A COPOSITIVE FORMULATION FOR THE STABILITY NUMBER OF INFINITE GRAPHS

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Abstract. In the last decade, copositive formulations have been proposed for a variety of combinatorial optimization problems, for example the stability number (independence number). In this paper, we generalize this approach to infinite graphs and show that the stability number of an infinite graph is the optimal solution of some infinite-dimensional copositive program. For this we develop a duality theory between the primal convex cone of copositive kernels and the dual convex cone of completely positive measures. We determine the extreme rays of the latter cone, and we illustrate this theory with the help of the kissing number problem.

1. INTRODUCTION

One way to deal with problems in combinatorial optimization is to (re-)formulate them as convex optimization problems. This is beneficial because it allows a geometric interpretation of the original combinatorial problem and because the convexity provides ways to find bounds or even to certify optimality of solutions.

In the last decade copositive formulations have been proposed for many NP-hard problems; see the survey of Dür [10] and references therein. In these formulations the hardness is entirely moved into the copositivity constraint. Therefore any progress in understanding this constraint immediately provides new insights for a variety of problems.

Bomze, Dür, de Klerk, Roos, Quist, and Terlaky [5] were the first to give a copositive formulation of an NP-hard combinatorial problem, namely the clique number of a graph. Similarly, de Klerk and Pasechnik [14] considered the stability number of a graph. The *stability number* of a finite, undirected, simple graph $G = (V, E)$ is

$$\alpha(G) = \max\{|S| : S \text{ is a stable set}\},$$

where $S \subseteq V$ is a *stable set* if for all $x, y \in S$ we have $\{x, y\} \notin E$. Finding the stability number of a graph is a fundamental problem in combinatorial optimization and has many applications. It is one of the most difficult NP-hard problems, in the sense that even providing an approximation of any reasonable quality is NP-hard, see Håstad [13]. In [14, Theorem 2.2] de Klerk and Pasechnik gave the following

copositive formulation: For $V = \{1, \ldots, n\}$,

$$\alpha(G) = \min \quad t$$
$$t \in \mathbb{R}, \ K \in \mathcal{COP}_n,$$
$$K(i,i) = t - 1 \qquad \text{for all } i \in V,$$
$$K(i,j) = -1 \qquad \text{for all } \{i,j\} \notin E,$$

where $\mathcal{COP}_n$ denotes the convex cone of copositive $n \times n$-matrices. Recall that a real symmetric matrix $K \in \mathbb{R}^{n \times n}$ is called *copositive* if

(1)
$$\sum_{i=1}^{n} \sum_{j=1}^{n} K(i,j) a_i a_j \geq 0 \ \text{ for all } a_1, \ldots, a_n \geq 0.$$

The space of symmetric matrices is equipped with the usual Frobenius trace inner product

$$\langle K, L \rangle = \text{trace}(KL).$$

With this inner product the dual of the copositive cone is the cone of *completely positive matrices*

$$\mathcal{CP}_n = \{L \in \mathbb{R}^{n \times n} : L \text{ symmetric}, \ \langle K, L \rangle \geq 0 \text{ for all } K \in \mathcal{COP}_n\}.$$

In [12, Theorem 3.1 (iii)] Hall and Newman determined the extreme rays of this cone. They showed that a matrix generates an extreme ray of $\mathcal{CP}_n$ if and only if it is of the form

$$aa^{\mathsf{T}} = \left(\sum_{i=1}^{n} a_i e_i\right)\left(\sum_{i=1}^{n} a_i e_i\right)^{\mathsf{T}} \quad \text{with } a_1, \ldots, a_n \geq 0,$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$. We write aa^{T} in this seemingly complicated form because it suggests a way to generalize this to the infinite setting.

The concept of stable sets in graphs is also useful in infinite graphs, for instance to model geometric packing problems in metric spaces. Let V be a compact metric space with probability measure ω, which is strictly positive on open sets, and distance function d. Finding the densest packing of balls with radius r in V is equivalent to finding the stability number of the graph $G = (V, E)$ where the vertices of G are the elements of the compact metric space V and $\{x, y\} \in E$ iff $d(x, y) \in (0, 2r)$. For example, one can formulate the kissing number problem in this way (see, e.g., the exposition by Pfender and Ziegler [20]): Take V to be the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : x^{\mathsf{T}} x = 1\}$, let ω be the normalized, induced Lebesgue measure, d the angular distance $d(x, y) = \arccos x^{\mathsf{T}} y$, and $r = \pi/6$.

The graphs defined above are compact topological packing graphs as introduced by de Laat and Vallentin [16]. The formal definition is as follows.

Definition 1.1. *A graph whose vertex set is a Hausdorff topological space is called a* topological packing graph *if each finite clique is contained in an open clique. An* open clique *is an open subset of the vertex set where every two vertices are adjacent.*

In the remainder of the paper we assume that $G = (V, E)$ is a compact topological packing graph where the vertex set V is metrizable. Compactness implies that the stability number of G is finite.

The aim of the present paper is to give a copositive formulation of the stability number of compact topological packing graphs, and thereby to initiate the study

of copositive formulations also for other combinatorial problems in a continuous setting.

In order to go from the finite to the infinite setting we have to provide the right infinite dimensional generalizations of copositive and completely positive matrices. For this we will apply classical notions and results from infinite dimensional convexity theory, see the books by Barvinok [2], by Simon [24], or Rudin [22].

On the copositive side, the natural generalization of finite $n \times n$-matrices (where rows and columns are indexed by $\{1, \ldots, n\}$) are real-valued continuous *Hilbert-Schmidt kernels* on the compact Hausdorff space V. The set of continuous Hilbert-Schmidt kernels is defined as follows:

$$C(V \times V)_{\text{sym}} = \{K \colon V \times V \to \mathbb{R} : K \text{ is symmetric and continuous}\}.$$

Symmetry means here that for all $x, y \in V$ we have $K(x, y) = K(y, x)$.

In analogy to (1) we call a kernel $K \in C(V \times V)_{\text{sym}}$ *copositive* if

$$(2) \qquad \int_V \int_V K(x, y) f(x) f(y) \, d\omega(x) d\omega(y) \geq 0 \ \text{ for all } f \in C(V)_{\geq 0},$$

where $C(V)_{\geq 0}$ is the convex cone of all nonnegative continuous functions on V. We denote the convex cone of copositive kernels by $\mathcal{COP}_V$.

Note that the set $\mathcal{COP}_V$ is independent of the choice of ω. This follows from Lemma 2.1 below. In the finite dimensional case, this corresponds to the fact that $K \in \mathcal{COP}_n$ if and only if $DKD \in \mathcal{COP}_n$ for any diagonal matrix D with strictly positive diagonal elements. In the finite setting, this scaling invariance is relevant when studying approximations of $\mathcal{COP}_n$, cf. [9].

Also note that a measure ω with the required property (strict positivity on open sets) always exists: Let V be compact and metrizable. Then V is separable, so there exists a sequence $(x_n)_{n \in \mathbb{N}}$ which is dense in V. Choose a sequence $a_n > 0$ with $\sum_{n \in \mathbb{N}} a_n = 1$ and for any set $A \subset V$ define $\omega(A) := \sum_{n \in \mathbb{N}} a_n \chi_A(x_n)$. Then ω is a probability measure with the required property and consequently the set $\mathcal{COP}_V$ is well-defined.

The dual space of $C(V \times V)_{\text{sym}}$ equipped with the supremum norm consists of all continuous linear functionals. By Riesz' representation theorem it can be identified with the space of symmetric, signed Radon measures $M(V \times V)_{\text{sym}}$ equipped with the total variation norm (see e.g. [3, Chapter 2.2]). A *signed Radon measure* is the difference of two Radon measures, where a *Radon measure* μ is a locally finite measure on the Borel algebra satisfying *inner regularity*: $\mu(B) = \sup\{\mu(C) : C \subseteq B, C \text{ compact}\}$ for each Borel set B. A Radon measure μ is *symmetric* if $\mu(A \times B) = \mu(B \times A)$ for all measurable $A, B \subseteq V$.

Let K be a continuous kernel and μ be a symmetric, signed Radon measure on $V \times V$. The duality is given by the pairing

$$(3) \qquad \qquad \langle K, \mu \rangle = \int_{V \times V} K(x, y) \, d\mu(x, y).$$

We endow the spaces with the weakest topologies compatible with the pairing: the weak topology on $C(V \times V)_{\text{sym}}$ and the weak* topology on $M(V \times V)_{\text{sym}}$. Then the dual cone of the cone $\mathcal{COP}_V$ of copositive kernels is a cone which we call the *cone of completely positive measures*,

$$\mathcal{CP}_V = \{\mu \in M(V \times V)_{\text{sym}} : \langle K, \mu \rangle \geq 0 \text{ for all } K \in \mathcal{COP}_V\}.$$

Using these definitions we can state our main theorem which gives a copositive formulation of the stability number of compact topological packing graphs.

Theorem 1.2. *Let $G = (V, E)$ be a compact topological packing graph. Then the stability number of G equals*

$$\alpha(G) = \inf \quad t$$

(P)
$$t \in \mathbb{R}, \ K \in \mathcal{COP}_V$$
$$K(x, x) = t - 1 \qquad \text{for all } x \in V$$
$$K(x, y) = -1 \qquad \text{for all } \{x, y\} \notin E.$$

The remainder of the paper is organized as follows: In Section 2 we analyze properties of the two infinite-dimensional cones $\mathcal{COP}_V$ and $\mathcal{CP}_V$; we give a characterization of copositive kernels and we determine the extreme rays of the cone $\mathcal{CP}_V$ of completely positive measures. In Section 3 we prove our main result, Theorem 1.2. There we first derive a completely positive formulation of the stability number — which we will denote by (D) —, that is the dual of (P). Then by proving that there is no duality gap between the primal and the dual we derive Theorem 1.2. We also give a version of Theorem 1.2 for the the weighted stability number. In Section 4 we provide an interpretation of our copositive formulation for the kissing number problem. Then we end by posing a question for possible future work.

2. Copositive kernels and completely positive measures

2.1. **Copositive kernels.** In (2) we defined a kernel to be copositive by integrating it with nonnegative continuous functions. Instead of using nonnegative continuous functions we can also define copositivity by means of finite nonnegative delta measures. For the larger class of positive definite kernels this is a classical fact which holds under the same assumptions on V and ω as imposed here, as realized for instance by Bochner [4, Lemma 1], see also Folland [11, Proposition 3.35]:

A kernel $K \in C(V \times V)_{\mathrm{sym}}$ is called *positive (semi-)definite* if

$$\int_V \int_V K(x, y) f(x) f(y) \, d\omega(x) d\omega(y) \geq 0 \ \text{ for all } f \in C(V),$$

where $C(V)$ denotes the space of continuous functions. Bochner [4] proved that a kernel $K \in C(V \times V)_{\mathrm{sym}}$ is positive semidefinite if and only if for any choice $x_1, \ldots, x_N$ of finitely many points in V, the matrix $(K(x_i, x_j))_{i,j=1}^N$ is positive semidefinite.

The following lemma shows that a similar characterization holds for copositive kernels. For the reader's convenience we provide a proof here.

Lemma 2.1. *Let V be a compact space with probability measure ω which is strictly positive on open sets. A kernel $K \in C(V \times V)_{\mathrm{sym}}$ is copositive if and only if for any choice of finitely many points $x_1, \ldots, x_N \in V$, the matrix $(K(x_i, x_j))_{i,j=1}^N$ is copositive.*

Proof. Since $V \times V$ is compact, the continuous function K is uniformly continuous and bounded on $V \times V$.

Suppose that for any choice $x_1, \ldots, x_N$ of finitely many points in V, the matrix $(K(x_i, x_j))_{i,j=1}^N$ is copositive. Let $\varepsilon > 0$ and $f \in C(V)_{\geq 0}$. Since K is uniformly continuous, we can partition V into a finite number of measurable sets $V_1, \ldots, V_N$ and find points $x_i \in V_i$ such that

(4) $|K(x, y) - K(x_i, x_j)| \leq \varepsilon \quad \text{for all } x \in V_i, \ y \in V_j.$

Set $a_i = \int_{V_i} f(x)\,d\omega(x)$. Then $a_i \geq 0$ and

$$\left| \sum_{i=1}^{N} \sum_{j=1}^{N} K(x_i, x_j) a_i a_j - \int_V \int_V K(x,y) f(x) f(y)\, d\omega(x) d\omega(y) \right|$$

$$= \left| \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{V_i} \int_{V_j} (K(x_i, x_j) - K(x,y)) f(x) f(y)\, d\omega(x) d\omega(y) \right|$$

$$\leq \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{V_i} \int_{V_j} |K(x_i, x_j) - K(x,y)|\, f(x) f(y)\, d\omega(x) d\omega(y)$$

$$\leq \varepsilon \int_V \int_V f(x) f(y)\, d\omega(x) d\omega(y) \to 0$$

as $\varepsilon \to 0$. One obtains

$$\int_V \int_V K(x,y) f(x) f(y)\, d\omega(x) d\omega(y) \geq 0,$$

and hence K is copositive.

Conversely, assume K is copositive. Then, inequality (2) holds also for the larger class of integrable functions $f \in L^1(V)$ which are nonnegative ω-almost everywhere. This follows since the family of nonnegative continuous functions lies dense in the family of ω-almost everywhere nonnegative L^1-functions. Let $x_1, \ldots, x_N \in V$ (we may assume the x_i's are pairwise different), let $a_1, \ldots, a_N \geq 0$ and let $\varepsilon > 0$. We construct disjoint open neighborhoods V_i of x_i such that (4) holds. By assumption, since ω is strictly positive on open sets, we have $\omega(V_i) > 0$. Thus the integrable function

$$f(x) = \begin{cases} \frac{a_i}{\omega(V_i)} & \text{if } x \in V_i, \\ 0 & \text{otherwise,} \end{cases}$$

is nonnegative on V, and $K(x_i, x_j) a_i a_j$ can be expressed as

$$K(x_i, x_j) a_i a_j = \int_{V_i} \int_{V_j} K(x_i, x_j) f(x) f(y)\, d\omega(x) d\omega(y).$$

Then

$$\left| \sum_{i=1}^{N} \sum_{j=1}^{N} K(x_i, x_j) a_i a_j - \int_V \int_V K(x,y) f(x) f(y)\, d\omega(x) d\omega(y) \right| \leq \varepsilon \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j.$$

By letting ε tend to zero, one obtains $\sum_{i=1}^{N} \sum_{j=1}^{N} K(x_i, x_j) a_i a_j \geq 0$, which concludes the proof. $\square$

An alternative characterization of copositive kernels was first noted by Pfender [19, Lemma 3.3] in the case when V is the unit sphere S^{n-1} and for copositive kernels which are invariant under the orthogonal group. In fact, this holds in general.

Lemma 2.2. *A kernel $K \in C(V \times V)_{\mathrm{sym}}$ is copositive if and only if for any choice of finitely many points $x_1, \ldots, x_N$ in V, the sum $\sum_{i=1}^{N} \sum_{j=1}^{N} K(x_i, x_j)$ is nonnegative.*

Proof. Assume that K is copositive and take any finite set of points $x_1, \ldots, x_N \in V$. Then choosing all a_i's in the previous lemma equal to one gives

$$\sum_{i=1}^{N} \sum_{j=1}^{N} K(x_i, x_j) \geq 0.$$

To show the converse, let $f \in C(V)_{\geq 0}$. By scaling we may assume that f is a probability density function on V, i.e. $\int_V f(x) \, d\omega(x) = 1$. Picking points $x_1, \ldots, x_N \in V$ independently at random from this distribution gives

$$0 \leq \mathbb{E}\left[\frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} K(x_i, x_j) \right]$$

$$= \frac{1}{N} \int_V K(x, x) f(x) \, d\omega(x) + \frac{N-1}{N} \int_V \int_V K(x, y) f(x) f(y) \, d\omega(x) d\omega(y).$$

By letting N tend to infinity, we see that the double integral is nonnegative, and hence K is copositive. $\qquad\square$

2.2. Completely positive measures.

In the finite setting, the rank-1-matrices

$$aa^{\mathsf{T}} = \left(\sum_{i=1}^{n} a_i e_i \right) \left(\sum_{i=1}^{n} a_i e_i \right)^{\mathsf{T}} \quad \text{with} \quad a_1, \ldots, a_n \geq 0,$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$, determine all extreme rays of the cone of completely positive matrices $\mathcal{CP}_n$. So we have an explicit description of this cone,

$$\mathcal{CP}_n = \operatorname{cone} \left\{ aa^{\mathsf{T}} : a_1, \ldots, a_n \geq 0 \right\}.$$

In a sense, this fact generalizes to the infinite setting as we shall show soon in Theorem 2.4.

We have to find the proper replacement of the rank-1-matrices aa^{T}. We will show next that delta measures of the form

$$(5) \qquad \sum_{i=1}^{N} a_i \delta_{x_i} \otimes \sum_{i=1}^{N} a_i \delta_{x_i} \quad \text{with} \quad x_1, \ldots, x_N \in V, \ a_1, \ldots, a_N \geq 0$$

defined by

$$\left\langle K, \sum_{i=1}^{N} a_i \delta_{x_i} \otimes \sum_{i=1}^{N} a_i \delta_{x_i} \right\rangle = \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j K(x_i, x_j) \quad \text{for } K \in C(V \times V)_{\text{sym}}$$

play a similar role as the rank-1-matrices.

Proposition 2.3. *The cone of completely positive measures equals*

$$\mathcal{CP}_V = \operatorname{cl} \operatorname{cone} \left\{ \sum_{i=1}^{N} a_i \delta_{x_i} \otimes \sum_{i=1}^{N} a_i \delta_{x_i} : N \in \mathbb{N}, \ x_i \in V, \ a_i \geq 0 \right\},$$

where the closure is taken with respect to the weak* topology.

Proof. One inclusion is straightforward: By definition, the cone $\mathcal{CP}_V$ of completely positive measures is closed, and by Lemma 2.1 delta measures of the form (5) lie in $\mathcal{CP}_V$.

For the other inclusion we use the Hahn-Banach theorem for locally convex topological vector spaces. For this note that $M(V \times V)_{\text{sym}}$ with the weak* topology

is a locally convex topological space, and all continuous linear functionals are given by $\langle K, \cdot \rangle$ for some $K \in C(V \times V)_{\text{sym}}$. Take

$$\mu \in M(V \times V)_{\text{sym}} \setminus \text{cl cone} \left\{ \sum_{i=1}^{N} a_i \delta_{x_i} \otimes \sum_{i=1}^{N} a_i \delta_{x_i} : N \in \mathbb{N}, \ x_i \in V, \ a_i \geq 0 \right\}.$$

By Hahn-Banach there exists a kernel $K \in C(V \times V)_{\text{sym}}$ such that $\langle K, \mu \rangle < 0$ and

$$\left\langle K, \sum_{i=1}^{N} a_i \delta_{x_i} \otimes \sum_{i=1}^{N} a_i \delta_{x_i} \right\rangle \geq 0 \quad \text{for all } N \in \mathbb{N}, x_i \in V \text{ and } a_i \geq 0 \ (i = 1, \ldots, N).$$

Hence, again by Lemma 2.1, the kernel K is copositive and therefore $\mu \notin \mathcal{CP}_V$. $\square$

However, it turns out that we really need to take the closure in the statement of Proposition 2.3. In particular, the set of extreme rays of the cone of completely positive measures is strictly larger than the set of delta measures given in the proposition. The set of extreme rays consists of all product measures of $\mu \times \mu$ where μ is a nonnegative measure on V:

Theorem 2.4. *A measure generates an extreme ray of the cone $\mathcal{CP}_V$ of completely positive measures if and only if it is a product measure of the form $\mu \otimes \mu$, where $\mu \in M(V)$ is a nonnegative measure on V.*

Before proving the theorem, we first want to describe our strategy. We start by cutting $\mathcal{CP}_V$ into compact convex slices $\lambda \mathcal{B}$ where $\mathcal{B}$ is the closure of the convex hull of all product measures of finitely supported probability measures (Lemma 2.5 and Proposition 2.6). Then we consider in the proof of Theorem 2.4 set $\mathcal{K}_1$ which consists of all product measures of all probability measures on V. It is clear that $\mathcal{B}$ equals cl conv $\mathcal{K}_1$. From Milman's converse of the Krein-Milman theorem (Theorem 2.7) we get immediately that extreme points of $\mathcal{B}$ are contained in $\mathcal{K}_1$. Proving the converse inclusion requires work. For this we rely on Choquet's theorem (Theorem 2.8).

The following general lemma is a slight variation of [2, Lemma III.2.10] where we do not use the convexity assumption.

Lemma 2.5. *Let $\mathcal{B}$ be a compact set in a topological vector space such that $0 \notin \mathcal{B}$. Then the set $\mathcal{K}$ defined by the union $\mathcal{K} = \bigcup_{\lambda \geq 0} \lambda \mathcal{B}$ is closed.*

Proof. We shall show that the complement of $\mathcal{K}$ is open. Let $u \notin \mathcal{K}$. Since $0 \notin \mathcal{B}$, there is a neighborhood W of 0 that does not intersect $\mathcal{B}$. Let U_1 be a neighborhood of u, and $\delta > 0$ such that $\alpha U_1 \subset W$ for all $|\alpha| < \delta$ (from the continuity of $(\alpha, x) \to \alpha x$ at $(0, u)$). Then $U_1 \cap \lambda \mathcal{B} = \emptyset$ for all $\lambda > 1/\delta$. The image of the compact set $[0, 1/\delta] \times \mathcal{B}$ by the continuous map $(\alpha, x) \to \alpha x$ is compact and is contained in $\mathcal{K}$. Hence there is a neighborhood U_2 of u that does not intersect the image. Then the intersection $U_1 \cap U_2$ is a neighborhood of u that does not intersect $\mathcal{K}$ which proves that $\mathcal{K}$ is closed. $\square$

Proposition 2.6. *The set*

$$\mathcal{B} = \text{cl conv} \left\{ \sum_{i=1}^{N} a_i \delta_{x_i} \otimes \sum_{i=1}^{N} a_i \delta_{x_i} : N \in \mathbb{N}, \ x_i \in V, \ a_i \geq 0, \ \sum_{i=1}^{N} a_i = 1 \right\}$$

is weak* compact and equality

$$\mathcal{CP}_V = \bigcup_{\lambda \geq 0} \lambda \mathcal{B}.$$

holds. Hence the extreme rays of $\mathcal{CP}_V$ are precisely the rays generated by the extreme points of $\mathcal{B}$.

Proof. The set $\mathcal{B}$ is closed by definition, so in order to prove the weak* compactness it suffices to show that $\mathcal{B}$ is contained in a compact set. But this is clear since

$$\mathcal{B} \subseteq \{\mu \in M(V \times V)_{\mathrm{sym}} : \mu(V \times V) \leq 1, \ \mu \geq 0\}$$

and the latter set is compact in the weak* topology by the Theorem of Banach-Alaoglu.

Since $\mathcal{B}$ does not contain the origin, the union $\bigcup_{\lambda \geq 0} \lambda \mathcal{B}$ is closed by Lemma 2.5 and so the desired equality follows by Proposition 2.3. $\qquad\square$

We cite two results from Choquet theory, see Phelps [21].

Theorem 2.7 (Milman's converse of the Krein-Milman theorem)**.** *Suppose that X is a compact convex subset of a locally convex space. Suppose further that $Z \subseteq X$, and that $X = \mathrm{cl}\,\mathrm{conv}\,Z$. Then the extreme points of X are contained in the closure of Z, i.e. $\mathrm{ex}\,X \subseteq \mathrm{cl}\,Z$.*

Theorem 2.8 (Choquet)**.** *Suppose that X is a metrizable compact convex subset of a locally convex space E, and let $x_0 \in X$. Then there exists a probability measure P on X which represents x_0, i.e.,*

$$u(x_0) = \int_X u(x)\,dP(x) \quad \text{for every continuous linear functional } u \text{ on } E,$$

and is supported by the extreme points of X, i.e., $P\big(X \setminus \mathrm{ex}\,X\big) = 0$.

Now we are ready to prove the main result of this section.

Proof of Theorem 2.4. We define the following two sets:

$$\mathcal{M}_1^+(V) = \{\mu \in M(V) : \mu \geq 0, \mu(V) = 1\}, \quad \mathcal{K}_1 = \{\mu \otimes \mu : \mu \in \mathcal{M}_1^+(V)\}.$$

We will show that

$$\mathrm{ex}\,\mathrm{cl}\,\mathrm{conv}\,\mathcal{K}_1 = \mathcal{K}_1.$$

The set $\mathcal{K}_1$ is weak* compact. Therefore, Milman's theorem (Theorem 2.7) gives the first inclusion

$$\mathrm{ex}\,\mathrm{cl}\,\mathrm{conv}\,\mathcal{K}_1 \subseteq \mathcal{K}_1.$$

To show the converse, assume that $\mu \otimes \mu \in \mathcal{K}_1$ can be written as $\mu \otimes \mu = \frac{1}{2}(\nu_1 + \nu_2)$ for some $\nu_1, \nu_2 \in \mathrm{cl}\,\mathrm{conv}\,\mathcal{K}_1$. Since $\mathcal{K}_1$ is weak* compact and weak* metrizable, it follows from Choquet's theorem (Theorem 2.8) that there exist probability measures P_1, P_2 on $\mathcal{M}_1^+$ such that for all $u \in \big(M(V \times V)_{\mathrm{sym}}\big)^*$ we have the representation

$$(6) \qquad\qquad u(\nu_i) = \int_{\mathcal{M}_1^+} u(\rho \otimes \rho)\,dP_i(\rho), \quad i = 1, 2.$$

Setting $P := \frac{1}{2}P_1 + \frac{1}{2}P_2$, we conclude that for all $F \in C(V \times V)_{\mathrm{sym}}$,

$$(\mu \otimes \mu)(F) = \int_{\mathcal{M}_1^+} (\rho \otimes \rho)(F)\,dP(\rho).$$

Since V is a compact metrizable space, the space $C(V)$ of continuous functions on V is separable. Therefore, there exists a countable dense subset H of $C(V)_{\geq 0}$.

Take $f \in H$, let $\mathbf{1}_V$ be the constant function equal to 1 on V, and consider

$$F := \tfrac{1}{2}(f \otimes \mathbf{1}_V) + \tfrac{1}{2}(\mathbf{1}_V \otimes f).$$

Then

$$(7) \qquad \mu(f) = (\mu \otimes \mu)(F) = \int_{\mathcal{M}_1^+} (\rho \otimes \rho)(F)\, dP(\rho) = \int_{\mathcal{M}_1^+} \rho(f)\, dP(\rho).$$

Similarly, consider $F' = f \otimes f$ to obtain

$$(8) \qquad \mu(f)^2 = (\mu \otimes \mu)(F') = \int_{\mathcal{M}_1^+} (\rho \otimes \rho)(F')\, dP(\rho) = \int_{\mathcal{M}_1^+} \rho(f)^2\, dP(\rho).$$

Now if $\mu(f) = 0$, then (7) gives that $\rho(f) = 0$ P-almost everywhere. If $\mu(f) > 0$, then combining (7) and (8) gives

$$\int_{\mathcal{M}_1^+} \frac{\rho(f)}{\mu(f)}\, dP(\rho) = 1 = \int_{\mathcal{M}_1^+} \frac{\rho(f)^2}{\mu(f)^2}\, dP(\rho),$$

which implies that there exists a set $N_f \subset \mathcal{M}_1^+$ with $P(N_f) = 0$ such that $\rho(f) = \mu(f)$ for all $\rho \in \mathcal{M}_1^+ \setminus N_f$. Set $N = \bigcup_{f \in H} N_f$ and since H is countable, we have $P(N) = 0$ and

$$\rho(f) = \mu(f) \quad \text{for all } \rho \in \mathcal{M}_1^+ \setminus N \text{ and for all } f \in H.$$

As H is dense in $C(V)_{\geq 0}$, we get $\rho = \mu$ for all $\rho \in \mathcal{M}_1^+ \setminus N$.

Since $0 \leq P_i(N) \leq 2P(N) = 0$, we obtain that for $i = 1, 2$ and for all $F \in C(V \times V)_{\text{sym}}$,

$$\nu_i(F) = \int_{\mathcal{M}_1^+ \setminus N} (\rho \otimes \rho)(F)\, dP_i(\rho)$$

$$= \int_{\mathcal{M}_1^+ \setminus N} (\mu \otimes \mu)(F)\, dP_i(\rho)$$

$$= (\mu \otimes \mu)(F) \int_{\mathcal{M}_1^+ \setminus N} dP_i(\rho)$$

$$= (\mu \otimes \mu)(F).$$

Hence, $\nu_1 = \nu_2 = \mu \otimes \mu$, which means that $\mu \otimes \mu \in \operatorname{ex} \operatorname{cl} \operatorname{conv} \mathcal{K}_1$, and the proof of the converse inclusion is complete.

The theorem now follows from $\operatorname{cl} \operatorname{conv} \mathcal{K}_1 = \mathcal{B}$ and Proposition 2.6. $\qquad \square$

3. Copositive formulation for the stability number of infinite graphs

In order to develop our copositive formulation of the stability number we make use of Kantorovich's approach to linear programming over cones in the framework of locally convex topological vector spaces. This theory is thoroughly explained in Barvinok [2, Chapter IV] and we follow his notation closely.

In Section 3.1 we cast the copositive problem (P) into the general framework of conic problems as studied by Barvinok, and using this general theory, we derive the dual of (P) which will turn out to be an infinite-dimensional completely positive problem. Then we prove our main theorem, Theorem 1.2, in two steps. In the first step, we show in Section 3.2 that the stability number of G equals the optimal value

of the dual problem. In particular we show that the optimum is attained. In the second step, Section 3.3, we establish the fact that there is no duality gap between primal and dual. In Section 3.4 we extend these results and give a copositive formulation for the weighted stability number.

3.1. **Primal-dual pair.** As before, let $G = (V, E)$ be a compact topological packing graph with metrizable vertex set. For this graph, the copositive problem (P) can be seen as a general conic problem of the form

$$(9) \qquad \begin{array}{ll} \inf & \langle x, c \rangle_1 \\ & x \in \mathcal{K}, \ Ax = b \end{array}$$

with the following notations:

$$x = (t, K) \in \mathbb{R} \times C(V \times V)_{\mathrm{sym}}$$
$$c = (1, 0) \in \mathbb{R} \times M(V \times V)_{\mathrm{sym}}$$
$$\langle \cdot, \cdot \rangle_1 : (\mathbb{R} \times C(V \times V)_{\mathrm{sym}}) \times (\mathbb{R} \times M(V \times V)_{\mathrm{sym}}) \to \mathbb{R}$$
$$\mathcal{K} = \mathbb{R}_{\geq 0} \times \mathcal{COP}_V$$
$$A : \mathbb{R} \times C(V \times V)_{\mathrm{sym}} \to C(V) \times C(\overline{E})$$
$$\quad A(t, K) = (x \mapsto K(x, x) - t, (x, y) \mapsto K(x, y))$$
$$b = (-1, -1) \in C(V) \times C(\overline{E}).$$

Here $\overline{E} = \{\{x, y\} : x \neq y, \{x, y\} \notin E\}$ is the complement of the edge set. Note that we can replace the constraint $t \in \mathbb{R}$ in (P) by $t \in \mathbb{R}_{\geq 0}$, since $t \geq 1$ holds automatically because diagonal elements of copositive kernels are nonnegative.

The dual problem of (9) is

$$(10) \qquad \begin{array}{ll} \sup & \langle b, y \rangle_2 \\ & c - A^* y \in \mathcal{K}^*. \end{array}$$

Applying this to our setting, it is not difficult to see that we need

$$\langle \cdot, \cdot \rangle_2 : (C(V) \times C(\overline{E})) \times (M(V) \times M(\overline{E})) \to \mathbb{R}$$
$$y = (\mu_0, \mu_1) \in M(V) \times M(\overline{E})$$
$$A^* : M(V) \times M(\overline{E}) \to \mathbb{R} \times M(V \times V)_{\mathrm{sym}}$$
$$\quad A^*(\mu_0, \mu_1) = (-\mu_0(V), \mu_0 + \mu_1)$$
$$\mathcal{K}^* = \mathbb{R}_{\geq 0} \times \mathcal{CP}_V.$$

The map A^* is the adjoint of A because

$$\langle A(t, K), (\mu_0, \mu_1) \rangle_2 = \int_V K(x, x) - t \, d\mu_0(x) + \int_{\overline{E}} K(x, y) \, d\mu_1(x, y)$$
$$= -t\mu_0(V) + \int_{V \times V} K(x, y) \, d(\mu_0 + \mu_1)(x, y)$$
$$= \langle (t, K), A^*(\mu_0, \mu_1) \rangle_1.$$

Above, when we add the measures μ_0 and μ_1, we consider them as measures defined on the product space $V \times V$, where we see the measure μ_0 as a measure defined on the diagonal $D = \{(x, x) : x \in V\}$.

With this, the dual of (P) is the completely positive program

$$
\begin{aligned}
\sup \quad & -\mu_0(D) - \mu_1(\overline{E}) \\
& \mu_0 \in M(D), \ \mu_1 \in M(\overline{E}) \\
& 1 + \mu_0(D) \geq 0 \\
& -\mu_0 - \mu_1 \in \mathcal{CP}_V.
\end{aligned}
$$

To simplify this dual, we define the *support* of a measure μ as follows:

$$
\operatorname{supp}\mu = (V \times V) \setminus O,
$$

where O is the inclusionwise largest open set with $\mu(O) = 0$. Note that O is given by

$$
O = \bigcup_{\substack{W \text{ open in } V \times V \\ \mu(W) = 0}} W.
$$

Then the *dual, completely positive program* equals

$$
\begin{aligned}
\sup \quad & \mu(V \times V) \\
& \mu \in \mathcal{CP}_V \\
& \mu(D) \leq 1 \\
& \operatorname{supp}\mu \subseteq D \cup \overline{E}.
\end{aligned}
$$

One can argue by scaling the inequality constraint $\mu(D) \leq 1$ can be replaced by the equality constraint $\mu(D) = 1$ and therefore we get

(D)
$$
\begin{aligned}
\sup \quad & \mu(V \times V) \\
& \mu \in \mathcal{CP}_V \\
& \mu(D) = 1 \\
& \operatorname{supp}\mu \subseteq D \cup \overline{E}.
\end{aligned}
$$

This completely positive program using measures is a generalization of the finite-dimensional completely positive program for finite graphs $G = (V, E)$, with $V = \{1, \ldots, n\}$, of de Klerk, Pasechnik [14]:

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} X(i,j) \\
& X \in \mathcal{CP}_n \\
& \sum_{i=1}^{n} X(i,i) = 1 \\
& X(i,j) = 0 \qquad \text{for all } \{i,j\} \in E.
\end{aligned}
$$

3.2. **Completely positive formulation.** We next show that the optimal value of problem (D) equals the stability number.

Theorem 3.1. *Let $G = (V, E)$ be a compact topological packing graph. Then the optimal value of the completely positive program (D) is attained and equals $\alpha(G)$.*

Proof. Let λ be the optimal value of (D). For the ease of notation we write α for $\alpha(G)$ in this proof.

Let $x_1, \ldots, x_\alpha \in V$ be a stable set in G of maximal cardinality. Then the measure

$$
\frac{1}{\alpha}\left(\sum_{i=1}^{\alpha} \delta_{x_i}\right) \otimes \left(\sum_{i=1}^{\alpha} \delta_{x_i}\right)
$$

is a feasible solution of (D) with objective value α. Hence, $\lambda \geq \alpha$.

In order to prove the reverse inequality we first show that set $\mathcal{F}_D$ of feasible solutions of (D) is weak* compact. For this define

$$\mathcal{F} = \{t(\mu_0 + \mu_1) : (\mu_0, \mu_1) \in S_1, t \in [1, \alpha]\},$$

where

$$S_1 = \{(\mu_0, \mu_1) \in M(D) \times M(\overline{E}) : \mu_0 + \mu_1 \in \mathcal{CP}_V, \mu_0(D) + \mu_1(\overline{E}) \leq 1\}.$$

By Theorem of Banach-Alaoglu, the set S_1 is weak* compact, so $\mathcal{F}$ is weak* compact as well.

Consider the convex cone

$$\mathcal{M}_G = \{\mu \in \mathcal{CP}_V : \operatorname{supp} \mu \subseteq D \cup \overline{E}\}.$$

It follows from Theorem 2.4 that the extreme rays of $\mathcal{M}_G$ are product measures $\rho \otimes \rho$. Furthermore, since G is a topological packing graph, the extreme rays of $\mathcal{M}_G$ have to be of the form

$$(11) \quad \left(\sum_{i=1}^{N} a_i \delta_{x_i}\right) \otimes \left(\sum_{i=1}^{N} a_i \delta_{x_i}\right) \quad \text{with} \quad a_i \geq 0, \; x_1, \ldots, x_N \text{ a stable set of } G,$$

because the restriction of $\rho \otimes \rho$ to D has finite support since for every point $x \in V$ there is an open neighborhood U of x with $(\rho \otimes \rho)(U \times U) \cap D = (\rho \otimes \rho)(\{(x,x)\})$.

Now let

$$\mu = \left(\sum_{i=1}^{N} a_i \delta_{x_i}\right) \otimes \left(\sum_{i=1}^{N} a_i \delta_{x_i}\right) \in \mathcal{F}_D$$

be a feasible solution of (D) which lies in an extreme ray of $\mathcal{M}_G$. We have

$$\mu(V \times V) = \left(\sum_{i=1}^{N} a_i\right)^2 \quad \text{and} \quad \mu(D) = \sum_{i=1}^{N} a_i^2 = 1.$$

Write $\mu = \nu_0 + \nu_1$ with $\nu_0 \in M(D)$ and $\nu_1 \in M(\overline{E})$. Let s be a real number such that $s(\nu_0 + \nu_1)(V \times V) = 1$. Then setting $\mu_0 = s\nu_0$, $\mu_1 = s\nu_1$, $t = \frac{1}{s}$, shows that $s \in [\frac{1}{\alpha}, 1]$ because of the Cauchy-Schwartz inequality

$$1 \geq s = \frac{1}{\left(\sum_{i=1}^{N} a_i\right)^2} \geq \frac{1}{N \sum_{i=1}^{N} a_i^2} = \frac{1}{N} \geq \frac{1}{\alpha}.$$

Hence $\mu \in \mathcal{F}$, and consequently $\mathcal{F}_D \subseteq \mathcal{F}$. This shows that $\mathcal{F}_D$ is weak* compact because $\mathcal{F}_D$ is closed.

Because of this compactness, the supremum of (D) is attained at an extreme point of $\mathcal{F}_D$. Suppose $(\sum_{i=1}^{N} a_i \delta_{x_i}) \otimes (\sum_{i=1}^{N} a_i \delta_{x_i})$ is a maximizer of (D). Then again by Cauchy-Schwartz we get that

$$\lambda = \left(\sum_{i=1}^{N} a_i\right)^2 \leq N \sum_{i=1}^{N} a_i^2 = N \leq \alpha,$$

and the claim of the theorem follows. $\square$

3.3. **Copositive formulation.** In this section, we prove our main result, Theorem 1.2, by showing that we have strong duality between (P) and (D).

Theorem 3.2. *There is no duality gap between the primal copositive program* (P) *and the dual completely positive program* (D). *In particular, the optimal value of both programs equals* $\alpha(G)$.

For the proof of this theorem we make use of a variant of the zero duality gap theorem of a primal-dual pair of conic linear programs, see Barvinok [2, Chapter IV.7.2]: By dualizing the statement of [2, Problem 3 in Chapter IV.7.2] we see that if the cone

$$\{(d - A^*y, \langle b, y \rangle_2) : y \in M(V) \times M(\overline{E}), d \in \mathbb{R}_{\geq 0} \times \mathcal{CP}_V\}$$

is closed in $\mathbb{R} \times M(V \times V)_{\mathrm{sym}} \times \mathbb{R}$, then there is no duality gap.

To show this closedness condition we need again Lemma 2.5 and the following lemma which is a slight modification of [2, Lemma IV.7.3].

Lemma 3.3. *Let V and W be topological vector spaces, let $\mathcal{K} \subseteq V$ be a cone such that there is a compact set $\mathcal{B} \subseteq V$ with $0 \notin \mathcal{B}$ and $\mathcal{K} = \bigcup_{\lambda \geq 0} \lambda \mathcal{B}$. Let $T : V \to W$ be a continuous linear transformation such that $\ker T \cap \mathcal{K} = \{0\}$. Then $T(\mathcal{K}) \subseteq W$ is a closed convex cone.*

Proof. Obviously $T(\mathcal{K})$ is a convex cone. The set $\mathcal{B}' = T(\mathcal{B})$ is compact, $0 \notin \mathcal{B}'$, and $T(\mathcal{K}) = \bigcup_{\lambda \geq 0} \lambda \mathcal{B}'$. Applying Lemma 2.5 gives that $T(\mathcal{K})$ is closed. $\square$

Now we are ready for the proof of the theorem:

Proof of Theorem 3.2. Consider the continuous linear transformation

$$T(d, y) = (d - A^*y, \langle b, y \rangle_2).$$

We have already seen that the cone has a compact base. Suppose (d, y) lie in the kernel of T. Then the condition $\langle b, y \rangle_2 = 0$ forces y to be zero. This forces $d = 0$ and we can apply Lemma 3.3 to complete the proof of the theorem. $\square$

3.4. **Copositive formulation for the weighted stability number.** In some situations one wishes to consider packing problems with different types of objects, having different sizes; for instance the problem of packing spherical caps having different radii as considered by de Laat, Oliveira, and Vallentin [15]. In these cases it is helpful to use a weighted version of the copositive problem formulation which is presented in the next theorem. We omit its proof here since it is completely analogous to the one of Theorems 3.2 and 3.1. The only difference is that we are now given a continuous weight function $w : V \to \mathbb{R}_{\geq 0}$ for the vertex set, and in our optimization problems we replace the objective function

$$\mu(V \times V) = \int_V \int_V d\mu(x, y) \qquad \text{by} \qquad \int_V \int_V \sqrt{w(x)w(y)} \, d\mu(x, y).$$

Theorem 3.4. *Let $G = (V, E)$ be a compact topological packing graph and let $w : V \to \mathbb{R}_{\geq 0}$ be a continuous weight function for the vertex set. Then the weighted stability number $\alpha_w(G)$ defined by*

$$\alpha_w(G) = \max \left\{ \sum_{x \in S} w(x) : S \text{ stable set of } G \right\}$$

has the following copositive formulation

$$
\begin{aligned}
\alpha_w(G) = \inf \quad & t \\
& t \in \mathbb{R}, \ K \in \mathcal{COP}_V \\
& K(x,x) = t - w(x) \qquad \textit{for all } x \in V \\
& K(x,y) = -\sqrt{w(x)w(y)} \quad \textit{for all } \{x,y\} \notin E.
\end{aligned}
$$

(12)

For the finite case, Bomze [6] showed that the maximum weight clique problem can be formulated as a standard quadratic problem. With the techniques from [5] this in turn can be written as a copositive problem of which (12) is the infinite counterpart.

4. Copositive formulation of the kissing number

In this section we give a copositive formulation of the kissing number problem. We show that in this case the copositive program can be equivalently transformed into a semi-infinite linear program. We start with the original copositive formulation:

$$
\begin{aligned}
\inf \quad & t \\
& t \in \mathbb{R}, \ K \in \mathcal{COP}_{S^{n-1}} \\
& K(x,x) = t - 1 \qquad \text{for all } x \in S^{n-1} \\
& K(x,y) = -1 \qquad \text{for all } x,y \in S^{n-1} \text{ with } x^\mathsf{T} y \in [-1, 1/2].
\end{aligned}
$$

Since the packing graph is invariant under the orthogonal group, also the copositive formulation is invariant under this group. By convexity we can restrict the copositive formulation above to copositive kernels which are invariant under the orthogonal group. So $K(x,y)$ only depends on the inner product $x^\mathsf{T} y$.

By Stone-Weierstrass we know that polynomials lie dense in $C([-1,1])$, so we approximate $K(x,y)$ by $\sum_{k=0}^d c_k (x^\mathsf{T} y)^k$. Then by Lemma 2.2, the copositivity condition $K \in \mathcal{COP}_{S^{n-1}}$ translates to

$$
(13) \qquad \sum_{i=1}^N \sum_{j=1}^N \sum_{k=0}^d c_k (x_i^\mathsf{T} x_j)^k \geq 0 \quad \text{for all } N \in \mathbb{N} \text{ and } x_1, \ldots, x_N \in S^{n-1}.
$$

The other constraints of the above copositive problem translate likewise, and observing that $x^\mathsf{T} x = 1$ for $x \in S^{n-1}$, we get the following semi-infinite linear program whose optimal value converges to the kissing number if the degree d tends to infinity:

$$
\begin{aligned}
\inf \quad & 1 + \sum_{k=0}^d c_k \\
& c_0, \ldots, c_d \in \mathbb{R} \\
& \sum_{i=1}^N \sum_{j=1}^N \sum_{k=0}^d c_k (x_i^\mathsf{T} x_j)^k \geq 0 \quad \text{for all } N \in \mathbb{N} \text{ and } x_1, \ldots, x_N \in S^{n-1} \\
& \sum_{k=0}^d c_k s^k \leq -1 \qquad \qquad \text{for all } s \in [-1, 1/2].
\end{aligned}
$$

We impose the condition $\sum_{k=0}^d c_k s^k \leq -1$ instead of $\sum_{k=0}^d c_k s^k = -1$ to make the problem feasible for finite degree d. It is easy to see that this relaxation is not effecting the optimal value when d tends to infinity.

Note here that all the difficulty of the problem lies in the copositivity constraint (13). In contrast to this, the other constraint $\sum_{k=0}^d c_k s^k \leq -1$ for all $s \in [-1, 1/2]$ is computationally relatively easy. Although it gives infinitely many

linear conditions on the coefficients c_k, it can be modeled equivalently as a semidefinite constraint using the sums of squares techniques for polynomial optimization; see for instance Parrilo [18] and Lasserre [17].

If, instead of requiring copositivity of the invariant kernel

$$(x, y) \mapsto \sum_{k=0}^{d} c_k (x^\mathsf{T} y)^k$$

we impose the weaker constraint that this kernel should be positive semidefinite, then things become considerably simpler. Using harmonic analysis on the unit sphere, by Schoenberg's theorem [23], one can identify this class of kernels explicitly, namely these are the kernels which can be written as

$$(x, y) \mapsto \sum_{k=0}^{d} g_k P_k^{((n-3)/2,(n-3)/2)} (x^\mathsf{T} y) \quad \text{with} \quad g_0, \ldots, g_d \geq 0,$$

where $P_k^{((n-3)/2,(n-3)/2)}$ is the Jacobi polynomial of degree k with parameters $((n-3)/2, (n-3)/2)$. Thus requiring this weaker constraint instead of the copositivity constraints yields the linear programming bound for the kissing number due to Delsarte, Goethals, and Seidel [8]. This bound is known to be tight in a few cases only, namely for $n = 1, 2, 8, 24$.

5. Future work

In this paper we gave a copositive formulation of the stability number of compact topological packing graphs. This condition guarantees in particular that all stable sets are finite. Sometimes one is also interested in stable sets which are infinite, measurable sets. For instance, what is the measurable stability number of the graph on the unit sphere where two vertices are adjacent whenever they are orthogonal? Semidefinite relaxations for problems of this kind have been proposed by Bachoc, Nebe, Oliveira, and Vallentin [1]. However, the bound which one can obtain by this method is very weak for the orthogonality graph on the unit sphere, and it is difficult to find additional valid inequalities to improve it significantly (see DeCorte and Pikhurko [7] for the case $n = 3$). For this reason we think that it would be interesting to derive a copositive formulation for this problem to be able to derive stronger bounds.

Acknowledgements

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