

ALTERNATIVES FOR TESTING TOTAL DUAL INTEGRALITY

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ABSTRACT. In this paper we provide characterizing properties of TDI systems, among others the following: a system of linear inequalities is TDI if and only if its coefficient vectors form a Hilbert basis, and there exists a test-set for the system's dual integer programs where all test vectors have positive entries equal to 1. Reformulations of this provide relations between computational algebra and integer programming and they contain Applegate, Cook and McCormick's sufficient condition for the TDI property and Sturmfels' theorem relating toric initial ideals generated by square-free monomials to unimodular triangulations. We also study the theoretical and practical efficiency and limits of the characterizations of the TDI property presented here.

In the particular case of set packing polyhedra our results correspond to endowing the weak perfect graph theorem with an additional, computationally interesting, geometric feature: the normal fan of the stable set polytope of a perfect graph can be refined into a regular triangulation consisting only of unimodular cones.

1. INTRODUCTION

A restricted draft concerning an earlier stage of this research has been reported in the IPCO 2007 conference proceedings [28].

Let $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n] \in \mathbb{Z}^{d \times n}$ and assume that A has rank d . With an abuse of notation the ordered set of vectors consisting of the columns of A will also be denoted by A . For every $\sigma \subseteq [n] := \{1, \dots, n\}$ we have the $d \times |\sigma|$ matrix A_σ given by the columns of A indexed by σ . Let $\text{cone}(A)$, $\mathbb{Z}A$ and $\mathbb{N}A$ denote the non-negative real, integer and non-negative integer span of A respectively and assume that $\mathbb{Z}A = \mathbb{Z}^d$.

Fixing $\mathbf{c} \in \mathbb{R}^n$, for each $\mathbf{b} \in \mathbb{R}^d$ the *linear program* (or *primal program*) $\text{LP}_{A,\mathbf{c}}(\mathbf{b})$ and its *dual program* $\text{DP}_{A,\mathbf{c}}(\mathbf{b})$ are defined by

$$\text{LP}_{A,\mathbf{c}}(\mathbf{b}) := \text{minimize } \{ \mathbf{c} \cdot \mathbf{x} : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0} \}$$

and $\text{DP}_{A,\mathbf{c}}(\mathbf{b}) := \text{maximize } \{ \mathbf{y} \cdot \mathbf{b} : \mathbf{y}A \leq \mathbf{c} \}$. Let $P_{\mathbf{b}}$ and $Q_{\mathbf{c}}$ denote the feasible regions of $\text{LP}_{A,\mathbf{c}}(\mathbf{b})$ and $\text{DP}_{A,\mathbf{c}}(\mathbf{b})$ respectively. Note that the linear program $\text{LP}_{A,\mathbf{c}}(\mathbf{b})$ is feasible if and only if $\mathbf{b} \in \text{cone}(A)$. We refer to Schrijver [33] for basic terminology, facts and notations about linear programming.

The corresponding *integer program* is defined as

$$\text{IP}_{A,\mathbf{c}}(\mathbf{b}) := \text{minimize } \{ \mathbf{c} \cdot \mathbf{x} : A\mathbf{x} = \mathbf{b}, \mathbf{x} \in \mathbb{N}^n \}.$$

Date: November 12, 2018.

E.O'S. was supported by a Fulbright grant and by NSF grants DMS-9983797 and DMS-0401047. A.S. was supported by the "Marie Curie Training Network" ADONET of the European Community.

We say that $\mathbf{c} \in \mathbb{R}^n$ is *generic* for A if the integer program $\text{IP}_{A,\mathbf{c}}(\mathbf{b})$ has a unique optimal solution for all $\mathbf{b} \in \mathbb{N}A$. If \mathbf{c} is generic then $Q_{\mathbf{c}} \neq \emptyset$ and each linear program $\text{LP}_{A,\mathbf{c}}(\mathbf{b})$ also has a unique optimal solution for all $\mathbf{b} \in \text{cone}(A)$ but the converse is not true in general. However, the converse clearly holds for TDI systems: the system $\mathbf{y}A \leq \mathbf{c}$ is *totally dual integral* (TDI) if $\text{LP}_{A,\mathbf{c}}(\mathbf{b})$ has an integer optimal solution $\mathbf{x} \in \mathbb{N}^n$ for each $\mathbf{b} \in \text{cone}(A) \cap \mathbb{Z}^d$. In other words, the system $\mathbf{y}A \leq \mathbf{c}$ defining $Q_{\mathbf{c}} \neq \emptyset$ is TDI exactly if the optimal values of $\text{LP}_{A,\mathbf{c}}(\mathbf{b})$ and of $\text{IP}_{A,\mathbf{c}}(\mathbf{b})$ coincide for all $\mathbf{b} \in \text{cone}(A) \cap \mathbb{Z}^d$. (If $Q_{\mathbf{c}} \neq \emptyset$, LP and DP are both feasible and bounded.) This is a slight twist of notation when compared to habits in combinatorial optimization: we defined the TDI property for the dual problem, in order to be in accordance with notations in computational algebra.

Totally dual integral (TDI) systems of linear inequalities play a central role in combinatorial optimization. The recognition of TDI systems has been recently proved to be coNP-complete by Ding, Feng and Zang [4] and this result has been sharpened to the recognition of explicitly given systems with only 0 – 1 coefficient vectors, and where the defined polyhedron has exactly one vertex (Hilbert basis testing by Pap [30]). Graph theory results of Chudnovsky, Cornuéjols, Liu, Seymour and Vušković [9] allow one to recognize TDI systems with 0 – 1 coefficient matrices and right hand sides. However, solving the corresponding dual pair of integer linear programs (including the coloration of perfect graphs) in polynomial time with combinatorial algorithms remains open even in this special case. The fixed dimension case has been solved long ago [7], whereas the fixed codimension case only recently [14], [22], the first using a generalization of integer programming, the second computer algebra, both starting from a characterization of Hilbert bases in [35].

A particular case where the recognition of TDI systems is still open, is the case of generic systems (Problem 3.2), which is slightly more general than the perfectness test (detecting perfection in a graph), and could be a possible start for an alternative, simpler algorithm for the latter. This paper wishes to contribute to testing TDI in cases that occur in integer programming and combinatorial optimization, and which usually do not belong to the extremities that have been understood so far.

In Section 2, characterizing properties of TDI systems are provided. Some of these properties involve tools from combinatorial optimization, some others from computational algebra. Section 3 specializes these results to integral set packing polytopes. Finally, Section 4 exhibits the possible alternatives and their relative efficiency for recognizing TDI systems. The remainder of this introduction is devoted to explaining the main results and providing some of the necessary background.

A collection of subsets $\{\sigma_1, \dots, \sigma_t\}$ of $[n]$ will be called a *regular subdivision* of A if there exists $\mathbf{c} \in \mathbb{R}^n$, and $\mathbf{z}_1, \dots, \mathbf{z}_t \in \mathbb{R}^d$, such that $\mathbf{z}_i \cdot \mathbf{a}_j = c_j$ for all $j \in \sigma_i$ and $\mathbf{z}_i \cdot \mathbf{a}_j < c_j$ for all $j \notin \sigma_i$. The sets $\sigma_1, \dots, \sigma_t$ are called the *cells* of the regular subdivision and the regular subdivision is denoted by $\Delta_{\mathbf{c}}(A) = \{\sigma_1, \dots, \sigma_t\}$ or simply $\Delta_{\mathbf{c}}$ when A is unambiguous.

Equivalently, regular subdivisions are simply capturing *complementary slackness* from linear programming. Namely, a feasible solution to $\text{LP}_{A,\mathbf{c}}(\mathbf{b})$ is optimal if and only if the support of the feasible solution is a subset of some cell of $\Delta_{\mathbf{c}}$. Geometrically, $\Delta_{\mathbf{c}}$ can be thought of as a partition of $\text{cone}(A)$ by the inclusionwise maximal ones among the cones $\text{cone}(A_{\sigma_1}), \dots, \text{cone}(A_{\sigma_t})$; each such cone is generated by the normal vectors of defining

inequalities of faces of $Q_{\mathbf{c}}$, each maximal cell indexes the set of normal vectors of inequalities satisfied with equality by a vertex (or minimal face) of $Q_{\mathbf{c}}$. The $\mathbf{z}_1, \dots, \mathbf{z}_t$ above are these vertices and so the regular subdivision $\Delta_{\mathbf{c}}$ is geometrically realized as the *normal fan* of $Q_{\mathbf{c}}$.

A regular subdivision of A is called a *triangulation* if the columns of each A_{σ_i} are linearly independent for all $i = 1, \dots, t$. Note that a regular subdivision $\Delta_{\mathbf{c}}$ is a triangulation if and only if every vertex is contained in exactly d facets; that is, the polyhedron $Q_{\mathbf{c}}$ is *simple*, or, *non-degenerate*. A triangulation $\Delta_{\mathbf{c}}$ is called *unimodular* if $\det(A_{\sigma_i}) = \pm 1$ for each maximal cell of $\Delta_{\mathbf{c}}$. A *refinement* of a subdivision $\Delta_{\mathbf{c}}$ of A is another subdivision $\Delta_{\mathbf{c}'}$ of A so that each cell of $\Delta_{\mathbf{c}'}$ is contained in some cell of $\Delta_{\mathbf{c}}$. A set $B \subset \mathbb{Z}^d$ is a *Hilbert basis* if $\mathbb{N}B = \text{cone}(B) \cap \mathbb{Z}^d$; we will say that a matrix B is a Hilbert basis if its columns form a Hilbert basis. Note that if for some $\mathbf{c} \in \mathbb{R}^n$ $\Delta_{\mathbf{c}}$ is a unimodular triangulation of A then Cramer's rule implies that A itself is a Hilbert basis.

Let $\text{IP}_{A,\mathbf{c}} := \{\text{IP}_{A,\mathbf{c}}(\mathbf{b}) : \mathbf{b} \in \mathbb{N}A\}$ denote the family of integer programs $\text{IP}_{A,\mathbf{c}}(\mathbf{b})$ having a feasible solution. Informally, a *test-set* for the family of integer programs $\text{IP}_{A,\mathbf{c}}$ is a finite collection of integer vectors, called *test vectors*, with the property that any non-optimal feasible solution to any integer program in this class can be improved (in objective value) by subtracting a test vector from it. Test-sets for the family $\text{IP}_{A,\mathbf{c}}$ were first introduced by Graver [18]. Graver's test set, called a *Graver basis* in the literature, is typically not a minimal test set.

A simple but helpful characterization of the TDI property in terms of the Hilbert basis property of regular subdivisions has been provided by Schrijver [33]. We prove another elementary characterization in Section 2 whose simplified version is the following:

Theorem 2.5 *The system $\mathbf{y}A \leq \mathbf{c}$ is TDI if and only if A is a Hilbert basis, and there exists a test-set for $\text{IP}_{A,\mathbf{c}}$ with all test vectors having positive entries equal to 1 and this positive support indexes either a linearly independent set or a minimally linear dependent set of columns of A .*

Establishing this theorem, some of its corollaries (see Section 2) have led us to three relevant earlier results that have been found in contexts different from ours, independently of one another:

- Applegate, Cook & McCormick's result [1, Theorem 2] which states that if A is a Hilbert basis then $\mathbf{y}A \leq \mathbf{c}$ is TDI if $\text{LP}_{A,\mathbf{c}}(\mathbf{b})$ has an integer optimum for every constraint vector \mathbf{b} that is a sum of linearly independent columns of A . However, our test-sets and their use for optimizing on IP are new.
- Hoşten and Sturmfels result [21, Theorem 1.1] is stated in a computer algebra context, but the special case when the gap is 0 can be seen to be equivalent to the above statement of Theorem 2.5. However, it is still easier to prove each of these two independent results in their own terms, directly.
- Eisenbrand and Shmonin [13] proved Hoşten and Sturmfels result in an elementary way, yet in the context of a generalization of integer programming called “parametric integer programming”. However, this work also uses more involved machinery than we do here.

In addition, none of these works characterizes the TDI property in terms of test-sets or solving the underlying integer programs in fixed dimension without using Lenstra's algorithm. As previously noted in [1], this allows us to deduce shortly Cook, Lovász & Schrijver's result [7] on testing for the TDI property in fixed dimension, when given the conditions of Theorem 2.5.

Another virtue of Theorem 2.5 is a useful reformulation to polynomial ideals: it generalizes a well-known algebraic result proved by Sturmfels [38, Corollary 8.9] relating toric initial ideals to unimodular triangulations. The basic connections between integer programming and computational algebra, knowledge of which will not be assumed here, was initiated by Conti and Traverso [5] and further studied from various viewpoints in [37], [38], [39], [40].

If A is a matrix whose first $d \times (n - d)$ submatrix is a $0 - 1$ matrix and whose last $d \times d$ submatrix is $-I_d$, and \mathbf{c} is all 1 except for the last d coordinates which are 0, then $\text{DP}_{A,\mathbf{c}}(\mathbf{b})$ is called a *set packing problem*, and $Q_{\mathbf{c}}$ a *set packing polytope*. In Section 3 we show that the converse of the following fact (explained at the end of Section 2) holds for normal fans of integral set packing polytopes: *if $\mathbf{c}, \mathbf{c}' \in \mathbb{R}^n$ are such that $\Delta_{\mathbf{c}'}$ is a refinement of $\Delta_{\mathbf{c}}$, where $\Delta_{\mathbf{c}'}$ is a unimodular triangulation, then $\mathbf{y}A \leq \mathbf{c}$ is TDI*. In general, the converse does not hold and the most that is known in this direction is the existence of just one full dimensional subset of the columns of A which is unimodular [16]. Not even a “unimodular covering” of a Hilbert basis may be possible [2]. However, the converse does hold for normal fans of integral set packing polytopes. More precisely, the main result of Section 3 is the following:

Theorem 3.1 *Given a set-packing problem defined by A and \mathbf{c} , $Q_{\mathbf{c}}$ has integer vertices if and only if there exists \mathbf{c}' such that $\Delta_{\mathbf{c}'}$ is a refinement of the normal fan $\Delta_{\mathbf{c}}$ of $Q_{\mathbf{c}}$, where $\Delta_{\mathbf{c}'}$ is a unimodular triangulation.*

The proof relies on the basic idea of Fulkerson's famous “pluperfect graph theorem” [17] stating that the integrality of such polyhedra implies their total dual integrality in a very simple “greedy” way. Chandrasekaran and Tamir [3] and Cook, Fonlupt and Schrijver [6] exploited Fulkerson's method by pointing out its lexicographic or advantageous Carathéodory feature. In [35, §4] it is noticed with the same method that the active rows of the dual of integral set packing polyhedra (the cells of their normal fan) have a unimodular subdivision, which can be rephrased as follows: *the normal fan of integral set packing polyhedra has a unimodular refinement*. However, the proof of the regularity of such a refinement appears for the first time in the present work.

These results offer four methods for recognizing TDI systems, explained, illustrated and compared to previously known ones as well, in Section 4. Particular attention will be given to the recognition of integral set packing polytopes in practice, through Theorem 3.1 above, and by using computational algebra packages like `Macaulay 2`.

2. TDI SYSTEMS

In this section we provide some new characterizations of TDI systems. We show the equivalence of five properties, three polyhedral (one of them is the TDI property) and two concern polynomial ideals. A third property is also equivalent to these in the generic case.

While the proofs of the equivalences of the three polyhedral properties use merely polyhedral arguments, the last among them – (iii) – has an appealing reformulation into the language of polynomial ideals. Therefore, we start this section by introducing the necessary background on polynomial ideals; namely, toric ideals, their initial ideals and Gröbner bases. The characterizations of TDI systems involving polynomial ideals are useful generalizations of known results in computational algebra. See [11] and [38] for further background.

An *ideal* I in a polynomial ring $R := \mathbf{k}[x_1, \dots, x_n]$ is an R -vector subspace with the property that $I \cdot R = I$. It was proven by Hilbert that every ideal is finitely generated. That is, given an ideal I there exists a finite set of polynomials $f_1, \dots, f_t \in I$ such that for every $f \in I$ there exists $h_1, \dots, h_t \in R$ with $f = h_1 f_1 + \dots + h_t f_t$. We call such a collection $f_1, \dots, f_t \in I$ a *generating set* for the ideal I and denote this by $I = \langle f_1, \dots, f_t \rangle$. For the monomials in R we write $\mathbf{x}^{\mathbf{u}} = x_1^{u_1} \cdots x_n^{u_n}$ for the sake of brevity. We call \mathbf{u} the *exponent vector* of $\mathbf{x}^{\mathbf{u}}$. A monomial $\mathbf{x}^{\mathbf{u}}$ is said to be *square-free* if $\mathbf{u} \in \{0, 1\}^n$. An ideal is called a *monomial ideal* if it has a generating set consisting only of monomials. For any ideal J of R , $\text{mono}(J)$ denotes the monomial ideal in R generated by the set of monomials in J . An algorithm for computing the generators of the monomial ideal $\text{mono}(J)$ can be found in [32, Algorithm 4.4.2].

Every weight vector $\mathbf{c} \in \mathbb{R}^n$ induces a partial order \succeq on the monomials in R via $\mathbf{x}^{\mathbf{u}} \succeq \mathbf{x}^{\mathbf{v}}$ if $\mathbf{c} \cdot \mathbf{u} \geq \mathbf{c} \cdot \mathbf{v}$. If $\mathbf{c} \in \mathbb{R}^n$ where 1 is the monomial of minimum \mathbf{c} -cost (that is, $\mathbf{c} \cdot \mathbf{u} \geq 0$ for every monomial $\mathbf{x}^{\mathbf{u}}$), then we can define initial terms and initial ideals. Given a polynomial $f = \sum_{\mathbf{u} \in \mathbb{N}^n} r_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \in I$ the *initial term* of f with respect to \mathbf{c} , is denoted by $\text{in}_{\mathbf{c}}(f)$, and equals the sum of all $r_{\mathbf{u}} \mathbf{x}^{\mathbf{u}}$ of f , where $\mathbf{c} \cdot \mathbf{u}$ is maximum. Note that we can always write a polynomial f as $f = \text{in}_{\mathbf{c}}(f) + \text{trail}_{\mathbf{c}}(f)$ in the obvious way. The *initial ideal* of I with respect to \mathbf{c} is defined as the ideal in R generated by the initial terms of the polynomials in I : $\text{in}_{\mathbf{c}}(I) := \langle \text{in}_{\mathbf{c}}(f) : f \in I \rangle$. A *Gröbner basis* of an ideal I with respect to \mathbf{c} , is a finite collection of elements g_1, \dots, g_s in I such that $\text{in}_{\mathbf{c}}(I) = \langle \text{in}_{\mathbf{c}}(g_1), \text{in}_{\mathbf{c}}(g_2), \dots, \text{in}_{\mathbf{c}}(g_s) \rangle$. Every Gröbner basis is a generating set for the ideal I .

If $\text{in}_{\mathbf{c}}(I)$ is a monomial ideal then a Gröbner basis is *reduced* if for every $i \neq j$, no term of g_i is divisible by $\text{in}_{\mathbf{c}}(g_j)$. The reduced Gröbner basis is unique. In this case, the set of monomials in $\text{in}_{\mathbf{c}}(I)$ equal $\{\mathbf{x}^{\mathbf{u}} : \mathbf{u} \in U\}$ with $U := D + \mathbb{N}^n$ where D is the set of exponent vectors of the monomials $\text{in}_{\mathbf{c}}(g_1), \text{in}_{\mathbf{c}}(g_2), \dots, \text{in}_{\mathbf{c}}(g_s)$. Dickson's lemma states that sets of the form $D + \mathbb{N}^n$, where D is arbitrary have only a finite number of minimal elements (with respect to coordinate wise inequalities). This is an alternative proof to Hilbert's result that every polynomial ideal is finitely generated. In this case, the Gröbner basis also provides a generalization of the Euclidean algorithm for polynomial rings with two or more variables called Buchberger's algorithm (see [11, Chapter 2, §7]). This algorithm solves the *ideal membership problem*: decide if a given polynomial is in an ideal or not. However, a Gröbner basis for an ideal can have many elements (compared to a minimal generating set for the ideal), and none of the related computations can be achieved in polynomial time.

If $\text{in}_{\mathbf{c}}(I)$ is not a monomial ideal then we can form a monomial initial ideal, $\text{in}_{\succ_{\mathbf{c}}}(I)$, as follows: fix an arbitrary *term order* independent of \mathbf{c} , (that is, a total ordering \succ of the vectors in \mathbb{N}^n satisfying $\mathbf{u} \succ \mathbf{0}$ for every $\mathbf{u} \in \mathbb{N}^n$, and if $\mathbf{u} \succ \mathbf{v}$ then $\mathbf{u} + \gamma \succ \mathbf{v} + \gamma$ for all $\gamma \in \mathbb{N}^n$). We use this term order to break ties: $\mathbf{x}^{\mathbf{u}} \succ_{\mathbf{c}} \mathbf{x}^{\mathbf{v}}$ if and only if $\mathbf{c} \cdot \mathbf{u} > \mathbf{c} \cdot \mathbf{v}$, or

$\mathbf{c} \cdot \mathbf{u} = \mathbf{c} \cdot \mathbf{v}$ and $\mathbf{u} \succ \mathbf{v}$. Clearly, “ $\succ_{\mathbf{c}}$ ” is also a term order and so $\text{in}_{\succ_{\mathbf{c}}}(I)$ is a monomial ideal. Such a tie-breaking will be needed in the proof of Lemma 2.2.

The *toric ideal* of A is the ideal $I_A = \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} : A\mathbf{u} = A\mathbf{v}, \mathbf{u}, \mathbf{v} \in \mathbb{N}^n \rangle$ and is called a *binomial ideal* since it is generated by polynomials having at most terms. Every reduced Gröbner basis of a toric ideal consists of binomials. A *toric initial ideal* is any initial ideal of a toric ideal.

Remark 2.1. It follows for (a not necessarily generic) \mathbf{c} , that the reduced Gröbner basis of a toric ideal I_A (with respect to $\succ_{\mathbf{c}}$) is of the form $\mathcal{G}_{\succ_{\mathbf{c}}} = \{\mathbf{x}^{\mathbf{u}_i^+} - \mathbf{x}^{\mathbf{u}_i^-} : i = 1, \dots, t\}$ and we can suppose that $\text{in}_{\mathbf{c}}(\mathbf{x}^{\mathbf{u}_i^+} - \mathbf{x}^{\mathbf{u}_i^-}) = \mathbf{x}^{\mathbf{u}_i^+}$ for $i = 1, \dots, s$ and $\text{in}_{\mathbf{c}}(\mathbf{x}^{\mathbf{u}_i^+} - \mathbf{x}^{\mathbf{u}_i^-}) = \mathbf{x}^{\mathbf{u}_i^+} - \mathbf{x}^{\mathbf{u}_i^-}$ for $i = s+1, \dots, t$. Furthermore, the set of polynomials $\mathcal{S}_{\succ} := \{\mathbf{x}^{\mathbf{u}_1^+}, \dots, \mathbf{x}^{\mathbf{u}_s^+}, \mathbf{x}^{\mathbf{u}_{s+1}^+} - \mathbf{x}^{\mathbf{u}_{s+1}^-}, \dots, \mathbf{x}^{\mathbf{u}_t^+} - \mathbf{x}^{\mathbf{u}_t^-}\}$ is a reduced Gröbner basis for $\text{in}_{\mathbf{c}}(I_A)$ with respect to the term order \succ cf. [38, Corollary 1.9]. Note that if \mathbf{c} were generic then $s = t$ and \mathcal{S}_{\succ} is simply the minimal generating set for $\text{in}_{\mathbf{c}}(I_A)$.

The following lemma is a natural connection between integer programming and toric initial ideals. It originally appeared in [32, Lemma 4.4.7] but we prove it here in order for this article to be self-contained.

Lemma 2.2. *For $A \in \mathbb{Z}^{d \times n}$ and $\mathbf{c} \in \mathbb{R}^n$ the monomial ideal $\text{mono}(\text{in}_{\mathbf{c}}(I_A))$ is equal to*

$$\langle \mathbf{x}^{\omega} : \omega \in \mathbb{N}^n \text{ is a non-optimal solution for } \text{IP}_{A,\mathbf{c}}(A\omega) \rangle.$$

Proof. It is straightforward to show that $\text{mono}(\text{in}_{\mathbf{c}}(I_A))$ contains the defined set: let ω be a non-optimal solution, and ω' an optimal solution to $\text{IP}_{A,\mathbf{c}}(A\omega)$. Then $\mathbf{x}^{\omega} - \mathbf{x}^{\omega'} \in I_A$ is a binomial having the monomial $\mathbf{x}^{\omega} \in \text{mono}(\text{in}_{\mathbf{c}}(I_A))$ as its initial term with respect to \mathbf{c} .

Suppose now $\mathbf{x}^{\omega} \in \text{in}_{\mathbf{c}}(I_A)$. That is, there exists a polynomial $f \in I_A$ with $\text{in}_{\mathbf{c}}(f) = \mathbf{x}^{\omega}$. We will prove that ω is a non-optimal solution for $\text{IP}_{A,\mathbf{c}}(A\omega)$.

Let \succ be an arbitrary term order and let \mathcal{S}_{\succ} be the reduced Gröbner basis of $\text{in}_{\mathbf{c}}(I_A)$ with respect to \succ as in Remark 2.1. We proceed by induction with respect to the minimum number of successive polynomial divisions of f by the elements of \mathcal{S}_{\succ} , and replacing f by the remainder after each division, until arriving at a 0 remainder. As noted in Remark 2.1 $\mathcal{S}_{\succ} = \{\mathbf{x}^{\mathbf{u}_1^+}, \dots, \mathbf{x}^{\mathbf{u}_s^+}, \mathbf{x}^{\mathbf{u}_{s+1}^+} - \mathbf{x}^{\mathbf{u}_{s+1}^-}, \dots, \mathbf{x}^{\mathbf{u}_t^+} - \mathbf{x}^{\mathbf{u}_t^-}\}$.

If first we divide f with a monomial in \mathcal{S}_{\succ} , say $\mathbf{x}^{\mathbf{u}_1^+}$, then $\mathbf{c} \cdot \mathbf{u}_1^+ > \mathbf{c} \cdot \mathbf{u}_1^-$, and $\text{in}_{\mathbf{c}}(f) = \mathbf{x}^{\omega}$ is divisible by $\mathbf{x}^{\mathbf{u}_1^+}$, so $\omega - \mathbf{u}_1^+ \geq \mathbf{0}$, $\mathbf{c} \cdot (\omega - (\mathbf{u}_1^+ - \mathbf{u}_1^-)) < \mathbf{c} \cdot \omega$, where $A(\omega - (\mathbf{u}_1^+ - \mathbf{u}_1^-)) = A\omega$. Therefore ω is also a non-optimal solution to $\text{IP}_{A,\mathbf{c}}(A\omega)$.

Otherwise, the first polynomial division is by a binomial in \mathcal{S}_{\succ} , say, $\mathbf{x}^{\mathbf{u}_t^+} - \mathbf{x}^{\mathbf{u}_t^-}$. Now recall that $f \in I_A$ is a polynomial with $\text{in}_{\mathbf{c}}(f) = \text{in}_{\succ_{\mathbf{c}}}(f)$ and so the resulting polynomial after the division can be written as $f' := \frac{\mathbf{x}^{\omega}}{\mathbf{x}^{\mathbf{u}_t^+}} \mathbf{x}^{\mathbf{u}_t^-} - \text{trail}_{\mathbf{c}}(f)$. The initial term $\text{in}_{\mathbf{c}}(f')$ is a monomial and equals $\frac{\mathbf{x}^{\omega}}{\mathbf{x}^{\mathbf{u}_t^+}} \mathbf{x}^{\mathbf{u}_t^-}$ since $\mathbf{c} \cdot (\omega - (\mathbf{u}_t^+ - \mathbf{u}_t^-))$ equals $\mathbf{c} \cdot \omega$, due to $\mathbf{c} \cdot \mathbf{u}_t^+ = \mathbf{c} \cdot \mathbf{u}_t^-$.

This implies that $\text{in}_{\mathbf{c}}(\text{trail}_{\mathbf{c}}(f))$ is strictly cheaper (with respect to \mathbf{c}) than $\frac{\mathbf{x}^{\omega}}{\mathbf{x}^{\mathbf{u}_t^+}} \mathbf{x}^{\mathbf{u}_t^-}$ and so $\frac{\mathbf{x}^{\omega}}{\mathbf{x}^{\mathbf{u}_t^+}} \mathbf{x}^{\mathbf{u}_t^-} \in \text{in}_{\mathbf{c}}(I_A)$. Since f' requires one less division than f by \mathcal{S}_{\succ} to arrive at 0

remainder then, by induction, we have that $\omega - (\mathbf{u}_t^+ - \mathbf{u}_t^-)$ is not an optimal solution to $\text{IP}_{A,\mathbf{c}}(A(\omega - (\mathbf{u}_t^+ - \mathbf{u}_t^-)))$ and so ω is not an optimal solution to $\text{IP}_{A,\mathbf{c}}(A\omega)$ either. \square

Remark 2.3. $\sigma \subseteq [n]$ is contained in a cell of $\Delta_{\mathbf{c}}$ if and only if $\sum_{i \in \sigma} \mathbf{e}_i \in \mathbb{R}^n$ is an optimal solution to $\text{LP}_{A,\mathbf{c}}(\mathbf{b}_\sigma)$ where $\mathbf{b}_\sigma := A(\sum_{i \in \sigma} \mathbf{e}_i)$. This happens in particular if $\sigma = \emptyset$. Indeed, then $\sum_{i \in \sigma} \mathbf{e}_i = \mathbf{0}$, and $\mathbf{b}_\sigma := A(\sum_{i \in \sigma} \mathbf{e}_i) = \mathbf{0}$. For this $\mathbf{b} = \mathbf{0} \in \mathbb{R}^d$, $\mathbf{0} \in \mathbb{R}^n$ is an optimal solution, otherwise $\text{LP}_{A,\mathbf{c}}(\mathbf{b})$ is not bounded for any \mathbf{b} , and $Q_{\mathbf{c}} = \emptyset$. We are not interested in such polyhedra, and we will avoid this situation by supposing $Q_{\mathbf{c}} \neq \emptyset$.

A *test-set* [18] for the family of integer programs $\text{IP}_{A,\mathbf{c}} := \{\text{IP}_{A,\mathbf{c}}(\mathbf{b}) : \mathbf{b} \in \mathbb{N}A\}$ is a collection of integer vectors $\{\mathbf{v}_i^+ - \mathbf{v}_i^- : A\mathbf{v}_i^+ = A\mathbf{v}_i^-, \mathbf{c} \cdot \mathbf{v}_i^+ > \mathbf{c} \cdot \mathbf{v}_i^-, \mathbf{v}_i^+, \mathbf{v}_i^- \in \mathbb{N}^n, i = 1, \dots, s\}$ with the property that for all feasible, non-optimal solution \mathbf{u} to $\text{IP}_{A,\mathbf{c}}(\mathbf{b})$ there exists an i , $1 \leq i \leq s$, such that $\mathbf{u} - (\mathbf{v}_i^+ - \mathbf{v}_i^-) \geq \mathbf{0}$. We can now state and prove our characterizations of TDI.

Example 2.4. Let us suppose $Q_{\mathbf{c}} \neq \emptyset$, that is, $\mathbf{0} \in \mathbb{R}^n$ is an optimal solution for $\text{LP}_{A,\mathbf{c}}(\mathbf{0})$, and show a particular test-set related to Remark 2.3. We will say that $\kappa \subseteq [n]$ is a *wheel* (with respect to A and \mathbf{c}), if $A(\sum_{i \in \kappa} \mathbf{e}_i) = \mathbf{0}$, $\sum_{i \in \kappa} \mathbf{c}_i > 0$ and for all $i \in \kappa$, $\kappa \setminus \{i\}$ is a subset of a cell (of $\Delta_{\mathbf{c}}(A)$).

For instance if $A = (\mathbf{a}_1, \dots, \mathbf{a}_d, -(\mathbf{a}_1 + \dots + \mathbf{a}_d))$, where $\mathbf{a}_1, \dots, \mathbf{a}_d$ are linearly independent integer vectors, and $\mathbf{c} \geq 0$, then $\Delta_{\mathbf{c}}(A)$ is a triangulation whose maximal cells are precisely $\{[d+1] \setminus \{i\} : i = 1, 2, \dots, d+1\}$. In this case the one-element set $\{(1, \dots, 1) \in \mathbb{R}^{d+1}\}$ is a test-set for $\text{IP}_{A,\mathbf{c}}$. Note that if $\text{cone}(A)$ is pointed then A has no wheel, regardless of the \mathbf{c} in question.

Theorem 2.5. Fix $A \in \mathbb{Z}^{d \times n}$, where A is a Hilbert basis, $\mathbf{c} \in \mathbb{R}^n$, and $Q_{\mathbf{c}} \neq \emptyset$. The following statements are equivalent:

- (i) The system $\mathbf{y}A \leq \mathbf{c}$ is TDI.
- (ii) The subconfiguration A_σ of A is a Hilbert basis for every cell σ in $\Delta_{\mathbf{c}}$.
- (iii) There exists a test-set for $\text{IP}_{A,\mathbf{c}}$ where all the positive coordinates are equal to 1, and each positive support is either the incidence vector of a linearly independent set of columns, or of a wheel; in the former case the negative support is a subset of a cell, in the latter case it is \emptyset .
- (iv) The monomial ideal $\langle \mathbf{x}^\omega : \omega \in \mathbb{N}^n \text{ is not an optimal solution for } \text{IP}_{A,\mathbf{c}}(A\omega) \rangle$ has a square-free generating set.
- (v) The monomial ideal generated by the set of monomials in $\text{in}_{\mathbf{c}}(I_A)$, that is, $\text{mono}(\text{in}_{\mathbf{c}}(I_A))$ has a square-free generating set.

The main content of the equivalence of (ii) and (v) is that the ‘‘Hilbert basis property’’ is equivalent to the existence of a ‘‘square-free generating set’’, extending a well-known result [38, Corollary 8.9] (see Corollary 2.7 below) to the non-generic case. We have recently found a similar extension in Hoşten and Sturmfels [21], equivalent to ours. (The latter paper provides an algorithm to compute the integer programming gap $\text{gap}_{A,\mathbf{c}} := \max\{\text{OPTIP}_{A,\mathbf{c}}(\mathbf{b}) - \text{OPTLP}_{A,\mathbf{c}}(\mathbf{b}) : \mathbf{b} \in \mathbb{N}A\}$, where OPTIP and OPTLP mean the optimal value of the corresponding programs. Note that the system $\mathbf{y}A \leq \mathbf{c}$ being TDI is equivalent to A being a Hilbert basis and $\text{gap}_{A,\mathbf{c}} = 0$. Hoşten and Sturmfels compute

$\text{gap}_{A,\mathbf{c}}$ by studying the “primary decomposition” of the monomial ideal $\text{mono}(\text{in}_{\mathbf{c}}(I_A))$. We present below our direct elementary argument.)

Note also that (i) implies that A is a Hilbert basis, since this latter is equivalent to the following property: if $\text{LP}_{A,\mathbf{c}}(\mathbf{b})$ has a feasible solution, then it also has an integer feasible solution; (ii) also implies it, since a cone that has a subdivision to Hilbert cones is itself a Hilbert basis. So this condition has to be added only to (iii), (iv) and (v); they may all be satisfied without A being a Hilbert basis, certifying then that $\mathbf{y}A \leq \mathbf{c}$ is not TDI, unless $Q_{\mathbf{c}} = \emptyset$. The condition $Q_{\mathbf{c}} \neq \emptyset$ makes sure that $\text{LP}_{A,\mathbf{c}}(\mathbf{b})$ is bounded, otherwise already the statement of (iii), (iv), (v) is not well defined.

Proof. (i) \Rightarrow (ii) : This is well-known from Schrijver’s work, (see for instance [33]), but we provide the (very simple) proof here for the sake of completeness: Suppose the system $\mathbf{y}A \leq \mathbf{c}$ is TDI, and let $\sigma \in \Delta_{\mathbf{c}}$. We show that A_{σ} is a Hilbert basis. Let $\mathbf{b} \in \text{cone}(A_{\sigma})$. Since the optimal solutions for $\text{LP}_{A,\mathbf{c}}(\mathbf{b})$ are exactly the non-negative combinations of the columns of A_{σ} with result \mathbf{b} , the TDI property means exactly that \mathbf{b} can also be written as a non-negative integer combination of columns in A_{σ} , as claimed.

(ii) \Rightarrow (iii) : Suppose (ii) holds true for $\Delta_{\mathbf{c}}$ of A . For every $\tau \subseteq [n]$ with τ not contained in any cell of $\Delta_{\mathbf{c}}$, let $\mathbf{b}_{\tau} := \sum_{i \in \tau} \mathbf{a}_i = A(\sum_{i \in \tau} \mathbf{e}_i)$. Since τ is not contained in any cell of $\Delta_{\mathbf{c}}$, there exists an optimal solution β_{τ} to $\text{LP}_{A,\mathbf{c}}(\mathbf{b}_{\tau})$ with $\mathbf{c} \cdot \beta_{\tau} < \mathbf{c} \cdot \sum_{i \in \tau} \mathbf{e}_i$. By the optimality of β_{τ} we must have $\text{supp}(\beta_{\tau}) \subseteq \sigma$ for some cell σ of $\Delta_{\mathbf{c}}(A)$.

If $\mathbf{b}_{\tau} \neq \mathbf{0}$, then $\sigma \neq \emptyset$, and by (ii) A_{σ} is a Hilbert basis. Therefore β_{τ} can be chosen to be an integral vector. On the other hand, if $\mathbf{b}_{\tau} = \mathbf{0}$, then by the condition $Q_{\mathbf{c}} \neq \emptyset$, $\mathbf{0}$ is an optimal solution for $\text{LP}_{A,\mathbf{c}}(\mathbf{b}_{\tau})$ (see Remark 2.3). Let

$$\begin{aligned} \mathcal{T}_{A,\mathbf{c}} := & \left\{ \sum_{i \in \tau} \mathbf{e}_i - \beta_{\tau} : A_{\tau} \text{ is linearly independent and } \tau \text{ is not contained in any cell of } \Delta_{\mathbf{c}} \right\} \\ & \bigcup \left\{ \sum_{i \in \kappa} \mathbf{e}_i : \kappa \text{ is a wheel (with respect to } A \text{ and } \mathbf{c}) \right\}. \end{aligned}$$

We claim that $\mathcal{T}_{A,\mathbf{c}}$ is a test-set for $\text{IP}_{A,\mathbf{c}}$. By construction every $\mathbf{t} \in \mathcal{T}_{A,\mathbf{c}}$ satisfies $A\mathbf{t} = \mathbf{0}$ and $\mathbf{c} \cdot \mathbf{t} > 0$, so we have to prove only the following:

Claim: *For every $\mathbf{b} \in \mathbb{Z}^d$ and feasible but not optimal solution ω of $\text{IP}_{A,\mathbf{c}}(\mathbf{b})$, there exists $\mathbf{t} \in \mathcal{T}_{A,\mathbf{c}}$ such that $\omega - \mathbf{t} \geq \mathbf{0}$.*

We have $A\omega = \mathbf{b}$, where $\text{supp}(\omega)$ is not contained in any cell σ of $\Delta_{\mathbf{c}}$.

Case 1. $\mathbf{b} := A\omega \neq \mathbf{0}$:

By basic linear programming (“Caratheodory’s theorem”) there exists $\tau \subseteq \text{supp}(\omega)$, so that A_{τ} is linearly independent, and $\mathbf{b} \in \text{cone}(A_{\tau})$. If every τ satisfying these three conditions is a subset of a cell, then the unique solution of the equation $A_{\tau}\mathbf{x}_{\tau} = \mathbf{b}$ is feasible for $\text{LP}_{A,\mathbf{c}}(\mathbf{b})$ and optimal, and again, by basic linear programming this would contradict that ω is not optimal. So τ can be chosen so that it satisfies these three properties, but *it is not contained in any cell of $\Delta_{\mathbf{c}}$* . Clearly, $\omega - (\sum_{i \in \tau} \mathbf{e}_i - \beta_{\tau}) \geq \mathbf{0}$, and $\mathbf{t} := \sum_{i \in \tau} \mathbf{e}_i - \beta_{\tau} \in \mathcal{T}_{A,\mathbf{c}}$, finishing the proof of the claim.

Case 2. $\mathbf{b} := A\omega = \mathbf{0}$:

Since then $\mathbf{0}$ is optimal, and ω is not, we have

$$\mathbf{c} \cdot \omega > 0. \quad (1)$$

(a) *every proper subset of $\text{supp}(\omega)$ is contained in some cell, and ω is a 0 – 1 vector.*

First, if there exists $\omega' \in \mathbb{N}^n$, $\omega' \leq \omega$, with ω' not optimal for $\mathbf{0} \neq \mathbf{b}' := A\omega'$, then we can apply Case 1 to ω' , and get $\mathbf{t} \in \mathcal{T}_{A,\mathbf{c}}$, $\omega - \mathbf{t} \geq \omega' - \mathbf{t} \geq 0$, and we are done. Second, if ω has a coordinate that is bigger than 1, say $\omega_1 > 1$, then Case 1 can be applied to $\omega' := \omega - e_1$.

We can now assume that (a) holds. Letting $\kappa := \text{supp}(\omega)$, we show:

(b) *every proper subset of A_κ is linearly independent.*

Suppose (b) is not true. Then for some $J \subsetneq \kappa$ there exists a coefficient vector $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, $\lambda_j \neq 0$ if $j \in J$, and $\lambda_j = 0$ if $j \notin J$. Moreover (by dividing with the highest coefficient, say that of the first coordinate), $\lambda_j \leq 1$ ($j \in [n]$), and $\lambda_1 = 1$, such that

$$A\lambda = \mathbf{0}.$$

Furthermore, we can partition $J = J^+ \cup J^-$ where $J^+ := \{j : \lambda_j > 0\}$ and $J^- := \{j : \lambda_j < 0\}$. Using this partition, we can rewrite λ in the obvious way as $\lambda = \lambda^+ - \lambda^-$ where $\lambda^+, \lambda^- \geq \mathbf{0}$ with the supports being J^+ and J^- respectively. Consequently, we have

$$A\lambda^+ = A\lambda^-$$

and according to (a) there exist cells of $\Delta_{\mathbf{c}}(A)$ containing J^+ and J^- so both λ^+ and λ^- are optimal solutions to the same linear program implying that:

$$\mathbf{c} \cdot \lambda^+ = \mathbf{c} \cdot \lambda^-. \quad (2)$$

By our choice of the size of the components in $\lambda = \lambda^+ - \lambda^-$, we have $\lambda_1 = 1$. In addition, $\lambda \leq \omega$ and $\omega_1 = 1$ and, since $\text{supp}(\omega - (\lambda^+ - \lambda^-)) \subseteq \kappa \setminus \{1\} \subsetneq \kappa$ then, by (a) and the fact that $A\lambda^+ = A\lambda^-$, $\omega - (\lambda^+ - \lambda^-)$ is an optimal solution to $\text{LP}_{A,\mathbf{c}}(\mathbf{b})$.

Recall that $\mathbf{0}$ is also an optimal solution to $\text{LP}_{A,\mathbf{c}}(\mathbf{b})$. Combining (1) and (2) above, we get that $\mathbf{c} \cdot (\omega - (\lambda^+ - \lambda^-)) > 0 = \mathbf{c} \cdot \mathbf{0}$ which contradicts the optimality of $\omega - (\lambda^+ - \lambda^-)$ and so no such linear dependent $J \subsetneq \kappa$ can exist. This proves (b).

Now by (a) and (b): κ is a wheel for A and \mathbf{c} , and as such $\sum_{i \in \kappa} \mathbf{e}_i \in \mathcal{T}_{A,\mathbf{c}}$, finishing the proof of the claim and of (ii) \Rightarrow (iii).

(iii) \Rightarrow (i) : Suppose (iii) holds, and $\mathbf{b} \in \text{cone}(A)$. Then $\text{LP}_{A,\mathbf{c}}(\mathbf{b})$ is feasible, and since $Q_{\mathbf{c}} \neq \emptyset$ the optimum is bounded. So by the condition that A is a Hilbert basis, $\text{IP}_{A,\mathbf{c}}(\mathbf{b})$ is also feasible and it also has a bounded minimum attained by $\omega \in \mathbb{N}^n$. Suppose for a contradiction that the optimal solution α/D to $\text{LP}_{A,\mathbf{c}}(\mathbf{b})$, ($\alpha \in \mathbb{N}^n$, D is a positive integer) satisfies $\mathbf{c} \cdot \alpha/D < \mathbf{c} \cdot \omega$. This also implies that $D\omega$ is not an optimal solution to $\text{IP}_{A,\mathbf{c}}(D\mathbf{b})$.

By (iii) there exists a $\gamma^+ - \gamma^- \in \mathcal{T}_{A,\mathbf{c}}$ with $\gamma^+ \in \{0, 1\}^n$ and $\gamma^- \in \mathbb{N}^n$ such that $\mathbf{c} \cdot (\gamma^+ - \gamma^-) > 0$ and $D\omega - (\gamma^+ - \gamma^-) \in \mathbb{N}^n$. Hence, $\text{supp}(\gamma^+) \subseteq \text{supp}(D\omega) = \text{supp}(\omega)$. Since the value of all elements in γ^+ is 0 or 1 then we also have $\omega \geq \gamma^+$, so $\omega - (\gamma^+ - \gamma^-) \in \mathbb{N}^n$ is also a feasible solution to $\text{IP}_{A,\mathbf{c}}(\mathbf{b})$ with $\mathbf{c} \cdot (\omega - (\gamma^+ - \gamma^-)) < \mathbf{c} \cdot \omega$, in contradiction to the optimality of ω .

(iii) \Leftrightarrow (iv) \Leftrightarrow (v): Both (iii) and (iv) can be reformulated as follows: If $\omega \in \mathbb{N}^n$ is not an optimal solution to $\text{IP}_{A,\mathbf{c}}(A\omega)$ then the vector $\omega' := \sum_{i \in \text{supp}(\omega)} \mathbf{e}_i$ is also a non-optimal solution to $\text{IP}_{A,\mathbf{c}}(A\omega')$. The equivalence of (iv) and (v) is a special case of Lemma 2.2. \square

The simple equivalence of (i) with (iii) is a possible alternative to Cook, Lovász & Schrijver's result [7], and (iii) could also be replaced by a characterization of Applegate, Cook & McCormick [1] stating that it is sufficient to check the property for functions \mathbf{b} that are small subsums of rows of A . For the sake of completeness and for later reference we present the corresponding computational corollary in our context:

Corollary 2.6. [1], [7] *Let the dimension d be fixed but the system $\mathbf{y}A \leq \mathbf{c}$, given as input, have an arbitrary number of inequalities, where A is a Hilbert basis. Then this system can be tested for the TDI property in polynomial time.*

Proof. We use the equivalence of (i) and (iii) and the construction of $\mathcal{T}_{A,\mathbf{c}}$ from above. Note that the wheels are straightforward to identify but they do not have to be identified for testing TDI-ness.

We can construct the non-wheels of $\mathcal{T}_{A,\mathbf{c}}$ or conclude that the system is not TDI in $O(n^d)$ time. Listing all the linearly independent (or generously all the d element) subsets τ of $\{1, \dots, n\}$ that are not subsets of cells in the fan. For each such subset we can form the vector $\mathbf{b}_\tau := A(\sum_{i \in \tau} \mathbf{e}_i)$ and identify its cell σ in the fan by solving $\text{LP}_{A,\mathbf{c}}(\mathbf{b}_\tau)$.

Next, for each such τ , with its corresponding optimal cell σ , do the following: we can either find an integer optimal solution β_τ to $\text{LP}_{A,\mathbf{c}}(\mathbf{b}_\tau)$ (which will have support in σ), or else no such integer solution exists in which case we conclude that $\mathbf{y}A \leq \mathbf{c}$ is not TDI, which can be done in polynomial time – see [20, §6.7] for example.

If $\text{LP}_{A,\mathbf{c}}(\mathbf{b}_\tau)$ has an integer solution then repeat the same test, as for \mathbf{b}_τ above, for $\mathbf{b}_\tau - r_i \mathbf{a}_i$ where $i \in \sigma$ and r_i is the largest positive integer for which the optimal solution to $\text{LP}_{A,\mathbf{c}}(\mathbf{b}_\tau - r_i \mathbf{a}_i)$ is in the same cell as σ . If $\text{LP}_{A,\mathbf{c}}(\mathbf{b}_\tau - r_i \mathbf{a}_i)$ has an integer optimum solution then repeat as above by replacing \mathbf{b}_τ with $\mathbf{b}_\tau - r_i \mathbf{a}_i$. If at any point during this procedure the updated \mathbf{b}_τ yields no integer optimal solution to $\text{LP}_{A,\mathbf{c}}(\mathbf{b}_\tau)$ then we immediately stop and conclude that $\mathbf{y}A \leq \mathbf{c}$ is not TDI. Otherwise, we add $\sum_{i \in \tau} \mathbf{e}_i - \beta_\tau$ to the test set.

We repeat this procedure for every such τ as above and if for none of the linearly independent sets of columns τ (different from the cells) we arrive at the conclusion that the system is not TDI, then we have in $\mathcal{T}_{A,\mathbf{c}}$ an element of the form $\sum_{i \in \tau} \mathbf{e}_i - \beta_\tau$ for every τ that is not subset of a cell, and we can conclude like in the proof of the theorem, that $\mathcal{T}_{A,\mathbf{c}}$ is a test-set. Therefore (iii) is satisfied, so according to Theorem 2.5, (i) is also satisfied, that is, $\mathbf{y}A \leq \mathbf{c}$ is TDI. If d is fixed, A has a polynomial number of linearly independent subsets of columns. \square

One consequence of this corollary is that the constructed test-set allows us to solve $\text{IP}_{A,\mathbf{c}}(\mathbf{b})$ in polynomial time without the use of Lenstra's algorithm.

Recall that we defined $\mathbf{c} \in \mathbb{R}^n$ to be *generic* with the first of the following conditions; the others are equivalent properties for toric ideals [40]:

- The integer program $\text{IP}_{A,\mathbf{c}}(\mathbf{b})$ has a unique optimal solution for all $\mathbf{b} \in \mathbb{N}A$.

- The toric initial ideal $\text{in}_{\mathbf{c}}(I_A)$ is a monomial ideal.
- There exists a reduced Gröbner basis $\{\mathbf{x}^{\mathbf{u}_1^+} - \mathbf{x}^{\mathbf{u}_1^-}, \dots, \mathbf{x}^{\mathbf{u}_s^+} - \mathbf{x}^{\mathbf{u}_s^-}\}$ of I_A with $\mathbf{c} \cdot \mathbf{u}_i^+ > \mathbf{c} \cdot \mathbf{u}_i^-$ for each $i = 1, \dots, s$.

Recall that if each $\text{IP}_{A,\mathbf{c}}(\mathbf{b})$ has a unique solution then so does $\text{LP}_{A,\mathbf{c}}(\mathbf{b})$. Hence in the generic case, assuming $Q_{\mathbf{c}} \neq \emptyset$, Cramer's rule tells us that $\Delta_{\mathbf{c}}$ being a unimodular triangulation of A implies that A is a Hilbert basis *without supposing it in advance*; and so (ii) implies (v) without any condition; to prove the converse statement in the generic case, (v) along with the assumption that $\mathbb{Z}A = \mathbb{Z}^d$ implies that A is a Hilbert basis, and then we can apply again Theorem 2.5 (v) implies (ii) to get:

Corollary 2.7. (Sturmfels) [38, Corollary 8.9] *Let $A \in \mathbb{Z}^{d \times n}$ and let $\mathbf{c} \in \mathbb{R}^n$ be generic with respect to A . Then $\Delta_{\mathbf{c}}$ is a unimodular triangulation if and only if the toric initial ideal $\text{in}_{\mathbf{c}}(I_A)$ is generated by square-free monomials.*

Theorem 2.5 is the result of generalizing Sturmfels' above result to arbitrary TDI systems. Still concerning generic \mathbf{c} it is worth to note the following result of Conti and Traverso which provides another connection between integer linear programming and Gröbner bases. Here we think of an element $\mathbf{x}^{\mathbf{v}^+} - \mathbf{x}^{\mathbf{v}^-}$ as a vector $\mathbf{v}^+ - \mathbf{v}^-$.

Proposition 2.8. (Conti-Traverso) [3] – see [41, Lemma 3] *If $\text{IP}_{A,\mathbf{c}}(\mathbf{b})$ has a unique optimal solution for every $\mathbf{b} \in \mathbb{N}A$ then the reduced Gröbner basis is a minimal test-set for the family of integer programs $\text{IP}_{A,\mathbf{c}}$.*

This proposition means for us that in the generic case the following (vi) can be added to Theorem 2.5: (vi) *The initial terms in the reduced Gröbner basis are square-free.* In particular, in the generic case of condition (iii) of Theorem 2.5 the unique inclusionwise minimal test-set is defined by the reduced Gröbner basis, which, by (vi) has only square-free terms initial terms. Even though Theorem 2.5 concerns general TDI systems and could be proved in elementary means, it was highly stimulated by the above results concerning the generic case.

As is typically the case in combinatorial optimization, the cost vector \mathbf{c} is not generic for A . However, there may be cases where one can slightly perturb \mathbf{c} to another cost vector that is generic for A , and where the TDI property for $\mathbf{y}A \leq \mathbf{c}$ can be more easily studied when \mathbf{c} is perturbed. More precisely, from the implication “(ii) implies (i)” we immediately get the following:

Proposition 2.9. *If $\mathbf{c}, \mathbf{c}' \in \mathbb{R}^n$ are such that $\Delta_{\mathbf{c}'}$ of A is a refinement of $\Delta_{\mathbf{c}}$ of A , where A_{σ} is a Hilbert basis for all $\sigma \in \Delta_{\mathbf{c}'}$, and in particular if $\Delta_{\mathbf{c}'}$ is a unimodular triangulation of A , then $\mathbf{y}A \leq \mathbf{c}$ is TDI.*

Remark 2.10. Having a regular unimodular refinement $\Delta_{\mathbf{c}'}$ of $\Delta_{\mathbf{c}}$ amounts to providing an integer point on the face of each $P_{\mathbf{b}}$, for all $\mathbf{b} \in \text{cone}(A) \cap \mathbb{Z}^d$, that is minimized by \mathbf{c} . This integer point is the vertex of $P_{\mathbf{b}}$ minimized by \mathbf{c}' and so having a regular unimodular refinement provides an integer point on each $P_{\mathbf{b}}$ in a uniform manner, dictated by \mathbf{c}' .

In the rest of the paper one of our favorite themes will be the use of Proposition 2.9. Clearly, the unimodular triangulation does not even need to be regular: in fact, a unimodular

cover of each of the cells of $\Delta_{\mathbf{c}}$ suffices for verifying that $\mathbf{y}A \leq \mathbf{c}$ is TDI. However, we are interested in cases when the converse of Proposition 2.9 is true. In general it is not. It is not even true that a Hilbert basis has a unimodular partition or a unimodular covering [2] and this counterexample inspires two more remarks. First, it cannot be expected that the equivalence of (i) and (v) can be reduced to Sturmfels' generic case. Secondly, it should be appreciated that the converse of this remark does hold in the important set packing special case, as we will see in the next section.

3. SET PACKING

Let a set packing problem be defined with a matrix A and vector \mathbf{c} , and recall $\mathbf{c} := (\mathbf{1}, \mathbf{0}) \in \mathbb{R}^n$, where the last d entries of \mathbf{c} are 0. If the set packing polytope $Q_{\mathbf{c}}$ has integer vertices then the matrix A and the polytope $Q_{\mathbf{c}}$ are said to be *perfect*. (We will not use the well-known equivalence of this definition with the integer values of optima: this will follow.) Lovász' (weak) perfect graph theorem [25] is equivalent to: *the matrix A defining a set packing polytope is perfect if and only if its first $(n - d)$ columns form the incidence vectors (indexed by the vertices) of the inclusionwise maximal complete subgraphs of a perfect graph.*

A polyhedral proof of the perfect graph theorem can be split into two parts: Lovász' *replication lemma* [25] and Fulkerson's *pluperfect graph theorem* [17]. The latter states roughly that a set packing polytope with integer vertices is described by a TDI system of linear inequalities. In this section we restate Fulkerson's result in a sharper form: there is a unimodular regular triangulation that refines the normal fan of any integral set packing polytope. We essentially repeat Fulkerson's proof, completing it with a part that shows unimodularity along the lines of the proof of [35, Theorem 3.1]. The following theorem contains the weak perfect graph theorem and endows it with an additional geometric feature.

Theorem 3.1. *Let $Q_{\mathbf{c}}$ be a set packing polytope defined by A and \mathbf{c} . Then there exists a vector $\varepsilon \in \mathbb{R}^n$ such that $\mathbf{c}' := (\mathbf{1}, \mathbf{0}) + \varepsilon$ defines a regular triangulation $\Delta_{\mathbf{c}'}$ refining $\Delta_{\mathbf{c}}$, and this triangulation is unimodular if and only if $Q_{\mathbf{c}}$ is perfect.*

We do not claim that the following proof of this theorem is novel. All essential ingredients except unimodularity are already included in the proof of Fulkerson's pluperfect graph theorem [17]. Fulkerson's proof suggests a greedy way of taking active rows with an integer coefficient (see below); this is often exploited to prove that some particular systems are TDI, let us only cite two papers the closest to ours, Cook, Fonlupt & Schrijver [6] and Chandrasekaran & Tamir [3]. The latter paper extensively used *lexicographically best* solutions, which is an important tool in linear programming theory. This idea was used in [35] to prove that the normal fan has a unimodular refinement. This same lexicographic perturbation is accounted for by the vector ε of Theorem 3.1, showing that the unimodular refinement is regular. This motivated the following problem, which contains the perfectness test (detecting perfection in a graph):

Problem 3.2. [36] *Given a $d \times n$ integer matrix A and an n dimensional integer vector \mathbf{c} , decide in polynomial time whether the normal fan of $Q_{\mathbf{c}}$ consists only of unimodular cones. Equivalently, can it be decided in polynomial time that $Q_{\mathbf{c}}$ is non-degenerate, and the determinant of A_{σ} is ± 1 for all $\sigma \in \Delta_{\mathbf{c}}$.*

Motivated by the perfectness test, the following problem might still be polynomially solvable.

Problem 3.3. *Given a $d \times n$ integer matrix A and an n dimensional integer vector \mathbf{c} , decide in polynomial time whether $\Delta_{\mathbf{c}}$ has a unimodular refinement $\Delta_{\mathbf{c}'}$.*

In the set packing case not only is the perfectness test more efficient, but also the linear programs and their duals can be solved in polynomial time (even if the algorithm uses nonlinear optimization and the ellipsoid method). Could this also be true in general ?

Problem 3.4. *Given a $d \times n$ integer matrix A and an n dimensional integer vector \mathbf{c} as input, is there a polynomial algorithm that finds at least one of the following as output: a solution to $\text{IP}_{A,\mathbf{c}}(\mathbf{b})$, or a “NO” answer to Problem 3.3 (or to Problem 3.2).*

All these problems can be solved in polynomial time in the set packing case: Problem 3.3 according to Theorem 3.1 for the moment only by using [9]; Problem 3.4 for the moment only by [20]. It remains an interesting question whether there are more efficient and conceptually simpler solutions to these problems.

We now prepare the proof of Theorem 3.1. It is a last step in a sharpening series of observations all having essentially the same proof. We begin with the proof of Fulkerson’s pluperfect graph theorem which will indicate what the \mathbf{c}' of Theorem 3.1 should be, and then finish by showing that $\Delta_{\mathbf{c}'}$ is a unimodular triangulation.

Assume that A is a perfect matrix for the remainder of this section and that $\mathbf{c} = (\mathbf{1}, \mathbf{0})$ as before. For all $\mathbf{b} \in \mathbb{Z}^d$ and $i \in [n]$ let

$$\lambda_{\mathbf{c},i}(\mathbf{b}) := \max\{x_i : \mathbf{x} \text{ is an optimal solution of } \text{LP}_{A,\mathbf{c}}(\mathbf{b})\}.$$

That is, $\lambda_{\mathbf{c},i}(\mathbf{b})$ is the largest value of x_i such that $\mathbf{c} \cdot \mathbf{x}$ is minimum under $\mathbf{x} \in P_{\mathbf{b}}$.

Remark 3.5. If σ is the minimal cell of $\Delta_{\mathbf{c}}$ such a $\mathbf{b} \in \text{cone}(A_{\sigma})$, then $\mathbf{b} - \lambda_{\mathbf{c},i}(\mathbf{b})\mathbf{a}_i \in \text{cone}(A_{\sigma'})$ where $\sigma' \in \Delta_{\mathbf{c}}$, $\sigma' \subseteq \sigma$ and the dimension of $\text{cone}(A_{\sigma'})$ is strictly smaller than that of $\text{cone}(A_{\sigma})$. Furthermore, $\mathbf{b} - \lambda\mathbf{a}_i \notin \text{cone}(A_{\sigma})$ if $\lambda > \lambda_{\mathbf{c},i}(\mathbf{b})$.

For all $\mathbf{b} \in \mathbb{Z}^d$ we show that $\lambda_{\mathbf{c},i}(\mathbf{b})$ is an integer for every $i = 1, \dots, n$. This is the heart of Fulkerson’s pluperfect graph theorem [17, Theorem 4.1]. We state this in the following lemma in a way that is most useful for our needs. Denote the common optimal value of $\text{LP}_{A,\mathbf{c}}(\mathbf{b})$ and $\text{DP}_{A,\mathbf{c}}(\mathbf{b})$ by $\gamma_{\mathbf{c}}(\mathbf{b})$. Note that $\gamma_{\mathbf{c}}$ is a monotone increasing function in all of the coordinates.

Lemma 3.6. *Suppose $\gamma_{\mathbf{c}}(\mathbf{b}) \in \mathbb{Z}$ for all $\mathbf{b} \in \mathbb{Z}^d$. If \mathbf{x} is an optimal solution to $\text{LP}_{A,\mathbf{c}}(\mathbf{b})$ with $x_l \neq 0$ for some $1 \leq l \leq n$, then there exists \mathbf{x}^* also optimal for $\text{LP}_{A,\mathbf{c}}(\mathbf{b})$, such that $x_l^* \geq 1$.*

Note that this lemma implies the integrality of $\lambda := \lambda_{\mathbf{c},l}(\mathbf{b})$ for all $l = 1, \dots, n$: if λ were not an integer then setting $\mathbf{b}' := \mathbf{b} - \lfloor \lambda \rfloor \mathbf{a}_l$ we have $\lambda_{\mathbf{c},l}(\mathbf{b}') = \{\lambda\}$ where $0 \leq \{\lambda\} := \lambda - \lfloor \lambda \rfloor < 1$, contradicting Lemma 3.6.

Proof. Suppose $\mathbf{x} \in P_{\mathbf{b}}$ with $\mathbf{c} \cdot \mathbf{x} = \gamma(\mathbf{b})$ and $x_l > 0$ for some $1 \leq l \leq n$. We can assume that $x_l < 1$ since otherwise $\mathbf{x}^* := \mathbf{x}$. We have two cases: either $1 \leq l \leq n-d$ or $n-d+1 \leq l \leq n$.

If $n - d + 1 \leq l \leq n$ then $\mathbf{a}_l = -\mathbf{e}_{l-(n-d)} \in \mathbb{R}^d$ and $c_l = 0$. In this case, we have $\gamma_{\mathbf{c}}(\mathbf{b}) = \gamma_{\mathbf{c}}(\mathbf{b} + x_l \mathbf{e}_{l-(n-d)})$ because replacing x_l by 0 in \mathbf{x} we get a solution of the same objective value for the right hand side $\mathbf{b} + x_l \mathbf{e}_{l-(n-d)}$ which gives $\gamma_{\mathbf{c}}(\mathbf{b}) \geq \gamma_{\mathbf{c}}(\mathbf{b} + x_l \mathbf{e}_{l-(n-d)})$. The reverse inequality follows from the (coordinate-wise) monotonicity of $\gamma_{\mathbf{c}}$. But then

$$\gamma_{\mathbf{c}}(\mathbf{b} + \mathbf{e}_{l-(n-d)}) \leq \gamma_{\mathbf{c}}(\mathbf{b} + x_l \mathbf{e}_{l-(n-d)}) + 1 - x_l = \gamma_{\mathbf{c}}(\mathbf{b}) + 1 - x_l,$$

and since $\gamma_{\mathbf{c}}(\mathbf{b} + \mathbf{e}_{l-(n-d)})$ is integer and $0 < 1 - x_l < 1$, we conclude that $\gamma_{\mathbf{c}}(\mathbf{b} + \mathbf{e}_{l-(n-d)}) = \gamma_{\mathbf{c}}(\mathbf{b})$.

So for any optimal $\mathbf{x}' \in P_{\mathbf{b} + \mathbf{e}_{l-(n-d)}}$ where $\mathbf{c} \cdot \mathbf{x}' = \gamma_{\mathbf{c}}(\mathbf{b})$, letting $\mathbf{x}^* := \mathbf{x}' + \mathbf{e}_{l-(n-d)} \in P_{\mathbf{b}}$ we have $\mathbf{c} \cdot \mathbf{x}^* \leq \gamma_{\mathbf{c}}(\mathbf{b})$ and so \mathbf{x}^* is optimal and $x_l^* \geq 1$.

Suppose now $1 \leq l \leq n - d$. Replacing x_l in \mathbf{x} by 0 we get a point in $P_{\mathbf{b} - x_l \mathbf{a}_l}$. This point has objective value $\mathbf{c} \cdot \mathbf{x} - x_l < \mathbf{c} \cdot \mathbf{x} = \gamma_{\mathbf{c}}(\mathbf{b})$, and so we have by monotonicity

$$\gamma(\mathbf{b} - \mathbf{a}_l) \leq \gamma(\mathbf{b} - x_l \mathbf{a}_l) < \gamma(\mathbf{b}).$$

Since the left and right hand sides are both integer values then $\gamma(\mathbf{b} - \mathbf{a}_l) \leq \gamma(\mathbf{b}) - 1$. Letting $\mathbf{x}^* := \mathbf{x}' + \mathbf{e}_l \in P_{\mathbf{b}}$ we have $\mathbf{c} \cdot \mathbf{x}^* \leq \gamma_{\mathbf{c}}(\mathbf{b}) - 1 + 1 = \gamma_{\mathbf{c}}(\mathbf{b})$, so \mathbf{x}^* is optimal, and $x_l^* \geq 1$. \square

Let us now define the appropriate \mathbf{c}' for the theorem, depending only on \mathbf{c} . Define $\mathbf{c}' := \mathbf{c} + \varepsilon \in \mathbb{R}^n$ where $\varepsilon_i := -(1/d^{d+2})^i$ for each $i = 1, \dots, n$. Note that the absolute value of the determinant of a $d \times d$ $\{-1, 0, 1\}$ -matrix cannot exceed d^d . It follows, by Cramer's rule, that the coefficients of linear dependencies between the columns of A are at most d^d in absolute value, and then the sum of absolute values of the coefficients between two solutions of an equation $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{b} \in \mathbb{R}^d$ can differ by at most a factor of d^{d+2} . After this observation, the following lemma is straightforward to verify

Lemma 3.7. (i) *Any optimal solution to $\text{LP}_{A, \mathbf{c}'}(\mathbf{b})$ is also optimal for $\text{LP}_{A, \mathbf{c}}(\mathbf{b})$.*
(ii) *If \mathbf{x}' and \mathbf{x}'' are both optimal solutions to $\text{LP}_{A, \mathbf{c}}(\mathbf{b})$ then \mathbf{x}' is lexicographically bigger than \mathbf{x}'' (that is, the first non-zero coordinate of $\mathbf{x}' - \mathbf{x}''$ is positive) if and only if $\mathbf{c}' \cdot \mathbf{x}' < \mathbf{c}' \cdot \mathbf{x}''$.*

Statement (i) of the lemma means that $\Delta_{\mathbf{c}'}$ refines $\Delta_{\mathbf{c}}$, and statement (ii) means that an optimal solution to $\text{LP}_{A, \mathbf{c}'}(\mathbf{b})$ is constructed by defining $\mathbf{b}^0 := \mathbf{b}$ and recursively

$$x_i := \lambda_{\mathbf{c}, i}(\mathbf{b}^{i-1}), \quad \mathbf{b}^i := \mathbf{b}^{i-1} - x_i \mathbf{a}_i \quad \text{for all } i = 1, \dots, n.$$

Furthermore, this optimum is unique and it follows that $\Delta_{\mathbf{c}'}$ is a triangulation. We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. The necessity of the condition is straightforward: each vertex $\mathbf{y} \in Q_{\mathbf{c}}$ satisfies the linear system (consisting of d equations in d unknowns) of the form $\mathbf{y} A_{\sigma'} = \mathbf{c}_{\sigma'}$ where σ' is a cell of $\Delta_{\mathbf{c}'}$, $\mathbf{c}_{\sigma'}$ is the subvector of \mathbf{c} indexed by σ' . Since the determinant of $A_{\sigma'}$ is ± 1 , \mathbf{y} must be an integer vector because of Cramer's rule.

Conversely, we will prove the assertion supposing only that $\gamma_{\mathbf{c}}(\mathbf{b})$ is integer for all $\mathbf{b} \in \mathbb{Z}^d$ and applying Lemma 3.6. Note that, by the already proven easy direction, we will have proved from this weaker statement that $Q_{\mathbf{c}}$ is perfect. Without loss of generality, suppose that $\mathbf{b} \in \mathbb{Z}^d$ cannot be generated by less than d columns of A . That is, the minimal cell σ of $\Delta_{\mathbf{c}}$ such that $\mathbf{b} \in \text{cone}(A_{\sigma})$ is a maximal cell of $\Delta_{\mathbf{c}}$. That is, $\text{cone}(A_{\sigma})$ is d -dimensional.

Because of Lemma 3.7(i), an optimal solution to $\text{LP}_{A, \mathbf{c}'}(\mathbf{b})$ will have support in σ and Lemma 3.7(ii) implies that such an optimal solution is constructed as follows: Let $s_1 := \min\{i : i \in \sigma\}$ and $x_{s_1} := \lambda_{\mathbf{c}, s_1}(\mathbf{b})$. Recursively, for $j = 2, \dots, d$ let s_j be the smallest element in σ indexing a column of A on the minimal face of $\text{cone}(A_\sigma)$ containing

$$\mathbf{b} - \sum_{i=1}^{j-1} x_{s_i} \mathbf{a}_{s_i}.$$

Since \mathbf{b} is in the interior of $\text{cone}(A_\sigma)$ then $x_{s_i} > 0$ for each $i = 1, \dots, d$, and by Lemma 3.6, these d x_{s_i} 's are integer. Moreover, since the dimension of $\text{cone}(A_{\sigma \setminus \{s_1, \dots, s_i\}})$ is strictly decreasing as $i = 2, \dots, d$ progresses, then $\mathbf{b} - \sum_{i=1}^d x_{s_i} \mathbf{a}_{s_i} = 0$, and setting $U := \{s_1, \dots, s_d\} \subseteq \sigma$, we have that the columns of A_U are linearly independent. Note that U is a cell of $\Delta_{\mathbf{c}'}$ and by Lemma 3.7(ii) every maximal cell of $\Delta_{\mathbf{c}'}$ arises in this fashion. We show that the matrix A_U has determinant ± 1 .

Suppose not. Then the inverse of the matrix A_U is non-integer, and from the matrix equation $(A_U)^{-1} A_U = I_d$ we see that there exists a unit vector $\mathbf{e}_j \in \mathbb{R}^d$ which is a noninteger combination of columns in A_U : $\sum_{i=1}^d x_{s_i} \mathbf{a}_{s_i} = \mathbf{e}_j$. Let \mathbf{z} be the vector

$$\mathbf{z} := \sum_{i=1}^d \{x_{s_i}\} \mathbf{a}_{s_i}.$$

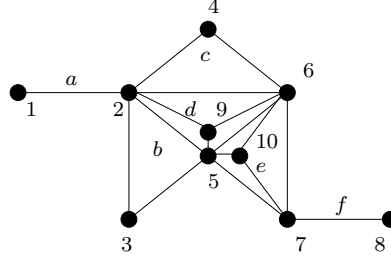
Clearly, $\mathbf{z} \in \text{cone}(A_U)$ and furthermore $\mathbf{z} \in \mathbb{Z}^d$ since it differs from \mathbf{e}_j by an integer combination of the columns of A_U . So Lemma 3.6 can be applied to $\mathbf{b} := \mathbf{z}$: letting $l := \min\{i : \{x_{s_i}\} \neq 0\}$ we see that $\lambda_{\mathbf{c}, s_l}(\mathbf{z}) = \{x_{s_l}\} < 1$ contradicting Lemma 3.6. Hence both A_U and $(A_U)^{-1}$ are integer, their determinant is ± 1 ; since A_U was an arbitrary maximal cell of $\Delta_{\mathbf{c}'}$, we conclude that $\Delta_{\mathbf{c}'}$ is unimodular. \square

The argument concerning the inverse matrix replaces the use of parallelepipeds (compare with [35, proof of Theorem 3.1]). Note that all the numbers in the definition of \mathbf{c}' are at most d^{d^2} , so they have a polynomial number of digits: the perturbed problem has polynomial size in terms of the original one, reducing the perfectness test to Problem 3.2.

Remark 3.8. Note that the ε of the perturbed vector \mathbf{c}' in Theorem 3.1 made no prescribed order on the columns of A . These regular refinements are known as *pulling* refinements [24] and so Theorem 3.1 complements the result of Ohsugi and Hibi [26] regarding pulling refinements, of not the normal fan of $Q_{\mathbf{c}}$ but, of the polytope $Q_{\mathbf{c}}$ itself. See [24] for more background on triangulations of polytopes.

4. COMPUTATION

In this section we wish to provide an illustration of how the results presented in this work lead to practical algorithms. We argue that the computational algebra methods for detecting the TDI property can be especially efficient in practice when there is a generic perturbation of the system in the sense of Proposition 2.9. We wonder if this practical efficiency is in some way related to the detection of perfection being in P [9] ? Could the perfection recognition problem be solved with a polynomial algorithm based on such

FIGURE 1. A chordal graph G with 6 maximal cliques on 10 vertices.

geometric ideas ? By analogy to the perfectness test, could a unimodular perturbation be found in polynomial time (Problem 3.3) ?

To show the ideas, we focus on one example of a set packing problem for a chordal graph G with 6 maximal cliques on 10 vertices. The coefficient matrix A has 10 rows and 16 columns and the cost vector \mathbf{c} has 16 entries; the first 6 being equal to 1, the last 10 of which equal 0 (see the Figure).

There are essentially five different ways in which we can detect the TDI property in the system $\mathbf{y}A \leq \mathbf{c}$ coming from the graph G , in finite time, and in polynomial time in fix dimension. The first two use Theorem 3.1: Indeed, one can test (even if the time will depend exponentially on the dimension) if the triangulation $\Delta_{\mathbf{c}'}$ is unimodular. The second possibility is to test if the (monomial) toric initial ideal $\text{in}_{\mathbf{c}'}(I_A)$ is generated by squarefree monomials, by computing a Gröbner basis.

There are two other ways using Theorem 2.5. Ignoring the generic perturbation arising from Theorem 3.1, we can simply study the original set packing system $\mathbf{y}A \leq \mathbf{c}$: a third algorithm can be based on statement (iii) of Theorem 2.5, or more precisely, on Corollary 2.6. A fourth would be to use statement (v) of Theorem 2.5.

A fifth possibility for the set packing special case is to use the polynomial algorithm of Chudnovsky, Cornuéjols, Liu, Seymour and Vušković [9] which (and actually its early predecessors for testing whether a graph is chordal) are certainly the best on this example and any chordal graph, since it provides an immediate positive answer through clique separation. However, the example will only contrast the methods expounded upon here.

The triangulation $\Delta_{\mathbf{c}'}$: Using `polymake` [15] we can compute the vertices of the simple polytope $Q_{\mathbf{c}'}$. We find that there are 288 vertices in all and for each of these vertices, we can find their 10 active facets. Finally, for each vertex the coordinates of the active facets give rise to a 10×10 matrix and we would need to check that each of these 288 square matrices have determinant ± 1 .

The initial ideal $\text{in}_{\mathbf{c}'}(I_A)$: The toric ideal I_A lives in $\mathbf{k}[a, \dots, f, v_1, \dots, v_{10}]$ where a, \dots, f correspond to the maximal cliques of G (the first 6 columns of A) and where v_1, \dots, v_{10} correspond to the vertices of G (the ordered columns of $-I_{10}$, the last 10 columns of A) as before. The toric ideal for set packing matrices A has a very simple generating set: there is one generator for every maximal clique in the graph, all of the form $zv_{i_1}v_{i_2} \cdots v_{i_r} - 1$, where z is the variable corresponding to the clique and $\{i_1, i_2, \dots, i_r\}$ are the vertices of that clique. See [27, §2.2] for justification.

This simple generating set enables quick computation of the toric initial ideals of A . Using `Macaulay 2` [19], we computed $\text{in}_{\mathbf{c}'}(I_A) = \langle fv_7v_8, ev_5v_6v_{10}, dv_2v_9, cv_4, bv_3, av_1 \rangle$ in less than one second on a standard desktop. The monomial toric initial ideal $\text{in}_{\mathbf{c}'}(I_A)$ has 6 generators and all are squarefree. Not only was this initial ideal computed quickly but furthermore, because of the small number of generators, we see that the TDI property is presented in a highly compact manner, unlike the triangulation $\Delta_{\mathbf{c}'}$ above.

The construction of $\mathcal{T}_{A,\mathbf{c}}$: To construct $\mathcal{T}_{A,\mathbf{c}}$ we would first need to find all sets of size 10 that are not cells in $\Delta_{\mathbf{c}}$. There are a total of $\binom{16}{10} = 8008$ candidate sets of size 10. We would then have to eliminate all 10-sets that are contained in some cell of $\Delta_{\mathbf{c}}$. Again, using `polymake` [15], we can find all 101 maximal cells of $\Delta_{\mathbf{c}}$. One such cell is the 14-set $\sigma := [16] \setminus \{8, 13\}$ and so all 10-sets that are contained in σ must be removed. Similarly, using the other 100 cells of $\Delta_{\mathbf{c}}$ we could find all 10-sets that are non-cells. We would then need to implement the remaining steps outlined in the algorithm of Corollary 2.6, a computationally challenging process that would need to be carried out for each of the non-cells. This task does not compare favourably to the computation of $\text{in}_{\mathbf{c}'}(I_A)$ above.

The monomial ideal $\text{mono}(\text{in}_{\mathbf{c}'}(I_A))$: There is an algorithm [32, Algorithm 4.4.2] for computing the generators of the monomial ideal $\text{mono}(\text{in}_{\mathbf{c}}(I_A))$ which involves four steps, all implemented in `Macaulay 2` [19]. The first is finding a generating set for the toric initial ideal $\text{in}_{\mathbf{c}}(I_A)$; one way to do this is to find a Gröbner basis of I_A with respect to \mathbf{c} . For our example, one generating set is $\{fv_7v_8, ev_5v_6v_{10} - fv_8, dv_2v_9 - ev_7v_{10}, cv_4 - dv_5v_9, bv_3 - dv_6v_9, av_1 - dv_5v_6v_9\}$ which consists of 1 monomial and 5 binomials in the polynomial ring $\mathbf{k}[a, \dots, f, v_1, \dots, v_{10}]$. Next, we must carry out a *multi-homogenization* procedure on this generating set to form the multi-homogenized ideal $\text{in}_{\mathbf{c}}(I_A)^{\text{homo}}$ in the polynomial ring $\mathbf{k}[a, \dots, f, v_1, \dots, v_{10}, A, \dots, F, V_1, \dots, V_{10}]$ consisting of 32 variables where the 16 new homogenizing variables are $A, \dots, F, V_1, \dots, V_{10}$. Finding generating sets for $\text{in}_{\mathbf{c}}(I_A)$ and $\text{in}_{\mathbf{c}}(I_A)^{\text{homo}}$ can be quickly done using `Macaulay 2`.

The third step is to find a Gröbner basis of $\text{in}_{\mathbf{c}}(I_A)^{\text{homo}}$ with respect to an elimination order. The fourth and final step would be to extract the monomials in the Gröbner basis of step 3 – this set of monomials generate the monomial ideal: $\text{mono}(\text{in}_{\mathbf{c}}(I_A)) = \langle av_1v_2, bcv_2v_3v_4, bv_2v_3v_5, cv_2v_4v_6, bf v_2v_3v_8, cf v_2v_4v_8, fv_7v_8, dv_2v_5v_6v_9, df v_2v_8v_9, aev_1v_7v_{10}, bcev_3v_4v_7v_{10}, bev_3v_5v_7v_{10}, cev_4v_6v_7v_{10}, ev_5v_6v_7v_{10} \rangle$. Like the computation of $\text{in}_{\mathbf{c}'}(I_A)$ above, this set of 14 generators was done in less than a second on a standard desktop and from it we can see the TDI property presented in a compact manner. Note that, from the equivalence of (iii) and (v) in Theorem 2.5, we also get the 14 κ 's and τ 's of $\mathcal{T}_{A,\mathbf{c}}$ from these generators.

We hope that this pedagogical example exhibits some of the utility of computational algebra methods in integer programming. In larger examples, like Padberg's *windmill* [29, last figure] (a graph with 21 maximal cliques and 20 vertices), the computation of $\text{mono}(\text{in}_{\mathbf{c}}(I_A))$ cannot be carried out on a standard desktop but computing $\text{in}_{\mathbf{c}'}(I_A)$ can be carried out in less than a second. However, it should be noted, that for both our example above and for Padberg's windmill, it is easy to observe that the graph is perfect via clique separation, a classical simple operation that preserves perfectness.

The ability to only attain $\text{in}_{\mathbf{c}'}(I_A)$ in the Padberg example was typical of other examples of large set packing problems. Could this or some other “geometric” (polyhedral) method

ever handle perfectness – a notion that can be defined in purely polyhedral terms – efficiently (in polynomial time) and in a less technical way ? In view of the NP-completeness of the general problem [4], [30], unimodular covering may provide a distinguishing clue for this 0 – 1 special case.

ACKNOWLEDGMENTS

The authors wish to thank Rekha Thomas for her valuable input and suggestions. Some more work related to the results of this article, including the computational experimentation, can be found in the first author's Ph.D. dissertation at the University of Washington. Thanks are also due to the developers of the computational packages used in this work.

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