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FULL LENGTH PAPER

Quadratic programming and combinatorial minimum weight product problems

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Abstract We present a fully polynomial time approximation scheme (FPTAS) for minimizing an objective $(a^{\mathrm{T}}x+\gamma)(b^{\mathrm{T}}x+\delta)$ under linear constraints $Ax \leq d$. Examples of such problems are combinatorial minimum weight product problems such as the following: given a graph $G = (V, E)$ and two edge weights $a, b : E \to \mathbb{R}_+$ find an $s - t$ path P that minimizes $a(P)b(P)$, the product of its edge weights relative to a and b.

Keywords Quadratic programming · Approximation scheme · Shortest path

Mathematics Subject Classification (2000) 90C20 · 90C26 · 90C27

1 Introduction

The problem of minimizing a quadratic objective function $x^{\mathrm{T}}Qx + c^{\mathrm{T}}x$ under m linear constraints $Ax \leq d$ is well-known to be NP-hard [4], even when Q has only a single negative eigenvalue [9]. In case Q is positive semidefinite, the problem can be solved efficiently ([6] or [12]). Here, we focus on the case where the objective is the product of two affine functions:

$$z^* = \min\ (a^{\mathrm{T}}x + \gamma)(b^{\mathrm{T}}x + \delta) \quad Ax \leq d. \tag{1.1}$$

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The complexity status of this (in general non-convex) problem is open (cf. [9]). We present a *fully polynomial time approximation scheme* (FPTAS) for this class. More precisely, we present an algorithm which correctly decides whether $z^* < 0$, $z^* = 0$ or $z^* > 0$ holds and, in addition, computes for any given $\varepsilon > 0$ an *ε-approximate solution*, i.e., a feasible solution of (1.1) whose objective value differs from the optimum z^* by at most $\varepsilon|z^*|$. (In case $z^* \leq 0$ we can even solve the problem exactly in polynomial time.) The running time of the algorithm is polynomially bounded in $1/\varepsilon$ and the size of (1.1).

In Sect. 3, we discuss possible applications of our result to combinatorial minimum weight product problems such as the following: given a graph $G = (V, E)$ with two non-negative edge weights $a, b : E \to \mathbb{R}_+$, find an $s - t$ path P minimizing $a(P)b(P)$, the product of its edge weights relative to a and b.

Remark Vavasis [11] presents an FPTAS for (more general) quadratic objectives $q(x) = x^{\mathrm{T}}Qx + c^{\mathrm{T}}x$ with a bounded number of negative eigenvalues. His work, however, is based on a different concept of "ε-approximate solution": In [11], a feasible x is ε-approximate if its objective value differs from the optimum z^* by at most

$$\varepsilon \left(\max_{Ax \leq d} q(x) - \min_{Ax \leq d} q(x) \right).$$

This concept of "ε-approximation" is not suited for the combinatorial applications that we discuss in Sect. 3. It is unclear whether our results can somehow be extended to the case of bounded number of negative eigenvalues.

2 The algorithm

Relative to (1.1), we consider the related system

$$\begin{aligned} \alpha - a^{\mathrm{T}}x - \gamma &= 0 \\ \beta - b^{\mathrm{T}}x - \delta &= 0 \\ \mathrm{A\,x} &\leq d \end{aligned} \tag{2.1}$$

of $m + 2$ (in-)equalities in variables $(\alpha, \beta, x) \in \mathbb{R}^{n+2}$.

Let $P \subseteq \mathbb{R}^{n+2}$ denote the polyhedron defined by (2.1) and let

$$\widehat{P} := \left\{ (\alpha, \beta) \mid \exists x : (\alpha, \beta, x) \text{ solves } (2.1) \right\} \subseteq \mathbb{R}^2$$

denote its projection into $\mathbb{R}^2$.

With $f(\alpha, \beta) := \alpha\beta$, our problem can thus be restated as

$$z^* = \min_{(\alpha,\beta) \in \widehat{P}} f(\alpha, \beta). \tag{2.2}$$

Note that $\widehat{P}$, being a projection of P, may have exponentially many describing inequalities. Yet we can clearly solve linear optimization problems over $\widehat{P}$, as these reduce to LP's over P. Indeed, for $c \in \mathbb{R}^2$, we have

$$\min_{\binom{\alpha}{\beta} \in \widehat{P}} c^{\mathrm{T}} \binom{\alpha}{\beta} \quad \widehat{=} \quad \min_{\begin{pmatrix}\alpha\\ \beta\\ x\end{pmatrix} \in P} (c^{\mathrm{T}}, 0^{\mathrm{T}}) \begin{pmatrix}\alpha\\ \beta\\ x\end{pmatrix}. \tag{2.3}$$

Equivalently (cf., e.g., [10]), we can efficiently solve the separation problem for $\widehat{P}$. As a consequence of this, we may apply the ellipsoid method to check whether $\widehat{P}$ is full-dimensional or not and, – in case it is not, – determine the (possibly infinite) line segment that equals $\widehat{P}$. Thus, in case $\widehat{P}$ is not full-dimensional, (2.2) reduces to a 1-dimensional quadratic problem, which is readily solved by elementary calculus.

In what follows, we therefore assume throughout that $\widehat{P}$ is full-dimensional. Next we can check (by means of linear programming) which of the four quadrants in $\mathbb{R}^2$ is (properly) intersected by $\widehat{P}$. This allows us to distinguish between the following three cases

$$z^* = 0, \quad z^* < 0 \quad \text{and} \quad z^* > 0,$$

which we treat separately.

$z^* = 0$. This is tantamount to $\widehat{P} \subseteq \mathbb{R}^2_+$ or $\widehat{P} \subseteq \mathbb{R}^2_-$ and $\widehat{P}$ intersecting (touching) one of the coordinate axes. Assume, say, that $\widehat{P} \subseteq \mathbb{R}^2_+$ and

$$\min_{\binom{\alpha}{\beta} \in \widehat{P}} \beta = 0.$$

In this case an optimal solution x^* of (2.2) is obtained by solving the second problem in (2.3) with $c^{\mathrm{T}} = (0, 1)$.

$z^* < 0$, i.e., $\widehat{P}$ contains some $(\alpha, \beta) \in \widehat{P}$ with $\alpha\beta < 0$.

Let $\widehat{P}^{\pm} := \left\{ \binom{\alpha}{\beta} \in \widehat{P} \mid \alpha \leq 0, \beta \geq 0 \right\}$ and $\widehat{P}^{\mp} := \left\{ \binom{\alpha}{\beta} \in \widehat{P} \mid \alpha \geq 0, \beta \leq 0 \right\}$. Then (2.2) basically reduces to two separate “convex” problems on $P^{\pm}$ resp. $P^{\mp}$. Indeed, for $z < 0$ let

$$L_z := \left\{ \binom{\alpha}{\beta} \mid \alpha < 0, \beta > 0, \alpha\beta \leq z \right\}$$

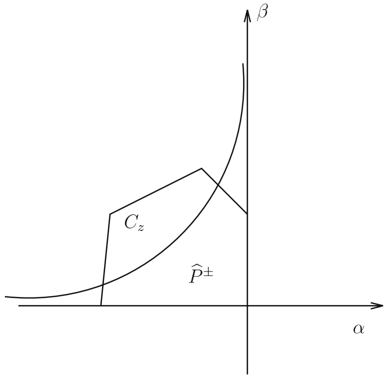

Fig. 1 L_z and C_z

and $C_z := L_z \cap \widehat{P}^{\pm}$ (cf. Fig. 1). Clearly,

$$\min_{\binom{\alpha}{\beta} \in \widehat{P}^{\pm}} f(\alpha, \beta) = \min\{z | C_z \neq \emptyset\} \tag{2.4}$$

holds. Now C_z is a convex set for $z < 0$ and it is straightforward to design a separation algorithm for C_z (cf., e.g., [3], Sect. 10.6). It is then routine work to verify that we may use the ellipsoid algorithm to determine (exactly) the optimum value $z^*_{\pm}$ in (2.4): Note that, due to the KKT-conditions, the optimum is achieved in a rational point $x^* = (\alpha^*, \beta^*)$ of polynomially bounded size, say, $size(x^*) \leq p$. So x^* can be computed exactly by rounding a sufficiently good approximation $x \in C_z$. Such a (sufficiently small) set $C_z \neq \emptyset$ is obtained by binary search for z^*.

Applying the same arguments to $\widehat{P}^{\mp}$ (in case this is non-empty), we obtain a corresponding $z^*_{\mp}$ and observe that $z^* = \min\left\{z^*_{\pm}, z^*_{\mp}\right\}$ solves (2.2).

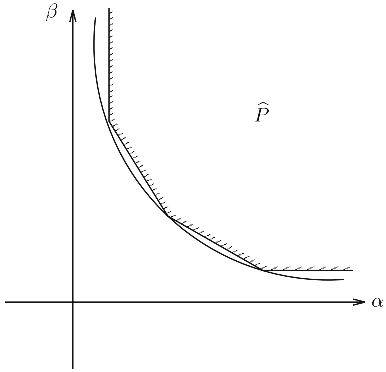

Fig. 2 $\widehat{P} \subseteq \mathbb{R}^2_+$ and the level curve l_z

$z^* > 0$. This case occurs when $\widehat{P} \subseteq \mathbb{R}^2_+$ (or $\widehat{P} \subseteq \mathbb{R}^2_-$) and $\widehat{P}$ does not touch any coordinate axes. This case may be considered as "essentially concave", as several local minima may exist (cf. Fig. 2). In what follows we assume w.l.o.g. that $\widehat{P} \subseteq \mathbb{R}^2_+$.

Lemma 2.1 *The minimum in* (2.2) *is achieved at a vertex of* $\widehat{P}$.

Proof This is an immediate consequence of the fact that $f(\alpha,\beta) = \alpha\beta$ is *quasi-concave* on $\mathbb{R}^2_+$ (cf. [2]), i.e., for any $x_1, x_2 \in \mathbb{R}^2_+$, f achieves its minimum on the line segment $[x_1, x_2]$ in one of the endpoints:

$$f(\lambda x_1 + (1-\lambda)x_2) \geq \min\{f(x_1), f(x_2)\}, \quad \lambda \in [0,1]. \tag{2.5}$$

$\square$

It is well-known that the vertices of a polyhedron have components with size polynomially bounded in the size of the describing system of inequalities. Thus the vertices of P and, the more, the vertices (α,β) of $\widehat{P}$ have size at most p with p polynomially bounded in the size of the problem instance (1.1). In particular, we conclude that

$$\underline{z} := 2^{-2p} \leq z^* \leq 2^{2p} =: \overline{z} \tag{2.6}$$

holds.

We seek to determine the value z^* approximately by binary search. Given $\underline{z} < z < \overline{z}$ we check whether $z^* < z$ holds (approximately) or not by approximating the level curve

$$\ell_z = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mid \alpha\beta = z, \alpha, \beta \geq 0 \right\}$$

by finitely many tangent lines at the points $(\alpha_k, \beta_k) = (\sqrt{z}(1+\varepsilon)^k, \sqrt{z}(1+\varepsilon)^{-k})$, $k = 0, \pm 1, \ldots, \pm K$, where $K > 0$ is chosen so that $\alpha_K > 2^p$ (hence K is polynomially bounded in $1/\varepsilon$ and the size of (1.1)).

More precisely, to determine whether $z^* < z$ holds approximately, we solve polynomially (in $1/\varepsilon$ and the size of (1.1)) many linear optimization problems

$$z_k = \min_{x \in \widehat{P}} (\beta_k, \alpha_k)x.$$

(Note that (β_k, α_k) is the gradient of $f(\alpha,\beta) = \alpha\beta$ in (α_k, β_k).)

Lemma 2.2 *If $z_k \leq 2z$ for some k, $|k| \leq K$, then $z^* \leq z$. If $z_k > 2z$ for all k, $|k| \leq K$, then $z \leq (1+\varepsilon)z^*$.*

Proof The first claim is obvious.

As to the second claim, assume $z_k > 2z$ for all k. Let $z^* = \alpha\beta$, $(\alpha,\beta) \in \widehat{P}$ and assume w.l.o.g. that $\alpha \geq \beta$. Let k be the smallest k such that $\alpha \leq \alpha_k$. If $k \leq 0$, then $\beta \leq \alpha \leq \alpha_0 = \beta_0$ implies

$$z_0 \leq \beta_0\alpha + \alpha_0\beta \leq 2\alpha_0\beta_0 = 2z,$$

a contradiction. Hence $k \geq 1$. By assumption, we have

$$2z < z_k \leq \beta_k\alpha + \alpha_k\beta \leq \beta_k\alpha_k + \alpha_k\beta = z + \alpha_k\beta.$$

Hence $\alpha_k\beta > z$, i.e., $\beta > \beta_k$. But then

$$\alpha\beta > \alpha_{k-1}\beta_k = (1+\varepsilon)^{-1}z,$$

as claimed. □

This enables us to perform a binary search for z^* on $[\underline{z}, \overline{z}]$, solving (2.2) approximately in time polynomially bounded in $1/\varepsilon$ and the size of (1.1): Start with $\underline{z} := 2^{-p}$ and $\overline{z} := 2^p$, where p bounds the size of the vertices of $\widehat{P}$. We then let $z := (\underline{z} + \overline{z})/2$ and apply Lemma 2.2. In case $z \leq (1+\varepsilon)z^*$, we replace $\underline{z}$ by $(1+\varepsilon)^{-1}z$. Else we replace $\overline{z}$ by z. Note that in the latter case the computation of z_k provides us with some $(\alpha, \beta) \in \widehat{P}$ such that $\alpha\beta \leq z$. The equivalent (in the sense of (2.3)) optimization problem over P provides us with a point $(\alpha, \beta, x) \in P$ satisfying $\alpha\beta \leq z$. So with each update of $\overline{z}$ we get some x satisfying $Ax \leq d$ and $(a^{\mathrm{T}}x+\gamma)(b^{\mathrm{T}}x+\delta) \in [\underline{z}, \overline{z}]$. Eventually, i.e., when $\underline{z} \geq (1+\varepsilon)^{-1}z$ or, equivalently, $\overline{z} \leq (1+\varepsilon)\underline{z}$, we will thus have exhibited a good approximate solution.

3 Minimum weight product problems

Every combinatorial minimum weight problem

$$\min \left\{ c^{\mathrm{T}}x \mid x \in D \right\}, \tag{3.1}$$

where $D \subseteq \{0,1\}^n$ has a corresponding *minimum ratio* version, where the objective $c^{\mathrm{T}}x$ is replaced by a quotient $p^{\mathrm{T}}x/q^{\mathrm{T}}x$ with $q > 0$. Probably the best known example is the so-called "tramp steamer problem", where $D \in \{0,1\}^E$ is the set of directed circuits through a given node in a digraph $G = (V, E)$ (cf., e.g., [1]). Typically such minimum ratio problems seek to model *multicriteria objective functions* (e.g., "maximize profit versus time"). Such minimum ratio versions are well-studied in the literature and it is known since long [8] that the minimum ratio version is (modulo polynomial time computation) at most as difficult as the original minimum weight problem.

In the context of multicriteria objectives it is often equally natural to consider other combinations of weights such as, e.g., *product versions* with objective $(a^{\mathrm{T}}x)(b^{\mathrm{T}}x)$. For example, if $D \subseteq \{0,1\}^E$ is the set of $s-t$ paths in a graph, then $a \in \mathbb{R}_+^E$ may define failure probabilities and $b \in \mathbb{R}_+^E$ may define edge costs [7]. In contrast to minimum ratio problems, however, such product versions of minimum weight problems appear to be more difficult in general.

Our FPTAS from Sect. 2 can be used to approximately solve minimum weight product problems in case $D \subseteq \{0,1\}^n$ is the vertex set of a polyhedron $Ax \leq d$ and we are able to solve (3.1) efficiently. Thus, for example, our result applies when D is the set of $s - t$ paths, spanning trees or perfect matchings in a graph. (Note that our arguments in Sect. 2 rely only on the assumption that we can efficiently optimize a linear objective over $Ax \leq d$.)

For simplicity, we restrict our discussion to minimum weight product $s - t$ paths as a generic example. Consider a directed graph $G = (V, E)$ with two given edge weights $a, b : E \to \mathbb{R}_+$ and assume we are to find an $s - t$ path p minimizing the product $a(p)b(p)$ of its edge weights relative to a and b. We first seemingly relax our problem, replacing the path p by an $s - t$ flow of value 1. Let $A \in \mathbb{R}^{n \times m}$ denote the node-arc incidence matrix of G and let $d \in \mathbb{R}^n$ have coordinates $d_s = 1$, $d_t = -1$ and $d_v = 0$ else. Then our relaxation can be written as

$$\begin{aligned} z^* = \min(a^{\mathrm{T}}x)(b^{\mathrm{T}}x) \\ Ax = d \\ x \geq 0. \end{aligned} \tag{3.2}$$

Clearly $z^* \geq 0$ holds. Furthermore, $z^* = 0$ holds only in the trivial case where an $s - t$ path p with $a(p) = 0$ or $b(p) = 0$ exists. Hence we may assume $z^* > 0$. In this case, the minimum in (3.2) is achieved at a vertex of the feasible region (due to Lemma 2.1), which corresponds to an $s - t$ path. So (3.2) is an exact restatement of our original problem.

As our FPTAS from Sect. 2 obtains the ε-approximate solution x of (3.2) via linear programming, we may assume w.l.o.g. that x is an $s - t$ path. (Alternatively, decompose $x = \lambda_1 x_1 + \cdots + \lambda_k x_k$ into a convex combination of $s - t$ paths x_i and use (2.5) to exhibit one of the $s - t$ paths x_i as an approximately optimal solution.)

We like to remark that a similar approach also works for slightly different objective functions like, e.g., $\widetilde{f}(x) = (a^{\mathrm{T}}x)\sqrt{b^{\mathrm{T}}x}$. All we need is that the level curves of $\widetilde{f}$ can be nicely approximated by piecewise linear functions.

We conclude our discussion by commenting on the complexity of the (exact) problem (3.2). As pointed out by one of the referees, the corresponding maximization problem is NP-hard (contains Longest Path as a special case). The complexity status of the minimization problem (3.2) is open and only two special cases are known to be efficiently solvable: $a = b$ (trivial) and $a = 1$ [7]. The latter also follows from our approach by observing that it suffices to approximate the level curves only in the points $\alpha_1 = 1, \ldots, \alpha_{n-1} = n - 1$, corresponding to the possible values $\alpha = a^{\mathrm{T}}x$ for an $s - t$ path $x \in \{0,1\}^E$.

Alternatively, the case $a = 1$ may also be settled directly by computing for each possible path length $k = 1, \ldots, n - 1$ the corresponding minimum b-weight b_k over all $s - t$ paths of length k (and observing that $z^* = \min_k kb_k$). The computation of b_k can be accomplished as follows. Let $V_0, \ldots, V_k$ be $k + 1$ copies of V and let G^k denote the directed graph on $V_0 \cup \cdots \cup V_k$ with arcs going from V_t

to V_{t+1}, joining vertices as in G. More precisely, the arc set E^k of G^k is given by

$$E^k = \left\{(i_t, j_{t+1}) \mid (i,j) \in E,\ 0 \leq t \leq k-1\right\},$$

where i_t is the copy of i in V_t.

The edge weights $b : E \to \mathbb{R}_+$ give rise to edge weights $b : E^k \to \mathbb{R}_+$ by setting $b_{i_t, j_{t+1}} = b_{ij}$. Now b_k, the minimum b-weight of an $s - t$ path of length k in G is simply the minimum b-weight of an $s_0 - t_k$ path in G^k.

For general $a, b : E \to \mathbb{R}_+$, (3.2) can be solved by computing all vertices of $\widehat{P} \subseteq \mathbb{R}^2$ that minimize a linear function $(\alpha, \beta) \to \alpha + \lambda\beta$, $\lambda \in \mathbb{R}_+$, over $\widehat{P}$. (Each local minimizer of $f(\alpha, \beta) = \alpha\beta$ over P must be such a vertex.) These vertices can be determined successively: let $\lambda > 0$ and consider the parametric minimum $s - t$ path problem with edge costs $c_\lambda = a + \lambda b$. For $\lambda > 0$ sufficiently small, a min cost $s - t$ path relative to cost c_λ will be an $s - t$ path x_0 that is minimal relative to a and, among all such a-minimal paths, has minimum b-weight. Standard sensitivity analysis then allows us to exhibit a largest interval $[\lambda_0 = 0, \lambda_1]$ such that x_0 is optimal relative to c_λ for each $\lambda \in [\lambda_0, \lambda_1]$. We then proceed by chosing $\lambda > \lambda_1$ sufficiently small and a min cost path x_1 relative to $c_\lambda = a + \lambda b$ to determine the next interval $[\lambda_1, \lambda_2]$ for which x_1 is optimal etc.

The running time of this procedure basically equals the number of *breakpoints* $\lambda_1, \lambda_2, \ldots$ in the parametric min cost $s - t$ path problem with parametrized cost function $c = a + \lambda b$, $\lambda \geq 0$. Gusfield [5] has shown that this number has a subexponential bound $O(n^{\log n})$. For this reason, we do not expect (3.2) to be NP-hard.

To make our presentation selfcontained we briefly sketch the argument from [5]. Let B_n denote the number of breakpoints in the parametric min cost path problem. Furthermore, we let B_n^k denote the number of breakpoints if only paths of length k are allowed, i.e., if we replace G by G^k as defined above.

For fixed k we estimate B_n^k as follows. Fix a node r in the middle layer $V_{\lfloor k/2 \rfloor}$ of G^k and let $B_n^k(r)$ denote the number of breakpoints if only $s - t$ paths through r are allowed. As λ varies, the costs of $s - r$ and $r - t$ paths in G^k vary independently. So we can conclude that

$$B_n^k(r) \leq 2B_n^{\lfloor k/2 \rfloor}.$$

This proves

$$B_n^k \leq \sum_r B_n^k(r) \leq 2nB_n^{\lfloor k/2 \rfloor}$$

and $B_n^k = O(n^{\log n})$ follows inductively. Hence also $B_n \leq \sum_k B_n^k = O(n^{\log n})$, as claimed.

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