# Robust Combining of Disparate Classifiers through Order Statistics 

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#### Abstract

Integrating the outputs of multiple classifiers via combiners or meta-learners has led to substantial improvements in several difficult pattern recognition problems. In the typical setting investigated till now, each classifier is trained on data taken or resampled from a common data set, or (almost) randomly selected subsets thereof, and thus experiences similar quality of training data. However, in certain situations where data is acquired and analyzed on-line at several geographically distributed locations, the quality of data may vary substantially, leading to large discrepancies in performance of individual classifiers. In this article we introduce and investigate a family of classifiers based on order statistics, for robust handling of such cases. Based on a mathematical modeling of how the decision boundaries are affected by order statistic combiners, we derive expressions for the reductions in error expected when such combiners are used. We show analytically that the selection of the median, the maximum and in general, the $i^{t h}$ order statistic improves classification performance. Furthermore, we introduce the trim and spread combiners, both based on linear combinations of the ordered classifier outputs, and show that they are quite beneficial in presence of outliers or uneven classifier performance. Experimental results on several public domain data sets corroborate these findings.


## 1 Introduction

Since different types of classifiers have different "inductive bias", one does not expect the generalization performance of two classifiers to be identical 22, 25 for difficult pattern recognition problems, even when they are both trained on the same data set. If only the "best" classifier is selected based on an estimation of the true generalization performance using a finite test set 60], valuable information contained in the results of the discarded classifiers may be lost. Such potential loss of information can be avoided if the outputs of all available classifiers are used in the final classification decision. This
concept has received a great deal of attention recently, and many methods for combining classifier outputs have been proposed 23, 27, 29, 42, 53]. Furthermore, diversity among classifiers has been actively promoted, by strategies such as bagging [8], arcing [9, 19, 20], boosting [18, 17, 45, 51, 52], and correlation control 2, 59, as a prelude to combining.

Approaches to pooling classifiers can be separated into two main categories: (i) simple combiners, e.g., voting [4, 12], Bayesian based weighted product rule 31], or averaging [41, 58], and, (ii) meta-learners, such as arbitration [11] or stacking [7, 61]. The simple combining methods are best suited for problems where the individual classifiers perform the same task, and have comparable success. However, such combiners are more susceptible to outliers and to unevenly performing classifiers. In the second category, either sets of combining rules, or full fledged classifiers acting on the outputs of the individual classifiers, are constructed [1, 30, 61. This type of combining is more general, but is vulnerable to all the problems associated with the added learning (e.g., overparameterizing, lengthy training time).

An implicit assumption in most combining schemes is that each classifier sees the same training data or resampled versions of the same data. If the individual classifiers are then appropriately chosen and trained properly, their performances will be (relatively) comparable in any region of the problem space. So gains from combining are derived from the diversity 32, 40 among classifiers rather that by compensating for weak members of the pool. However, in real life, there are situations where individual classifiers may not have access to the same data. Such conditions arise in certain data mining, sensor fusion and electrical logging (oil services) problems where there are large variabilities in the data which is acquired locally and needs to be processed in (near) real time at geographically separated places 13. These conditions create a pool of classifiers that may have significant variations in their overall performance. Moreover, they may lead to conditions where individual classifiers have similar average performance, but substantially different performance over different parts of the input space.

In such cases, combining is still desirable, but neither simple combiners nor metalearners are particularly well-suited for the type of problems that arise. For example, the simplicity of averaging the classifier outputs is appealing, but the prospect of one poor classifier corrupting the combiner makes this a risky choice. Weighted averaging of classifier outputs appears to provide some flexibility [28, 37]. Unfortunately, the weights are still assigned on a per classifier basis rather than a per sample or per class basis. If a classifier is accurate only in certain areas of the input space, this scheme fails to take advantage of the variable accuracy of the classifier in question. Using a meta learner that provides different weights for different patterns can potentially solve this problem, but at a considerable cost. In particular, the off-line training of a meta-learner using substantial amount of data outputted by geographically distributed classifiers, may not be feasible. In addition to providing robustness, the order statistic combiners presented in this work also aim at bridging the gap between simplicity and generality by allowing the flexible selection of classifiers without the associated cost of training meta-classifiers.

Section 2 summarizes the relationship between classifier errors and decision boundaries and provides the necessary background for mathematically analyzing order statistic combiners 58. Section 3 introduces simple order statistic combiners. Based on these concepts, in Section 4 we propose two powerful combiners, trim and spread, and derive the amount of error reduction associated with each. In Section 5 we present the performance of order statistic combiners on Proben1/UCI benchmarks 43. Section 6
discusses the implications of using linear combinations of order statistics as a strategy for pooling the outputs of individual classifiers.

## 2 Error Characterization in a Single Classifier

In this section we summarize the approach and results of 58$]$, that quantify the effect of inaccuracies in estimating a posterior class probabilities on the classification error for a single classifier. This background is needed to characterize and understand the impact of order statistics combiners, as described in Sections 3 and 4.

It is well known that, given one-of- $L$ desired outputs and sufficient training samples reflecting the class priors, the outputs of certain classifiers trained to minimize a mean square or cross-entropy error criteria, approximate the a posteriori probability densities of the corresponding classes [47, 49]. Based on this result, one can model the $i$ th output of the $m$ th such classifier as:

$$
\begin{equation*}
f_{i}^{m}(x)=p_{i}(x)+\epsilon_{i}^{m}(x), \tag{1}
\end{equation*}
$$

where $p_{i}(x)$ is the true posterior for $i$ th class on input $x$, and $\epsilon_{i}^{m}(x)$ is the error of the $m$ th classifier in estimating that posterior.


Figure 1: Error regions associated with approximating the a posteriori probabilities [58].
Now, let us decompose the error into two parts: $\epsilon_{i}^{m}(x)=\beta_{i}^{m}+\eta_{i}^{m}(x)$. The first component does not vary with the input, and provides an offset, or systematic error for each class. The second component gives the variability from that systematic error, for each $x$ in each class, and has zero mean and variance $\sigma_{\eta_{i}^{m}(x)}^{2}$. These two components of the error are similar to the bias and variance decomposition for a quadratic loss function given in [22], although they are at the individual input level. We will therefore refer to classifiers as "biased" and "unbiased" implying $\beta_{k}^{m} \neq 0$ for some $k, m$, and

[^0]$\beta_{k}^{m}=0, \forall k, m$, respectively. Let $b^{m}$ denote the offset between the ideal class boundary, $x^{*}$ (based on $\left.p_{i}(x)=p_{j}(x)\right)$ and the realized boundary, $x_{b}^{m}$ (based on $\left.f_{i}^{m}(x)=f_{j}^{m}(x)\right)$, as shown in Figure 11 58. This boundary offset $\left(b^{m}=x_{b}^{m}-x^{*}\right)$ has mean and variance given respectively by:
\[

$$
\begin{equation*}
\beta^{m}=\frac{\beta_{i}^{m}-\beta_{j}^{m}}{s} \tag{2}
\end{equation*}
$$

\]

and

$$
\begin{equation*}
\sigma_{b^{m}}^{2}=\frac{\sigma_{\eta_{i}^{m}(x)}^{2}+\sigma_{\eta_{j}^{m}(x)}^{2}}{s^{2}} \tag{3}
\end{equation*}
$$

where $s=p_{j}^{\prime}\left(x^{*}\right)-p_{i}^{\prime}\left(x^{*}\right)$ as introduced in 58.
Let us further denote the probability density function of this boundary offset by $f_{b}(x)$. The expected model error associated with the selection of a particular classifier $m$, can then be expressed as:

$$
\begin{equation*}
E_{\text {model }}^{m}=\int_{-\infty}^{\infty} A(b) f_{b}(b) d b \tag{4}
\end{equation*}
$$

where $A(b)=\int_{x^{*}}^{x^{*}+b}\left(p_{j}(x)-p_{i}(x)\right) d x$ is the error due to the selection of a particular decision boundary. In general, it is not possible to obtain the density function for the boundary offset without making assumptions on the distributions of the errors. However, a first order approximation, derived in 55], leads to:

$$
\begin{equation*}
E_{\text {model }}^{m}=\int_{-\infty}^{\infty} \frac{1}{2} b^{2} s f_{b}(b) d b \tag{5}
\end{equation*}
$$

Let us define the first and second moments of the boundary offset as follows:

$$
\mathcal{M}_{1}=\int_{-\infty}^{\infty} x f_{b}(x) d x \quad \text { and } \quad \mathcal{M}_{2}=\int_{-\infty}^{\infty} x^{2} f_{b}(x) d x
$$

If the individual classifiers are unbiased, the offset $b^{m}$ of a single classifier has $\mathcal{M}_{1}=0$ and $\mathcal{M}_{2}=\sigma_{b^{m}}^{2}$, leading to:

$$
\begin{equation*}
E_{\text {model }}^{m}=\frac{s \mathcal{M}_{2}}{2}=\frac{s \sigma_{b^{m}}^{2}}{2} \tag{6}
\end{equation*}
$$

Now, if the classifiers are biased, the variance of $b$ is left unchanged (given by Equation (3), but the mean becomes $\beta=\frac{\beta_{i}-\beta_{j}}{s}$. In other words, we have $\mathcal{M}_{1}=\beta^{m}$ and $\sigma_{b^{m}}^{2}=\mathcal{M}_{2}-\mathcal{M}_{1}{ }^{2}$, leading to the following model error:

$$
\begin{equation*}
E_{\text {model }}^{m}(\beta)=\frac{s \mathcal{M}_{2}}{2}=\frac{s}{2}\left(\sigma_{b^{m}}^{2}+\left(\beta^{m}\right)^{2}\right) \tag{7}
\end{equation*}
$$

To emphasize the distinction between biased and unbiased classifiers, the model error will be given as a function of $\beta$ for biased classifiers. A more detailed derivation of class boundaries and error regions is presented in 58]. For analyzing the error regions after combining and comparing them to the single classifier case, one needs to determine how the first and second moments of the boundary distributions are affected by combining. The following sections focus on obtaining those values for various combiners.

## 3 Combining Multiple Classifiers through Order Statistics

### 3.1 Basic Concepts

In this section, we briefly discuss some basic concepts and properties of order statistics. Let $X$ be a random variable with probability density function $f_{X}(\cdot)$, and cumulative distribution function $F_{X}(\cdot)$. Let $\left(X_{1}, X_{2}, \cdots, X_{N}\right)$ be a random sample drawn from this distribution. Now, let us arrange them in non-decreasing order, providing:

$$
X_{1: N} \leq X_{2: N} \leq \cdots \leq X_{N: N}
$$

The $i$ th order statistic denoted by $X_{i: N}$, is the $i$ th value in this progression. The cumulative distribution function for the smallest and largest order statistic can be obtained by noting that:

$$
F_{X_{N: N}}(x)=P\left(X_{N: N} \leq x\right)=\Pi_{i=1}^{N} P\left(X_{i: N} \leq x\right)=\left[F_{X}(x)\right]^{N}
$$

and:

$$
\begin{aligned}
F_{X_{1: N}}(x) & =P\left(X_{1: N} \leq x\right)=1-P\left(X_{1: N} \geq x\right)=1-\Pi_{i=1}^{N} P\left(X_{i: N} \geq x\right) \\
& =1-\left(1-\Pi_{i=1}^{N} P\left(X_{i: N} \leq x\right)=1-\left[1-F_{X}(x)\right]^{N}\right.
\end{aligned}
$$

The corresponding probability density functions can be obtained from these equations. In general, for the $i$ th order statistic, the cumulative distribution function gives the probability that exactly $i$ of the chosen $X$ 's are less than or equal to $x$. The probability density function of $X_{i: N}$ is then given by 14:

$$
\begin{equation*}
f_{X_{i: N}}(x)=\frac{N!}{(i-1)!(N-i)!}\left[F_{X}(x)\right]^{i-1}\left[1-F_{X}(x)\right]^{N-i} f_{X}(x) \tag{8}
\end{equation*}
$$

This general form however, cannot always be computed in closed form. Therefore, obtaining the expected value of a function of $x$ using Equation 8 is not always possible. However, the first two moments of the density function are widely available for a variety of distributions [3]. These moments can be used to compute the expected values of certain specific functions, e.g., polynomials of order less than two.

### 3.2 Combining Unbiased Classifiers through Order Statistics

Now, let us turn our attention to order statistics (OS) combiners. For a given input $x$, let the network outputs of each of the $N$ classifiers for each class $i$ be ordered in the following manner:

$$
f_{i}^{1: N}(x) \leq f_{i}^{2: N}(x) \leq \cdots \leq f_{i}^{N: N}(x)
$$

Then one constructs the $k$ th order statistic combiner, by selecting the $k$ th ranked output for each class $\left(f_{i}^{k: N}(x)\right)$, as representing its posterior 57.

In particular, max, med and min combiners are defined as follows:

$$
\begin{align*}
& f_{i}^{\max }(x)=f_{i}^{N: N}(x),  \tag{9}\\
& f_{i}^{\text {med }}(x)
\end{align*}=\left\{\begin{array}{ll}
\frac{f_{i}^{\frac{N}{2}: N}(x)+f_{i}^{\frac{N}{2}+1: N}(x)}{2} & \text { if } N \text { is even }  \tag{10}\\
f_{i}^{\frac{N+1}{2}: N}(x) & \text { if } N \text { is odd },
\end{array}, ~ \begin{array}{ll}
f_{i}^{\min }(x) & =f_{i}^{1: N}(x) . \tag{11}
\end{array}\right.
$$

These three combiners are relevant because they represent important qualitative interpretations of the output space. Selecting the maximum combiner is equivalent to selecting the class with the highest posterior. Indeed, since the network outputs approximate the class a posteriori distributions, selecting the maximum reduces to selecting the classifier that is the most "certain" of its decision. The drawback of this method however is that it can be compromised by a single classifier that repeatedly provides high values. The selection of the minimum combiner follows a similar logic, but focuses on classes that are unlikely to be correct, rather than on the correct class. Thus, this combiner eliminates less likely classes by basing the decision on the lowest value for a given class. This combiner suffers from the same ills as the max combiner. However, it is less dependent on a single error, since it performs a min-max operation, rather than a max-max2. The median classifier on the other hand considers the most "typical" representation of each class. For highly noisy data, this combiner is more desirable than either the min or max combiners since the decision is not compromised as much by a single large error.

The analysis that follows does not depend on the particular order statistic chosen. Therefore, we will denote all OS combiners by $f_{k}^{o s}(x)$ and derive the model error, $E_{\text {model }}^{o s}$. The network output provided by $f_{k}^{o s}(x)$ is given by:

$$
\begin{equation*}
f_{k}^{o s}(x)=p_{k}(x)+\epsilon_{k}^{o s}(x), \tag{12}
\end{equation*}
$$

Let us first investigate the zero-bias case $\left(\beta_{k}=0, \forall k\right)$, where we get $\epsilon_{k}^{o s}(x)=\eta_{k}^{o s}(x)$. Proceeding as in Section 2, the boundary $b^{o s}$ is shown to be:

$$
\begin{equation*}
b^{o s}=\frac{\eta_{i}^{o s}\left(x_{b}\right)-\eta_{j}^{o s}\left(x_{b}\right)}{s} \tag{13}
\end{equation*}
$$

For i.i.d. $\eta_{k}$ 's, the first two moments will be identical for each class. Moreover, taking the order statistic will shift the mean of both $\eta_{i}^{o s}$ and $\eta_{j}^{o s}$ by the same amount, leaving the mean of the difference unaffected. Therefore, $b^{o s}$ will have zero mean, and variance:

$$
\begin{equation*}
\sigma_{b^{o s}}^{2}=\frac{2 \sigma_{\eta_{k}^{o s}}^{2}}{s^{2}}=\frac{2 \alpha \sigma_{\eta_{k}^{m}}^{2}}{s^{2}}=\alpha \sigma_{b^{m}}^{2}, \tag{14}
\end{equation*}
$$

where $\alpha$ is a reduction factor that depends on the order statistic and on the distribution of $b$. For most distributions, $\alpha$ can be found in tabulated form [3]. For example, Table 1 provides $\alpha$ values for all order statistic combiners, up to 10 classifiers, for a Gaussian distribution [3, 50]. (Because this distribution is symmetric, the $\alpha$ values of $l$ and $k$ where $l+k=N+1$ are identical, and listed in parenthesis).

Returning to the error calculation, we have: $\mathcal{M}_{1}^{o s}=0$, and $\mathcal{M}_{2}^{o s}=\sigma_{b^{o s}}^{2}$, providing:

$$
\begin{equation*}
E_{\text {model }}^{o s}=\frac{s \mathcal{M}_{2}^{o s}}{2}=\frac{s \sigma_{b^{o s}}^{2}}{2}=\frac{s \alpha \sigma_{b^{m}}^{2}}{2}=\alpha E_{\text {model }}^{m} \tag{15}
\end{equation*}
$$

Equation 15 shows that the reduction in the error due to using the OS combiner instead of the $m$ th classifier is directly related to the reduction in the variance of the boundary offset $b$. Since the means and variances of order statistics for a variety of distributions are widely available in tabular form, the reductions can be readily quantified.

[^1]Table 1: Reduction factors $\alpha$ for the Gaussian Distribution, based on [50].

| N | $k$ | $\alpha$ | N | $k$ | $\alpha$ | N | $k$ | $\alpha$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 1.00 | 6 | 2 (5) | . 280 | 9 | 1 (9) | . 357 |
| 2 | 1 (2) | . 682 |  | 3 (4) | . 246 |  | 2 (8) | . 226 |
| 3 | 1 (3) | . 560 | 7 | 1 (7) | . 392 |  | 3 (7) | . 186 |
|  | 2 | . 449 |  | 2 (6) | . 257 |  | 4 (6) | . 171 |
| 4 | 1 (4) | . 492 |  | 3 (5) | . 220 |  | 5 | . 166 |
|  | 2 (3) | . 360 |  | 4 | . 210 | 10 | 1 (10) | . 344 |
|  | 1 (5) | . 448 | 8 | 1 (8) | . 373 |  | 2 (9) | . 215 |
| 5 | 2 (4) | . 312 |  | 2 (7) | . 239 |  | 3 (8) | . 175 |
|  | 3 | . 287 |  | 3 (6) | . 201 |  | 4 (7) | . 158 |
| 6 | 1 (6) | . 416 |  | 4 (5) | . 187 |  | 5 (6) | . 151 |

### 3.3 Combining Biased Classifiers through Order Statistics

In this section, we analyze the error regions for biased classifiers. Let us return our attention to $b^{o s}$. First, note that the error terms can no longer be studied separately, since in general $(a+b)^{o s} \neq a^{o s}+b^{o s}$. We will therefore need to specify the mean and variance of the result of each operation Equation 13 becomes:

$$
\begin{equation*}
b^{o s}=\frac{\left(\beta_{i}+\eta_{i}\left(x_{b}\right)\right)^{o s}-\left(\beta_{j}+\eta_{j}\left(x_{b}\right)\right)^{o s}}{s} . \tag{16}
\end{equation*}
$$

Let $\overline{\beta_{k}}=\frac{1}{N} \sum_{m=1}^{N} \beta_{k}^{m}$ be the mean of classifier biases. Since $\eta_{k}^{m}$ 's have zero-mean, $\beta_{k}+\eta_{k}\left(x_{b}\right)$ has first moment $\overline{\beta_{k}}$ and variance $\sigma_{\eta_{k}^{m}}^{2}+\sigma_{\beta_{k}^{m}}^{2}$, with $\sigma_{\beta_{k}^{m}}^{2}=E\left[\left(\beta_{k}^{m}\right)^{2}\right]-{\overline{\beta_{k}}}^{2}$, where [.] denotes the expected value operator.

Taking a specific order statistic of this expression will modify both moments. The first moment is given by $\overline{\beta_{k}}+\mu^{o s}$, where $\mu^{o s}$ is a shift which depends on the order statistic chosen, but not on the class. Then, the first moment of $b^{o s}$ is given by:

$$
\begin{equation*}
\frac{\left(\bar{\beta}_{i}+\mu^{o s}\right)-\left(\bar{\beta}_{j}+\mu^{o s}\right)}{s}=\frac{\bar{\beta}_{i}-\bar{\beta}_{j}}{s}=\bar{\beta} . \tag{17}
\end{equation*}
$$

Note that the bias term represents an "average bias" since the contributions due to the order statistic are removed. Therefore, reductions in bias cannot be obtained from a table similar to Table 1 .

Now, let us turn our attention to the variance. Since $\beta_{k}^{m}+\eta_{k}^{m}\left(x_{b}\right)$ has variance $\sigma_{\eta_{k}^{m}}^{2}+\sigma_{\beta_{k}^{m}}^{2}$, it follows that $\left(\beta_{k}+\eta_{k}\left(x_{b}\right)\right)^{o s}$ has variance $\sigma_{\eta_{k}^{o s}}^{2}=\alpha\left(\sigma_{\eta_{k}^{m}}^{2}+\sigma_{\beta_{k}^{m}}^{2}\right)$, where $\alpha$ is the factor discussed in Section 3.2. Therefore, the variance of $b^{o s}$ is given by:

$$
\begin{align*}
\sigma_{b^{o s}}^{2} & =\frac{\sigma_{\eta_{i}^{o s}}^{2}+\sigma_{\eta_{j}^{o s}}^{2}}{s^{2}}=\frac{2 \alpha \sigma_{\eta_{i}^{m}}^{2}}{s^{2}}+\frac{\alpha\left(\sigma_{\beta_{i}^{m}}^{2}+\sigma_{\beta_{j}^{m}}^{2}\right)}{s^{2}} \\
& =\alpha\left(\sigma_{b^{m}}^{2}+\sigma_{\beta^{m}}^{2}\right) \tag{18}
\end{align*}
$$

where $\sigma_{\beta^{m}}^{2}=\frac{\sigma_{\beta_{i}^{m}+}^{2}+\sigma_{\beta_{j}^{m}}^{2}}{s^{2}}$ is the variance introduced by the systematic errors of different classifiers.

[^2]We have now obtained the first and second moments of $b^{o s}$, and can compute the model error. Namely, we have $\mathcal{M}_{1}^{o s}=\bar{\beta}$ and $\sigma_{b^{o s}}^{2}=\mathcal{M}_{2}^{o s}-\left(\mathcal{M}_{1}^{o s}\right)^{2}$, leading to:

$$
\begin{align*}
E_{\text {model }}^{o s}(\beta) & =\frac{s}{2} \mathcal{M}_{2}^{o s}=\frac{s}{2}\left(\sigma_{b^{o s}}^{2}+\bar{\beta}^{2}\right)  \tag{19}\\
& =\frac{s}{2}\left(\alpha\left(\sigma_{b^{m}}^{2}+\sigma_{\beta^{m}}^{2}\right)+\bar{\beta}^{2}\right) . \tag{20}
\end{align*}
$$

The reduction in the error is more difficult to assess in this case. By writing the error as:

$$
E_{\text {model }}^{o s}(\beta)=\alpha \frac{s}{2}\left(\sigma_{b}^{2}+\left(\beta^{m}\right)^{2}\right)+\frac{s}{2}\left(\alpha \sigma_{\beta}^{2}+\bar{\beta}^{2}-\alpha\left(\beta^{m}\right)^{2}\right),
$$

we get:

$$
\begin{equation*}
E_{\text {model }}^{o s}(\beta)=\alpha E_{\text {model }}^{m}(\beta)+\frac{s}{2}\left(\alpha \sigma_{\beta}^{2}+\bar{\beta}^{2}-\alpha\left(\beta^{m}\right)^{2}\right) . \tag{21}
\end{equation*}
$$

Analyzing the error reduction in the general case requires knowledge about the bias introduced by each classifier. Unlike regression problems where the bias and variance contributions to the error are additive and well-understood, in classification problems their interaction is more complex 21]. Indeed it has been observed that ensemble methods do more than simply reduce the variance 52].

Based on these observations and Equation 21, let us analyze extreme cases. For example, if each classifier has the same bias, $\sigma_{\beta}^{2}$ is reduced to zero and $\bar{\beta}=\beta^{m}$. In this case the error reduction can be expressed as:

$$
E_{\text {model }}^{o s}(\beta)=\frac{s}{2}\left(\alpha \sigma_{b}^{2}+\left(\beta^{m}\right)^{2}=\alpha E_{\text {model }}^{m}(\beta)+\frac{s(1-\alpha)}{2}\left(\beta^{m}\right)^{2},\right.
$$

where $\alpha$ balances the two contributions to the error. A small value for $\alpha$ will reduce the first component of the error (mainly variance), while leaving the second term untouched. The net effect will be very similar to results obtained for regression problems. In this case, it is important to reduce classifier bias before combining (e.g., by using an overparametrized model).

If on the other hand, the biases produce a zero mean variable, we obtain $\bar{\beta}=0$. In this case, the model error becomes:

$$
E_{\text {model }}^{o s}(\beta)=\alpha E_{\text {model }}^{m}(\beta)+\frac{s \alpha}{2}\left(\sigma_{\beta^{m}}^{2}-\left(\beta^{m}\right)^{2}\right)
$$

and the error reduction will be significant if the second term is small or negative. In fact, if the variation among the biases is small relative to their magnitude, the error will be reduced more than in the unbiased cases. If however, the variation is large compared to the magnitude, the error reduction will be minimal. Furthermore, if $\alpha$ is large and the biases are small and highly varied, it is possible for this combiner to do worse than the individual classifiers, which is a danger not present for regression problems. This observation very closely parallels results reported in 21.

## 4 Linear Combining of Ordered Classifier Outputs

In the previous section, we derived error reductions when the class posteriors are directly estimated through the ordered classifier outputs. Since simple averaging has also been shown to provide benefits, in this section, we investigate the combinations of averaging and order statistics for pooling classifier outputs.

### 4.1 Spread Combiner

The first linear combination of ordered classifier outputs we study focuses on extrema. As discussed in Section 3.2, the maximum and minimum of a set of classifier outputs carry specific meanings. Indeed, the maximum can be viewed as the class for which there is the most evidence. Similarly, the minimum deletes classes with little evidence. In order to avoid a single classifier from having too large of an impact on the eventual output, these two values can be averaged to yield the spread combiner. This combiner strikes a balance between the positive and negative evidence, leading to a more robust combiner than either of them.

### 4.1.1 Spread Combiner for Unbiased Classifiers:

For a classifier without bias, the spread combiner is formally defined as:

$$
\begin{equation*}
f_{i}^{s p r}(x)=\frac{1}{2}\left(f_{i}^{1: N}(x)+f_{i}^{N: N}(x)\right)=p\left(c_{i} \mid x\right)+\eta_{i}^{s p r}(x), \tag{22}
\end{equation*}
$$

where:

$$
\eta_{i}^{s p r}(x)=\frac{1}{2}\left(\eta_{i}^{1: N}(x)+\eta_{i}^{N: N}(x)\right)
$$

The variance of $\eta_{i}^{s p r}(x)$ is given by:

$$
\begin{equation*}
\sigma_{\eta_{i}^{s p r}}^{2}=\frac{1}{4} \sigma_{\eta_{i}^{1: N}(x)}^{2}+\frac{1}{4} \sigma_{\eta_{i}^{N: N}(x)}^{2}+\frac{1}{2} \operatorname{cov}\left(\eta_{i}^{1: N}(x), \eta_{i}^{N: N}(x)\right) . \tag{23}
\end{equation*}
$$

where $\operatorname{cov}(\cdot, \cdot)$ represents the covariance between two variables (even when the $\eta_{i}{ }^{6}$ s are independent, ordering introduces correlations). Note that because of the ordering, the variances in the first two terms of Equation 23 can be expressed in terms of the individual classifier variances. Furthermore, the covariance between two order statistics can also be determined in tabulated form for given distributions. Table 2 provides these values for a Gaussian distribution based on [50]. This expression can be further simplified for symmetric distributions where $\sigma_{\eta^{1: N}}^{2}=\sigma_{\eta^{N: N}}^{2}$ (e.g., Gaussian noise model) and leads to:

$$
\begin{equation*}
\sigma_{\eta_{i}^{s p r}}^{2}=\frac{1}{2}\left(\alpha_{1: N}+B_{1, N: N}\right) \sigma_{\eta_{i}(x)}^{2} \tag{24}
\end{equation*}
$$

where $\alpha_{m: N}$ is the variance of the $m$ th ordered sample and $B_{m, l: N}$ is the covariance between the $m$ th and $l$ th ordered samples, given that the initial samples had unit variance 50. Because this is a symmetric distribution, the $\beta$ values are also symmetric (e.g., $\beta_{1,2: 5}=\beta_{4,5: 5}$ ).

Then, using Equation 3, the variance of the boundary offset $b^{s p r}$ can be calculated:

$$
\begin{align*}
\sigma_{b^{s p r}}^{2} & =\frac{\sigma_{\eta_{i} s p r}^{2}+\sigma_{\eta_{j} s p r}^{2}}{s^{2}} \\
& =\frac{1}{2}\left(\alpha_{1: N}+B_{1, N: N}\right) \sigma_{b}^{2} \tag{25}
\end{align*}
$$

Finally, through Equation 6, we can obtain the reduction in the model error due to the spread combiner:

$$
\begin{equation*}
\frac{E_{\text {model }}^{s p r}}{E_{\text {model }}}=\frac{\alpha_{1: N}+B_{1, N: N}}{2} \tag{26}
\end{equation*}
$$

Table 2: Some Reduction Factors $B$ for the Gaussian Distribution, based on [50].

| N | k,l | B | N | k,l | $B$ | N | k,l | $B$ | N | k,l | $B$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | 1,2 | . 318 |  | 2,3 | . 189 |  | 1,4 | . 095 |  | 1,6 | . 059 |
| 3 | 1,2 | . 276 | 6 | 2,4 | . 140 |  | 1,5 | . 075 |  | 1,7 | . 049 |
|  | 1,3 | . 165 |  | 2,5 | . 106 |  | 1,6 | . 060 |  | 1,8 | . 040 |
| 4 | 1,2 | . 246 |  | 3,4 | . 183 |  | 1,7 | . 048 |  | 1,9 | . 031 |
|  | 1,3 | . 158 |  | 1,2 | . 196 |  | 1,8 | . 037 |  | 2,3 | . 154 |
|  | 1,4 | . 105 |  | 1,3 | . 132 |  | 2,3 | . 163 |  | 2,4 | . 117 |
|  | 2,3 | . 236 |  | 1,4 | . 099 | 8 | 2,4 | . 123 |  | 2,5 | . 093 |
| 5 | 1,2 | . 224 |  | 1,5 | . 077 |  | 2,5 | . 098 |  | 2,6 | . 077 |
|  | 1,3 | . 148 |  | 1,6 | . 060 |  | 2,6 | . 079 | 9 | 2,7 | . 063 |
|  | 1,4 | . 106 | 7 | 1,7 | . 045 |  | 2,7 | . 063 |  | 2,8 | . 052 |
|  | 1,5 | . 074 |  | 2,3 | . 175 |  | 3,4 | . 152 |  | 3,4 | . 142 |
|  | 2,3 | . 208 |  | 2,4 | . 131 |  | 3,5 | . 121 |  | 3,5 | . 114 |
|  | 2,4 | . 150 |  | 2,5 | . 102 |  | 3,6 | . 098 |  | 3,6 | . 093 |
| 6 | 1,2 | . 209 |  | 2,6 | . 080 |  | 4,5 | . 149 |  | 3,7 | . 077 |
|  | 1,3 | . 139 |  | 3,4 | . 166 | 9 | 1,2 | . 178 |  | 4,5 | . 137 |
|  | 1,4 | . 102 |  | 3,5 | . 130 |  | 1,3 | . 121 |  | 4,6 | . 113 |
|  | 1,5 | . 077 |  | 1,2 | . 186 |  | 1,4 | . 091 |  |  |  |
|  | 1,6 | . 056 | 8 | 1,3 | . 126 |  | 1,5 | . 073 |  |  |  |

Based on Equation 26 and Tables 1 and 22, Table 3 displays the error reductions provided by the spread combiner for a Gaussian noise model (for comparison purposes, the error reduction for the min and max combiners is also provided. Note that for the Gaussian distribution, the error reduction of $\min$ is equal to that of $\max$.).

Table 3: Error Reduction Factors for the Spread, min and max Combiners with Gaussian Noise Model.

| N | spread | min or $\max$ |
| :---: | :---: | :---: |
| 2 | .500 | .682 |
| 3 | .362 | .560 |
| 4 | .299 | .492 |
| 5 | .261 | .448 |
| 6 | .236 | .416 |
| 7 | .219 | .392 |
| 8 | .205 | .373 |
| 9 | .194 | .357 |
| 10 | .186 | .344 |

### 4.1.2 Spread Combiner for Biased Classifiers:

Now, if the classifier biases are non-zero, the spread combiner's output is given by:

$$
\begin{equation*}
f_{i}^{s p r}(x)=\frac{1}{2}\left(f_{i}^{1: N}(x)+f_{i}^{N: N}(x)\right)=p\left(c_{i} \mid x\right)+\left(\eta_{i}(x)+\beta_{i}\right)^{s p r} . \tag{27}
\end{equation*}
$$

In that case, the boundary offset is given by:

$$
\begin{equation*}
b^{s p r}=\frac{\left(\beta_{i}+\eta_{i}\left(x_{b}\right)\right)^{s p r}-\left(\beta_{j}+\eta_{j}\left(x_{b}\right)\right)^{s p r}}{s} \tag{28}
\end{equation*}
$$

which after expanding each term and regrouping can be expressed as:

$$
\begin{align*}
b^{s p r}= & \frac{\left(\beta_{i}+\eta_{i}\left(x_{b}\right)\right)^{1: N}-\left(\beta_{j}+\eta_{j}\left(x_{b}\right)\right)^{1: N}}{2 s} \\
& +\frac{\left(\beta_{i}+\eta_{i}\left(x_{b}\right)\right)^{N: N}-\left(\beta_{j}+\eta_{j}\left(x_{b}\right)\right)^{N: N}}{2 s} \tag{29}
\end{align*}
$$

The first moment of $b^{s p r}$ can be obtained by analyzing each term of Equation 29 . In fact, the offset introduced by the first and $n$th order statistic for classes $i$ and $j$ will cancel each other out, leaving only the average bias between the min and max components of the error (as in Equation 17), given by $\beta^{s p r}=\frac{\beta_{i}^{1: N}-\beta_{j}^{1: N}+\beta_{i}^{N: N}-\beta_{j}^{N: N}}{s}$.

The variance of $b^{s p r}$ needs to be derived from Equation 29. Proceeding as in Equation 18, the variance of the spread combiner can be expressed as:

$$
\begin{equation*}
\sigma_{b^{s p r}}^{2}=\left(\frac{1}{4} \alpha_{1: N}+\frac{1}{4} \alpha_{N: N}+\frac{1}{2} B_{1, N: N}\right)\left(\sigma_{b^{m}}^{2}+\sigma_{\beta^{m}}^{2}\right) . \tag{30}
\end{equation*}
$$

For a symmetric distribution (where $\alpha_{1: N}=\alpha_{N: N}$ ), we obtain the following error:

$$
\begin{align*}
E_{\text {model }}^{s p r}(\beta)= & \frac{s}{2} \mathcal{M}_{2}=\frac{s}{2}\left(\sigma_{b^{s p r}}^{2}+\mathcal{M}_{1}^{2}\right) \\
= & \left.\frac{s}{2}\left(\frac{1}{2} \alpha_{1: N}+\frac{1}{2} B_{1, N: N}\right)\left(\sigma_{b^{m}}^{2}+\sigma_{\beta^{m}}^{2}\right)+\left(\beta^{s p r}\right)^{2}\right) \\
= & \frac{1}{2}\left(\alpha_{1: N}+B_{1, N: N}\right) E_{\text {model }}(\beta)+ \\
& \frac{s}{4}\left(\alpha_{1: N}+B_{1, N: N}\right)\left(\sigma_{\beta^{m}}^{2}-\left(\beta^{m}\right)^{2}\right)+\frac{s}{2}\left(\beta^{s p r}\right)^{2} \tag{31}
\end{align*}
$$

which is very similar to Equation 21, where the value of $\alpha$ for a single order statistic is now replaced by $\frac{\alpha_{1: N}+B_{1, N: N}}{2}$, since the mean of the first and $n$th order statistic is used in the posterior estimate.

### 4.2 Trimmed Means

Instead of actively using the extreme values as was the case with the spread combiner, one can base the posterior estimate around the median values. However, instead of selecting one classifier output as was done for $f^{m e d}$, one can use multiple classifiers whose outputs are "typical." In this scheme, only a certain fraction of all available classifiers are used for a given pattern. The main advantage of this method over weighted averaging is that the set of classifiers which contribute to the combiner vary from pattern to pattern. Furthermore, they do not need to be determined externally, but are a function of the current pattern and the classifier responses to that pattern.

### 4.2.1 Trimmed Mean Combiner for Unbiased Classifiers:

Let us formally define the trimmed mean combiner $\left(\beta_{k}=0, \forall k\right)$ as follows:

$$
\begin{equation*}
f_{i}^{\text {trim }}(x)=\frac{1}{N_{2}-N_{1}+1} \sum_{m=N_{1}}^{N_{2}} f_{i}^{m: N}(x)=p\left(c_{i} \mid x\right)+\eta_{i}^{\text {trim }}(x) \tag{32}
\end{equation*}
$$

where:

$$
\eta_{i}^{t r i m}(x)=\frac{1}{N_{2}-N_{1}+1} \sum_{m=N_{1}}^{N_{2}} \eta_{i}^{m}(x)
$$

The variance of $\eta_{i}^{t r i m}(x)$ is given by:

$$
\begin{align*}
\sigma_{\eta_{i}^{t r i m}}^{2} & =\frac{1}{\left(N_{2}-N_{1}+1\right)^{2}} \sum_{l=N_{1}}^{N_{2}} \sum_{m=N_{1}}^{N_{2}} \operatorname{cov}\left(\eta_{i}^{m: N}(x), \eta_{i}^{l: N}(x)\right) \\
& =\frac{1}{\left(N_{2}-N_{1}+1\right)^{2}}\left(\sum_{m=N_{1}}^{N_{2}} \sigma_{\eta_{i}^{m: N}(x)}^{2}+\sum_{m=N_{1}}^{N_{2}} \sum_{l>m}^{N_{2}} 2 \operatorname{cov}\left(\eta_{i}^{m: N}(x), \eta_{i}^{l: N}(x)\right)\right) \tag{33}
\end{align*}
$$

Again, using the factors in Tables 11 and 2, Equation 33 can be further simplified. Note that because the Gaussian distribution is symmetric, the covariance between the $k$ th and $l$ th ordered samples is the same as that between the $N+1-k$ th and $N+1-l$ th ordered samples. Therefore, Equation 33 leads to:

$$
\begin{align*}
\sigma_{\eta_{i}^{t r i m}}^{2} & =\frac{1}{\left(N_{2}-N_{1}+1\right)^{2}} \sum_{m=N_{1}}^{N_{2}} \alpha_{m: N} \sigma_{\eta_{i}(x)}^{2} \\
& +\frac{2}{\left(N_{2}-N_{1}+1\right)^{2}} \sum_{m=N_{1}}^{N_{2}} \sum_{l>m} B_{m, l: N} \sigma_{\eta_{i}(x)}^{2} \tag{34}
\end{align*}
$$

where $\alpha_{m: N}$ is the variance of the $m$ th ordered sample and $B_{m, l: N}$ is the covariance between the $m$ th and $l$ th ordered samples, given that the initial samples had unit variance [50]. Using the theory highlighted in Section 2, and Equation 34, we obtain the following model error reduction:

$$
\begin{equation*}
\frac{E_{\text {model }}^{\text {trim }}}{E_{\text {model }}}=\frac{1}{\left(N_{2}-N_{1}+1\right)^{2}}\left(\sum_{m=N_{1}}^{N_{2}} \alpha_{m: N}+2 \sum_{m=N_{1}}^{N_{2}} \sum_{l>m} B_{m, l: N}\right) \tag{35}
\end{equation*}
$$

Based on Equation 35 and Tables 1 and 2 , we have generated a sample trim combiner reduction table. Because there are many possibilities for $N_{1}$ and $N_{2}$, a table that exhaustively provides all reduction values is not practical. In this sample table we have selected $N_{1}=2$ and $N_{2}=N-1$, that is, averaging after the lowest and highest values have been removed. For comparison purposes the reduction factors of the averaging combiner for $N$ and $N-2$ classifiers are also provided (for i.i.d. classifiers the reduction factors are $1 / \mathrm{N}$ as derived in 58]; similar results were obtained for regression problems 42). As these numbers demonstrate, although $N-2$ classifiers are used in the trim combiner, selectively weeding out undesirable classifiers provides reduction factors significantly better than simply averaging $N-2$ arbitrary classifiers. The trim combiner provides reduction factors comparable the the $N$ classifier ave combiner without being susceptible to corruption by one particularly faulty classifier.

### 4.2.2 Trimmed mean Combiner for Biased Classifiers:

Now, if the classifier biases are non-zero, the trimmed mean combiner's output is given by:

$$
\begin{equation*}
f_{i}^{t r i m}(x)=\frac{1}{N_{2}-N_{1}+1} \sum_{m=N_{1}}^{N_{2}} f_{i}^{m: N}(x)=p\left(c_{i} \mid x\right)+\left(\eta_{i}(x)+\beta_{i}\right)^{\text {trim }} \tag{36}
\end{equation*}
$$

Table 4: Error Reduction Factors for Trim and two corresponding ave Combiners with Gaussian Noise Model.

| N | ave (for N ) | trim (for $N_{1}=2 ; N_{2}=N-1$ ) | ave (for $N-2$ ) |
| :---: | :---: | :---: | :---: |
| 3 | .333 | .449 | 1.00 |
| 4 | .250 | .298 | .500 |
| 5 | .200 | .227 | .333 |
| 6 | .167 | .184 | .250 |
| 7 | .143 | .155 | .200 |
| 8 | .125 | .134 | .167 |
| 9 | .111 | .113 | .143 |

In that case the boundary offset is given by:

$$
\begin{equation*}
b^{\text {trim }}=\frac{\left(\beta_{i}+\eta_{i}\left(x_{b}\right)\right)^{t r i m}-\left(\beta_{j}+\eta_{j}\left(x_{b}\right)\right)^{\text {trim }}}{s} . \tag{37}
\end{equation*}
$$

The first moment of $b^{\text {trim }}$ can be obtained from a manner similar to that of the spread combiner. Indeed, each mean offset introduced by a specific order statistic for class $i$ will be offset by the one introduced for class $j$. Only the trimmed mean of the biases will remain, giving the first moment of $b^{\text {trim }}$ :

$$
\begin{equation*}
\beta^{\text {trim }}=\frac{1}{N_{2}-N_{1}+1} \sum_{m=N_{1}}^{N_{2}} \frac{\beta_{i}^{m: N}-\beta_{j}^{m: N}}{s} . \tag{38}
\end{equation*}
$$

In deriving the variance of $b^{\text {trim }}$, we follow the same steps as in Sections 3.3 and 4.1.1. The resulting boundary variance is similar to Equation 18, but the since the reduction is due to the linear combination of multiple ordered outputs, $\alpha$ is replaced by $\mathcal{A}$, where:

$$
\begin{equation*}
\mathcal{A}=\frac{1}{\left(N_{2}-N_{1}+1\right)^{2}}\left(\sum_{m=N_{1}}^{N_{2}} \alpha_{m: N}+2 \sum_{m=N_{1}}^{N_{2}} \sum_{l>m} B_{m, l: N}\right) . \tag{39}
\end{equation*}
$$

The model error reduction in this case is given by:

$$
\begin{align*}
E_{\text {model }}^{\text {trim }}(\beta) & =\frac{s}{2} \mathcal{M}_{2}=\frac{s}{2}\left(\sigma_{b^{\text {trim }}}^{2}+\mathcal{M}_{1}^{2}\right) \\
& =\frac{s}{2}\left(\mathcal{A}\left(\sigma_{b^{m}}^{2}+\sigma_{\beta^{m}}^{2}\right)+\left(\beta^{s p r}\right)^{2}\right) \\
& =\mathcal{A} E_{\text {model }}(\beta)+\frac{s}{2}\left(\mathcal{A}\left(\sigma_{\beta^{m}}^{2}-\left(\beta^{m}\right)^{2}\right)+\left(\beta^{s p r}\right)^{2}\right) . \tag{40}
\end{align*}
$$

Once again we need to look at the interaction between the two parts of the error reduction. The first term provides the error reduction compared to the model error of an individual classifier. The smaller $\mathcal{A}$ is, the more error reduction there will be. In the second term, on the other hand, a small value for $\mathcal{A}$ is only useful if the variability in the individual biases is higher than the biases themselves $\left(\sigma_{\beta^{m}}^{2}>\left(\beta^{m}\right)^{2}\right)$.

## 5 Experimental Results

The order statistics-based combining methods proposed in this article are tailored for situations where:

1. individual classifier performance is uneven and class dependent;
2. it is not possible (insufficient data, high amount of noise) to fine tune the individual classifiers without using computationally expensive methods.

Such situations occur, for example, in electrical logging while drilling for oil, where data from certain well sites almost completely misses out on portions of the problem space, and in imaging from airborne platforms where the classifiers receive inputs from different satellites and/or different types of sensors (e.g., thermal, optical, SAR). While we have seen such data from Schlumberger, Austin, and NASA, Houston, unfortunately the data sets are not standard or public domain. So, in this article we restrict ourselves to public domain datasets and simulate such variability by using "early stopping" i.e., prematurely terminating the training of the individual classifiers Thus combining results are first reported for the case where only half the classifiers are finely tuned. This procedure produces an artificially created quality variation in the pool of classifiers.

For the experiments reported below, we used a multi-layer perceptron (MLP) with a single hidden layer, whose weights were randomly initialized for each run. All classification results reported in this article are test set error rates averaged over 20 runs, along with the $95 \%$ confidence intervals. Several types of simple combiners such as averaging, weighted averaging, voting, median, products, weighted products (Bayesian), using Dempster-Schafer theory of evidence, and entropy-based averaging, have been proposed in the literature. However, on a wide variety of data sets, it has been observed that simple averaging usually provides results comparable to any of these techniques (and, surprisingly, often better than most of them) [26, 59]. For this reason, in this study, we use the average combiner as a representative of simple combiners, for comparison purposes.

The first two data sets (Tables 國 and are based on underwater sonar signals. From the original sonar signals of four different underwater objects (porpoise sound, cracking ice and two different whale sounds), two feature sets are extracted 24:

- WOC: a 25-dimensional feature set, consisting of Gabor wavelet coefficients, temporal descriptors and spectral measurements; and,
- RDO: a 24-dimensional feature set, consisting of reflection coefficients based on both short and long time windows, and temporal descriptors.

For both feature sets, an MLP with 50 hidden units was used. These data sets are available at URL http://www.lans.ece.utexas.edu. Further details about this 4-class problem can be found in [24, 59].

The next six data sets (Tables 6 and 8) were selected from the Proben1/UCI benchmarks 43. The Proben1 benchmarks are particular training, validation and test splits of the UCI data sets which are available from URL http://www.ics.uci.edu/~ mlearn/MLRepository.htm. The results presented in this article are based on the first training, validation and test partition discussed in 43, where half the data is used for training, and a quarter each for validation and testing purposes. Briefly these data sets, and the corresponding single layer feed-forward neural network architectures are5:

[^3]- Cancer: a 9-dimensional, 2-class data set based on breast cancer data 34, with 699 patterns; an MLP with 10 hidden units;
- Card: a 51-dimensional, 2-class data set based on credit approval decision 44, with 690 patterns; an MLP with 20 hidden units;
- Diabetes: an 8-dimensional data set with two classes based on personal data from 768 Pima Indians obtained from the National institute of Diabetes and Digestive and Kidney Diseases 54; an MLP with 10 hidden units;
- Gene: a 120-dimensional data set with two classes, based on the detection of splice junctions in DNA sequences [39, with 3175 patterns; an MLP with 20 hidden units;
- Glass: a 9-dimensional, 6-class data set based on the chemical analysis of glass splinters, with 214 patterns; an MLP with 15 hidden units; and,
- Soybean: an 82-dimensional, 19-class problem 38 with 683 patterns; an MLP with 40 hidden units.

Table 5: Combining Results in the Presence of High Variability in Individual Classifier Performance for the Sonar Data (\% misclassified $\pm 95 \%$ confidence interval).

| Data | N | Ave | Max | Min | Spread | Trim $\left(N_{1}-N_{2}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| RDO | 4 | $11.57 \pm .22$ | $11.94 \pm .25$ | $11.52 \pm .40$ | $11.04 \pm .19$ | $11.34 \pm .28(3-4)$ |
| $13.32 \pm 1.66$ | 8 | $11.64 \pm .18$ | $11.47 \pm .22$ | $11.29 \pm .27$ | $11.51 \pm .18$ | $12.30 \pm .17(4-5)$ |
| WOC | 4 | $8.80 \pm .18$ | $7.84 \pm .20$ | $9.31 \pm .24$ | $8.54 \pm .12$ | $8.43 \pm .26(3-4)$ |
| $12.07 \pm 2.23$ | 8 | $8.82 \pm .17$ | $7.68 \pm .23$ | $8.91 \pm .13$ | $8.24 \pm .22$ | $7.81 \pm .16(7-8)$ |

Table 6: Combining Results in the Presence of High Variability in Individual Classifier Performance for the Proben1/UCI Benchmarks (\% misclassified $\pm 95 \%$ confidence interval).

| Data | N | Ave | Max | Min | Spread | Trim $\left(N_{1}-N_{2}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Cancer | 4 | $1.38 \pm .13$ | $1.38 \pm .13$ | $1.38 \pm .13$ | $1.38 \pm .13$ | $1.32 \pm .13(2-3)$ |
| $1.49 \pm .39$ | 8 | $1.32 \pm .12$ | $1.44 \pm .14$ | $1.44 \pm .14$ | $1.44 \pm .14$ | $1.32 \pm .12(2-6)$ |
| Card | 4 | $13.60 \pm .22$ | $13.37 \pm .22$ | $13.49 \pm .21$ | $13.37 \pm .22$ | $13.60 \pm .15(3-4)$ |
| $14.33 \pm .36$ | 8 | $13.66 \pm .19$ | $13.08 \pm .14$ | $13.02 \pm .14$ | $12.97 \pm .12$ | $13.20 \pm .18(7-8)$ |
| Diabetes | 4 | $25.26 \pm .37$ | $25.00 \pm .46$ | $25.00 \pm .42$ | $25.00 \pm .42$ | $25.26 \pm .37(3-4)$ |
| $26.09 \pm 1.27$ | 8 | $24.84 \pm .36$ | $25.05 \pm .33$ | $25.05 \pm .33$ | $25.05 \pm .33$ | $24.84 \pm .30(6-8)$ |
| Gene | 4 | $12.90 \pm .23$ | $12.90 \pm .26$ | $12.94 \pm .25$ | $12.66 \pm .21$ | $12.67 \pm .22(3-4)$ |
| $15.01 \pm .78$ | 8 | $12.89 \pm .22$ | $12.76 \pm .24$ | $12.41 \pm .10$ | $12.43 \pm .22$ | $12.56 \pm .20(7-8)$ |
| Glass | 4 | $33.77 \pm .27$ | $40.19 \pm .72$ | $33.21 \pm .44$ | $33.21 \pm .44$ | $33.77 \pm .27(2-3)$ |
| $42.78 \pm .75$ | 8 | $33.96 \pm .06$ | $39.43 \pm .27$ | $33.77 \pm .27$ | $33.40 \pm .41$ | $33.77 \pm .27(1-6)$ |
| Soybean | 4 | $7.76 \pm .11$ | $7.94 \pm .14$ | $12.88 \pm .39$ | $7.71 \pm .15$ | $7.82 \pm .18(3-4)$ |
| $10.71 \pm 1.69$ | 8 | $7.65 \pm .00$ | $7.82 \pm .13$ | $13.41 \pm .53$ | $7.71 \pm .15$ | $7.65 \pm .00(4-8)$ |

Tables 5 and 6 present the combining results for the Proben1 benchmarks and the underwater acoustic data sets respectively, when the individual classifier performance
was highly variable. The misclassification percentage for individual classifiers are reported in the first column. For the trimmed mean combiner, we also provide $N_{1}$ and $N_{2}$, the upper and lower cutting points in the ordered average used in Equation 32 , obtained through the validation set.

On the Sonar data, the results indicate that when the individual classifier performance is highly variable, order statistics-based combiners (particularly the spread combiner) provide better classification results than simple combiners. This performance improvement is obtained without sacrificing the simplicity of the combiner. On the UCI/Proben1 benchmarks, the order statistics based combiners provide better classification performance on three of the six sets studied (no statistically significant differences were detected among the various combiners in the remaining data sets). One important thing to note, however, is that in all eight data sets studied, the order statistics based combiners performed at least as well as the simple combiner, implying that no risk is taken by using this method.

A close inspection of these results reveals that using either the max or min combiner can provide better classification rates than ave, but it is difficult to determine which of the two will be more successful given a data set. A validation set may be used to select one over the other, but in that case, potentially precious training data is used solely for determining which combiner to use. The use of the spread combiner removes this dilemma by consistently providing results that are comparable to, or better than, the best of the max-min duo. It is important to note that the min combiner performs poorly on the Soybean data. Because this data set has 19 outputs, the posterior estimates of unlikely classes become extremely small and highly inaccurate. Basing decisions on such spurious values compromises the combiner's performance. Notice, however, that the spread combiner is not adversely affected by this phenomenon.

Table 7: Combining Results with Fine-Tuned Classifiers for the Sonar Data (\% misclassified $\pm 95 \%$ confidence interval).

| Data | N | Ave | Max | Min | Spread | Trim $\left(N_{1}-N_{2}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| RDO | 4 | $9.26 \pm .32$ | $9.67 \pm .20$ | $9.45 \pm .19$ | $9.33 \pm .20$ | $9.28 \pm .28(2-3)$ |
| $9.95 \pm .36$ | 8 | $8.94 \pm .06$ | $9.62 \pm .16$ | $9.36 \pm .15$ | $9.48 \pm .18$ | $8.92 \pm .10(1-6)$ |
| WOC | 4 | $7.05 \pm .12$ | $7.31 \pm .15$ | $7.44 \pm .17$ | $7.31 \pm .16$ | $7.05 \pm .16(2-3)$ |
| $7.47 \pm .21$ | 8 | $7.17 \pm .08$ | $7.19 \pm .12$ | $7.41 \pm .16$ | $7.22 \pm .07$ | $7.07 \pm .10(2-6)$ |

When there is ample data, and all the classifiers are finely tuned (i.e., a validation set is used to determine the stopping time that yields the best generalization performance), simple combiners are expected to be adequate. However, it is not always possible to determine whether all conditions that lead to such an ideal situation are satisfied. Therefore, it is important to know whether the trimmed mean and spread combiners presented in this article perform worse than simple combiners under such conditions. To that end, we have combined finely tuned feed forward neural networks using the methods proposed in this article and compared the results with the traditional averaging method. In this new set of experiments, all the conditions favor the averaging combiner (i.e., all possible difficulties for the average combiner have been removed). The results displayed in Tables 7 and 8 indicate that, even under such circumstances, both the spread and trim combiners provide results that are comparable to those obtained by the
ave combiner. Furthermore, even under such conditions, the order statistics combiners provide statistically significant improvements on two data sets.

Table 8: Combining Results with Fine-Tuned Classifiers for the Proben1/UCI Benchmarks ( $\%$ misclassified $\pm 95 \%$ confidence interval).

| Data | N | Ave | Max | Min | Spread | Trim $\left(N_{1}-N_{2}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Cancer | 4 | $0.69 \pm .11$ | $0.69 \pm .11$ | $0.69 \pm .11$ | $0.69 \pm .11$ | $0.69 \pm .11(2-3)$ |
| $.69 \pm .11$ | 8 | $0.69 \pm .11$ | $0.57 \pm .01$ | $0.57 \pm .01$ | $0.57 \pm .01$ | $0.57 \pm .11(7-8)$ |
| Card | 4 | $13.14 \pm .23$ | $12.91 \pm .11$ | $13.02 \pm .23$ | $12.91 \pm .11$ | $13.14 \pm .23(2-3)$ |
| $13.87 \pm .36$ | 8 | $13.14 \pm .23$ | $12.79 \pm .01$ | $12.79 \pm .01$ | $12.79 \pm .01$ | $12.80 \pm .01(7-8)$ |
| Diabetes | 4 | $23.33 \pm .29$ | $23.23 \pm .30$ | $23.33 \pm .24$ | $23.23 \pm .30$ | $23.33 \pm .29(3-4)$ |
| $23.52 \pm .35$ | 8 | $22.92 \pm .23$ | $23.23 \pm .34$ | $23.12 \pm .34$ | $23.23 \pm .34$ | $22.92 \pm .23(4-8)$ |
| Gene | 4 | $12.41 \pm .21$ | $12.46 \pm .24$ | $12.51 \pm .18$ | $12.41 \pm .17$ | $12.41 \pm .12(3-4)$ |
| $13.49 \pm .21$ | 8 | $12.26 \pm .14$ | $12.46 \pm .18$ | $12.16 \pm .08$ | $12.11 \pm .19$ | $12.16 \pm .09(1-6)$ |
| Glass | 4 | $32.08 \pm .01$ | $32.45 \pm .36$ | $32.08 \pm .01$ | $32.08 \pm .01$ | $32.08 \pm .01(3-6)$ |
| $32.26 \pm .27$ | 8 | $32.08 \pm .01$ | $32.08 \pm .01$ | $32.08 \pm .01$ | $32.08 \pm .01$ | $32.08 \pm .01(3-6)$ |
| Soybean | 4 | $7.06 \pm .00$ | $7.18 \pm .11$ | $8.12 \pm .77$ | $7.06 \pm .00$ | $7.06 \pm .00(3-6)$ |
| $7.36 \pm .43$ | 8 | $7.06 \pm .00$ | $7.18 \pm .05$ | $9.06 \pm .82$ | $7.06 \pm .00$ | $7.06 \pm .00(3-6)$ |

## 6 Conclusion

In this article we present and analyze combiners based on order statistics. These combiners blend the simplicity of averaging with the generality of meta-learners. They are particularly effective if there are significant variations among component classifiers in at least some parts of the joint input-output space. Variations can arise when the individual training sets cannot be considered as random samples from a common universal data set. Examples of such cases include real-time data acquisition and classification from geographically distributed sources or data mining problems with large databases, where random subsampling is computationally expensive and practical methods lead to non-random subsamples [6]. Furthermore, The robustness of order statistics combiners is also helpful when certain individual classifiers experience catastrophic failures (e.g., due to faulty sensors).

The analytical framework provided in this paper quantifies the reductions in error achieved when an order statistics based ensemble is used. It also shows that the two methods for linear combination of order statistics introduced in this paper provide more reliable estimates of the true posteriors than any of the individual order statistic combiners.

The experimental results of Section 5 indicate that when there is high variability among the classifiers, the order statistics-based combiners significantly outperform simple combiners, whereas in the absence of such variability these combiners perform no worse. Thus the family of order statistic combiners is able to extract an appropriate amount of information from the individual classifier outputs without requiring tuning additional parameters as in meta-learners, and without being substantially affected by outliers.

A future endeavor, which will be helpful for this work as well as for the study of classification based on very large datasets in general, is to obtain a suite of public domain datasets which are intrinsically partitioned into segments with varying quality. Though such situations sometimes occur in practice (for example in oil logging data [10] and mortgage scoring [36]; both data sets proprietary), they are not represented in the standard, venerable databases such as UCI, ELENA and Statlog typically used by the academic community. Perhaps the recent CRoss-Industry Standard Process for Data Mining (CRISP-DM) initiative will provide a satisfactory solution to this problem in the near future.

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[^0]:    ${ }^{1}$ This and other related papers can be downloaded from URL http://www.lans.ece.utexas.edu

[^1]:    ${ }^{2}$ Recall that the pattern is ultimately assigned to the class with the highest combined output.

[^2]:    ${ }^{3}$ Since the exact distribution parameters of $b^{o s}$ are not known, we use the sample mean and the sample variance.

[^3]:    ${ }^{4}$ In all the experiments reported here, "high variability" among classifiers refers to classifiers being trained exactly half as long as the "fine tuned" classifiers.
    ${ }^{5}$ After deciding on a single hidden layered architecture, the number of hidden units was determined experimentally.

