# A Complementarity Partition Theorem for Multifold Conic Systems 

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#### Abstract

Consider a homogeneous multifold convex conic system $$
A x=0, x \in K_{1} \times \cdots \times K_{r}
$$ and its alternative system $$
A^{\mathrm{T}} y \in K_{1}^{*} \times \cdots \times K_{r}^{*},
$$ where $K_{1}, \ldots, K_{r}$ are regular closed convex cones. We show that there is canonical partition of the index set $\{1, \ldots, r\}$ determined by certain complementarity sets associated to the most interior solutions to the two systems. Our results are inspired by and extend the GoldmanTucker Theorem for linear programming.


Key words strict complementarity, Goldman-Tucker Theorem, conic feasibility system, multifold conic system

## 1 Introduction

Assume $K \subseteq \mathbb{R}^{n}$ is a closed convex cone and $A \in \mathbb{R}^{m \times n}$. Consider the homogeneous conic system

$$
\begin{equation*}
A x=0, \quad x \in K, \tag{P}
\end{equation*}
$$

and its alternative system

$$
\begin{equation*}
A^{\mathrm{T}} y \in K^{*}, \tag{D}
\end{equation*}
$$

[^0]where $K^{*} \in \mathbb{R}^{n}$ is the dual of $K^{*}$. It is immediate that any solutions $\bar{x}$ and $\bar{y}$ to (P) and (D) respectively are complementary, that is, they satisfy $\bar{y}^{\mathrm{T}}(A \bar{x})=0$. In particular, if either (P) or (D) has a strict feasible solution, then the other one only has trivial solutions. In the special case when $K=\mathbb{R}_{+}^{n}$, a stronger related property holds. As a consequence of GoldmanTucker Theorem [3], there always exist solutions $\bar{x}$ and $\bar{y}$ to ( $\mathbb{P}$ ) and (D) respectively such that $\bar{x}+A^{\mathrm{T}} \bar{y} \in \mathbb{R}_{++}^{n}$. Such pairs of strictly complementary solutions are associated to a canonical partition $B \cup N=\{1, \ldots, n\}$ of the index set $\{1, \ldots, n\}$ (see Proposition $\square$ below). The partition sets $B$ and $N$ correspond to the most interior solutions to (D) and (D) respectively. Furthermore, there is a nice geometric interpretation of the sets $B, N$ (see Proposition 2 below).

We present a generalization of the above strict complementary results to more general conic systems. To that end, we consider the case when the cone $K$ is the direct product of $r$ lower-dimensional regular closed convex cones. That is, we assume

$$
\begin{equation*}
K=K_{1} \times \cdots \times K_{r}, \tag{1}
\end{equation*}
$$

where $K_{i} \subseteq \mathbb{R}^{n_{i}}$ is a regular closed convex cone for $i=1, \ldots, r$. Throughout the sequel we shall use $I$ to denote the set $I=\{1, \ldots, r\}$ and $n$ to denote the dimension $n=\sum_{i=1}^{r} n_{i}$.

Following the terminology introduced in [2] we call the conic systems (P) and (D) multifold when the cone $K$ is as in (1). This type of multifold structure is common in optimization. Formulations for linear programming (LP), second-order conic programming (SOCP) and semidefinite programming (SDP) problems generally lead to feasibility problems of this form. Our first main result (Theorem (1) shows that there are some canonical subsets $B, N$ and $B_{0}, N_{0}$ of $I$ associated to certain geometric properties of the problems ( P ) and (D). These sets generalize the partition sets $B, N$ in the case $K=\mathbb{R}_{+}^{n}$. Our second main result (Theorem (2) shows that there exists a unique canonical partition of the index set $I$ associated to the most interior solutions to (B) and (D).

The paper is organized as follows. Section 2 provides the foundation for our work, namely the existence of strictly complementary solutions to (P), (D) when $K=\mathbb{R}_{+}^{n}$. Section 3 presents our main results, namely Theorem [1 and Theorem[2. Section 4 discusses in more detail the special case of secondorder conic systems. Section 5 concludes the paper with some final remarks.

## 2 Strict Partition for Polyhedral Homogeneous Systems

To motivate and state our main results, we first consider the special case when $K=\mathbb{R}_{+}^{n}$ in (P), (D). In this case the conic systems become

$$
\begin{equation*}
A x=0, \quad x \geq 0 ; \tag{2}
\end{equation*}
$$

and

$$
\begin{equation*}
A^{\mathrm{T}} y \geq 0, \tag{3}
\end{equation*}
$$

where $A \in \mathbb{R}^{m \times n}$. This can be considered as a special case of a multifold conic system with $r=n$ and $K_{i}=\mathbb{R}_{+}$in (11). Hence throughout this section we have $I=\{1, \ldots, n\}$. Furthermore, for notational convenience, we shall write $A=\left[\begin{array}{lll}a_{1} & \cdots & a_{n}\end{array}\right] \in \mathbb{R}^{m \times n}$. In other words, $a_{i} \in \mathbb{R}^{n}$ is the $i$-th column of $A$. The following result is a consequence of the Goldman-Tucker Theorem for linear programming [3].

Proposition 1. Consider the pair of feasibility problems (2) and (3) for a given $A \in \mathbb{R}^{m \times n}$. For a unique partition $B \cup N=I$ of the index set $I$ there exist solutions $\bar{x}$ to (2) and $\bar{y}$ to (3) satisfying

$$
\bar{x}_{B}>0, \quad A_{N}^{\mathrm{T}} \bar{y}>0,
$$

where we have used the standard notation: $\bar{x}_{B}>0$ means $\bar{x}_{i}>0$ for all $i \in B$, and $A_{N}^{\mathrm{T}} \bar{y}>0$ means $a_{i}^{\mathrm{T}} \bar{y}>0$ for all $i \in N$.

The partition sets $B, N$ in Proposition 1 can be described is several ways. The three descriptions of $B, N$ displayed in Proposition 2 below lay the foundation for our main work. In the sequel we use the following convenient notation. For a convex cone $C \subseteq \mathbb{R}^{d}$, let $\operatorname{Lin}(C) \subseteq \mathbb{R}^{d}$ denote the lineality space of $C$, that is, the largest linear subspace contained in $C$. Observe that because $C$ is a convex cone, $\operatorname{Lin}(C)=\{x \mid x,-x \in C\}$.

Proposition 2. The sets $B, N$ in Proposition 1 can be described as

$$
\begin{gather*}
B=\left\{i \in I \mid \exists x: A x=0, x \geq 0, x_{i}>0\right\}, \\
N=\left\{i \in I \mid \exists y: A^{\mathrm{T}} y \geq 0, a_{i}^{\mathrm{T}} y>0\right\} . \tag{4}
\end{gather*}
$$

These sets can also be described as

$$
\begin{gather*}
B=\left\{i \in I \mid A^{\mathrm{T}} y \geq 0 \Rightarrow a_{i}^{\mathrm{T}} y=0\right\},  \tag{5}\\
N=\left\{i \in I \mid A x=0, x \geq 0 \Rightarrow x_{i}=0\right\} .
\end{gather*}
$$

And they can also be described as

$$
\begin{align*}
& B=\left\{i \in I \mid a_{i} \in \operatorname{Lin}\left(A \mathbb{R}_{+}^{n}\right)\right\}, \\
& N=\left\{i \in I \mid a_{i} \notin \operatorname{Lin}\left(A \mathbb{R}_{+}^{n}\right)\right\} . \tag{6}
\end{align*}
$$

The description (6) of the sets $B, N$ has an interestcolorreding geometric interpretation. What determines if a particular index $i$ belongs to $B$ or $N$ is whether the corresponding $i$-th column $a_{i}$ lies in the lineality space of the cone $A \mathbb{R}_{+}^{n}$. This geometric interpretation has an interesting extension to multifold conic systems as Theorem 1 below shows. Proposition 2 is a consequence of Farkas Lemma and is also a special case of Theorem $\begin{aligned} & \text { below. }\end{aligned}$

## 3 A Canonical Partition for Multifold Conic Systems

Consider now the general conic systems (D), (D) where $A \in \mathbb{R}^{m \times n}$ and the cone $K \subseteq \mathbb{R}^{n}$ is as in (1). For notational convenience, write $A=$ $\left[\begin{array}{lll}A_{1} & \cdots & A_{r}\end{array}\right]$, where $A_{i} \in \mathbb{R}^{m \times n_{i}}$ is the $i$-th block of the matrix $A$.

Our main results generalize Proposition 1 and Proposition 2 to multifold conic systems. Motivated by (4), define

$$
\begin{gather*}
B=\left\{i \in I \mid \exists x: A x=0, x \in K, x_{i} \in \operatorname{int} K_{i}\right\}, \\
N=\left\{i \in I \mid \exists y: A^{\mathrm{T}} y \in K^{*}, A_{i}^{\mathrm{T}} y \in \operatorname{int} K_{i}^{*}\right\} . \tag{7}
\end{gather*}
$$

Likewise, motivated by (5), define

$$
\begin{gather*}
B_{0}=\left\{i \in I \mid A^{\mathrm{T}} y \in K^{*} \Rightarrow A_{i}^{\mathrm{T}} y=0\right\},  \tag{8}\\
N_{0}=\left\{i \in I \mid A x=0, x \in K \Rightarrow x_{i}=0\right\} .
\end{gather*}
$$

We are now ready to state our main results. The following theorem establishes a characterization of the index sets $B, B_{0}, N, N_{0}$ in terms of the geometry of the sets $A K$ and $A_{i} K_{i}$. In the statement below, $\overline{A K}$ denotes the closure of $A K$.

Theorem 1. (i) The sets $B, N$ defined in (17) can also be described as

$$
\begin{gather*}
B=\left\{i \in I \mid \text { ri } A_{i} K_{i} \cap \operatorname{Lin}(A K) \neq \emptyset\right\}, \\
N=\left\{i \in I \mid A_{i}\left(K_{i} \backslash\{0\}\right) \cap \operatorname{Lin}(\overline{A K})=\emptyset\right\} . \tag{9}
\end{gather*}
$$

(ii) The sets $B_{0}, N_{0}$ defined in (8) can also be described as

$$
\begin{gather*}
B_{0}=\left\{i \in I \mid \text { ri } A_{i} K_{i} \cap \operatorname{Lin}(\overline{A K}) \neq \emptyset\right\}, \\
N_{0}=\left\{i \in I \mid A_{i}\left(K_{i} \backslash\{0\}\right) \cap \operatorname{Lin}(A K)=\emptyset\right\} . \tag{10}
\end{gather*}
$$

To ease exposition, we defer the proof of Theorem 1 to the end of this Section.

Observe that in the case when $K$ is a polyhedral cone, we have $A K=$ $\overline{A K}$. Thus for $K$ polyhedral Theorem 1 yields $B=B_{0}$ and $N=N_{0}$. In particular Proposition 2 readily follows from Theorem 1 .

The next theorem generalizes Proposition 1. It shows that there is a unique canonical partition of the index set $I$ into six complementarity subsets of indices.

Theorem 2. For a unique partition $B \cup B^{\prime} \cup N \cup N^{\prime} \cup C \cup O=I$ of the index set I the following three properties hold:
(i) There exists a solution $\bar{x}$ to (P) such that

$$
\bar{x}_{i} \in \operatorname{int} K \text { for all } i \in B \text { and } x_{i} \neq 0 \text { for all } i \in B^{\prime} \cup C \text {, }
$$

(ii) There exists a solution $\bar{y}$ to (D) such that

$$
A_{i}^{\mathrm{T}} \bar{y} \in \operatorname{int} K_{i}^{*} \text { for all } i \in N \text {, and } A_{i}^{\mathrm{T}} \bar{y} \neq 0 \text { for all } i \in N^{\prime} \cup C .
$$

(iii) For any solutions $x$ to (D) and $y$ to (D) we have

$$
x_{i}=0 \text { for all } i \in N^{\prime} \cup N \cup O \text { and } A_{i}^{\mathrm{T}} y=0 \text { for all } i \in B \cup B^{\prime} \cup O .
$$

Proof. Take $B, N$ and $B_{0}, N_{0}$ as in (7) and (8) respectively, and let
$B^{\prime}:=B_{0} \backslash\left(B \cup N_{0}\right) ; \quad N^{\prime}=N_{0} \backslash\left(N \cup B_{0}\right) ; \quad O=B_{0} \cap N_{0} ; \quad C=I \backslash\left(B_{0} \cup N_{0}\right)$.
The sets $B, B^{\prime}, C, N, N^{\prime}, O$ comprise a partition of $I$ because by Theorem $\mathbb{1}$ $B \subseteq B_{0}, N \subseteq N_{0}$, and also $B \cap N_{0}=N \cap B_{0}=\emptyset$.

We next prove part (i). By Theorem 1 (ii), for every $i \notin N_{0}$ there exists a solution $x^{(i)}$ to (P) such that $x_{i}^{(i)} \in K_{i} \backslash\{0\}$. Hence $x_{N_{0}}=\sum_{i \in I \backslash N_{0}} x^{(i)}$ is solution to ( (P) and for every $i \notin N_{0}$ we have $x_{i} \neq 0$ (since $K_{i}$ is pointed). By the definition of $B$, for each $i \in B$ there exists a solution $\bar{x}^{(i)}$ to (P) such that $x_{i}^{(i)} \in \operatorname{int} K_{i}$. Then $x_{B}=\sum_{i \in B} x^{(i)}$ is solution to (P) and $\left(x_{B}\right)_{i} \in \operatorname{int} K_{i}$ (by
4. Lemma A.2.1.6]). Therefore, again by the pointedness of each $K_{i}$ and by [4, Lemma A.2.1.6], the point $\bar{x}=x_{B}+x_{N_{0}}$ is a solution to (P) such that $\bar{x}_{i} \in \operatorname{int} K$ for all $i \in B$ and $x_{i} \neq 0$ for all $i \in B^{\prime} \cup C$. An analogous argument proves part (ii). Part (iii) follows directly from the definition (8) of $B_{0}=B \cup B^{\prime} \cup O$ and $N_{0}=N \cup N^{\prime} \cup O$.

The uniqueness of the partition is proven as follows. First, observe that if (i) and (ii) hold, then by construction $B, N$ must be as in (7). Likewise if (iii) holds, then $B_{0}, N_{0}$ must be as in (8). Therefore if (i), (ii), and (iii) hold, the sets $B, B^{\prime}, C, N, N^{\prime}, O$ must be as in (11).

The Venn diagram representing the relations between the subsets of $B$, $B^{\prime}, N, N^{\prime}, C$ and $O$ of $I$ is given in Fig. 1. In Section 5.1 we provide


Figure 1: Partition of $I$ into six disjoint sets based on $B, N, B_{0}$ and $N_{0}$
an example of a second-order conic programming problem for which all six sets are nonempty. It should be noted that a six-set partition for secondorder conic programs similar to the one suggested here was mentioned in [1, Section 6]. However, there was no prior characterization of this partition along the lines of Theorem [1,

We conclude this section with the proof of Theorem 1. Our proof relies on the following separation lemma. Although this result is likely known, we were not able to locate it in the literature in this exact form.

Lemma 1. Let $K_{1}, K_{2} \subseteq \mathbb{R}^{n}$ be closed convex cones such that $K_{1} \cap K_{2}=\{0\}$ and $\operatorname{Lin}\left(K_{2}\right)=\{0\}$. Then $K_{1}$ and $K_{2}$ can be strictly separated in the following sense. There exists $s \in \mathbb{R}^{n}$ such that

$$
\begin{equation*}
\langle s, y\rangle \leq 0 \quad \forall y \in K_{1}, \quad\langle s, y\rangle>0 \quad \forall y \in K_{2} \backslash\{0\} . \tag{12}
\end{equation*}
$$

Proof. Let $C:=\left\{x \in K_{2} \mid\|x\|=1\right\}$. Since $K_{2}$ is closed and Lin $K_{2}=\{0\}$, the set co $C$ is compact and $0 \notin$ co $C$. In particular $K_{1} \cap$ co $C=\emptyset$. Hence, by [4, Corol. A.4.1.3], there exists a point $s \in \mathbb{R}^{n}$ such that

$$
\begin{equation*}
\sup _{y \in K_{1}}\langle s, y\rangle<\min _{y \in \operatorname{co} C}\langle s, y\rangle . \tag{13}
\end{equation*}
$$

Since $0 \in K_{1}$ we have $\sup _{y \in K_{1}}\langle s, y\rangle \geq\langle s, 0\rangle=0$. Thus from (13) and the fact that $K_{1}$ is a cone it follows that

$$
\sup _{y \in K_{1}}\langle s, y\rangle=0<\min _{y \in C}\langle s, y\rangle,
$$

and (12) readily follows.

## Proof of Theorem 1 .

(i) We first show $B \supseteq\left\{i \in I \mid\right.$ ri $\left.A_{i} K_{i} \cap \operatorname{Lin}(A K) \neq \emptyset\right\}$. Assume $i \in$ $I$ is such that ri $\left(A_{i} K_{i}\right) \cap \operatorname{Lin}(A K) \neq \emptyset$. By [4, Prop. A.2.1.12], $\operatorname{ri}\left(A_{i} K_{i}\right)=A_{i}\left(\right.$ ri $\left.K_{i}\right)=A_{i}\left(\operatorname{int} K_{i}\right)$. Hence there exists $\bar{x}_{i} \in \operatorname{int} K_{i}$ such that $A_{i} x_{i} \in \operatorname{Lin}(A K)$. Thus $-A_{i} x_{i}=A x^{\prime}$ for some $x^{\prime} \in K$. Let $x \in K$ be defined by putting $x_{j}=x_{j}^{\prime}$ for $j \neq i$ and $x_{i}=x_{i}^{\prime}+\bar{x}_{i}$. By [4, Lemma A.2.1.6], it follows that $x$ is a solution to ( (P) and $x_{i} \in \operatorname{int} K_{i}$. Thus $i \in B$.

Next, we show $B \subseteq\left\{i \in I \mid\right.$ ri $\left.A_{i} K_{i} \cap \operatorname{Lin}(A K) \neq \emptyset\right\}$. Assume $i \in B$. Hence there exists $x \in K$ such that $x_{i} \in \operatorname{int} K_{i}=$ ri $K_{i}$ and $A x=0$. By [4, Prop. A.2.1.12],

$$
\begin{equation*}
A_{i} x_{i} \in \operatorname{ri} A_{i} K_{i} . \tag{14}
\end{equation*}
$$

Let $x^{\prime} \in \mathbb{R}^{n}$ be defined by putting $x_{j}^{\prime}=0$ for $j \neq i$ and $x_{i}^{\prime}=x_{i}$. We have $\bar{x}=x-x^{\prime} \in K$ and so $-A_{i} x_{i}=A \bar{x} \in A K$. But $A_{i} x_{i} \in A_{i} K_{i} \subset$ $A K$ as well, therefore

$$
\begin{equation*}
A_{i} x_{i} \in \operatorname{Lin}(A K) \tag{15}
\end{equation*}
$$

From (14) and (15) we have ri $A_{i} K_{i} \cap \operatorname{Lin}(A K) \neq \emptyset$.
Now we show $N \supseteq\left\{i \in I \mid A_{i}\left(K_{i} \backslash\{0\}\right) \cap \operatorname{Lin}(\overline{A K})=\emptyset\right\}$. Assume $i \in I$ is such that $A_{i}\left(K_{i} \backslash\{0\}\right) \cap \operatorname{Lin}(\overline{A K})=\emptyset$. Since $A_{i} K_{i} \subseteq \overline{A K}$, this yields

$$
\operatorname{Lin}\left(A_{i} K_{i}\right)=\{0\} \text { and }-A_{i}\left(K_{i} \backslash\{0\}\right) \cap \overline{A K}=\emptyset
$$

Therefore by Lemma 1 applied to $K_{1}=\overline{A K}$ and $K_{2}=-A_{i} K_{i}$, there exists a nonzero $y \in \mathbb{R}^{m}$ such that $y^{\mathrm{T}} A x \geq 0 \quad \forall x \in K$ and
$y^{\mathrm{T}}\left(-A_{i} x_{i}\right)<0 \quad \forall x_{i} \in K_{i} \backslash\{0\}$. In particular, $y$ is a solution to (D) and $A_{i}^{\mathrm{T}} y \in \operatorname{int} K_{i}^{*}$.
Next we show $N \subseteq\left\{i \in I \mid A_{i}\left(K_{i} \backslash\{0\}\right) \cap \operatorname{Lin}(\overline{A K})=\emptyset\right\}$. To that end, we show the contrapositive. Assume $i \in I$ is such that $A_{i}\left(K_{i} \backslash\right.$ $\{0\}) \cap \operatorname{Lin}(\overline{A K}) \neq \emptyset$. Then there exists $x_{i} \in K_{i} \backslash\{0\}$ such that $A_{i} x_{i},-A_{i} x_{i} \in \overline{A K}$. Hence for any solution $y$ to (D) we have $y^{\mathrm{T}} A_{i} x_{i} \geq$ 0 and $\quad y^{\mathrm{T}}\left(-A_{i} x_{i}\right) \geq 0$ so $y^{\mathrm{T}} A_{i} x_{i}=0$. Since $x_{i} \in K_{i} \backslash\{0\}$, this implies that $A_{i}^{\mathrm{T}} y \notin \operatorname{int} K_{i}^{*}$. Consequently $i \notin N$.
(ii) We first show $B_{0} \supseteq\left\{i \in I\right.$ : ri $\left.A_{i} K_{i} \cap \operatorname{Lin}(\overline{A K}) \neq \emptyset\right\}$. Assume that ri $A_{i} K_{i} \cap \operatorname{Lin}(\overline{A K}) \neq \emptyset$. Then by [4, Prop. A.2.1.12] there exists $x_{i} \in \operatorname{int} K_{i}$ such that $A_{i} x_{i},-A_{i} x_{i} \in \overline{A K}$. Therefore, as in the previous paragraph, it follows that $y^{\mathrm{T}} A_{i} x_{i}=0$ for any solution $y$ to (D). Since $x_{i} \in \operatorname{int} K_{i}$, this implies that $A_{i}^{\mathrm{T}} y=0$ for any solution $y$ to (D). Thus $i \in B_{0}$.
We next show $B_{0} \subseteq\left\{i \in I\right.$ : ri $\left.A_{i} K_{i} \cap \operatorname{Lin}(\overline{A K}) \neq \emptyset\right\}$. Assume $i \in B_{0}$. Then for all solutions $y$ to (D) and all $x_{i} \in K_{i}$ we have $\left(A_{i} x_{i}\right)^{\mathrm{T}} y=0$. Thus $A_{i} x_{i},-A_{i} x_{i} \in \overline{A K}$ for all $x_{i} \in K_{i}$, and hence $A_{i} K_{i} \subset \operatorname{Lin}(\overline{A K})$.

We now show $N_{0} \supseteq\left\{i \in I: A_{i}\left(K_{i} \backslash\{0\}\right) \cap \operatorname{Lin}(A K)=\emptyset\right\}$. To that end, we show the contrapositive. Assume $i \in I$ is such that there exists a solution $x$ to (P) with $x_{i} \neq 0$. Since $-A_{i} x_{i}=\sum_{j \neq i} A_{j} x_{j} \in A K$, we have $A_{i} x_{i},-A_{i} x_{i} \in A K$ with $x_{i} \neq 0$. Hence $A_{i} x_{i} \in A_{i}\left(K_{i} \backslash\{0\}\right) \cap$ $\operatorname{Lin}(A K)$.

We finally show $N_{0} \subseteq\left\{i \in I: A_{i}\left(K_{i} \backslash\{0\}\right) \cap \operatorname{Lin}(A K)=\emptyset\right\}$. Again we show the contrapositive. Assume $i \in I$ is such that $A_{i}\left(K_{i} \backslash\{0\}\right) \cap$ $\operatorname{Lin}(A K) \neq \emptyset$. Then there exists $x_{i} \in K_{i} \backslash\{0\}$ such that $A_{i} x_{i},-A_{i} x_{i} \in$ $A K$. In particular, for some $x^{\prime} \in K$ we have $-A_{i} x_{i}=A x^{\prime}$. Then the point $\bar{x} \in K$ defined by putting $\bar{x}_{j}=x_{j}^{\prime}$ for $j \neq i$ and $\bar{x}_{i}=x_{i}^{\prime}+x_{i}$ is a solution to ( (P) with $\bar{x}_{i} \neq 0$ (because $K_{i}$ is pointed).

## 4 Second-Order Conic Systems

Consider the special case when the cone $K$ in (D),(D) is a cartesian product of Lorentz cones. In other words,

$$
\begin{equation*}
K=\mathcal{L}_{n_{1}-1} \times \cdots \times \mathcal{L}_{n_{r}-1}, \tag{16}
\end{equation*}
$$

where

$$
\mathcal{L}_{n_{i}-1}=\left\{\left(x_{0}, \bar{x}\right) \in \mathbb{R}^{n_{i}} \mid x_{0} \geq\|\bar{x}\|\right\}, i=1, \ldots, r .
$$

Here $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^{n_{i}}$. We shall put, by convention, $\mathcal{L}_{0}=$ $\mathbb{R}_{+}$when $n_{i}=1$. Also, for $d \geq 1$ we will let $\mathbb{B}_{d} \subseteq \mathbb{R}^{d}$ denote the Euclidean closed unit ball in $\mathbb{R}^{d}$ centered at zero.

For each $i \in I$ assume the $i$-th block $A_{i} \in \mathbb{R}^{m \times n_{i}}$ of $A$ is of the form

$$
A=\left[\begin{array}{ll}
A_{i 0} & \bar{A}_{i}
\end{array}\right], A_{i 0} \in \mathbb{R}^{m}, \bar{A}_{i} \in \mathbb{R}^{m \times\left(n_{i}-1\right)} .
$$

In other words, $A_{i 0}$ denotes the first column of $A_{i}$, and $\bar{A}_{i}$ denotes the block of remaining $n_{i}-1$ columns. Put

$$
E_{i}= \begin{cases}A_{i 0}+\bar{A}_{i} \mathbb{B}_{n_{i}-1}, & \text { if } n_{i}>1,  \tag{17}\\ A_{i 0}, & \text { if } n_{i}=1 .\end{cases}
$$

Observe that $A K=$ cone $\operatorname{co}_{i \in I}\left\{E_{i}\right\}$. Theorem $\mathbb{1}$ can now be stated in a way that more closely resembles (6) in Proposition 2.

Proposition 3. Consider the pair of multifold conic systems (P), (D). Assume $K$ is as in (16) and $E_{i}, i \in I$ are as in (17). Then
(i) The sets $B, N$ defined in (7) satisfy

$$
\begin{gathered}
B=\left\{i \in I \mid \text { ri } E_{i} \cap \operatorname{Lin}(A K) \neq \emptyset\right\}, \\
N=\left\{i \in I \mid E_{i} \cap \operatorname{Lin}(\overline{A K})=\emptyset\right\} .
\end{gathered}
$$

(ii) The sets $B_{0}, N_{0}$ defined in (8) satisfy

$$
\begin{gathered}
B_{0}=\left\{i \in I \mid \text { ri } E_{i} \cap \operatorname{Lin}(\overline{A K}) \neq \emptyset\right\}, \\
N_{0}=\left\{i \in I \mid E_{i} \cap \operatorname{Lin}(A K)=\emptyset\right\} .
\end{gathered}
$$

Proof. This readily follows from Theorem 1 and the construction of the sets $E_{i}, i \in I$.

We now discuss an example of a second-order feasibility system where all six sets $B, N, B^{\prime}, N^{\prime}, C, O$ in the partition of Theorem 2 are nonempty.
Example 1. Let $K=\mathbb{R}_{+} \times \mathbb{R}_{+} \times \mathbb{R}_{+} \times \mathcal{L}_{1} \times \mathcal{L}_{1} \times \mathcal{L}_{3} \subseteq \mathbb{R}^{11}$ and

$$
A=\left[\begin{array}{lllllrlllll}
1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{array}\right]
$$

In this case,

$$
\begin{gathered}
E_{1}=\{(1,0,0)\}, E_{2}=\{(0,1,0)\}, E_{3}=\{(0,0,1)\}, E_{4}=\operatorname{co}\{(1,0,0),(1,0,2)\}, \\
E_{5}=\operatorname{co}\{(0,-1,0),(2,-1,0)\}, E_{6}=\{(0,0,1)\}+\mathbb{B}_{3} .
\end{gathered}
$$

Thus

$$
\begin{gathered}
A K=\left\{(x, y, z) \in \mathbb{R}^{3} \mid z>0\right\} \cup\left\{(x, y, z) \in \mathbb{R}^{3} \mid z=0, x \geq 0\right\} \\
\overline{A K}=\left\{(x, y, z) \in \mathbb{R}^{3} \mid z \geq 0\right\}
\end{gathered}
$$

and

$$
\operatorname{Lin}(A K)=\{0\} \times \mathbb{R} \times\{0\} ; \quad \operatorname{Lin}(\overline{A K})=\mathbb{R} \times \mathbb{R} \times\{0\}
$$

Figure 2 shows the sets $\operatorname{Lin}(A K), \operatorname{Lin}(\overline{A K}), E_{1}, \ldots, E_{6}$.


Figure 2: Geometric interpretation of the partition in Example 1
From Proposition 3 we readily get

$$
B=\{2\}, N=\{3\}, B_{0}=\{1,2,5\}, N_{0}=\{1,3,4\} .
$$

Hence in this case the partition sets of Theorem 2 are

$$
O=\{1\}, B=\{2\}, N=\{3\}, N^{\prime}=\{4\}, B^{\prime}=\{5\}, C=\{6\}
$$

We note that in this small example the systems $A x=0, x \in K$ and $A^{\mathrm{T}} y \in K$ can be solved directly. We obtain the following parametric families of solutions to ( P ) and (D) respectively:

$$
x=(0, \lambda, 0,0,0, \lambda,-\lambda, \mu, 0,0,-\mu), \quad \lambda \geq 0, \mu \geq 0
$$

and

$$
y=(0,0, \gamma), \quad \gamma \geq 0
$$

The correctness of the partition $O=\{1\}, B=\{2\}, N=\{3\}, N^{\prime}=$ $\{4\}, B^{\prime}=\{5\}, C=\{6\}$ can then be directly verified.

## 5 Some Final Remarks

### 5.1 Geometric interpretation of Theorem 1

Proposition 3 can be stated in a form that holds more generally. Consider the general multifold systems (D), (D). Assume $K$ is as in (11) where each $K_{i} \subseteq \mathbb{R}^{n_{i}}, i \in I$ is regular. Furthermore, assume $B_{i}$ be a compact convex subset of $K_{i}$ such that $0 \notin B_{i}$ and $K_{i}=\operatorname{cone} B_{i}$ for $i \in I$. Put

$$
\begin{equation*}
E_{i}=A_{i} B_{i}, i \in I \tag{18}
\end{equation*}
$$

Observe that $A K=$ cone $\operatorname{co}_{i \in I}\left\{E_{i}\right\}$. Theorem 1 can now be stated as follows.

Theorem 3. Consider the pair of multifold conic systems (I), (D). Assume $K$ is as in (1) and $E_{i}, i \in I$ are as in (18). Then
(i) The sets $B, N$ defined in (7) satisfy

$$
\begin{gathered}
B=\left\{i \in I \mid \operatorname{ri} E_{i} \cap \operatorname{Lin}(A K) \neq \emptyset\right\}, \\
N=\left\{i \in I \mid E_{i} \cap \operatorname{Lin}(\overline{A K})=\emptyset\right\} .
\end{gathered}
$$

(ii) The sets $B_{0}, N_{0}$ defined in (8) satisfy

$$
\begin{gathered}
B_{0}=\left\{i \in I \mid \text { ri } E_{i} \cap \operatorname{Lin}(\overline{A K}) \neq \emptyset\right\}, \\
N_{0}=\left\{i \in I \mid E_{i} \cap \operatorname{Lin}(A K)=\emptyset\right\} .
\end{gathered}
$$

Remark 1. The alternate descriptions for the sets $B, B_{0}$ in Theorem 3 can also be stated as follows.

$$
\begin{aligned}
\text { ri } E_{i} \cap \operatorname{Lin}(A K) \neq \emptyset & \Leftrightarrow \text { ri } E_{i} \subseteq \operatorname{Lin}(A K), \\
\text { ri } E_{i} \cap \operatorname{Lin}(\overline{A K}) \neq \emptyset & \Leftrightarrow \text { ri } E_{i} \subseteq \operatorname{Lin}(\overline{A K}) .
\end{aligned}
$$

### 5.2 Some observations on polyhedral systems

While for the polyhedral feasibility problem strict complementarity always holds (Proposition (1), one might ask: what happens if each lower-dimensional cone in a multifold system is itself a product of nonnegative orthants? Since a linear image of a polyhedral set is closed, from Theorem 2 it follows that $B=B_{0}$ and $N=N_{0}$. Hence, we have only three possible complementarity sets: $B, N$ and $C=I \backslash(B \cup N)$. Any problem with both $B$ and $N$ nonempty could alternatively be considered as a multifold problem with a single cone. In this case its only index would be in $C$. Therefore, there are polyhedral systems with nonempty $C$. However, for any polyhedral system the partition sets $B^{\prime}, N^{\prime}$ and $O$ are always empty.

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