time i yields $S(i) - K > -K$. Hence, 'exercising' at time $I = \mathcal{J} + 2$ will never be optimal and so the holder will always exercise at least $N_{\min}$ rights, i.e. will buy at least $N_{\min}$ packages. We emphasize that this formulation of a swing option is much closer to reality than the formulation as a standard multiple stopping problem, which is considered e.g. by Carmona and Dayanik (2008), Carmona and Touzi (2008) and Meinshausen and Hambly (2004). In the latter formulation the holder is restricted to buying at most one package per time.

We denote by $\mathcal{S}_i$ the set of $\{i, \dots, I\}$-valued stopping times. Given a fixed constraint c and a number n of exercise rights we define $\mathcal{S}_i^n$ to be the set of all n-tuples $(\tau_1, \dots, \tau_n) \subset \mathcal{S}_i$ such that

$$\sum_{\nu=1}^{n} \mathbf{1}_{\{\tau_\nu = j\}} \leq c(j)$$

for all $j = i, \dots, I$. Hence, $\mathcal{S}_i^n$ contains all exercise policies $(\tau_1, \dots, \tau_n)$ which obey the constraint c at time i and later, if n rights are left to be exercised at time i. This set is always nonempty, because the strategy $\tau_\nu = I$ for all $\nu = 1, \dots, n$ belongs to this set. A fair price of this multi-exercise option under the volume constraint c with n exercise rights at time i can then be represented by the stopping problem

$$Y^{*,n}(i) := \operatorname*{esssup}_{(\tau_1,\dots,\tau_n) \in \mathcal{S}_1^n} \sum_{\nu=1}^{n} E[Z(\tau_\nu) | \mathcal{F}_i].$$

Consequently, the marginal value for receiving an additional nth exercise right at time i is

$$\Delta^n Y^*(i) = Y^{*,n}(i) - Y^{*,n-1}(i),$$

with the convention $Y^{*,0}(i) = 0$ As a first result we give a recursive construction of the optimal policy for n exercise rights under the constraint and thereby show that the marginal value $\Delta^n Y^*(i)$ can be represented by a standard optimal stopping problem (with a single exercise right) with respect to a modified cash-flow.

Given an adapted stochastic process X and stopping times $(\tau_1, \dots, \tau_n) \in \mathcal{S}_i^n$ we define a modified adapted process

$$X^{[\tau_1,\dots,\tau_n]}(j) = \begin{cases} X(j), & \sum_{\nu=1}^n \mathbf{1}_{\{\tau_\nu = j\}} < c(j) \\ b_{\min}, & \text{otherwise} \end{cases}, \quad j = i, \dots, I.$$

We will apply the convention that $X^{[\tau_1,\dots,\tau_n]}(j) = X(j)$ for $n = 0$. Moreover, $\mathcal{Y}^*(j; X)$ denotes the Snell envelope for the standard optimal stopping problem with one exercise right with respect to the process X.

We then define recursively, for $n = 1, \dots, N$ and $i = 0, \dots, I$

$$\tau_n^*(i) = \inf\left\{ j = i, \dots, I;\ Z^{[\tau_1^*(i),\dots,\tau_{n-1}^*(i)]}(j) \geq \mathcal{Y}^*(j; Z^{[\tau_1^*(i),\dots,\tau_{n-1}^*(i)]}) \right\}.$$

Note that, by standard results on optimal stopping,

$$E[Z^{[\tau_1^*(i),\dots,\tau_{n-1}^*(i)]}(\tau_n^*(i))|\mathcal{F}_i] = \operatorname*{esssup}_{\tau \in \mathcal{S}_i} E[Z^{[\tau_1^*(i),\dots,\tau_{n-1}^*(i)]}(\tau)|\mathcal{F}_i],$$

and $\tau_n^*(i)$ is the smallest stopping time with this optimality property. Moreover,

$$Z^{[\tau_1^*(i),\dots,\tau_{n-1}^*(i)]}(\tau_n^*(i)) = Z(\tau_n^*(i)),$$

as it can never be optimal to exercise the modified cash-flow $Z^{[\tau_1^*(i),\dots,\tau_{n-1}^*(i)]}$, if it takes value $b_{\min}$ thanks to assumption (1). Hence, it is evident that the family $(\tau_1^*(i), \dots, \tau_n^*(i))$ belongs to $\mathcal{S}_i^n$. The following theorem states that it is optimal for the original multiple stopping problem with n rights under the constraint c.

Theorem 2.2. *For all $n = 1, \dots, N$, $i = 0, \dots, I$, we have $(\tau_1^*(i), \dots, \tau_n^*(i)) \in \mathcal{S}_i^n$, and this exercise strategy is optimal in the sense that*

$$Y^{*,n}(i) = \sum_{\nu=1}^n E[Z(\tau_\nu^*(i))|\mathcal{F}_i]$$

In particular, the marginal value satisfies

$$\Delta^n Y^*(i) = E[Z(\tau_n^*(i))|\mathcal{F}_i] = \operatorname*{esssup}_{\tau \in \mathcal{S}_i} E[Z^{[\tau_1^*(i),\dots,\tau_{n-1}^*(i)]}(\tau)|\mathcal{F}_i].$$

The theorem validates the basic intuition that optimal exercise times for n rights are also optimal for $n+1$ rights and then only the additional $(n+1)$th right must be chosen in an optimal way at a still-allowed time point. The proof is given in Section 5 for the case of the unit volume constraint and in Section 6 for general constraints.

Now recall that the cash-flow $Z^{[\tau_1,\ldots,\tau_n]}(i)$ becomes unattractive for the investor at time i when all available exercise rights at time i are already used by the policy $(\tau_1,\ldots,\tau_n)$. As such strategy will typically be suboptimal, one expects that the $(n+1)$th right can be used in a more profitable way than when the first n rights were exercised optimally. This is stated in the following theorem.

Theorem 2.3. *For all $n=1,\ldots,N$,$i=0,\ldots,I$, it holds that*

$$\Delta^n Y^*(i) \leq \operatorname*{esssup}_{\tau\in\mathcal{S}_i} E[Z^{[\tau_1,\ldots,\tau_{n-1}]}(\tau)|\mathcal{F}_i]$$

for all $(\tau_1,\ldots,\tau_{n-1})\in\mathcal{S}_i^{n-1}$.

For the proof we refer again to Section 5 for the unit constraint and Section 6 for the general case.

By combining Theorems 2.2 and 2.3 with the dual formulation for standard stopping problems due to Rogers (2002) and Haugh and Kogan (2004), we obtain the following dual representation for the marginal value of the multi-exercise option under volume constraints.

Theorem 2.4. *For all $n=1,\ldots,N$, it holds that*

$$\Delta^n Y^*(0) = \inf_{(\tau_1,\ldots,\tau_{n-1})\in\mathcal{S}_0^{n-1}} \inf_{M\in\mathcal{H}_0} E[\max_{i=0,\ldots,I}(Z^{[\tau_1,\ldots,\tau_{n-1}]}(i)-M(i))].$$

where $\mathcal{H}_0$ denotes the set of all martingales M with $M(0)=0$. Moreover, the minimizer is given by $(\tau_1,\ldots,\tau_{n-1})=(\tau_1^*(0),\ldots,\tau_{n-1}^*(0))$ as constructed above and $M(i)=M^{*,n}(i)$, where $M^{*,n}(i)$ is the martingale part of the Doob decomposition of $\mathcal{Y}^*(i;Z^{[\tau_1^*(0),\ldots,\tau_{n-1}^*(0)]})$.

Proof. On the one hand we have,

$$\Delta^n Y^*(0) \leq \sup_{\tau\in\mathcal{S}_0} E[Z^{[\tau_1,\ldots,\tau_{n-1}]}(\tau)] \leq E[\max_{i=0,\ldots,I}(Z^{[\tau_1,\ldots,\tau_{n-1}]}(i)-M(i))]$$

for all $(\tau_1,\ldots,\tau_{n-1})\in\mathcal{S}_i^{n-1}$ and $M\in\mathcal{H}_0$ by Theorem 2.3 above and Theorem 2.1 in Rogers (2002) applied to the cash-flow $Z^{[\tau_1,\ldots,\tau_{n-1}]}$. On the other hand, applying Theorem 2.1 in Rogers (2002) to the cash-flow $Z^{[\tau_1^*(0),\ldots,\tau_{n-1}^*(0)]}$, we obtain from Theorem 2.2 that

$$\begin{aligned}\Delta^n Y^*(0) &= \sup_{\tau\in\mathcal{S}_0} E[Z^{[\tau_1^*(0),\ldots,\tau_{n-1}^*(0)]}(\tau)]\\ &= E[\max_{i=0,\ldots,I}(Z^{[\tau_1^*(0),\ldots,\tau_{n-1}^*(0)]}(i)-M^{*,n}(i))],\end{aligned}$$

where $M^{*,n}(i)$ is, as defined above, the martingale part of the Doob decomposition of $\mathcal{Y}^*(i; Z^{[\tau_1^*(0),\ldots,\tau_{n-1}^*(0)]})$. $\square$

In the case of the unit volume constraint basically the same dual formulation is due to Meinshausen and Hambly (2004). The only difference is their representation of an optimal martingale. The following theorem links the Snell envelope $\mathcal{Y}^*(i; Z^{[\tau_1^*(0),\ldots,\tau_{n-1}^*(0)]})$ to the marginal value of the multi-exercise option under volume constraints. On the one hand this result is useful for a numerical implementation in a Markovian setting as we shall explain in the next section. On the other hand it demonstrates that, for the unit constraint, the optimal martingale in Theorem 2.4 actually coincides with the one in the paper by Meinshausen and Hambly (2004).

Theorem 2.5. *For all $n = 1, \ldots, N-1$, $0 \leq i \leq j \leq I$, it holds that*

$$\mathcal{Y}^*(j; Z^{[\tau_1^*(i),\ldots,\tau_n^*(i)]}) = \Delta^{n+1-\mathcal{E}_n^*(j-1;i)} Y^*(j)$$

where

$$\mathcal{E}_n^*(j; i) = \sum_{\nu=0}^{n} \mathbf{1}_{\{\tau_\nu^*(i) \leq j\}}$$

is the number of optimal stopping times (starting at time i) under n exercise rights which occur no later than time j.

The proof will be given in Section 7.

3 A Monte Carlo algorithm

We now explain how the results from the previous section can be applied to devise a numerical algorithm for computing lower and upper price bounds in a Markovian setting. The procedure generalizes the algorithm by Meinshausen and Hambly (2004) from the unit constraint to general volume constraints.

To this end suppose that $(X(i), \mathcal{F}_i)$ is a Markovian process with values in $\mathbb{R}^D$ and

$$Z(i) = h(i, X(i)), \quad c(i) = a(i, X(i))$$

for deterministic functions h, a. From a straightforward dynamic programming formulation[1]

$$\begin{aligned} Y^{*,n}(I) &= nh(I, X(I)) \\ Y^{*,n}(i) &= \max_{0 \leq \nu \leq a(i,X(i)) \wedge n} \nu h(i, X(i)) + E[Y^{*,n-\nu}(i+1)|\mathcal{F}_i], \end{aligned}$$

[1] This dynamic programming formulation and an algorithm for computing lower price bounds, similar to the one we present below, can already be found in the slides entitled ‘Modelling and Pricing in Electricity Markets’ by Ben Hambly, November 27, 2006.

for $n = 1, \ldots, N$, we obtain by induction that

$$Y^{*,n}(i) = y^{*,n}(i, X(i))$$

with

$$\begin{aligned} y^{*,n}(I, x) &= nh(I, x) \\ y^{*,n}(i, x) &= \max_{0 \le \nu \le a(i,x) \wedge n} \nu h(i, x) + E[y^{*,n-\nu}(i+1, X(i+1))|X(i) = x]. \end{aligned}$$

We denote, for $n = 1, \ldots, N$ and $i = 0, \ldots, I-1$,

$$\begin{aligned} q^{*,n}(i, x) &= E[y^{*,n}(i+1, X(i+1))|X(i) = x], \\ \Delta^n q^*(i, x) &= q^{*,n}(i, x) - q^{*,n-1}(i, x). \end{aligned}$$

Thanks to Theorem 2.5 we obtain that

$$\begin{aligned} \tau_n^*(0) &= \inf\{i \ge 0;\ [h(i, X(i)) \ge \Delta^{n-\mathcal{E}_{n-1}^*(i)} q^*(i, X(i))] \\ &\qquad \text{and } [\mathcal{E}_{n-1}^*(i) - \mathcal{E}_{n-1}^*(i-1) < a(i, X(i))]\}. \end{aligned}$$

Here and in the remainder of the section, we apply the convention $\mathcal{E}_{n-1}^*(i) = \mathcal{E}_{n-1}^*(i; 0)$.

Similarly, setting,

$$\Delta^n y^*(i, x) = y^{*,n}(i, x) - y^{*,n-1}(i, x),$$

we get

$$M^{*,n}(i) = \sum_{j=1}^{i} \Delta^{n-\mathcal{E}_{n-1}^*(j-1)} y^*(j, X(j)) - \Delta^{n-\mathcal{E}_{n-1}^*(j-1)} q^*(j-1, X(j-1)).$$

We now suppose that we have Λ independent copies (simulated paths) $(X_\lambda(i),\ i = 0, \ldots, I)$, $\lambda = 1, \ldots, \Lambda$, of the Markovian process $(X(i),\ i = 0, \ldots, I)$ at our disposal. Moreover, we assume that we are given an approximation $\hat{q}^{*,n}(i, x)$ of $q^{*,n}(i, x)$. This approximation can e.g. be pre-calculated by applying least-squares Monte Carlo with a different set of paths to the dynamic program as suggested by Tsitsiklis and Van Roy (2001) and Longstaff and Schwartz (2001) in the context of a single exercise right.

Then, we can define, recursively for $n = 1, \ldots, N$, starting with $\hat{\mathcal{E}}_{0,\lambda}^*(i) = 0$,

$$\begin{aligned} \hat{\tau}_{n,\lambda}^* &= \inf\{i \ge 0;\ [h(i, X_\lambda(i)) \ge \Delta^{n-\hat{\mathcal{E}}_{n-1,\lambda}^*(i)} \hat{q}^*(i, X_\lambda(i))] \\ &\qquad \text{and } [\hat{\mathcal{E}}_{n-1,\lambda}^*(i) - \hat{\mathcal{E}}_{n-1,\lambda}^*(i-1) < a(i, X_\lambda(i))]\} \\ \hat{\mathcal{E}}_{n,\lambda}^*(i) &= \sum_{\nu=0}^{n} \mathbf{1}_{\{\hat{\tau}_{\nu,\lambda}^* \le i\}}. \end{aligned}$$

A lower biased estimator for the multi-exercise option with n rights under the volume constraint c is then given by

$$\hat{Y}_{\text{low}}^{n} = \frac{1}{\Lambda}\sum_{\lambda=1}^{\Lambda}\sum_{\nu=1}^{n} h(\hat{\tau}_{\nu,\lambda}^{*}, X_{\lambda}(\hat{\tau}_{\nu,\lambda}^{*})).$$

In order to construct an upper biased estimator we define $\hat{y}^{*,n}(i,x)$ via

$$\begin{aligned} \hat{y}^{*,n}(I,x) &= nh(I,x) \\ \hat{y}^{*,n}(i,x) &= \max_{0\le\nu\le a(i,x)\wedge n} \nu h(i,x) + \hat{q}^{*,n-\nu}(i,x), \end{aligned}$$

and set

$$\Delta^{n}\hat{y}^{*}(i,x) = \hat{y}^{*,n}(i,x) - \hat{y}^{*,n-1}(i,x).$$

We now simulate Λ' new independent copies of X, which we denote, for notational simplicity, again by X_{λ}. As described above, one constructs $\hat{\tau}_{n,\lambda}^{*}$ and $\hat{\mathcal{E}}_{n,\lambda}^{*}(i)$ along each path. For every $\lambda = 1,\dots,\Lambda'$ and $i = 0,\dots,I-1$, one then generates M independent copies $X_{\lambda,\mu}(i+1)$, $\mu = 1,\dots,M$, with the law of $X(i+1)$ conditional on $\{X(i) = X_{\lambda}(i)\}$. Then, an estimator for $\Delta^{n}q^{*}(i, X_{\lambda}(i))$ is given by

$$\Delta^{n}\hat{Q}_{\lambda}^{*}(i) = \frac{1}{M}\sum_{\mu=1}^{M}\Delta^{n}\hat{y}^{*}(i+1, X_{\lambda,\mu}(i+1)), \quad \lambda = 1,\dots,\Lambda'.$$

Hence, we obtain

$$\hat{M}^{*,n,\lambda}(i) = \sum_{j=1}^{i}\Delta^{n-\hat{\mathcal{E}}_{n-1,\lambda}^{*}(j-1)}\hat{y}^{*}(j, X_{\lambda}(j)) - \Delta^{n-\hat{\mathcal{E}}_{n-1,\lambda}^{*}(j-1)}\hat{Q}_{\lambda}^{*}(j-1)$$

as estimator for $M^{*,n}(i)$ along the path X_{λ} for $\lambda = 1,\dots,\Lambda'$. Then we consider the estimator

$$\hat{Y}_{\text{up}}^{n} = \frac{1}{\Lambda'}\sum_{\lambda=1}^{\Lambda'}\sum_{\nu=1}^{n}\max_{i=0,\dots,I}(Z_{\lambda}^{[\hat{\tau}_{1,\lambda}^{*},\dots,\hat{\tau}_{\nu-1,\lambda}^{*}]}(i) - \hat{M}^{*,\nu,\lambda}(i)),$$

where, of course,

$$Z_{\lambda}^{[\hat{\tau}_{1,\lambda}^{*},\dots,\hat{\tau}_{n-1,\lambda}^{*}]}(i) = \begin{cases} h(i, X_{\lambda}(i)), & \hat{\mathcal{E}}_{n-1,\lambda}^{*}(i) - \hat{\mathcal{E}}_{n-1,\lambda}^{*}(i-1) < a(i, X_{\lambda}(i)) \\ b_{\min}, & \text{otherwise.} \end{cases}$$

Note that the use of nested simulation for estimating the martingale in the dual approach is originally due to Andersen and Broadie (2004) in the context of a single exercise right. It ensures that the estimator $\hat{Y}_{\text{up}}^{n}$ for the price of the multi-exercise option with n rights has a positive bias.

Remark 3.1. When the original filtration $(\mathcal{F}_i)$ satisfies $\mathcal{F}_i = \mathcal{G}_{t_i}$ and $(\mathcal{G}_t)$ is the filtration generated by a Brownian motion, the use of nested simulation can be avoided. Instead one can apply the estimator by Belomestny et al. (2008), which is based upon the martingale representation property of the Brownian motion and least-squares Monte-Carlo, and preserves the martingale property.

4 Numerical experiments

We apply the numerical algorithm to an off-peak swing contract in an electricity market as described in Example 2.1 above. For simplicity we assume that no minimum number of rights must be exercised, i.e. we set $N_{\min} = 0$. As in Meinshausen and Hambly (2004) we choose a toy model of AR(1)-type for the logarithmic off-peak spot price $S(i)$ for a package of electricity, i.e.

$$\log(S(i)) = (1-k)(\log(S(i-1)) - \mu) + \mu + \sigma\epsilon(i), \quad S(0) = 1,$$

where $\epsilon(i)$ are i.i.d standard Gaussian random variables. We consider a period of 1000 days (i.e. $\mathcal{I} = 999$) and the parameters

$$\sigma = 0.5, \quad k = 0.9, \quad \mu = 0, \quad K = 0.$$

This very simplistic model is a discrete version of the exponential Gaussian Ornstein-Uhlenbeck process which was suggested as a model for electricity prices by Lucia and Schwartz (2002). It is however a main advantage of the proposed algorithm that it can be generically applied to any Markovian model. In particular, it is applicable to the non-Gaussian exponential Ornstein-Uhlenbeck models which have been proposed by Benth et al. (2007) and Hambly et al. (2008). For more information on the modeling of electricity prices we refer to the recent monograph by Benth et al. (2008).

In order to obtain the numerical results, reported below, we proceed as follows: The approximation of the continuation values within the dynamic program are pre-calculated by a least-squares regression with 1000 paths and the two basis functions $\psi_1(s) = 1$ and $\psi_2(s) = s$. The lower biased estimator is calculated with $\Lambda = 20000$ simulated paths. For the upper estimator we apply $\Lambda' = 2000$ outer paths and $M = 50$ inner paths.

Table 1 reports bounds for the marginal price of the nth exercise right for up to 50 rights for an off-peak swing option, i.e. in a situation, where one right may be exercised on business days and two rights on Saturdays or Sundays. Precisely, Table 1 shows the lower and upper estimators

$$\begin{aligned}
\Delta^n \hat{Y}_{\text{low}} &= \frac{1}{\Lambda}\sum_{\lambda=1}^{\Lambda} h(\hat{\tau}^*_{n,\lambda}, X_\lambda(\hat{\tau}^*_{n,\lambda})), \\
\Delta^n \hat{Y}_{\text{up}} &= \frac{1}{\Lambda'}\sum_{\lambda=1}^{\Lambda'} \max_{i=0,\dots,I} (Z_\lambda^{[\hat{\tau}^*_{1,\lambda},\dots,\hat{\tau}^*_{n-1,\lambda}]}(i) - \hat{M}^{*,n,\lambda}(i)),
\end{aligned}$$

Table 1: Numerical results for the marginal price for the nth right of a off-peak swing option

n	$\Delta^n \hat{Y}_{\text{low}}$	$\Delta^n \hat{Y}_{\text{up}}$	95% confidence interval
1	4.768	4.788	[4.755, 4.797]
2	4.380	4.403	[4.370, 4.411]
3	4.118	4.136	[4.109, 4.143]
4	3.945	3.961	[3.937, 3.968]
5	3.798	3.820	[3.791, 3.826]
10	3.368	3.393	[3.362, 3.397]
15	3.126	3.148	[3.121, 3.152]
20	2.952	2.977	[2.947, 2.981]
25	2.824	2.845	[2.821, 2.849]
30	2.713	2.739	[2.710, 2.743]
35	2.630	2.650	[2.627, 2.653]
40	2.551	2.575	[2.548, 2.579]
45	2.486	2.510	[2.483, 2.513]
50	2.424	2.450	[2.421, 2.454]

for the marginal values and a 95% confidence interval. We observe that, despite the large number of time steps (1000) and the large number of exercise rights, the relative difference between the negative- and positive-biased estimator does not exceed 1% (except for the case $n = 50$, where it is 1.1%). Note that the confidence interval could easily be shrunken further by increasing the number of outer paths Λ and Λ'. However, no matter how large the number of simulated outer paths is, there will remain a bias between negative- and positive biased estimator due to the error of approximating the continuation values by regression. Figure 1 compares the marginal price of the off-peak swing option with the unit constraint case, which was considered by Meinshausen and Hambly (2004). As expected, the marginal prices under the unit constraint decrease significantly faster than the marginal prices, when a second right can be exercised at weekends.

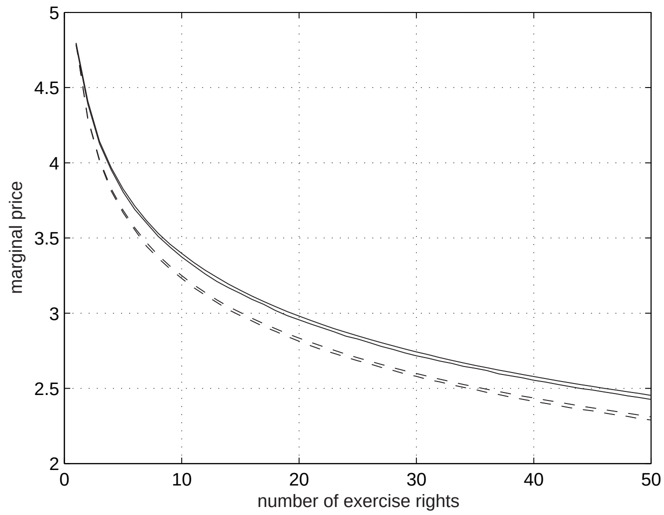

Figure 1: Comparison of the 95% confidence interval for the marginal price of the swing option under unit constraint (dashed lines) and with a second exercise right at weekends (solid lines).

5 Proof of Theorems 2.2 and 2.3 under the unit volume constraint

In this section we consider the case of the unit volume constraint

$$c(i) = 1,\ i = 0, \ldots, I-1, \quad c(i) = N,$$

hence at most one right can be exercised at a time (with exception of the terminal time) and the problem reduces to a standard multiple stopping problem as studied e.g. by Meinshausen and Hambly (2004).

It is well known that the Snell envelope $Y^{*,n}(i)$ satisfies in this situation

$$Y^{*,n}(i) = \sum_{\nu=1}^{n} E[Z(\sigma_\nu^{*,n}(i))|\mathcal{F}_i]$$

where

$$\begin{aligned} \sigma_0^{*,n}(i) &= i-1 \\ \sigma_\nu^{*,n}(i) &= \inf\{j \geq \sigma_{\nu-1}^{*,n}(i)+1;\ Z(j) \geq E[\Delta^{n+1-\nu}Y^*(j+1)|\mathcal{F}_j]\}, \end{aligned}$$

see e.g. Haggstrom (1967). Here we apply the convention that all time indices and stopping times are always truncated at I, e.g. j is to be read as $\min\{j, I\}$. In fact, $\sigma_\nu^{*,n}(i)$ is the smallest stopping time, at which it can be optimal to exercise the νth of n rights, where n is the number of rights remaining at time i. Consequently, $\{\sigma_1^{*,n}(i), \ldots, \sigma_n^{*,n}(i)\}$ is the 'smallest' optimal family for exercising the cash-flow $Z(j)$ n times starting from time i.

We shall first study some properties of the stopping times $\sigma_\nu^{*,n}(i)$.

Proposition 5.1. *In the case of unit volume constraint we have, for all $n = 1, \ldots, N$ and $i = 0, \ldots, I$,*

$$\{\sigma_1^{*,n-1}(i), \ldots, \sigma_{n-1}^{*,n-1}(i)\} \subset \{\sigma_1^{*,n}(i), \ldots, \sigma_n^{*,n}(i)\},$$

i.e. whenever it is optimal to exercise one of $(n-1)$ rights, it will also be optimal to exercise one of n rights.

For the proof we require the following straightforward lemma.

Lemma 5.2. *(i) For all $1 \leq \nu \leq n \leq N$ it holds that*

$$\sigma_\nu^{*,n}(i) = \sigma_1^{*,n+1-\nu}(\sigma_{\nu-1}^{*,n}(i) + 1)$$

(ii) For all $0 \leq \nu \leq n \leq N-1$ it holds that

$$\sigma_\nu^{*,n}(i) \leq \sigma_{\nu+1}^{*,n+1}(i).$$

Proof. (i) is a direct consequence of the definition, since

$$\begin{aligned}
\sigma_\nu^{*,n}(i) &= \inf\{j \geq \sigma_{\nu-1}^{*,n}(i) + 1;\ Z(j) \geq E[\Delta^{n+1-\nu} Y^*(j+1) | \mathcal{F}_j]\} \\
&= \inf\{j \geq \sigma_{\nu-1}^{*,n}(i) + 1;\ Z(j) \geq E[\Delta^{(n+1-\nu)+1-1} Y^*(j+1) | \mathcal{F}_j]\} \\
&= \sigma_1^{*,n+1-\nu}(\sigma_{\nu-1}^{*,n}(i) + 1).
\end{aligned}$$

(ii) is proved by induction on $\nu = 0, \ldots, n$. For $\nu = 0$ we clearly have,

$$\sigma_0^{*,n}(i) = i - 1 < i \leq \sigma_1^{*,n+1}(i).$$

Now suppose the assertion is already proved for $\nu - 1$. As $\sigma_1^{*,n}(i)$ is increasing in i for all n, we obtain from (i) and the inductive hypothesis,

$$\sigma_\nu^{*,n}(i) = \sigma_1^{*,n+1-\nu}(\sigma_{\nu-1}^{*,n}(i) + 1) \leq \sigma_1^{*,n+1-\nu}(\sigma_\nu^{*,n+1}(i) + 1) = \sigma_{\nu+1}^{*,n+1}(i).$$

□

We now give the

Proof of Proposition 5.1. It suffices to show the following two assertions:

(a) $\sigma_\nu^{*,n+1}(i) \leq \sigma_\nu^{*,n}(i)$, i.e. the νth of $(n+1)$ rights is never exercised later than the νth of n rights.

(b) If $\sigma_\nu^{*,n+1}(i) < \sigma_\nu^{*,n}(i)$, then $\sigma_{\nu+1}^{*,n+1}(i) = \sigma_\nu^{*,n}(i)$. I.e., if the νth of $(n+1)$ rights is exercised strictly earlier than the νth of n rights, then the $(\nu+1)$th of $(n+1)$ rights will be exercised at the same time as the νth of n rights.

We first prove (a) by induction on ν. For $\nu = 0$ the claim is obvious. Now suppose that it is already shown for some $\nu \leq n-1$. Using the fact that the marginal value $\Delta^n Y(i)$ is decreasing in n by Proposition 5.2 in Meinshausen and Hambly (2004), we obtain from the inductive hypothesis

$$\begin{aligned}\sigma_{\nu+1}^{*,n+1}(i) &= \inf\{j \geq \sigma_\nu^{*,n+1}(i)+1;\ Z(j) \geq E[\Delta^{n+1-\nu}Y^*(j+1)|\mathcal{F}_j]\} \\ &\leq \inf\{j \geq \sigma_\nu^{*,n}(i)+1;\ Z(j) \geq E[\Delta^{n+1-\nu}Y^*(j+1)|\mathcal{F}_j]\} \\ &\leq \inf\{j \geq \sigma_\nu^{*,n}(i)+1;\ Z(j) \geq E[\Delta^{n-\nu}Y^*(j+1)|\mathcal{F}_j]\} \\ &= \sigma_{\nu+1}^{*,n}(i).\end{aligned}$$

We now show assertion (b). Hence, we assume that $\sigma_\nu^{*,n+1}(i) < \sigma_\nu^{*,n}(i)$. Lemma 5.2 yields,

$$\sigma_{\nu-1}^{*,n}(i)+1 \leq \sigma_\nu^{*,n+1}(i)+1 \leq \sigma_\nu^{*,n}(i) = \sigma_1^{*,n+1-\nu}(\sigma_{\nu-1}^{*,n}(i)+1).$$

We now apply the operator $\sigma_1^{*,n+1-\nu}(\cdot)$ to this inequality. As $\sigma_1^{*,n+1-\nu}(j)$ is increasing in j and $\sigma_1^{*,n+1-\nu}(j) = \sigma_1^{*,n+1-\nu}(\sigma_1^{*,n+1-\nu}(j))$ we get

$$\sigma_1^{*,n+1-\nu}(\sigma_\nu^{*,n+1}(i)+1) = \sigma_1^{*,n+1-\nu}(\sigma_{\nu-1}^{*,n}(i)+1).$$

Now the left-hand side coincides with $\sigma_{\nu+1}^{*,n+1}(i)$ and the right-hand side coincides with $\sigma_\nu^{*,n}(i)$ by Lemma 5.2, (i). Thus, the proof is complete. □

We are now in a position to state and prove a version of Theorem 2.2 for the case of the unit volume constraint.

Theorem 5.3. *In the case of the unit volume constraint, we have for all $n = 1, \ldots, N$, $i = 1, \ldots, I$,*

$$\{\sigma_1^{*,n}(i), \ldots, \sigma_n^{*,n}(i)\} = \{\tau_1^*(i), \ldots, \tau_n^*(i)\}.$$

In particular,

$$Y^{*,n}(i) = \sum_{\nu=1}^n E[Z(\tau_\nu^*(i))|\mathcal{F}_i] = \sum_{\nu=1}^n \operatorname*{esssup}_{\tau_\nu \in \mathcal{S}_i} E[Z^{[\tau_1^*(i),\ldots,\tau_{\nu-1}^*(i)]}(\tau_\nu)|\mathcal{F}_i].$$

Proof. The proof is by induction on n. Recall that $\tau_1^*(i) = \sigma_1^{*,1}(i)$ by definition. Now assume that the assertions are already proved for some $n < N$. We define a new stopping time by

$$\sigma_{n+1}^{*,n}(i) = \inf\{j \geq i;\ (\forall_{\nu=1,\ldots,n}\ \sigma_\nu^{*,n}(i) \neq j) \text{ and } (\exists_{\mu=1,\ldots,n+1}\ \sigma_\mu^{*,n+1}(i) = j)\}.$$

Then Proposition 5.1 implies

$$\{\sigma_1^{*,n}(i), \ldots, \sigma_{n+1}^{*,n}(i)\} = \{\sigma_1^{*,n+1}(i), \ldots, \sigma_{n+1}^{*,n+1}(i)\}.$$

Consequently, by the inductive hypothesis and the definition of $\tau_{n+1}^*(i)$

$$\begin{aligned}
Y^{*,n+1}(i) &= \sum_{\nu=1}^{n+1} E[Z(\sigma_\nu^{*,n+1}(i))|\mathcal{F}_i] = \sum_{\nu=1}^{n+1} E[Z(\sigma_\nu^{*,n}(i))|\mathcal{F}_i] \\
&= Y^{*,n}(i) + E[Z^{[\sigma_1^{*,n}(i),\ldots,\sigma_n^{*,n}(i)]}(\sigma_{n+1}^{*,n}(i))|\mathcal{F}_i] \\
&= Y^{*,n}(i) + E[Z^{[\tau_1^*(i),\ldots,\tau_n^*(i)]}(\sigma_{n+1}^{*,n}(i))|\mathcal{F}_i] \\
&\leq Y^{*,n}(i) + \operatorname*{esssup}_{\tau\in\mathcal{S}_i} E[Z^{[\tau_1^*(i),\ldots,\tau_n^*(i)]}(\tau)|\mathcal{F}_i] \\
&= \sum_{\nu=1}^{n} E[Z(\tau_\nu^*(i))|\mathcal{F}_i] + E[Z^{[\tau_1^*(i),\ldots,\tau_n^*(i)]}(\tau_{n+1}^*(i))|\mathcal{F}_i] \\
&= \sum_{\nu=1}^{n+1} E[Z(\tau_\nu^*(i))|\mathcal{F}_i].
\end{aligned}$$

In particular, the inequality turns into an identity and we get

$$Y^{*,n+1}(i) = \sum_{\nu=1}^{n+1} E[Z(\tau_\nu^*(i))|\mathcal{F}_i] = \sum_{\nu=1}^{n+1} \operatorname*{esssup}_{\tau_\nu\in\mathcal{S}_i} E[Z^{[\tau_1^*(i),\ldots,\tau_{\nu-1}^*(i)]}(\tau_\nu)|\mathcal{F}_i]$$

and

$$E[Z^{[\tau_1^*(i),\ldots,\tau_n^*(i)]}(\sigma_{n+1}^{*,n}(i))|\mathcal{F}_i] = \operatorname*{esssup}_{\tau\in\mathcal{S}_i} E[Z^{[\tau_1^*(i),\ldots,\tau_n^*(i)]}(\tau)|\mathcal{F}_i].$$

As $\tau_{n+1}^*(i)$ is the smallest optimizer for this stopping problem, we obtain, $\tau_{n+1}^*(i) \leq \sigma_{n+1}^{*,n}(i)$. Now note that (using the inductive hypothesis again)

$$\{\sigma_1^{*,n}(i), \ldots, \sigma_n^{*,n}(i), \tau_{n+1}^*(i)\} = \{\tau_1^*(i), \ldots, \tau_{n+1}^*(i)\}$$

is an optimal stopping family for the multiple stopping problem with $(n+1)$ rights and $\{\sigma_1^{*,n}(i), \ldots, \sigma_{n+1}^{*,n}(i)\} = \{\sigma_1^{*,n+1}(i), \ldots, \sigma_{n+1}^{*,n+1}(i)\}$ is the smallest optimal one for the same problem. Hence, $\tau_{n+1}^*(i) \geq \sigma_{n+1}^{*,n}(i)$. Consequently, we have $\tau_{n+1}^*(i) = \sigma_{n+1}^{*,n}(i)$, which completes the proof.

□

We now turn to the proof of Theorem 2.3 in the case of the unit volume constraint. The following proposition covers the special case $n = 2$.

Proposition 5.4. *Under the unit volume constraint, it holds that*

$$\operatorname*{esssup}_{\sigma\in\mathcal{S}_i} E[Z^{[\tau_1^*(i)]}(\sigma)|\mathcal{F}_i] \leq \operatorname*{esssup}_{\sigma\in\mathcal{S}_i} E[Z^{[\tau]}(\sigma)|\mathcal{F}_i]$$

for all $i = 0, \ldots, I$ and $\tau \in \mathcal{S}_i$.

Proof. We fix an $i = 0, \ldots, I$ and a stopping time $\tau \in \mathcal{S}_i$. Then we consider a new stopping time $\bar{\sigma} \in \mathcal{S}_i$ defined by

$$\bar{\sigma} = \inf\{j \geq i;\ ((\tau_2^*(i) = j) \text{ and } (\tau \neq j)) \text{ or } ((\tau_1^*(i) = j) \text{ and } (\tau \neq j))\}.$$

Then, $Z^{[\tau]}(\bar{\sigma}) = Z(\bar{\sigma})$ and $\bar{\sigma} \in \{\tau_1^*(i), \tau_2^*(i)\}$. Hence,

$$\begin{aligned} E[Z^{[\tau]}(\bar{\sigma})|\mathcal{F}_i] &= E[Z(\bar{\sigma})|\mathcal{F}_i] \\ &= E[\mathbf{1}_{\{\tau_2^*(i)=\bar{\sigma}\}} Z(\tau_2^*(i))|\mathcal{F}_i] + E[\mathbf{1}_{\{\tau_2^*(i)\neq\bar{\sigma}\}} Z(\tau_1^*(i))|\mathcal{F}_i] \\ &= (I) + (II). \end{aligned}$$

As $\{\tau_2^*(i) \neq \bar{\sigma}\} = \{\tau_1^*(i) = \bar{\sigma}\} \cap \{\tau_1^*(i) \neq \tau_2^*(i)\} \in \mathcal{F}_{\tau_1^*(i)} \cap \mathcal{F}_{\tau_2^*(i)}$, we obtain from the optimality of $\tau_1^*(j)$,

$$\begin{aligned} (II) &= E[\mathbf{1}_{\{\tau_2^*(i)\neq\bar{\sigma}\}} \mathbf{1}_{\{\tau_2^*(i)\leq\tau_1^*(i)\}} E[Z(\tau_1^*(i))|\mathcal{F}_{\tau_2^*(i)}]|\mathcal{F}_i] \\ &\quad + E[\mathbf{1}_{\{\tau_2^*(i)\neq\bar{\sigma}\}} \mathbf{1}_{\{\tau_2^*(i)>\tau_1^*(i)\}} E[Z(\tau_1^*(i))|\mathcal{F}_{\tau_1^*(i)}]|\mathcal{F}_i] \\ &= E[\mathbf{1}_{\{\tau_2^*(i)\neq\bar{\sigma}\}} \mathbf{1}_{\{\tau_2^*(i)\leq\tau_1^*(i)\}} E[Z(\tau_1^*(\tau_2^*(i)))|\mathcal{F}_{\tau_2^*(i)}]|\mathcal{F}_i] \\ &\quad + E[\mathbf{1}_{\{\tau_2^*(i)\neq\bar{\sigma}\}} \mathbf{1}_{\{\tau_2^*(i)>\tau_1^*(i)\}} E[Z(\tau_1^*(\tau_1^*(i)))|\mathcal{F}_{\tau_1^*(i)}]|\mathcal{F}_i] \\ &\geq E[\mathbf{1}_{\{\tau_2^*(i)\neq\bar{\sigma}\}} \mathbf{1}_{\{\tau_2^*(i)\leq\tau_1^*(i)\}} E[Z(\tau_2^*(i))|\mathcal{F}_{\tau_2^*(i)}]|\mathcal{F}_i] \\ &\quad + E[\mathbf{1}_{\{\tau_2^*(i)\neq\bar{\sigma}\}} \mathbf{1}_{\{\tau_2^*(i)>\tau_1^*(i)\}} E[Z(\tau_2^*(i))|\mathcal{F}_{\tau_1^*(i)}]|\mathcal{F}_i] \\ &= E[\mathbf{1}_{\{\tau_2^*(i)\neq\bar{\sigma}\}} Z(\tau_2^*(i))|\mathcal{F}_i]. \end{aligned}$$

Therefore,

$$\begin{aligned} E[Z^{[\tau]}(\bar{\sigma})|\mathcal{F}_i] &\geq E[Z(\tau_2^*(i))|\mathcal{F}_i] = E[Z^{[\tau_1^*(i)]}(\tau_2^*(i))|\mathcal{F}_i] \\ &= \operatorname*{esssup}_{\sigma\in\mathcal{S}_i} E[Z^{[\tau_1^*(i)]}(\sigma)|\mathcal{F}_i]. \end{aligned}$$

□

For general n we obtain:

Theorem 5.5. *Under the unit volume constraint, it holds that*

$$\operatorname*{esssup}_{\sigma\in\mathcal{S}_i} E[Z^{[\tau_1^*(i),\ldots,\tau_{n-1}^*(i)]}(\sigma)|\mathcal{F}_i] \leq \operatorname*{esssup}_{\sigma\in\mathcal{S}_i} E[Z^{[\tau_1,\ldots,\tau_{n-1}]}(\sigma)|\mathcal{F}_i]$$

for all $n = 2, \ldots, N$, $i = 0, \ldots, I$ *and* $(\tau_1, \ldots, \tau_{n-1}) \in \mathcal{S}_i^{n-1}$.

Proof. The proof is by induction on n with the case $n = 2$ already treated in the previous proposition. Now suppose that $(\tau_1, \ldots, \tau_n) \in \mathcal{S}_i^n$. We can assume without loss of generality that

$$\tau_1 \in \{\tau_1^*(i), \ldots, \tau_{n-1}^*(i)\} \ \Rightarrow\ \tau_1 = I. \tag{2}$$

Otherwise we can construct stopping times

$$\begin{aligned} \tilde{\tau}_1 &= \inf\{j \geq i;\ (\exists_{\nu=1,\ldots,n}\ \tau_\nu = j) \text{ and } (\forall_{\mu=1,\ldots,n-1}\ \tau_\mu^*(i) \neq j)\}, \\ \tilde{\tau}_d &= \inf\{j \geq i;\ (\exists_{\nu=1,\ldots,n}\ \tau_\nu = j) \text{ and } (\forall_{\mu=1,\ldots,d-1}\ \tilde{\tau}_\mu \neq j)\},\ d = 2, \ldots, n. \end{aligned}$$

Then,

$$\tilde{\tau}_1 \in \{\tau_1^*(i), \dots, \tau_{n-1}^*(i)\} \;\Rightarrow\; \tilde{\tau}_1 = I,$$

and, for all $j = i, \dots, I$,

$$Z^{[\tau_1,\dots,\tau_n]}(j) = Z^{[\tilde{\tau}_1,\dots,\tilde{\tau}_n]}(j).$$

Assuming (2) we have

$$Z^{[\tau_1]}(\tau_\nu^*(i)) = Z(\tau_\nu^*(i))$$

for all $\nu = 1, \dots, n-1$. Thus, we easily observe that $(\tau_1^*(i), \dots, \tau_{n-1}^*(i))$ is also the smallest family of optimal stopping times for exercising the cash-flow $Z^{[\tau_1]}(j)$ $(n-1)$ times starting from time i. Applying the inductive hypothesis to the cash-flow $Z^{[\tau_1]}(j)$ yields

$$\begin{aligned}
& \operatorname*{esssup}_{\sigma \in \mathcal{S}_i} E[(Z^{[\tau_1^*(i),\dots,\tau_{n-1}^*(i)]})^{[\tau_1]}(\sigma)|\mathcal{F}_i] \\
= \; & \operatorname*{esssup}_{\sigma \in \mathcal{S}_i} E[(Z^{[\tau_1]})^{[\tau_1^*(i),\dots,\tau_{n-1}^*(i)]}(\sigma)|\mathcal{F}_i] \\
\le \; & \operatorname*{esssup}_{\sigma \in \mathcal{S}_i} E[(Z^{[\tau_1]})^{[\tau_2,\dots,\tau_n]}(\sigma)|\mathcal{F}_i] \\
= \; & \operatorname*{esssup}_{\sigma \in \mathcal{S}_i} E[(Z^{[\tau_1,\dots,\tau_n]}(\sigma)|\mathcal{F}_i]. \qquad (3)
\end{aligned}$$

Finally, we can apply the Proposition 5.4 to the cash-flow $Z^{[\tau_1^*(i),\dots,\tau_{n-1}^*(i)]}(j)$ and obtain from the optimality of $\tau_n^*(i)$ for the single stopping problem with this cash-flow,

$$\begin{aligned}
& \operatorname*{esssup}_{\sigma \in \mathcal{S}_i} E[Z^{[\tau_1^*(i),\dots,\tau_n^*(i)]}(\sigma)|\mathcal{F}_i] \\
= \; & \operatorname*{esssup}_{\sigma \in \mathcal{S}_i} E[(Z^{[\tau_1^*(i),\dots,\tau_{n-1}^*(i)]})^{[\tau_n^*(i)]}(\sigma)|\mathcal{F}_i] \\
\le \; & \operatorname*{esssup}_{\sigma \in \mathcal{S}_i} E[(Z^{[\tau_1^*(i),\dots,\tau_{n-1}^*(i)]})^{[\tau_1]}(\sigma)|\mathcal{F}_i].
\end{aligned}$$

Combining this estimate with (3) yields the assertion.

□

6 Proofs of Theorems 2.2 and 2.3 in the general case

In this section we explain how Theorems 2.2 and 2.3 in the general case can be reduced to the unit volume constraint case, which was considered in the previous section. Recall, that we are interested in the price of the multi-exercise option for $n = 1, \dots, N$ exercise rights under the volume constraint

$c(i)$. Hence, we can and shall assume without loss of generality that the $\mathcal{F}_i$-adapted, positive, integer-valued process c is bounded by N.

We next construct an auxiliary multiple stopping problem with unit volume constraint. We define

$$\begin{aligned}
K(j) &= \{jN, \ldots, (j+1)N-1\}, \quad j = 0, \ldots, I-1, \quad K(I) = \{IN\}, \\
\hat{\mathcal{F}}_i &= \mathcal{F}_j \text{ for } i \in K(j),\ j = 0, \ldots, I, \\
\hat{Z}(i) &= \begin{cases} Z(j), & i \in K(j) \text{ and } i \leq c(j) + jN - 1 \\ b_{\min}, & i \in K(j) \text{ and } i > c(j) + jN - 1, \end{cases} \\
\hat{c}(i) &= \begin{cases} 1, & i = 0, \ldots, IN-1 \\ N, & i = IN. \end{cases}
\end{aligned}$$

Obviously, the process $(\hat{Z}(i),\ i = 0, \ldots, IN)$ is $\hat{\mathcal{F}}_i$-adapted. We denote by $\hat{\mathcal{S}}_i$ the set of all $\{i, \ldots, IN\}$-valued $\hat{\mathcal{F}}_i$-stopping times, and by $\hat{\mathcal{S}}_i^n$ the set of n-tuples $(\sigma_1, \ldots, \sigma_n) \subset \hat{\mathcal{S}}_i$, which satisfy the unit volume constraint $\hat{c}(i)$, i.e

$$\sigma_\nu = \sigma_d \text{ for some } \nu \neq d \ \Rightarrow\ \sigma_\nu = IN. \tag{4}$$

We will now examine the relationship between the original multiple stopping problem with constraint c

$$\operatorname*{esssup}_{(\tau_1,\ldots,\tau_n)\in\mathcal{S}_i^n} \sum_{\nu=1}^{n} E[Z(\tau_\nu)|\mathcal{F}_i]$$

and the multiple stopping problem under the unit volume constraint $\hat{c}$

$$\operatorname*{esssup}_{(\sigma_1,\ldots,\sigma_n)\in\hat{\mathcal{S}}_i^n} \sum_{\nu=1}^{n} E[\hat{Z}(\sigma_\nu)|\hat{\mathcal{F}}_i].$$

To this end we first explore how to map stopping rules in $\mathcal{S}_i^n$ into stopping rules in $\hat{\mathcal{S}}_{iN}^n$. Suppose $\mathfrak{T} = (\tau_1, \ldots, \tau_n) \in \mathcal{S}_i^n$. We then define a family of random times $\mathfrak{S} = (\sigma_1^{\mathfrak{T}}, \ldots, \sigma_n^{\mathfrak{T}})$ via

$$\sigma_\nu^{\mathfrak{T}} = \min\{k \in K(\tau_\nu);\ \forall_{d<\nu}\ \sigma_d^{\mathfrak{T}} \neq k\} \wedge IN.$$

Clearly, the family $\mathfrak{S}$ satisfies (4) and by construction $\sigma_\nu^{\mathfrak{T}} \geq iN$ for all $\nu = 1, \ldots, n$. We claim that $\mathfrak{S} \in \hat{\mathcal{S}}_{iN}^n$, for which it remains to verify that $\sigma_\nu^{\mathfrak{T}}$ are $\hat{\mathcal{F}}_j$-stopping times. This follows by observing that the right-hand side of the decomposition ($j = i, \ldots, I-1$; $k = 0, \ldots, N-1$)

$$\{\sigma_\nu^{\mathfrak{T}} = jN + k\} = \{\tau_\nu = j\} \cap \bigcap_{\kappa=0}^{k-1} \bigcup_{d=1}^{\nu-1} \{\sigma_d^{\mathfrak{T}} = jN + \kappa\}$$

belongs to $\hat{\mathcal{F}}_{jN+k}$ by induction on ν and taking into account that τ_ν is a $\mathcal{F}_j$-stopping time.

The next proposition states that the original and the auxiliary stopping problems are basically equivalent.

Proposition 6.1. *For all $i =, \dots, I$ and $n = 1, \dots, N$ we have,*

$$\operatorname*{esssup}_{(\tau_1,\dots,\tau_n)\in \mathcal{S}_i^n} \sum_{\nu=1}^{n} E[Z(\tau_\nu)|\mathcal{F}_i] = \operatorname*{esssup}_{(\sigma_1,\dots,\sigma_n)\in \hat{\mathcal{S}}_{iN}^n} \sum_{\nu=1}^{n} E[\hat{Z}(\sigma_\nu)|\hat{\mathcal{F}}_{iN}].$$

Proof. Suppose $\mathfrak{T} = (\tau_1, \dots, \tau_n) \in \mathcal{S}_i^n$ is arbitrary but fixed. By the admissibility of this stopping family we know that the number of stopping times in $\mathfrak{S} = (\sigma_1^{\mathfrak{T}}, \dots, \sigma_n^{\mathfrak{T}})$ such that $\sigma_\nu^{\mathfrak{T}} \in K(j)$ is less or equal to $c(j)$ for all $j = i, \dots, I$. Hence, from the construction of $\mathfrak{S}$ and the definition of $\hat{Z}$ we deduce that $Z(\tau_\nu) = \hat{Z}(\sigma_\nu^{\mathfrak{T}})$ for all $\nu = 1, \dots, n$. Consequently,

$$\sum_{\nu=1}^{n} E[Z(\tau_\nu)|\mathcal{F}_i] = \sum_{\nu=1}^{n} E[\hat{Z}(\sigma_\nu^{\mathfrak{T}})|\hat{\mathcal{F}}_{iN}]. \tag{5}$$

As $\mathfrak{T} \in \mathcal{S}_i^n$ was arbitrary, the left-hand side of the asserted inequality is dominated by the right-hand side.

For proving the reverse inequality let us assume that $\mathfrak{S}^* = (\hat{\sigma}_1^*, \dots, \hat{\sigma}_n^*) \in \hat{\mathcal{S}}_{iN}^n$ is an optimizer for the problem on the right-hand side of the assertion. By the optimality of $\mathfrak{S}^*$ we obtain that $\hat{Z}(\hat{\sigma}_\nu^*) > b_{\min}$ and, thus,

$$\hat{Z}(\hat{\sigma}_\nu^*) = Z(j) \text{ on } \{\hat{\sigma}_\nu^* \in K(j)\}.$$

Moreover, the number of stopping times in $\mathfrak{S}^* = (\hat{\sigma}_1^*, \dots, \hat{\sigma}_n^*)$ such that $\hat{\sigma}_\nu^* \in K(j)$ is less or equal to $c(j)$ for all $j = i, \dots, I$. We now define random times τ_ν with values in $\{j = i, \dots, I\}$ by

$$\tau_\nu = j \;\Leftrightarrow\; \sigma_\nu^* \in K(j).$$

By the previous considerations, it is obvious that $(\tau_1, \dots, \tau_\nu) \in \mathcal{S}_i^n$ and $\hat{Z}(\hat{\sigma}_\nu^*) = Z(\tau_\nu)$ for all $\nu = 1, \dots, N$. Hence,

$$\sum_{\nu=1}^{n} E[Z(\tau_\nu)|\mathcal{F}_i] = \sum_{\nu=1}^{n} E[\hat{Z}(\hat{\sigma}_\nu^*)|\hat{\mathcal{F}}_{iN}],$$

which proves, by the optimality of $(\hat{\sigma}_1^*, \dots, \hat{\sigma}_n^*)$, the reverse inequality. □

Suppose now that $\mathfrak{T} = (\tau_1, \dots, \tau_n) \in \mathcal{S}_i^n$ for some $n < N$, and recall that, for $j = i, \dots, I$,

$$Z^{[\tau_1,\dots,\tau_n]}(j) = \begin{cases} Z(j), & \sum_{\nu=1}^{n} \mathbf{1}_{\{\tau_\nu = j\}} < c(j) \\ b_{\min}, & \text{otherwise.} \end{cases}$$

We set $\mathfrak{S} = (\sigma_1^{\mathfrak{T}}, \dots, \sigma_n^{\mathfrak{T}}) \in \hat{\mathcal{S}}_{iN}^n$ and, for $j = iN, \dots, IN$,

$$\begin{aligned} \hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}(j) &= \begin{cases} \hat{Z}(j), & \sum_{\nu=1}^{n} \mathbf{1}_{\{\sigma_\nu^{\mathfrak{T}} = j\}} < \hat{c}(j) \\ b_{\min}, & \text{otherwise} \end{cases} \\ &= \begin{cases} \hat{Z}(j), & \forall_{\nu=1,\dots,n}\, \sigma_\nu^{\mathfrak{T}} \neq j \text{ or } j = IN \\ b_{\min}, & \text{otherwise.} \end{cases} \end{aligned}$$

Note that $\hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}(IN) = Z(I) = Z^{[\tau_1,\dots,\tau_n]}(I)$ and, for $j = i, \dots, I-1$ and $k \in K(j)$

$$\begin{aligned}
\hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}(k) &= \begin{cases} Z(j), & k \le c(j) + jN - 1 \text{ and } \forall_{\nu=1,\dots,n}\ \sigma_\nu^{\mathfrak{T}} \neq k \\ b_{\min}, & \text{otherwise} \end{cases} \\
&= \begin{cases} Z(j), & \sum_{\nu=1}^n \mathbf{1}_{\{\tau_\nu = j\}} + jN \le k \le c(j) + jN - 1 \\ b_{\min}, & \text{otherwise.} \end{cases}
\end{aligned} \tag{6}$$

Hence $\hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}(k) = Z(j)$ implies $\sum_{\nu=1}^n \mathbf{1}_{\{\tau_\nu = j\}} < c(j)$ and, thus, we get $Z^{[\tau_1,\dots,\tau_n]}(j) = Z(j)$. In particular,

$$\hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}(k) \le Z^{[\tau_1,\dots,\tau_n]}(j), \quad k \in K(j). \tag{7}$$

Moreover, if $\hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}(c(j) + jN - 1) = b_{min}$, then $\sum_{\nu=1}^n \mathbf{1}_{\{\tau_\nu = j\}} \ge c(j)$, and consequently $Z^{[\tau_1,\dots,\tau_n]}(j) = b_{\min}$. Therefore,

$$\hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}(c(j) + jN - 1) = Z^{[\tau_1,\dots,\tau_n]}(j). \tag{8}$$

We now denote by $\hat{\mathcal{Y}}^*(j; \hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]})$, $j = iN, \dots, IN$, the Snell envelope for the single optimal stopping problem with cash-flow $\hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}$ with respect to the set of stopping times $\hat{\mathcal{S}}_{iN}^n$. As before, $\mathcal{Y}^*(j; Z^{[\tau_1,\dots,\tau_n]})$ is the Snell envelope for the cash-flow $Z^{[\tau_1,\dots,\tau_n]}$ in the original setting.

The following proposition is the key observation in order to transfer the results from the previous section to the general case.

Proposition 6.2. *Suppose that $\mathfrak{T} = (\tau_1, \dots, \tau_n) \in \mathcal{S}_i^n$ for $n = 0, \dots, N-1$ and $i = 0, \dots, I$. Then, for all $j = i, \dots, I$ and $k \in K(j)$,*

$$\hat{\mathcal{Y}}^*(k; \hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}) = \begin{cases} \mathcal{Y}^*(j; Z^{[\tau_1,\dots,\tau_n]}), & k \le c(j) + jN - 1 \\ E[\mathcal{Y}^*(j+1; Z^{[\tau_1,\dots,\tau_n]}) | \mathcal{F}_j], & k > c(j) + jN - 1. \end{cases}$$

In particular,

$$\hat{\mathcal{Y}}^*(jN; \hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}) = \mathcal{Y}^*(j; Z^{[\tau_1,\dots,\tau_n]}).$$

Proof. The proof is by backward induction on j. For $j = I$, we have $K(I) = \{IN\}$ and

$$\begin{aligned}
\hat{\mathcal{Y}}^*(IN; \hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}) &= \hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}(IN) = Z(I) = Z^{[\tau_1,\dots,\tau_n]}(I) \\
&= \mathcal{Y}^*(I; Z^{[\tau_1,\dots,\tau_n]}).
\end{aligned}$$

Suppose now that the assertion is already proved on the time segment $K(j)$ for some $j = i+1, \dots, I$. We show the assertion on the segment $K(j-1)$ by backward induction on $k = jN-1, \dots, (j-1)N$.

For $k = jN - 1$ we obtain from (6) and (8) that

$$\hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}(jN-1) = \begin{cases} Z^{[\tau_1,\dots,\tau_n]}(j-1), & c(j-1) = N \\ b_{\min}, & \text{otherwise.} \end{cases}$$

Thus,

$$\begin{aligned}
& \hat{\mathcal{Y}}^*(jN-1;\hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}) \\
= \; & \max\left\{\hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}(jN-1),\ E[\hat{\mathcal{Y}}^*(jN;\hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]})|\hat{\mathcal{F}}_{jN-1}]\right\} \\
= \; & \begin{cases} \max\left\{Z^{[\tau_1,\dots,\tau_n]}(j-1),\ E[\mathcal{Y}^*(j;Z^{[\tau_1,\dots,\tau_n]})|\mathcal{F}_{j-1}]\right\}, & c(j-1)=N \\ \max\left\{b_{\min},\ E[\mathcal{Y}^*(j;Z^{[\tau_1,\dots,\tau_n]})|\mathcal{F}_{j-1}]\right\}, & c(j-1)<N \end{cases} \\
= \; & \begin{cases} \mathcal{Y}^*(j-1;Z^{[\tau_1,\dots,\tau_n]}), & jN-1 \le c(j-1)+(j-1)N-1 \\ E[\mathcal{Y}^*(j;Z^{[\tau_1,\dots,\tau_n]})|\mathcal{F}_{j-1}], & jN-1 > c(j-1)+(j-1)N-1. \end{cases}
\end{aligned}$$

We now assume that the assertion is already verified for some $k \in K(j-1)$ and $k > (j-1)N$. Then,

$$\begin{aligned}
& \hat{\mathcal{Y}}^*(k-1;\hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}) \\
= \; & \max\left\{\hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}(k-1),\ E[\hat{\mathcal{Y}}^*(k;\hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]})|\hat{\mathcal{F}}_{k-1}]\right\} \\
= \; & \max\left\{\hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}(k-1),\ \hat{\mathcal{Y}}^*(k;\hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]})\right\}.
\end{aligned}$$

We shall distinguish three cases. On the set $\{k > c(j-1)+(j-1)N\}$ we get

$$\begin{aligned}
\hat{\mathcal{Y}}^*(k-1;\hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}) &= \max\left\{b_{\min},\ E[\mathcal{Y}^*(j;Z^{[\tau_1,\dots,\tau_n]})|\mathcal{F}_{j-1}]\right\} \\
&= E[\mathcal{Y}^*(j;Z^{[\tau_1,\dots,\tau_n]})|\mathcal{F}_{j-1}]
\end{aligned}$$

by the inductive hypothesis and (6). On the set $\{k = c(j-1)+(j-1)N\}$ we have, by the inductive hypothesis and (8),

$$\begin{aligned}
\hat{\mathcal{Y}}^*(k-1;\hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}) &= \max\left\{Z^{[\tau_1,\dots,\tau_n]}(j-1),\ E[\mathcal{Y}^*(j;Z^{[\tau_1,\dots,\tau_n]})|\mathcal{F}_{j-1}]\right\} \\
&= \mathcal{Y}^*(j-1;Z^{[\tau_1,\dots,\tau_n]}).
\end{aligned}$$

Finally, on $\{k < c(j-1)+(j-1)N\}$, the inductive hypothesis and (7) imply

$$\begin{aligned}
Z^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}(k-1) &\le Z^{[\tau_1,\dots,\tau_n]}(j-1) \le \mathcal{Y}^*(j-1;Z^{[\tau_1,\dots,\tau_n]}) \\
&= \hat{\mathcal{Y}}^*(k;\hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}),
\end{aligned}$$

whence

$$\hat{\mathcal{Y}}^*(k-1;\hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}) = \hat{\mathcal{Y}}^*(k;\hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}) = \mathcal{Y}^*(j-1;Z^{[\tau_1,\dots,\tau_n]}).$$

□

As a corollary we can relate the smallest optimal stopping times for both problems.

Corollary 6.3. *Under the assumptions of Proposition 6.2 let*

$$\begin{aligned}\tau^* &= \inf\{j = i, \dots, I;\ Z^{[\tau_1,\dots,\tau_n]}(j) \geq \mathcal{Y}^*(j; Z^{[\tau_1,\dots,\tau_n]})\} \\ \sigma^* &= \inf\{j = iN, \dots, IN;\ \hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}(j) \geq \hat{\mathcal{Y}}^*(j; \hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]})\}.\end{aligned}$$

Then,

$$\sigma^* = N\tau^* + \sum_{\nu=1}^{n} \mathbf{1}_{\{\sigma_\nu^{\mathfrak{T}} \in K(\tau^*)\}}.$$

In particular, denoting $\mathfrak{T}^* = (\tau_1, \dots, \tau_n, \tau^*)$, we have

$$(\sigma_1^{\mathfrak{T}}, \dots, \sigma_n^{\mathfrak{T}}, \sigma^*) = (\sigma_1^{\mathfrak{T}^*}, \dots, \sigma_{n+1}^{\mathfrak{T}^*}).$$

Proof. Fix some $k = iN, \dots, IN$ and choose j such that $k \in K(j)$. Then, by (6), (7) and Proposition 6.2,

$$\begin{aligned} & \hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}(k) \geq \hat{\mathcal{Y}}^*(k; \hat{Z}^{[\sigma_1^{\mathfrak{T}},\dots,\sigma_n^{\mathfrak{T}}]}) \\ \Leftrightarrow\ & \left(\sum_{\nu=1}^{n} \mathbf{1}_{\{\tau_\nu = j\}} + jN \leq k \leq c(j) + jN - 1\right) \\ & \text{and } \left(Z^{[\tau_1,\dots,\tau_n]}(j) \geq \mathcal{Y}^*(j; Z^{[\tau_1,\dots,\tau_n]})\right).\end{aligned}$$

As

$$\sum_{\nu=1}^{n} \mathbf{1}_{\{\tau_\nu = j\}} = \sum_{\nu=1}^{n} \mathbf{1}_{\{\sigma_\nu^{\mathfrak{T}} \in K(j)\}},$$

we observe that $\sigma^* = N\tau^* + \sum_{\nu=1}^{n} \mathbf{1}_{\{\sigma_\nu^{\mathfrak{T}} \in K(\tau^*)\}}$. □

Corollary 6.4. *For $n = 1, \dots, N$ and $i = 0, \dots, I$ we define $\mathfrak{T}^{*,n}(i) = (\tau_1^*(i), \dots, \tau_n^*(i))$. Then, $(\sigma_1^*(i), \dots, \sigma_N^*(i))$ given by*

$$\sigma_n^*(i) = \inf\{j = iN, \dots, IN;\ \hat{Z}^{[\sigma_1^*(i),\dots,\sigma_{n-1}^*(i)]}(j) \geq \hat{\mathcal{Y}}^*(j; \hat{Z}^{[\sigma_1^*(i),\dots,\sigma_{n-1}^*(i)]})\},$$

satisfy $\sigma_\nu^*(i) = \sigma_\nu^{\mathfrak{T}^{*,n}(i)}$ for all $1 \leq \nu \leq n \leq N$.

Proof. The proof is by induction on n. For $n = 1$ it directly follows from the previous corollary. Using the inductive hypothesis we obtain

$$\begin{aligned} & \sigma_{n+1}^*(i) \\ = {} & \inf\{j = iN, \dots, IN;\ \hat{Z}^{[\sigma_1^*(i),\dots,\sigma_n^*(i)]}(j) \geq \hat{\mathcal{Y}}^*(j; \hat{Z}^{[\sigma_1^*(i),\dots,\sigma_n^*(i)]})\} \\ = {} & \inf\{j = iN, \dots, IN;\ \hat{Z}^{[\sigma_1^{\mathfrak{T}^{*,n}(i)},\dots,\sigma_n^{\mathfrak{T}^{*,n}(i)}]}(j) \geq \hat{\mathcal{Y}}^*(j; \hat{Z}^{[\sigma_1^{\mathfrak{T}^{*,n}(i)},\dots,\sigma_n^{\mathfrak{T}^{*,n}(i)}]})\}.\end{aligned}$$

Now the inductive hypothesis and the previous corollary imply

$$\begin{aligned}(\sigma_1^*(i), \dots, \sigma_{n+1}^*(i)) &= (\sigma_1^{\mathfrak{T}^{*,n}(i)}, \dots, \sigma_n^{\mathfrak{T}^{*,n}(i)}, \sigma_{n+1}^*(i)) \\ &= (\sigma_1^{\mathfrak{T}^{*,n+1}(i)}, \dots, \sigma_{n+1}^{\mathfrak{T}^{*,n+1}(i)}).\end{aligned}$$

□

We are now in the position to prove Theorems 2.2 and 2.3 in the general case.

Proof of Theorem 2.2. With the notation from the previous corollary, we obtain from Proposition 6.1, Theorem 5.3, Corollary 6.4, and (5) that

$$\begin{aligned} & \operatorname*{esssup}_{(\tau_1,\ldots,\tau_n)\in\mathcal{S}_i^n} \sum_{\nu=1}^n E[Z(\tau_\nu)|\mathcal{F}_i] = \operatorname*{esssup}_{(\sigma_1,\ldots,\sigma_n)\in\hat{\mathcal{S}}_{iN}^n} \sum_{\nu=1}^n E[\hat{Z}(\sigma_\nu)|\hat{\mathcal{F}}_{iN}] \\ = \quad & \sum_{\nu=1}^n E[\hat{Z}(\sigma_\nu^{\mathfrak{T}^{*,n}(i)})|\hat{\mathcal{F}}_{iN}] = \sum_{\nu=1}^n E[Z(\tau_\nu^*(i))|\mathcal{F}_i]. \end{aligned}$$

□

Proof of Theorem 2.3. Suppose $\mathfrak{T}=(\tau_1,\ldots,\tau_n)\in\mathcal{S}_i^n$. Proposition 6.2, Theorem 5.5 and Corollary 6.4 yield

$$\begin{aligned} & \operatorname*{esssup}_{\tau\in\mathcal{S}_i} E[Z^{[\tau_1,\ldots,\tau_n]}(\tau)|\mathcal{F}_i] = \operatorname*{esssup}_{\sigma\in\hat{\mathcal{S}}_{iN}} E[\hat{Z}^{[\sigma_1^{\mathfrak{T}},\ldots,\sigma_n^{\mathfrak{T}}]}(\sigma)|\hat{\mathcal{F}}_{iN}] \\ \geq \quad & \operatorname*{esssup}_{\sigma\in\hat{\mathcal{S}}_{iN}} E[\hat{Z}^{[\sigma_1^{\mathfrak{T}^{*,n}(i)},\ldots,\sigma_n^{\mathfrak{T}^{*,n+1}(i)}]}(\sigma)|\hat{\mathcal{F}}_{iN}] \\ = \quad & \operatorname*{esssup}_{\tau\in\mathcal{S}_i} E[Z^{[\tau_1^*(i),\ldots,\tau_n^*(i)]}(\tau)|\mathcal{F}_i]. \end{aligned}$$

□

7 Proof of Theorem 2.5

In this section we prove the alternative representation of the Snell envelope $\mathcal{Y}^*(j;Z^{[\tau_1^*(i),\ldots,\tau_n^*(i)]})$, which is stated in Theorem 2.5.

As a preparation, we reorder for a given $n=1,\ldots,N$, the stopping times $(\tau_1^*(i),\ldots,\tau_n^*(i))$. Precisely, we define a stopping family $(\tau_1^{*,n}(i),\ldots,\tau_n^{*,n}(i))$ by the properties

$$\{\tau_1^*(i),\ldots,\tau_n^*(i)\} = \{\tau_1^{*,n}(i),\ldots,\tau_n^{*,n}(i)\} \tag{9}$$

and $\tau_\nu^{*,n}(i)\leq\tau_{\nu+1}^{*,n}(i)$.

Lemma 7.1. *For every $n=1,\ldots,N$, it holds that $(\tau_1^{*,n}(i),\ldots,\tau_n^{*,n}(i))$ is the smallest optimal exercise rule for the multi-exercise option under constraints, starting at time i, i.e. if $(\tilde{\tau}_1^{*,n}(i),\ldots,\tilde{\tau}_n^{*,n}(i))$ is another optimal exercise rule with $\tilde{\tau}_\nu^{*,n}(i)\leq\tilde{\tau}_{\nu+1}^{*,n}(i)$ for all $\nu=1,\ldots,n-1$, then $\tau_\nu^{*,n}(i)\leq\tilde{\tau}_\nu^{*,n}(i)$ for all $\nu=1,\ldots,n$.*

Proof. We define, with the notation of the previous section,

$$\mathfrak{S}^{*,n} = (\sigma_1^{\mathfrak{T}_n^*}, \dots, \sigma_n^{\mathfrak{T}_n^*})$$

and

$$\tilde{\mathfrak{S}}^{*,n} = (\sigma_1^{\tilde{\mathfrak{T}}_n^*}, \dots, \sigma_n^{\tilde{\mathfrak{T}}_n^*}),$$

where $\mathfrak{T}_n^* = (\tau_1^{*,n}(i), \dots, \tau_n^{*,n}(i))$ and analogously for the tilded expressions. By the proof of Proposition 6.1 we observe that the families $\mathfrak{S}^{*,n}$ and $\mathfrak{S}^{*,n}$ are optimal for the auxiliary multiple stopping problem under the unit constraint. Thanks to (9) and Corollary 6.4 we know that $\mathfrak{S}^{*,n}$ is the smallest optimal exercise policy for the auxiliary problem, whence

$$\sigma_\nu^{\mathfrak{T}_n^*} \le \sigma_\nu^{\tilde{\mathfrak{T}}_n^*}$$

for all $\nu = 1, \dots, n$. The construction of these stopping times immediately implies that $\tau_\nu^{*,n}(i) \le \tilde{\tau}_\nu^{*,n}(i)$ for all $\nu = 1, \dots, n$ as well. $\square$

Lemma 7.2. *Suppose $n = 1, \dots, N$ and $0 \le i \le j \le I$. Then,*

$$\tau_{\nu+k}^{*,n}(i) = \tau_k^{*,n-\nu}(j) \ \text{ on } \ \{\tau_\nu^{*,n}(i) < j \le \tau_{\nu+1}^{*,n}(i)\}$$

for all $k = 1, \dots, n-\nu$.

Proof. For fixed (n, i, j) we consider the stopping rules

$$\sigma_k = \begin{cases} \tau_k^{*,n}(i), & \tau_k^{*,n}(i) < j \\ \tau_k^{*,n-\nu}(j), & \tau_k^{*,n}(i) \ge j \text{ and } \tau_\nu^{*,n}(i) < j \le \tau_{\nu+1}^{*,n}(i) \end{cases}$$

for $k = 1, \dots, n$. Then,

$$\begin{aligned} & E[\sum_{k=1}^n Z(\sigma_k)|\mathfrak{F}_i] \\ = \;& E[\sum_{\nu=1}^n \mathbf{1}_{\{\tau_\nu^{*,n}(i) < j \le \tau_{\nu+1}^{*,n}(i)\}} (\sum_{k=1}^{n-\nu} E[Z(\tau_k^{*,n-\nu}(j))|\mathfrak{F}_j] + \sum_{l=1}^{\nu} Z(\tau_l^{*,n}(i)))|\mathfrak{F}_i] \\ \ge \;& E[\sum_{\nu=1}^n \mathbf{1}_{\{\tau_\nu^{*,n}(i) < j \le \tau_{\nu+1}^{*,n}(i)\}} (\sum_{k=1}^{n-\nu} E[Z(\tau_{\nu+k}^{*,n}(i))|\mathfrak{F}_j] + \sum_{l=1}^{\nu} Z(\tau_l^{*,n}(i)))|\mathfrak{F}_i] \\ = \;& E[\sum_{k=1}^n Z(\tau_k^{*,n}(i))|\mathfrak{F}_i] \end{aligned} \tag{10}$$

Hence, the inequality in (10) turns into an identity and the family $(\sigma_1, \dots, \sigma_n)$ is optimal for the multi-exercise option under constraints, starting from time i. In view of the previous lemma we deduce that $\sigma_k \ge \tau_k^{*,n}$ for all $k = 1, \dots, n$. In particular,

$$\tau_{\nu+k}^{*,n}(i) \le \tau_k^{*,n-\nu}(j) \ \text{ on } \ \{\tau_\nu^{*,n}(i) < j \le \tau_{\nu+1}^{*,n}(i)\}$$

for all $k = 1, \ldots, n-\nu$. Since identity holds in (10), we obtain, on $\{\tau_\nu^{*,n}(i) < j \le \tau_{\nu+1}^{*,n}(i)\}$

$$\sum_{k=1}^{n-\nu} E[Z(\tau_k^{*,n-\nu}(j))|\mathcal{F}_j] = \sum_{k=1}^{n-\nu} E[Z(\tau_{\nu+k}^{*,n}(i))|\mathcal{F}_j].$$

Hence, the previous lemma yields

$$\tau_{\nu+k}^{*,n}(i) \ge \tau_k^{*,n-\nu}(j) \quad \text{on} \quad \{\tau_\nu^{*,n}(i) < j \le \tau_{\nu+1}^{*,n}(i)\}$$

for all $k = 1, \ldots, n-\nu$, and the proof is complete. □

We are now ready to give the proof of Theorem 2.5.

Proof of Theorem 2.5. Thanks to the previous lemma, (9) and Theorem 2.2 we get

$$\begin{aligned}
& \mathcal{Y}^*(j; Z^{[\tau_1^*(i),\ldots,\tau_n^*(i)]}) = \operatorname*{esssup}_{\tau\in\mathcal{S}_j} E[Z^{[\tau_1^*(i),\ldots,\tau_n^*(i)]}(\tau)|\mathcal{F}_j] \\
= \ & \operatorname*{esssup}_{\tau\in\mathcal{S}_j} \sum_{\nu=1}^{n} \mathbf{1}_{\{\tau_\nu^{*,n}(i)<j\le\tau_{\nu+1}^{*,n}(i)\}} E[Z^{[\tau_{\nu+1}^{*,n}(i),\ldots,\tau_n^{*,n}(i)]}(\tau)|\mathcal{F}_j] \\
= \ & \sum_{\nu=1}^{n} \mathbf{1}_{\{\tau_\nu^{*,n}(i)<j\le\tau_{\nu+1}^{*,n}(i)\}} \operatorname*{esssup}_{\tau\in\mathcal{S}_j} E[Z^{[\tau_{\nu+1}^{*,n}(i),\ldots,\tau_n^{*,n}(i)]}(\tau)|\mathcal{F}_j] \\
= \ & \sum_{\nu=1}^{n} \mathbf{1}_{\{\tau_\nu^{*,n}(i)<j\le\tau_{\nu+1}^{*,n}(i)\}} \operatorname*{esssup}_{\tau\in\mathcal{S}_j} E[Z^{[\tau_1^{*,n-\nu}(j),\ldots,\tau_{n-\nu}^{*,n-\nu}(j)]}(\tau)|\mathcal{F}_j] \\
= \ & \sum_{\nu=1}^{n} \mathbf{1}_{\{\tau_\nu^{*,n}(i)<j\le\tau_{\nu+1}^{*,n}(i)\}} \operatorname*{esssup}_{\tau\in\mathcal{S}_j} E[Z^{[\tau_1^*(j),\ldots,\tau_{n-\nu}^*(j)]}(\tau)|\mathcal{F}_j] \\
= \ & \sum_{\nu=1}^{n} \mathbf{1}_{\{\tau_\nu^{*,n}(i)<j\le\tau_{\nu+1}^{*,n}(i)\}} \Delta^{n+1-\nu} Y^*(j).
\end{aligned}$$

As $\mathbf{1}_{\{\tau_\nu^{*,n}(i)<j\le\tau_{\nu+1}^{*,n}(i)\}} = \mathbf{1}_{\{\mathcal{E}_n^*(j-1;i)=\nu\}}$, the assertion is proved. □

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