# CONVEX ANALYSIS IN GROUPS AND SEMIGROUPS: A SAMPLER 

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#### Abstract

We define convexity canonically in the setting of monoids. We show that many classical results from convex analysis hold for functions defined on such groups and semigroups, rather than only vector spaces. Some examples and counter-examples are also discussed.


## Part I: Basic convex analysis

## 1. Introduction

The notion of convexity is classical [Roc97], and heavily used in diverse contexts [BV10, Chapter 1]. While normally considered in the concrete setting of vector spaces - either $\mathbb{R}^{d}$ or infinite dimensional - it has often been examined in very general axiomatic form, see [BHT82] and [vdV93]. In the vector space case, $x$ is said to be a convex combination of $x_{1}, \ldots, x_{n}$ if there exist $\alpha_{1}, \ldots, \alpha_{n} \in(0,1)$ such that

$$
\begin{equation*}
x=\sum_{i=1}^{n} \alpha_{i} x_{i}, \sum_{i=1}^{n} \alpha_{i}=1 . \tag{1.1}
\end{equation*}
$$

If we assume for a moment that $\alpha_{i}$ is of the form $\alpha_{i}=\frac{m_{i}}{\sum_{i=1}^{n} m_{i}}$ where $m_{1}, \ldots, m_{n} \in \mathbb{N}=$ $\{1,2, \ldots\}$, then (1.1) becomes

$$
\begin{equation*}
m x=\sum_{i=1}^{n} m_{i} x_{i}, m=\sum_{i=1}^{n} m_{i} \tag{1.2}
\end{equation*}
$$

In (1.1) we must be able to define $\alpha x$ for $\alpha \in \mathbb{R}$ and $x \in X$. More generally, (1.1) can be used whenever $X$ is a module. On the other hand, in (1.2) we use only the additive structure of $X$, i.e., we may assume that $X$ is merely an additive semigroup. (See Section 2 for the exact definitions.) Using (1.2), we show how one can build a canonical theory of convexity for additive groups and semigroups. We refer the reader to [Mur03, vdV93] for more information on abstract convexity in all its manifestations. Some aspects of convex analysis in a more abstract setting have also been studied in [Ham05, JLMS07, LMS04]. Note that in [LMS04] for example, it is only required that a function is convex over geodesic curves (in this case, in the Heisenberg group). Thus, the various notions of convexity do not always coincide. See also Remark 1 in [LMS04].

In a similar fashion to (1.2), one can define convex functions on additive groups and semigroups (again, see Section 2). It is then natural to ask whether one may obtain useful

[^0]analogues of known results for convex functions. It turns out that under only minimal assumptions on the underlying monoid or group, it is possible to reconstruct many classical results from the theory of convex functions such as Hahn-Banach type theorems, Fenchel duality, certain constrained optimisation results, and more. We dedicate Section 3 to exhibiting concrete examples of groups and their convex sets and convex hulls. It turns out that even in simple examples, the structure of convex sets is subtle and can differ significantly from the structure of convex sets in vector spaces.

The rest of the paper is dedicated to generalising classical results of the theory of convexity to more general settings. While many of the results presented here hold when we assume that the underlying space is a module (see Section 2.2), for the sake of concreteness we formulate most of the results for groups and semigroups. In Section 4 we discuss the interpolation of subadditive and convex function. In short, the question (say, in the convex case) is: given two functions $f$ and $g$ with $g \leq f$ and $f$, and $-g$ are convex, can we find an affine function $a$ such that $g \leq a \leq f$. Such questions were studied in [MO53] and generalised in [Kau66]. We show that interpolation is possible for convex functions on semigroups which are semidivisible (see Section 2.2).

Part II of this paper (Sections 5 and 6 ) is dedicated to the study of convex operators between (semi)groups. We define some well known and widely used notions, such as directional derivatives and conjugate functions in the groups setting. In Section 5, we show that some of the best known results, such as the the max formula, sandwich theorems and Fenchel type duality theorems extend to this general setting. Finally, in Section 6 we briefly discuss optimisation over groups before making some concluding remarks in Section 7.

## 2. Convex basics

We define convex sets and functions and examine some basic properties.
2.1. Convexity in algebraic structures. A semiring is a commutative semigroup under addition and a semigroup under multiplication. A (left) semimodule over a semiring is a commutative monoid (i.e., semigroup), satisfying all axioms of a module over a ring except the existence of an additive inverse.

Definition 2.1 (Convex set in semimodule). Assume that $X$ is a semimodule over a semiring $R$, and $A \subseteq X$. Let $r_{1}, \ldots, r_{n} \in R \backslash\{0\}$, and $x_{1}, \ldots, x_{n} \in A$. Assume that there exists $x \in X$ satisfying

$$
r x=\sum_{i=1}^{n} r_{i} x_{i}, r=\sum_{i=1}^{n} r_{i} .
$$

If $x \in A$ for every choice of $n \in \mathbb{N}, r_{1}, \ldots, r_{n} \in R \backslash\{0\}$ and $x_{1}, \ldots, x_{n} \in A$, then $A$ is said to be convex.

Herein we always assume that $\mathbb{N}=\{1,2, \ldots\}$, i.e., all positive integers. If $R$ is a ring, not just a semiring, then we assume it is equipped with a compatible partial order, i.e., that we have $r+r_{1} \leq r+r_{2}$ whenever $r_{1} \leq r_{2}$ and $r \cdot r_{1} \leq r \cdot r_{2}$ whenever $r_{1} \leq r_{2}$ and $r \geq 0$, and in Definition 2.1, we take only elements that are strictly positive. In particular, if $R$ is a field with a compatible partial order, $R_{+}$is the collection of all positive elements, and
$r_{1}, \ldots, r_{n} \in R_{+} \backslash\{0\}$, then we have

$$
\sum_{i=1}^{n} r_{i}=r \Longrightarrow \sum_{i=1}^{n} \frac{r_{i}}{r}=1, r x=\sum_{i=1}^{n} r_{i} x_{i} \Longrightarrow x=\sum_{i=1}^{n} \frac{r_{i}}{r} x_{i},
$$

which gives the standard definition of convexity (e.g., over $\mathbb{R}$ or $\mathbb{Q}$ ). As in vector spaces, we can also define convex cones.

Definition 2.2 (Convex cone in semimodule). A set $A \subseteq X$ is said to be a convex cone if in Definition 2.1 the assumption $\sum_{i=1}^{n} r_{i}=r$ is not imposed.

Every commutative group is a module over the $\mathbb{Z}$. Herein, we will focus on additive groups and semigroups. By a monoid we mean an additive semigroup with a unit. As noted in [Ham05], a monoid with a nontrivial idempotent element cannot be embedded in a group. Clearly every monoid is a semimodule over the semiring $\mathbb{Z}_{+}$. Thus, the elements in Definition 2.1 are positive integers, denoted $m_{j}$ instead of $r_{j}$.

For a general commutative group, one cannot always solve the equation

$$
\begin{equation*}
\left(\sum_{i=1}^{n} m_{i}\right) x=\sum_{i=1}^{n} m_{i} x_{i} . \tag{2.1}
\end{equation*}
$$

Yet, equation (2.1) is very useful in some cases. Thus, we recall the following.
Definition 2.3 (Divisible group). An additive group $X$ is said to be divisible if for every $n \in \mathbb{N}, n X=X$. Alternatively, $X$ is divisible if for every $y \in X$ and for every $n \in \mathbb{N}$, there exists $x \in X$ such that $n x=y$.
Definition 2.4 (Semidivisible group). An additive group is said to be $p$-semidivisible is there exists $p \in \mathbb{N}$ prime such that $p X=X$, and $X$ is said to be semidivisible if it is $p$-semidivisible for some prime $p$.

We can similarly define divisible and semidivisible monoids, as well as divisible and semidivisible semimodules. In particular, all divisible submodules and divisible submonoids are convex cones. A notion which is stronger than the above two is the following.

Definition 2.5 (Uniquely divisible group). An additive group $X$ is said to be uniquely divisible if for every $n \in \mathbb{N}$ and for every $y \in X$, there exists a unique $x \in X$ such that satisfies $n x=y$. Alternatively, $X$ is said to be uniquely divisible if it is divisible and for every $n \in \mathbb{N}$, the map $x \mapsto n x$ is an injective map.

Similarly, we can consider the following notion.
Definition 2.6 (Uniquely divisible monoid). A monoid $X$ is said to be uniquely divisible if it is divisible and for every $n \in \mathbb{N}$, the map $x \mapsto n x$ is an injective map.

Note that in monoids, singletons are convex if and only if the monoid is uniquely semidivisible, since we want $\sum_{i=1}^{n} m_{i} x=\left(\sum_{i=1}^{n} m_{i}\right) x$ to be the same as $\left(\sum_{i=1}^{n} m_{i}\right) y$ if and only if $x=y$. Divisibility and semidivisibility are important for the structure theory of infinite abelian groups. See for example [Fuc70, Rob96]. We also refer the reader to [KTW11,Law10] for some more recent examples relating to divisible groups.

Remark 2.1. A subgroup of a divisible group need not be divisible, or even semidivisible. As a simple example, take $X=\mathbb{R}$ and $\mathbb{Z} \subseteq X$.

Remark 2.2 (Divisibility in abelian groups). It is known that every abelian group is a subgroup of a divisible group. Moreover, the quotient of a divisible group is again divisible, e.g., $\mathbb{R} / \mathbb{Z}$ and $\mathbb{Q} / \mathbb{Z}$. Also, the torsion subgroup $T_{G}$ (of all elements of finite order) is divisible and the quotient $G / T_{G}$ is a $\mathbb{Q}$-vector space. Finally, the divisible groups are exactly the injective abelian groups.
Remark 2.3. If $X$ is $p$-semidivisible, i.e., $p X=X$ then for every $l \in \mathbb{N}$ we have $p^{l} X=$ $p^{l-1}(p X)=p^{l-1} X=\cdots=p X=X$.

Remark 2.4. Assume $X=n X$ for some $n \in \mathbb{N}$. Write $n=p_{1}^{m_{1}} \cdots p_{l}^{m_{l}}$. Then $X=$ $p_{1}^{m_{1}} \cdots p_{l}^{m_{l}} X=p_{1}\left(p_{1}^{m_{1}-1} \cdots p_{l}^{m_{l}}\right) X \subseteq p_{1} X \subseteq X$ and so $X=p_{1} X$. Thus, for us the assumption that $p$ is prime in Definition 2.4 plays no significant rôle.

As mentioned above, convexity has an entirely axiomatic approach. We refer the reader to [vdV93] for more information about this rich topic. We will present only the basic definitions and the return to the more concrete case of convexity in algebraic structures.

Definition 2.7 (Convexity). A collection $\mathcal{C}$ of subsets of a set $X$ is said to be a convexity (also an alignment), if it contains the empty set and is closed under intersections and directed unions.

It is straightforward to check the convex sets defined by Definition 2.1 form a convexity. Given the Definition 2.7, we can also define the convex hull.

Definition 2.8 (Convex hull). If $A \subseteq X$, define

$$
\operatorname{conv}(A)=\bigcap_{\substack{A \subseteq B \\ B \text { convex }}} B
$$

The convex hull is a closure operator, i.e., it satisfies the following: 1. $A \subseteq B \Longrightarrow$ $\operatorname{conv}(A) \subseteq \operatorname{conv}(B) ; 2 . A \subseteq \operatorname{conv}(A) ; 3 \cdot \operatorname{conv}(\operatorname{conv}(A))=\operatorname{conv}(A) ; 4 \cdot \operatorname{conv}(\emptyset)=\emptyset ; 5$. Closure under intersections and directed unions.

In the case of monoids, we have the following concrete result.
Proposition 2.1 (Convex hull in monoid). If $X$ is a monoid and $A \subseteq X$, the convex hull of $A$ is given by

$$
\begin{equation*}
\operatorname{conv}(A)=\left\{x \in X \mid m x=\sum_{i=1}^{n} m_{i} x_{i}, x_{i} \in A, m_{i} \in \mathbb{N}, m=\sum_{i=1}^{n} m_{i}\right\} . \tag{2.2}
\end{equation*}
$$

Proof. Clearly the set on the right side of (2.2) is convex and contains $A$. If $A \subseteq B$ and $B$ is convex, then $B$ contains the set on the right side of (2.2).

A map $T: X_{1} \rightarrow X_{2}$ between two monoids is said to be additive if $T\left(x_{1}+x_{2}\right)=T x_{1}+T x_{2}$ for all $x_{1}, x_{2} \in X_{1}$. It is well known that a linear image of a convex set in a vector space is again convex. We establish a similar fact for additive bijections between monoids.

Proposition 2.2 (Convexity under additive bijection). Assume that $X_{1}, X_{2}$ are monoids and $T: X_{1} \rightarrow X_{2}$ is an additive bijection. If $A \subseteq X_{1}$ is convex, then $T A \subseteq X_{2}$ is convex.
Proof. Assume that $m, m_{1}, \ldots, m_{n} \in \mathbb{N}$ and $y_{1}, \ldots, y_{n} \in T A, y \in X_{2}$ are such that $m y=$ $\sum_{i=1}^{n} m_{i} y_{i}, m=\sum_{i=1}^{n} m_{i}$. Since $T$ is onto, there exists $x \in X_{1}$ such that $T x=y$. Since
$y_{1}, \ldots, y_{n} \in T A$, there exist $x_{1}, \ldots, x_{n} \in A$ such that $y_{1}=T x_{1}, \ldots, y_{n}=T x_{n}$. Hence, we have $T(m x)=m T x=m y=\sum_{i=1}^{n} m_{i} y_{i}=\sum_{i=1}^{n} m_{i} T x_{i}=T\left(\sum_{i=1}^{n} m_{i} x_{i}\right)$. Since $T$ is injective, we have $m x=\sum_{i=1}^{n} m_{i} x_{i}$. Since $x_{1}, \ldots, x_{n} \in A$ and $A$ is convex, it follows that $x \in X$. Thus, $y=T x \in T A$ and $T A$ is convex.

Remark 2.5. If $X_{1}$ is divisible then in the proof of Proposition 2.2 we always have $y$ such that $m y=\sum_{i=1}^{n} m_{i} y_{i}$. If $T$ is additive and $A$ is convex, we must have $y \in T A$. Hence, in this case we need not assume that $T$ is a bijection.

For the inverse image, we have a more general result.
Proposition 2.3 (Convexity under inverse additive map). Assume that $X_{1}$ and $X_{2}$ are monoids and $T: X_{1} \rightarrow X_{2}$ is additive. Assume that $A \subseteq X_{2}$ is convex. Then $T^{-1} A \subseteq X_{1}$ is convex.

Proof. Assume that $x_{1}, \ldots, x_{n} \in T^{-1} A=\{x \mid T x \in A\}, m, m_{1}, \ldots, m_{n} \in \mathbb{N}$ and $x \in X_{1}$ are such that $m x=\sum_{i=1}^{n} m_{i} x_{i}, m=\sum_{i=1}^{n} m_{i}$. Since $x_{1}, \ldots, x_{n} \in T^{-1} A$, we have $T x_{1}, \ldots, T x_{n} \in$ $A$. Since $T$ is additive, we have $\sum_{i=1}^{n} m_{i} T x_{i}=T\left(\sum_{i=1}^{n} m_{i} x_{i}\right)=T(m x)=m T x$. Since $A$ is convex, we have $T x \in A$. Thus, $x \in T^{-1} A$, which completes the proof.

As we shall see, studying convexity in such a general setting also brings about a better understanding of this notion in the standard setting of vector spaces. One complaint about convexities is that there are too many of them and that in different settings one has to adjoin many additional axioms. This is one more motivation for the current study.
2.2. Classes of functions. Here we consider several classes of functions defined on semimodules, particularly on monoids, classes which are well studied in the vector spaces setting. In order to define convex functions, we need to consider an ordered semimodule, i.e., a semimodule with a partial order $\leq$. Given a semimodule $X$ over a semiring $R$, we say that a partial order $\leq$ is compatible with the module operations, if $r x_{1} \leq r x_{2}, x+x_{1} \leq x+x_{2}$ for all $x \in X, r \in R$, whenever $x_{1} \leq x_{2}$.

Definition 2.9 (Convex function). Let $X, Y$ be a semimodules over a semiring $R$. Assume that $Y$ is equipped with a compatible partial order $\leq$. A function $f: X \rightarrow Y$ is said to be convex if for every $n \in \mathbb{N}$, every $r_{1}, \ldots, r_{n} \in R \backslash\{0\}$ and every $x_{1}, \ldots, x_{n} \in X$,

$$
\begin{equation*}
r f(x) \leq \sum_{i=1}^{n} r_{i} f\left(x_{i}\right) \tag{2.3}
\end{equation*}
$$

for every $x$ satisfying,

$$
r x=\sum_{i=1}^{n} r_{i} x_{i}, r=\sum_{i=1}^{n} r_{i} .
$$

$f: X \rightarrow Y$ is said to be concave if $-f$ is convex. Clearly the sum of two convex functions is convex.

Remark 2.6. As in Definition 2.1, if we have modules over a ring rather that over a semiring, we assume we have a partial order on the ring that is compatible with the ring operations, and then in Definition 2.9, we consider only strictly positive elements from the ring.

Remark 2.7. We often consider a maximal element in $Y, \infty$. Also, in the case where $Y$ is a module, not just a semimodule, we may also consider a minimal element $-\infty$. In order for (2.3) to make sense, we assume for a convex function that $\infty-\infty=0 \cdot \infty=\infty$. $\diamond$

Definition 2.10 (Affine function). Let $X, Y$ be semimodules over a semiring $R$. Then $f: X \rightarrow Y$ is said to be affine if for every $n \in \mathbb{N}$, every $r_{1}, \ldots, r_{n} \in R \backslash\{0\}$ and every $x_{1}, \ldots, x_{n} \in X$,

$$
r f(x)=\sum_{i=1}^{n} r_{i} f\left(x_{i}\right),
$$

whenever $x \in X$ satisfies,

$$
r x=\sum_{i=1}^{n} r_{i} x_{i}, r=\sum_{i=1}^{n} r_{i} .
$$

Clearly every affine function is both convex and concave. For an affine function, we again cannot allow it to attain $\pm \infty$.

We can, however, consider the following notion.
Definition 2.11 (Generalised affine function). Assume that $X, Y$ are semimodules over a semiring $R$. Possibly $Y$ contains a maximal element $\infty$ or a minimal element $-\infty$. A function $f: X \rightarrow Y \cup\{ \pm \infty\}$ is said to be generalized affine if it is both convex and concave.

Generalised affine functions are either affine or 'very' infinite.
Proposition 2.4. Assume that $X$ and $Y$ are groups, and $a: X \rightarrow Y \cup\{ \pm \infty\}$ is generalised affine. Then either $a$ is everywhere finite, or $a=+\infty$, or $a=-\infty$, or a attains both values $+\infty$ and $-\infty$.

Proof. Assume that $a$ is not everywhere finite, and that it is not identically $+\infty$ or $-\infty$. Assume for example that there exist $x_{1}, x_{2} \in X$ such that $a\left(x_{1}\right)>\alpha$ for all $\alpha \in \mathbb{R}$ and $a\left(x_{2}\right)$ is finite. We have $2 x_{2}=\left(x_{2}+\left(x_{1}-x_{2}\right)\right)+\left(x_{2}-\left(x_{1}-x_{2}\right)\right)=x_{1}+\left(2 x_{2}-x_{1}\right)$, and so since $a$ is concave we have $2 a\left(x_{2}\right) \geq a\left(x_{1}\right)+a\left(2 x_{2}-x_{1}\right)>\alpha+a\left(2 x_{2}-x_{1}\right)$. Therefore we must have $a\left(2 x_{2}-x_{1}\right)=-\infty$. If we assume $a\left(x_{1}\right)=-\infty$ rather than $+\infty$, the proof is similar.

Definition 2.12 (Subadditive function). Assume that $X, Y$ are semimodules over a semiring $R$, and assume that $Y$ is equipped with a partial order $\leq$. A function $f: X \rightarrow Y \cup\{ \pm \infty\}$ is said to be subadditive if for every $x, y \in X$,

$$
f(x+y) \leq f(x)+f(y)
$$

The function $x \mapsto \sqrt{x}$ is subadditive on $[0,+\infty)$ but not convex. As we will mostly be concerned with groups and monoids, we now focus on functions with subadditive properties over $\mathbb{N}$.

Definition 2.13 ( $\mathbb{N}$-sublinear functions). Assume that $X, Y$ are semimodules over a semiring $R$, and assume that $Y$ is equipped with a partial order $\leq$. A function $f: X \rightarrow Y \cup\{ \pm \infty\}$ is said to be $\mathbb{N}$-sublinear if it is subadditive and in addition it is positively homogeneous, i.e., $f(m x)=m f(x)$ for every $x \in X$ and every $m \in \mathbb{N} \cup\{0\}$.
Definition 2.14 (Generalised $\mathbb{N}$-linear function). Assume that $X, Y$ are as in Definition 2.13. A function $f: X \rightarrow Y \cup\{ \pm \infty\}$ is said to be generalised $\mathbb{N}$-linear if both $f$ and $-f$ are $\mathbb{N}$ sublinear.

If $f$ is a generalised $\mathbb{N}$-linear function and $f$ is finite, then for every choice of positive integers $m_{1}, \ldots, m_{n} \in \mathbb{N}$, we have $f\left(\sum_{i=1}^{n} m_{i} x_{i}\right)=\sum_{i=1}^{n} m_{i} f\left(x_{i}\right)$. The functions that satisfy this property are exactly the additive functions on semimodules over $\mathbb{Z}_{+}$.

If $f$ is $\mathbb{N}$-sublinear and $m x=\sum_{i=1}^{n} m_{i} x_{i}$, where $m=\sum_{i=1}^{n} m_{i}$, then

$$
m f(x)=f(m x)=f\left(\sum_{i=1}^{n} m_{i} x_{i}\right) \leq \sum_{i=1}^{n} f\left(m_{i} x_{i}\right)=\sum_{i=1}^{n} m_{i} f\left(x_{i}\right) .
$$

In particular, every $\mathbb{N}$-sublinear function on a monoid is convex. Also we have the following.
Proposition 2.5. Assume that $X$ is a monoid, $(Y, \leq)$ a monoid with a compatible lattice order $\leq$, and $f_{1}, \ldots, f_{k}: X \rightarrow Y \cup\{ \pm \infty\}$ are convex ( $\mathbb{N}$-sublinear, subadditive). Then the function $\max \left\{f_{1}, \ldots, f_{k}\right\}$ is also convex $(\mathbb{N}$-sublinear, subadditive).

Proof. If $f_{1}, \ldots, f_{k}$ are convex and $m, m_{1}, \ldots, m_{n} \in \mathbb{N}, x, x_{1}, \ldots, x_{n} \in X$ are such that $m x=\sum_{i=1}^{n} m_{i} x_{i}, m=\sum_{i=1}^{n} m_{i}$, then

$$
\begin{aligned}
m \cdot \max _{1 \leq j \leq k}\left\{f_{j}(x)\right\} & =\max _{1 \leq j \leq k}\left\{m f_{j}(x)\right\} \\
& \leq \max _{1 \leq j \leq k}\left\{\sum_{i=1}^{n} m_{i} f_{j}\left(x_{i}\right)\right\} \\
& \stackrel{(*)}{\leq} \sum_{i=1}^{n} m_{i} \cdot \max _{1 \leq j \leq k}\left\{f_{j}\left(x_{i}\right)\right\} .
\end{aligned}
$$

In $(*)$ we used the fact that $\leq$ is a lattice order, compatible with the group operations on $Y$. The case of sublinear or subadditive functions is easy. We omit the proof.

Proposition 2.6. Assume that $X, Y$ are monoids. Then it suffices in Definition 2.9 that $r=p^{l}$ for a fixed prime $p$ and all $l \in \mathbb{N}$.

Proof. Indeed, if $r \neq p^{l}$, then there exists $l \in \mathbb{N}$ such that $r<p^{l}$. Thus,

$$
\left(p^{l}-r\right) x+\sum_{i=1}^{n} r_{i} x_{i}=p^{l} x
$$

By the convexity property,

$$
p^{l} f(x) \leq\left(p^{l}-r\right) f(x)+\sum_{i=1}^{n} r_{i} f\left(x_{i}\right),
$$

which gives

$$
r f(x) \leq \sum_{i=1}^{n} r_{i} f\left(x_{i}\right)
$$

as required.
Proposition 2.6 implies the following.
Proposition 2.7. Assume that $X, Y$ are monoids. Assume that $f: X \rightarrow Y$ is subadditive and there exists $p \in \mathbb{N}$ such that $f(p x)=p f(x)$ for every $x \in X$, then $f$ is convex. If $Y$ is a group, then $f$ is in fact $\mathbb{N}$-sublinear.

Proof. By Proposition 2.6, it is enough to assume in Definition 2.9 that $r=p^{l}, l \in \mathbb{N}$. Assume then that $p^{l} x=\sum_{i=1}^{n} m_{i} x_{i}$. We have,

$$
p^{l} f(x) \stackrel{(*)}{=} f\left(p^{l} x\right)=f\left(\sum_{i=1}^{n} m_{i} x_{i}\right) \stackrel{(* *)}{\leq} \sum_{i=1}^{n} m_{i} f\left(x_{i}\right)
$$

where in $(*)$ we used the homogeneity assumption on $f$, and in $(* *)$ we used the subadditivity of $f$. To prove the second assertion, let $m \in \mathbb{N}$. Then there exist $m^{\prime}, l \in \mathbb{N}$ such that $m+m^{\prime}=p^{l}$. Thus, we have

$$
\left(m+m^{\prime}\right) f(x)=f\left(\left(m+m^{\prime}\right) x\right) \leq f(m x)+m^{\prime} f(x) \leq\left(m+m^{\prime}\right) f(x)
$$

Thus, we have

$$
\left(m+m^{\prime}\right) f(x)=f(m x)+m^{\prime} f(x),
$$

and since $Y$ is a group, this implies that $f(m x)=m f(x)$ for all $m \in \mathbb{N}$ and all $x \in X$. This complete the proof.
2.3. Properties of convex functions. It is well known that a convex function on a (semi)normed vector space is continuous at $x_{0}$ if and only if $f$ is bounded from above in a neighbourhood of $x_{0}$. If the space is normed, we derive a Lipschitz condition. See [BV10, Zăl02]. We establish a similar fact for convex functions on topological monoids into $[-\infty, \infty]$. For a set $B \subseteq X$ in an additive group and $m \in \mathbb{N}$ define $\frac{1}{m} B=\{x \mid m x \in B\}$. It is straightforward to show that if $B$ is convex, $\frac{1}{m} B$ is convex for all $m \in \mathbb{N}$. Also, a set $B \subseteq X$ is said to be symmetric if $-B=B$. Again, if $B$ is symmetric, then $\frac{1}{m} B$ is symmetric. We have the following.

Proposition 2.8. Let $X$ be an additive group, $f: X \rightarrow[-\infty, \infty]$ a convex function, and assume that there is a symmetric $B \subseteq X$ and $M \in \mathbb{R}$ such that $f(x) \leq f\left(x_{0}\right)+M$ for all $x \in x_{0}+B$. Then for every $y \in \frac{1}{m} B$, we have $\left|f\left(x_{0}+y\right)-f\left(x_{0}\right)\right| \leq \frac{M}{m}$.
Proof. First, note that if $u \in B$ then $-u \in B$ and by convexity we have $2 f\left(x_{0}\right) \leq f\left(x_{0}+\right.$ $u)+f\left(x_{0}-u\right) \leq f\left(x_{0}+u\right)+f\left(x_{0}\right)+M$ and so $f\left(x_{0}+u\right) \geq f\left(x_{0}\right)-M$. If $f\left(x_{0}\right)=-\infty$ then $f=-\infty$ on $x_{0}+B$. Assume then that $f\left(x_{0}\right)>-\infty$. Let $y \in \frac{1}{m} B$. Then there exists $u \in B$ such that $m y=u$. Thus, we have $m\left(x_{0}+y\right)=\left(x_{0}+u\right)+(m-1) x_{0}$ and then using convexity of $f$ gives $m f\left(x_{0}+y\right) \leq f\left(x_{0}+u\right)+(m-1) f\left(x_{0}\right) \leq M$. This gives $f\left(x_{0}+y\right)-f\left(x_{0}\right) \leq \frac{1}{m}\left(f\left(x_{0}+u\right)-f\left(x_{0}\right)\right) \leq \frac{M}{m}$. Also, by convexity, we have $f\left(x_{0}\right)-f\left(x_{0}+y\right) \leq$ $f\left(x_{0}-y\right)-f\left(x_{0}\right) \leq \frac{M}{m}$, which completes the proof.

In a topological group the group operations are continuous, and we obtain:
Corollary 2.1 (Continuity). Assume that $X$ is a topological group and $f: X \rightarrow[-\infty, \infty]$ is convex. Then $f$ is bounded from above in around $x_{0}$ if and only if $f$ is continuous at $x_{0}$.

We next show convex minorants inherit continuity of a majorant.
Corollary 2.2 (Minorants). Assume that $X$ is a topological group and $f, g: X \rightarrow[-\infty, \infty]$. Suppose that $g$ is bounded above in a neighbourhood of $x_{0}, f$ is a convex minorant of $g$ and $f\left(x_{0}\right)$ is finite. Then $f$ is continuous at $x_{0}$.

Proposition 2.9 (Three-slope lemma for monoids). Let $X$ be a monoid, and $x, x_{1}, x_{2} \in X$, $m_{1}, m_{2} \in \mathbb{N}$ such that $\left(m_{1}+m_{2}\right) x=m_{1} x_{1}+m_{2} x_{2}$. Then for any convex function $f: X \rightarrow$ $(-\infty, \infty]$ we have

$$
\frac{f(x)-f\left(x_{1}\right)}{m_{2}} \leq \frac{f\left(x_{2}\right)-f\left(x_{1}\right)}{m_{1}+m_{2}} \leq \frac{f\left(x_{2}\right)-f\left(x_{1}\right)}{m_{1}} .
$$

Proof. By convexity, we have $\left(m_{1}+m_{2}\right) f(x) \leq m_{1} f\left(x_{1}\right)+m_{2} f\left(x_{2}\right)$, from which both inequalities follow easily.

Except in a divisible setting we do not capture convexity using only three points - we can not induct.

Proposition 2.10 (Monotone composition). Assume that $X$ is a monoid. If $f: X \rightarrow$ $(-\infty, \infty]$ is sublinear and increasing and $g: X \rightarrow(-\infty,+\infty)$ is convex and non-decreasing, then $f \circ g$ is also convex.
Proof. Assume that $m x=\sum_{i=1}^{n} m_{i} x_{i}, m_{i} \in \mathbb{N}, m=\sum_{i=1}^{n} m_{i}$. Then,

$$
m f(g(x))=f(m g(x)) \leq f\left(m_{1} g\left(x_{1}\right)+\cdots+m_{n} g\left(x_{n}\right)\right) \leq \sum_{i=1}^{n} m_{i} f\left(g\left(x_{i}\right)\right)
$$

as required.
Remark 2.8 (Midpoint convexity and measurability). It is well known that measurability forces a midpoint convex function on $\mathbb{R}$ to be convex and an additive function to be linear. There are certainly analogous results to be discovered in appropriate monoids, see for example [Ros09].
2.4. Operations on functions. We next extend some well-known vector operations on convex and subadditive functions.

Definition 2.15 (Subadditive and sublinear minorants). Assume that $X$ is a monoid and $f: X \rightarrow(-\infty, \infty]$. Define

$$
p(x)=\inf \left\{\sum_{i=1}^{n} f\left(x_{i}\right) \mid \sum_{i=1}^{n} x_{i}=x, n \in \mathbb{N}\right\}
$$

Then $p$ is the largest function satisfying $p \leq f$ and also $p(x+y) \leq p(x)+p(y)$. Define also

$$
p o(x)=\inf \left\{\left.\frac{p(m x)}{m} \right\rvert\, m \in \mathbb{N}\right\},
$$

where $p$ is defined as above.
Now po is positively homogeneous as we have

$$
\begin{aligned}
p o\left(m_{0} x\right) & =m_{0} \inf \left\{\left.\frac{1}{m_{0} m} \sum_{i=1}^{n} f\left(x_{i}\right) \right\rvert\, m \in \mathbb{N}, \sum_{i=1}^{n} x_{i}=m_{0} x\right\} \\
& =m_{0} \inf \left\{\left.\frac{1}{m} \sum_{i=1}^{n} f\left(x_{i}\right) \right\rvert\, m \in \mathbb{N}, \sum_{i=1}^{n} x_{i}=x\right\}
\end{aligned}
$$

where the last equality holds since for every $x_{1}, \ldots, x_{n} \in X$, we can choose $x_{1}^{\prime}, \ldots, x_{n^{\prime}}^{\prime} \in X$ satisfying $\frac{1}{m} \sum_{i=1}^{n} f\left(x_{i}\right)=\frac{1}{m_{0} m} \sum_{i=1}^{n^{\prime}} f\left(x_{i}^{\prime}\right)$. Also po is subadditive since

$$
\frac{1}{m_{1}} \sum_{i=1}^{n} f\left(x_{i}\right)+\frac{1}{m_{2}} \sum_{i=1}^{n^{\prime}} f\left(x_{i}^{\prime}\right)=\frac{1}{m_{1} m_{2}}\left(\sum_{i=1}^{n} m_{2} f\left(x_{i}\right)+\sum_{i=1}^{n^{\prime}} m_{1} f\left(x_{i}^{\prime}\right)\right)
$$

where $\sum_{i=1}^{n} x_{i}=m_{1} x, \sum_{i=1}^{n^{\prime}} x_{i}^{\prime}=m_{2} y$. Choosing a finite index set $I$ which is $m_{2}$ copies of each $x_{i}$ for $1 \leq i \leq n$ and $m_{1}$ copies of each $x_{i}^{\prime}$ for $1 \leq i \leq n^{\prime}$ we get $\sum_{i \in I} x_{i}=m_{1} m_{2}(x+y)$. Thus,

$$
\frac{1}{m_{1}} \sum_{i=1}^{n} f\left(x_{i}\right)+\frac{1}{m_{2}} \sum_{i=1}^{n^{\prime}} f\left(x_{i}^{\prime}\right)=\frac{1}{m_{1} m_{2}} \sum_{i \in I} f\left(x_{i}\right) \geq p o(x+y) .
$$

Taking infima over $m_{1}, m_{2}$ implies that $p o$ is sublinear.
Definition 2.16 ( $\mathbb{N}$-Sublinear minorant). Assume that $X$ is a monoid and $f, g: X \rightarrow$ $(-\infty, \infty]$. Define

$$
f \wedge g(x)=\inf \left\{\left.\frac{n_{1} f\left(x_{1}\right)+n_{2} g\left(x_{1}\right)}{n} \right\rvert\, n_{1} x_{1}+n_{2} x_{2}=n x\right\} .
$$

It is straightforward to check that if $f, g$ are $\mathbb{N}$-sublinear, so is $f \wedge g$.

## 3. Examples

Example 3.1 (Vector spaces). If $X$ is a real vector space, then by definition, $x \in \operatorname{conv}(A)$ if for every $n \in \mathbb{N}$, every $\alpha_{1}, \ldots, \alpha_{n} \in(0,1)$ and every $x_{1}, \ldots, x_{n} \in A$,

$$
\left(\sum_{i=1}^{n} \alpha_{i}\right) x=\sum_{i=1}^{n} \alpha_{i} x_{i}
$$

Taking $\beta_{i}=\frac{\alpha_{i}}{\sum \alpha_{i}}>0$, this is equivalent to

$$
x=\sum_{i=1}^{n} \beta_{i} x_{i}, \sum_{i=1}^{n} \beta_{i}=1
$$

which is the standard definition of a convex hull in a vector space over $\mathbb{R}$.
Example 3.2 ( $\mathbb{R}$ as a $\mathbb{Q}$-module). Consider $X=\mathbb{R}$ as a vector space over $\mathbb{Q}$. In such case $x \in \operatorname{conv}(A)$ if for every $n \in \mathbb{N}$, every $q_{1}, \ldots, q_{n} \in \mathbb{Q}_{+} \backslash\{0\}$ and every $x_{1}, \ldots, x_{n} \in A$,

$$
q x=\sum_{i=1}^{n} q_{i} x_{i}, q=\sum_{i=1}^{n} q_{i}
$$

which is equivalent to

$$
x=\sum_{i=1}^{n} q_{i}^{\prime} x_{i}, \sum_{i=1}^{n} q_{i}^{\prime}=1, q_{i}^{\prime} \in[0,1] \cap \mathbb{Q}
$$

i.e., we take only rational convex combinations.

We now present examples of monoids and of the behaviour of the hull operator.

Example 3.3 (The lattice $\mathbb{Z}^{d}$ ). Consider $X=\mathbb{Z}^{d}$ with the addition induced from $\mathbb{R}^{d}$. For every $A \subseteq X$, we have

$$
\begin{equation*}
\operatorname{conv}_{\mathbb{Z}^{d}}(A)=\operatorname{conv}_{\mathbb{R}^{d}}(A) \cap \mathbb{Z}^{d} \tag{3.1}
\end{equation*}
$$

where $\operatorname{conv}_{\mathbb{R}^{d}}(A)$ is the standard convex hull of $A$ in $\mathbb{R}^{n}$. To see this, first note that if $x \in \operatorname{conv}_{\mathbb{Z}^{d}}(A)$, then there exist $x_{1}, \ldots, x_{n} \in A$, and $m_{1}, \ldots, m_{n}, m \in \mathbb{N}$ such that $m x=$ $\sum_{i=1}^{n} m_{i} x_{i}, m=\sum_{i=1}^{n} m_{i}$. This implies that

$$
x=\sum_{i=1}^{n} \frac{m_{i}}{m} x_{i}, \sum_{i=1}^{n} \frac{m_{i}}{m}=1
$$

which means that $x \in \operatorname{conv}_{\mathbb{R}^{d}}(A)$, and so $\operatorname{conv}_{\mathbb{Z}^{d}}(A) \subseteq \operatorname{conv}_{\mathbb{R}^{d}}(A) \cap \mathbb{Z}^{d}$. To prove to other inclusion, use induction on the dimension. If $d=1$, and $x \in \operatorname{conv}_{\mathbb{R}}(A) \cap \mathbb{Z}$, then $x$ is an integer which is also a convex combination of two other integers $x_{1}, x_{2}$. Therefore, we can write $x=q_{1} x_{1}+q_{2} x_{2}$ with $q_{1}, q_{2} \in \mathbb{Q}$, and so there exist $m_{1}, m_{2}, m \in \mathbb{Z}$ such that $m x=m_{1} x_{1}+m_{2} x_{2}$ and $m=m_{1}+m_{2}$. To prove the general case, assume that $x \in \operatorname{conv}_{\mathbb{R}^{d}}(A) \cap \mathbb{Z}^{d}$. Then there exist $x_{1}, \ldots, x_{n} \in A$ and $\alpha_{1}, \ldots, \alpha_{n} \geq 0$ with $\sum_{i=1}^{n} \alpha_{i}=1$ such that $x=\sum_{i=1}^{n} \alpha_{i} x_{i}$. By Carathéodory's Theorem [Mat02], we can write $x=\sum_{i=1}^{n^{\prime}} \alpha_{j} x_{j}$, where $n^{\prime} \leq d+1$ (we might have to rearrange the points $\left.x_{1}, \ldots, x_{n}\right)$. If $\operatorname{dim}\left(\operatorname{span}\left\{x_{1}, \ldots, x_{n^{\prime}}\right\}\right)<d$, use the induction hypothesis to conclude that we can write $x=\sum_{i=1}^{n^{\prime}} q_{i} x_{i}$, with $q_{i} \in \mathbb{Q}$. Otherwise, we have the following linear system.

$$
\left[\begin{array}{cccc}
x_{1} & x_{2} & \ldots & x_{d+1} \\
1 & 1 & \ldots & 1
\end{array}\right]\left[\begin{array}{c}
\alpha_{1} \\
\vdots \\
\alpha_{d+1}
\end{array}\right]=x
$$

where $x_{1}, \ldots, x_{d+1}$ are written as column vectors. In this case, one can show that the system has a unique solution. Thus the matrix is invertible. Since the matrix has integer coefficients, it follows that the $q_{i}$ 's are rational. And so once again we can write $x=\sum_{1}^{n^{\prime}} q_{i} x_{i}$ with $q_{i} \in \mathbb{Q}$, which implies that $x \in \operatorname{conv}_{\mathbb{Z}^{d}}(A)$.
Example 3.4 (General lattices in $\mathbb{R}^{d}$ ). We say that $v_{1}, \ldots, v_{k} \in \mathbb{R}^{d}$ are independent over $\mathbb{Z}$ if

$$
\sum_{i=1}^{k} m_{i} v_{i}=0, m_{i} \in \mathbb{Z} \Longrightarrow m_{i}=0
$$

Assume that $\Gamma=\operatorname{span}_{\mathbb{Z}}\left\{v_{1}, \ldots, v_{k}\right\}$, where $v_{1}, \ldots, v_{k} \in \mathbb{R}^{d}$ are independent over $\mathbb{Z}$. Let $T: \mathbb{R}^{k} \rightarrow \mathbb{R}^{d}$ be defined as

$$
T\left(\alpha_{1}, \ldots, \alpha_{k}\right)=\sum_{i=1}^{k} \alpha_{i} v_{i}
$$

$T$ is linear and $T\left(\mathbb{Z}^{k}\right)=\Gamma$. Also, since $v_{1}, \ldots, v_{k}$ are independent over $\mathbb{Z}$, it follows that $\left.T\right|_{\mathbb{Z}^{k}}$ is invertible. Finally, since $\Gamma$ is a $\mathbb{Z}$-module, it follows from Proposition 2.1 that

$$
\operatorname{conv}_{\Gamma}(A)=\left\{\sum_{i=1}^{n} q_{i} a_{i} \mid a_{i} \in A, q_{i} \in \mathbb{Q} \cap[0,1], \sum_{i=1}^{n} q_{i}=1\right\}
$$

Hence,

$$
\begin{aligned}
\operatorname{conv}_{\Gamma}(A) & =T\left(\operatorname{conv}_{\mathbb{Z}^{k}}\left(T^{-1} A\right)\right) \\
& \stackrel{(*)}{=} T\left(\operatorname{conv}_{\mathbb{R}^{k}}\left(T^{-1} A\right) \cap \mathbb{Z}^{k}\right) \\
& \stackrel{(* *)}{=} T\left(\operatorname{conv}_{\mathbb{R}^{k}}\left(T^{-1} A\right)\right) \cap T \mathbb{Z}^{k} \\
& \stackrel{(* * *)}{=} \operatorname{conv}_{\mathbb{R}^{d}}(A) \cap \Gamma,
\end{aligned}
$$

where in $(*)$ we used Example 3.3, in $(* *)$ we used the invertibility of $T$ over $\mathbb{Z}^{k}$, and in $(* * *)$ we used the linearity of $T$.
Example 3.5 (Dyadic rationals). Let $X$ be the rational numbers of the form $\frac{m}{2^{n}}$, where $m, n \in \mathbb{Z}$. We have that $X$ is 2 -semidivisible as $X=2 X$, since $\frac{m}{2^{n}}=2 \frac{m}{2^{n-1}}$, but for any odd number $k$ we do not have $1=k \cdot \frac{m}{2^{n}}$. Thus, $X$ is not divisible.
Example 3.6 (Arctan semigroup). Let $X=([0, \infty), \oplus)$ with addition defined by

$$
a \oplus b=\frac{a+b}{1+a b} .
$$

Note that if $a, b \neq 0$ then $a \oplus b=\frac{1}{a} \oplus \frac{1}{b}$. The unit is 0 as $a \oplus 0=a$. Also, for all $a \geq 0$, $a \oplus 1=1$. Hence, $\operatorname{conv}(\{0\})=\{0\}$ and $\operatorname{conv}(\{1\})=\{1\}$. For every $a>0$ we have $a \oplus a=\frac{1}{a} \oplus \frac{1}{a}$. Thus, if $a \neq 1$ then $\frac{1}{a} \in \operatorname{conv}(\{a\})$. This means that $\{0\}$ and $\{1\}$ are the only convex singletons. Also, since $a \oplus 1=1$ for every $a \in X$, then for every $A \subseteq X$, we have

$$
\operatorname{conv}(A \cup\{1\})=\operatorname{conv}(A) \cup\{1\}
$$

Finally, note that for every $a \geq 0$, we have

$$
3 a=a \oplus a \oplus a=\frac{3 a+a^{3}}{1+3 a^{2}},
$$

and the function $a \mapsto \frac{3 a+a^{3}}{1+3 a^{2}}$ is onto $[0, \infty)$. Thus, $X$ is 3 -semidivisible. On the other hand, $a \oplus a=\frac{2 a}{1+a^{2}} \leq 1$, and so $X$ is not divisible. In fact is is divisible precisely for all odd numbers.

The next example illustrates that finding convex or affine functions on a group is solving potentially subtle functional equations and inequalities
Example 3.7 (Hyperbolic group). Let $X_{p}$ be the collection of all $2 \times 2$ symmetric matrices of the form $e^{\frac{2 \pi i l}{p}} M(\theta)$, where $M(\theta)=\left[\begin{array}{cc}\cosh (\theta) & \sinh (\theta) \\ \sinh (\theta) & \cosh (\theta)\end{array}\right], \theta \in \mathbb{R}$ and $0 \leq l \leq p-1$. Then $X_{p}$ is a group under the standard matrix multiplication, as we have

$$
\left(e^{\frac{2 \pi i l_{1}}{p}} M\left(\theta_{1}\right)\right) \cdot\left(e^{\frac{2 \pi i l_{2}}{p}} M\left(\theta_{2}\right)\right)=e^{\frac{2 \pi i\left(l_{1}+l_{2}\right)}{p}} M\left(\theta_{1}+\theta_{2}\right)
$$

In particular, the group is commutative. Also, if $p \mid n$, we have that $M(\theta)^{n}=\left(e^{\frac{2 \pi i l}{p}} M(\theta)\right)^{n}=$ $M(n \theta)$ for all $0 \leq l \leq p-1$. Thus, in this case we have $n X_{p} \subsetneq X_{p}$. Otherwise, if $p \nmid n$, then we have $\left(e^{\frac{2 \pi l}{p}} M(\theta)\right)^{n}=e^{\frac{2 \pi n l}{p}} M(n \theta)$. Since $\theta \mapsto n \theta$ and $e^{\frac{2 \pi l}{p}} \mapsto e^{\frac{2 \pi n l}{p}}$ is one-to-one and onto (the second since $p \nmid n$ ), it follows that in this case $n X_{p}=X_{p}$. Altogether, we conclude that $X_{p}$ is $n$-divisible if and only if $p \nmid n$.

Next, we would like to show that it is easy to produce convex functions on the group $X_{p}$. Indeed, if $f: \mathbb{R} \rightarrow \mathbb{R}$ is a convex function then defining $F\left(e^{\frac{2 \pi i l}{p}} M(\theta)\right)=f(\theta)$ is also convex. To see this, for $m_{1}, \ldots, m_{n} \in \mathbb{N}$ and $x_{1}, \ldots, x_{n}, x \in X$ satisfying $m x=\sum_{i=1}^{n} m_{i} x_{i}$, $m=\sum_{i=1}^{n} m_{i}$, assume that $x=e^{\frac{2 \pi i l}{p}} M(\theta), x_{1}=e^{\frac{2 \pi i l_{1}}{p}} M\left(\theta_{1}\right), \ldots, x_{n}=e^{\frac{2 \pi i l_{n}}{p}} M\left(\theta_{n}\right)$. Thus, we have

$$
\begin{equation*}
e^{\frac{2 \pi i m l}{p}} M(m \theta)=e^{\frac{2 \pi i}{p} \sum_{j=1}^{n} m_{j} l_{j}} M\left(m_{1} \theta_{1}+\ldots m_{n} \theta_{n}\right) . \tag{3.2}
\end{equation*}
$$

Note that if $e^{\frac{2 \pi i l}{p}} M(\theta)$ is the identity matrix, then $l=\theta=0$. Therefore, if $e^{\frac{2 \pi i l_{1}}{p}} M\left(\theta_{1}\right)=$ $e^{\frac{2 \pi i l_{1}}{p}} M\left(\theta_{1}\right)$, then $l_{1}=l_{2}$ and $\theta_{1}=\theta_{2}$. In particular, (3.2) implies that $m \theta=\sum_{i=1}^{n} m_{i} \theta_{i}$. Hence, we have

$$
m F(x)=m f(\theta) \leq \sum_{i=1}^{n} m_{i} f\left(\theta_{i}\right)=\sum_{i=1}^{n} m_{i} F\left(x_{i}\right)
$$

Note that restriction to $M(\theta)$ (determinant one) is a divisible subgroup. Also, consider the group

$$
X_{\mathbb{R}}=\left\{e^{i t} M(\theta) \mid t, \theta \in \mathbb{R}\right\}
$$

again with the standard multiplication. Then $X_{\mathbb{R}}$ is a divisible group, since for every $t, \theta \in \mathbb{R}$ and every $n \in \mathbb{N}$, we have

$$
e^{i t} M(\theta)=\left(e^{i \frac{t}{n}} M(\theta / n)\right)^{n}
$$

Note that for every $p, X_{p}$ is a semidivisible subgroup of $X_{\mathbb{R}}$. Finally, note that if we consider $X_{\mathbb{R}}$ as a topological space, equipped with the topology induced from $\mathbb{R}^{4}$, then $X_{\mathbb{R}}$ is connected since we can write $X_{\mathbb{R}}=\Phi\left(\mathbb{R}^{2}\right)$, where $\Phi:(t, \theta) \mapsto e^{i t} M(\theta)$ is continuous. See [BG15] for a more detailed discussion on convexity in topological groups.

Example 3.8 (Finite groups). If $X$ is a finite group then by the pigeon hole principle there exists $m \in \mathbb{N}$ such that $m x=0=m \cdot 0$. Thus $x \in \operatorname{conv}(\{0\})$ for every $x \in X$. Hence, $X$ and $\emptyset$ are the only convex sets in $X$.

Example 3.9 (Circle group). Let $\mathbb{T}=\mathbb{R} / \mathbb{Z}$ with the standard coset addition. In this case, if $x=[m / n] m, n \in \mathbb{N}$ then $n x=[0]$. Thus,

$$
\operatorname{conv}(\{0\})=\{x \in \mathbb{T} \mid x \text { has finite order }\}
$$

Also, for every $x \in X, x+y \in \operatorname{conv}(\{x\})$ for every $y \in X$ which is of finite order. Thus, there are no convex singletons in $X$.

Example 3.10 (Prüfer group). This is a subgroup of the circle group $\mathbb{T}$, which is given by

$$
\mathbb{Z}\left(p^{\infty}\right)=\left\{\exp \left(2 \pi i m / p^{n}\right) \mid m, n \in \mathbb{N} \cup\{0\}\right\}
$$

i.e., all $p^{n}$-th roots of unity. Every element in this group has a finite order and so by the previous example (and also by example 3.8), the only two convex sets are $\emptyset$ and the entire group. It is also known that $\mathbb{Z}\left(p^{\infty}\right)$ is divisible. To see this, note that it is enough to show that $X=q X$ for every prime $q$. Let $x=\exp \left(2 \pi i m / p^{n}\right)$. If $n=0$ then $x=1=1^{q}$. Assume then that $n>0$. If $q=p$ then $x=y^{q}$ where $y=\exp \left(2 \pi i m / p^{n+1}\right)$. If $q \neq p$ then since the
greatest common divisor of $p^{n}$ and $q$ is 1 , there exist $a, b \in \mathbb{Z}$ such that $a p^{n}+b q=1$. So $x=x^{a p^{n}+b q}=x^{a p^{n}} x^{b q}=x^{b q}$. Choosing $y=x^{b}$, then $x=y^{q}$, as needed.

Example 3.11 (Extensions of $\mathbb{Q}$ ). Consider $X=\mathbb{Q}+\theta \mathbb{Q}$, where $\theta$ is irrational, with the addition operation then the mapping $\Phi: a+\theta b \mapsto(a, b)$ is a group homomorphism from $X$ to $\mathbb{Q}^{2}$. Thus

$$
\operatorname{conv}_{X}(A)=\Phi^{-1}\left(\operatorname{conv}_{\mathbb{Q}^{2}}(\Phi(A))\right)
$$

Similarly, we can consider extensions of $\mathbb{Q}$ be any number of algebraically independent numbers.

Example 3.12 (Half line with multiplication). If $X=((0, \infty), \cdot)$, this semigroup is isomorphic to $(\mathbb{R},+)$ via $x \mapsto \log (x)$. Thus,

$$
\begin{equation*}
\operatorname{conv}_{X}(A)=\exp \left(\operatorname{conv}_{(\mathbb{R},+)}(\log (A))\right) \tag{3.3}
\end{equation*}
$$

If instead we choose $X=([0, \infty), \cdot)$, then if $0 \in A$, we have

$$
\operatorname{conv}_{X}(A)=\{0\} \cup \exp \left(\operatorname{conv}_{(\mathbb{R},+)}(\log (A))\right)
$$

if $0 \notin A$ then (3.3) still holds.
Example 3.13 ( $\sigma$-algebras with symmetric differences). Given a set $S$, let $X$ be a $\sigma$-algebra of subsets of $S$. For $A, B \in X$, let $A+B=A \triangle B=(A \cup B) \backslash(A \cap B)$. Clearly $A \triangle B=B \triangle A$. Also, note that for every $A \in \mathcal{F}, A \triangle \emptyset=A$, and $A \triangle A=\emptyset$. Thus, $\emptyset$ is the additive unit and $A=-A$. It also follows that from every $A \in X$ and $n \in \mathbb{N}, 2 n A=\emptyset$ and $(2 n-1) A=(2 n A) \triangle A=\emptyset \triangle A=A$. Thus, $2 n X=\{\emptyset\} \subsetneq X$ and $(2 n-1) X=X$, and so $X$ is $(2 n-1)$-semidivisible but not $2 n$-semidivisible. Next, assume that $A_{1}, \ldots, A_{n}, A \in X$ and $m_{1}, \ldots, m_{n}, m \in \mathbb{N}$ are such that $m A=\sum_{i=1}^{n} m_{i} A_{i}$ and $m=\sum_{i=1}^{n} m_{i}$. Then by the above arguments we have in fact

$$
m A=\sum_{i: 2 \nmid m_{i}} m_{i} A_{i}=\sum_{i: 2 \nmid m_{i}} A_{i} .
$$

Thus, if $\mathcal{A} \subseteq X$, then we can write

$$
\operatorname{conv}(\mathcal{A})=\left\{A \subseteq X \mid A=\sum_{i=1}^{n} A_{i}, A_{i} \in \mathcal{A}, n \in \mathbb{N}\right\}
$$

Note that we always have $\emptyset \in \operatorname{conv}(\mathcal{A})$ since $A+A=\emptyset=2 \emptyset$. This group can also be studied as a topological group. See [BG15].

## 4. Interpolation of scalar-valued functions

We begin with a slight extension of a seminal result.
Theorem 4.1 (Kaufman [Kau66]). Let $X$ be a monoid and $f, g: X \rightarrow[-\infty, \infty)$ satisfying $g \leq f$, where $f$ and $-g$ are subadditive. Then there exists a function $a: X \rightarrow \mathbb{R}$ which is additive and satisfies $g \leq a \leq f$.

Theorem 4.1 is a generalization of Kaufman's Hahn-Banach result which itself extends the seminal result by Mazur and Orlicz [MO53]. Under the assumption that $X$ is semidivisible, the following holds.


Figure 1. Separation in groups
Theorem 4.2 (Interpolation of convex functions). Assume that $X$ is a semidivisible monoid, and $f: X \rightarrow[-\infty, \infty]$ and $-g: X \rightarrow[-\infty, \infty]$ are convex. Then there exists a function $a: X \rightarrow[-\infty, \infty]$ which is generalised affine and satisfies $g \leq a \leq f$.

We illustrate the two results in Figure 1.
Proof. First, since $f,-g$ are convex and $g \leq f$, we have

$$
\begin{align*}
& m f(x) \geq \sum_{i=1}^{n} m_{i} g\left(x_{i}\right), \\
& m x=\sum_{i=1}^{n} m_{i} x_{i}, m=\sum_{i=1}^{n} m_{i} . \tag{4.1}
\end{align*}
$$

If $f=g$, then $f$ is generalised affine and the proof is complete. Assume then that there exists $x_{0} \in X$ and $r \in \mathbb{R}$ such that $f\left(x_{0}\right)>r>g\left(x_{0}\right)$. In such case, either we have

$$
\begin{equation*}
m f(x) \geq m_{0} r+\sum_{i=1}^{n} m_{i} g\left(x_{i}\right) \tag{4.2}
\end{equation*}
$$

whenever we have

$$
m x=m_{0} x_{0}+\sum_{i=1}^{n} m_{i} x_{i}, m=m_{0}+\sum_{i=1}^{n} m_{i}
$$

or else

$$
\begin{equation*}
\left(m^{\prime}-m_{0}^{\prime}\right) f(y)+m_{0}^{\prime} r \geq \sum_{i=1}^{n^{\prime}} m_{i}^{\prime} g\left(y_{i}\right) \tag{4.3}
\end{equation*}
$$

whenever we have

$$
m_{0}^{\prime} x_{0}+\left(m^{\prime}-m_{0}^{\prime}\right) y=\sum_{i=1}^{n^{\prime}} m_{i}^{\prime} y_{i}, m^{\prime}=\sum_{i=1}^{n} m_{i}^{\prime}, m_{0}^{\prime} \leq m^{\prime}
$$

To see this, assume that neither (4.2) nor (4.3) hold. Multiplying (4.2) by $m^{\prime}$ and (4.3) by $m$, we can find integers $m_{0}, \ldots, m_{n}, m_{0}^{\prime}, \ldots, m_{n^{\prime}}^{\prime} \in \mathbb{N}$ and elements $x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{n^{\prime}} \in X$ satisfying

$$
\begin{align*}
& m=m_{0}+\sum_{i=1}^{n} m_{i}, m x=m_{0} x_{0}+\sum_{i=1}^{n} m_{i} x_{i}  \tag{4.4}\\
& m^{\prime}=\sum_{i=1}^{n^{\prime}} m_{i}^{\prime}, m_{0}^{\prime} \leq m^{\prime}, m_{0}^{\prime} x_{0}+\left(m^{\prime}-m_{0}^{\prime}\right) y=\sum_{i=1}^{n^{\prime}} m_{i}^{\prime} y_{i} \tag{4.5}
\end{align*}
$$

such that

$$
\begin{aligned}
m_{0}^{\prime} \sum_{i=1}^{n} m_{i} g\left(x_{i}\right)+m_{0} \sum_{i=1}^{n^{\prime}} m_{i}^{\prime} g\left(y_{i}\right) & >m_{0}^{\prime} m f(x)+m_{0}\left(m^{\prime}-m_{0}^{\prime}\right) f(y) \\
& \geq\left(m_{0}^{\prime} m+m_{0}\left(m^{\prime}-m_{0}^{\prime}\right)\right) f(z)
\end{aligned}
$$

where $z$ satisfies

$$
\begin{align*}
\left(m_{0}^{\prime} m+m_{0}\left(m^{\prime}-m_{0}^{\prime}\right)\right) z & =m_{0}^{\prime} m x+m_{0}\left(m^{\prime}-m_{0}^{\prime}\right) y \\
& \stackrel{(4.4)}{=} m_{0}^{\prime} m_{0} x_{0}+m_{0}^{\prime} \sum_{i=1}^{n} m_{i} x_{i}+m_{0}\left(m^{\prime}-m_{0}^{\prime}\right) y \\
& \stackrel{(4.5)}{=} m_{0} \sum_{i=1}^{n^{\prime}} m_{i}^{\prime} y_{i}+m_{0}^{\prime} \sum_{i=1}^{n} m_{i} x_{i} . \tag{4.6}
\end{align*}
$$

Such $z$ always exists since $X$ is semidivisible, i.e., $X=p^{l} X$ for some prime $p$ and $l \in \mathbb{N}$ and by Proposition 2.6 we may assume that $m_{0}^{\prime} m+m_{0}\left(m^{\prime}-m_{0}^{\prime}\right)=p^{l}$. Now, we have

$$
\begin{array}{rlrl}
m_{0}^{\prime} \sum_{i=1}^{n} m_{i}+m_{0} \sum_{i=1}^{n^{\prime}} m_{i}^{\prime} \stackrel{(4.4) \wedge(4.5)}{=} & m_{0}^{\prime}\left(m-m_{0}\right)+m_{0} m^{\prime} \\
& = & m_{0}^{\prime} m-m_{0}^{\prime} m_{0}+m_{0} m^{\prime} \\
& = & m_{0}^{\prime} m+m_{0}\left(m^{\prime}-m_{0}^{\prime}\right) .
\end{array}
$$

Hence, we have

$$
\begin{align*}
\left(m_{0}^{\prime} m+m_{0}\left(m^{\prime}-m_{0}^{\prime}\right)\right) f(z) & \stackrel{(*)}{\geq}\left(m_{0}^{\prime} m+m_{0}\left(m^{\prime}-m_{0}^{\prime}\right)\right) g(z) \\
& \stackrel{(* *)}{\geq} m_{0}^{\prime} \sum_{i=1}^{n} m_{i} g\left(x_{i}\right)+m_{0} \sum_{i=1}^{n^{\prime}} m_{i}^{\prime} g\left(y_{i}\right), \tag{4.7}
\end{align*}
$$

where in $(*)$ we used the fact that $g \leq f$ and in $(* *)$ we used the fact that $g$ is concave. Now, (4.7) is a contradiction to (4.1). Thus, we must have that either (4.2) or (4.3) hold. Assume first that (4.2) holds. Define

$$
\begin{equation*}
h(x)=\sup \left[\frac{1}{k}\left(k_{0} r+\sum_{i=1}^{n} k_{i} g\left(x_{i}\right)\right)\right] \tag{4.8}
\end{equation*}
$$

where the supremum is taken over all $k, k_{0}, k_{1}, \ldots, k_{n} \in \mathbb{N}$ and $y_{1}, \ldots, y_{n} \in X$ such that $k x=k_{0} x_{0}+\sum_{i=1}^{n} k_{i} y_{i}$ and $k=k_{0}+\sum_{i=1}^{n} k_{i}$. By choosing $k_{1}=\cdots=k_{n}=0$, we have
$h\left(x_{0}\right) \geq r>g\left(x_{0}\right)$. Since $g$ is concave we also have that $h \geq g$, and by (4.2) it follows that $h \leq f$. Next, we would like to show that $h$ is concave, and that (4.1) holds for $h$ instead of $g$. To show the concavity, let $m_{1}, \ldots, m_{n} \in \mathbb{N}$, and $x_{1}, \ldots, x_{n}, x \in X$ such that $m x=\sum_{i=1}^{n} m_{i} x_{i}$ and $m=\sum_{i=1}^{n} m_{i}$. Let $\epsilon>0$, and for each $1 \leq i \leq n$, choose $k_{i}, k_{i, 0}, \ldots, k_{i, n_{i}} \in \mathbb{N}$ and $y_{i, 1}, \ldots, y_{i, n^{\prime}} \in X$ such that $k_{i} x=k_{i, 0} x_{0}+\sum_{j=1}^{n_{i}} k_{i, j} y_{i, j}, k_{i}=k_{i, 0}+\sum_{j=1}^{n_{i}} k_{i, j}$ such that

$$
\begin{equation*}
k_{i} h\left(x_{i}\right)-\frac{k_{i} \epsilon}{m} \leq k_{i, 0} r+\sum_{j=1}^{n_{i}} k_{i, j} g\left(y_{i, j}\right) . \tag{4.9}
\end{equation*}
$$

Now, we have

$$
\begin{aligned}
\left(m \prod_{i=1}^{n} k_{i}\right) x & =\sum_{i=1}^{n}\left(m_{i} \prod_{j \neq i} k_{j}\right) k_{i} x_{i} \\
& =\sum_{i=1}^{n}\left(m_{i} \prod_{j \neq i} k_{j}\right)\left(k_{i, 0} x_{0}+\sum_{j=1}^{n_{i}} k_{i, j} y_{i, j}\right) \\
& =\sum_{i=1}^{n}\left(m_{i} k_{i, 0} \prod_{j \neq i} k_{j}\right) x_{0}+\sum_{i=1}^{n}\left(m_{i} \prod_{j \neq i} k_{j}\right)\left(\sum_{j=1}^{n_{i}} k_{i, j} y_{i, j}\right) .
\end{aligned}
$$

Also, we have

$$
m \prod_{i=1}^{n} k_{i}=\sum_{i=1}^{n}\left(m_{i} k_{i, 0} \prod_{j \neq i} k_{j}\right)+\sum_{i=1}^{n}\left(m_{i} \prod_{j \neq i} k_{j}\right) \sum_{j=1}^{n_{i}} k_{i, j} .
$$

Thus, by the definition of $h$ (4.8), we have

$$
\begin{aligned}
& m h(x) \geq \frac{1}{\prod_{i=1}^{n} k_{i}}\left[\sum_{i=1}^{n}\left(m_{i} k_{i, 0} \prod_{j \neq i} k_{j}\right) r+\sum_{i=1}^{n}\left(m_{i} \prod_{j \neq i} k_{j}\right) \sum_{j=1}^{n_{i}} k_{i, j} g\left(y_{i, j}\right)\right] \\
& =\frac{1}{\prod_{i=1}^{n} k_{i}} \sum_{i=1}^{n} m_{i} \prod_{j \neq i} k_{j}\left[k_{i, 0} r+\sum_{j=1}^{n_{i}} k_{i, j} g\left(y_{i, j}\right)\right] \\
& \stackrel{(4.9)}{\geq} \frac{1}{\prod_{i=1}^{n} k_{i}} \sum_{i=1}^{n}\left(m_{i} \prod_{j \neq i} k_{j}\right)\left(k_{i} h\left(x_{i}\right)-\frac{k_{i} \epsilon}{m}\right) \\
& =\sum_{i=1}^{n} m_{i} h\left(x_{i}\right)-\epsilon .
\end{aligned}
$$

Since $\epsilon$ is arbitrary, it follows that $h$ is concave. Finally, we would like to show that if $m x=\sum_{i=1}^{n} m_{i} x_{i}, m=\sum_{i=1}^{n}$, then $\sum_{i=1}^{n} m_{i} h\left(x_{i}\right) \leq m f(x)$. This follows from the fact that $h$ is concave together with the fact that $h \leq f$. The existence and the properties of $h$ show that $g$ is not the maximal element in the class of all concave functions that satisfy (4.1). Analogously, if (4.3) holds, define

$$
\begin{equation*}
h^{\prime}(x)=\inf \left[\frac{1}{k^{\prime}}\left(k_{0}^{\prime} r+\left(k^{\prime}-k_{0}^{\prime}\right) f(y)\right)\right] \tag{4.10}
\end{equation*}
$$



Figure 2. Failure of finite affine separation
where the infimum is taken over all $k_{0}^{\prime} \geq 0, k^{\prime} \in \mathbb{N}$, and $y \in X$ such that $k^{\prime} x=k_{0}^{\prime} x_{0}+$ $\left(k^{\prime}-k_{0}^{\prime}\right) y$. If $k^{\prime}=k_{0}^{\prime}$ we define the right side of (4.10) to be $r$. Choosing $k^{\prime}=k_{0}^{\prime}$ gives $h^{\prime}\left(x_{0}\right) \leq r<f\left(x_{0}\right)$ and choosing $k_{0}=0$ gives $h^{\prime}(x) \leq f(x)$ for all $x \in X$. Since (4.3) holds and $g$ is concave, we also have that $g \leq h^{\prime}$ and (4.1) holds with $h^{\prime}$ instead of $f$. Also, in an analogous way to the previous case, one can show that $h^{\prime}$ is convex. To conclude the proof, define the following ordered set $\mathcal{D}$ of all pairs of the form $\left(h, h^{\prime}\right)$, where $h$ is concave, $h^{\prime}$ is convex, and (4.1) holds if we replace $g$ by $h$ or $f$ by $h^{\prime}$. Define the partial order on $\mathcal{D}$ to be $\left(h, h^{\prime}\right) \leq\left(w, w^{\prime}\right) \Longleftrightarrow h \leq w$ and $w^{\prime} \leq h^{\prime}$. Since $(g, f) \in \mathcal{D}$, this chain is non-empty and therefore has a maximal element. By the above consideration we conclude the maximal element is generalised affine.

Remark 4.1. Note that we used the semidivisibility only to show that either (4.2) or (4.3) must hold. We did not use this fact again in the proof.

Remark 4.2. Similarly, the results hold if we work in a semimodule.
Remark 4.3. In general we cannot expect the affine function $a$ to be better than generalised affine in Theorem 4.2, even if $X$ is a vector space. This is illustrated by the example of $f(x)=-\sqrt{x}$ if $x \geq 0$ and $f(x)=-\infty$ if $x<0$ and $g(x)=-f(-x)$, where $X=\mathbb{R}$. The only separator comes from letting $a$ to be $+\infty$ when $x>0,-\infty$ when $x<0$ and 0 when $x=0$. See Figure 2.

On the other hand, using Proposition 2.4, we have the following.
Corollary 4.1. Assume that $X$ is a group. If either $f$ or $g$ is everywhere finite and the other function is somewhere finite, then $a$ is finite and affine.

The vector space version of the following result is used in [Hol75] as the basis for HahnBanach theory. Once established, one imposes additional core conditions on $A, B$ to show $\mathrm{cl} C \cap \mathrm{cl} D$ is a separating half-space. Here one uses the algebraic closure. We take a different (more modern) approach in the next section.

Corollary 4.2 (Stone's lemma for monoids). Assume that $X$ is a semidivisible monoid and $A, B \subseteq X$ are disjoint convex sets. Then there exist $C, D \subseteq X$ disjoint and convex such that $A \subseteq C, B \subseteq D$ and $C \cup D=X$.

Proof. Let $f=\iota_{A}, g=-\iota_{B}$, where

$$
\iota_{A}(x)=\left\{\begin{array}{ll}
0 & x \in A \\
\infty & x \notin A
\end{array},\right.
$$

and similarly for $\iota_{B}$. Then $f,-g: X \rightarrow[-\infty, \infty]$ are convex. Use Theorem 4.2 to deduce the existence of a generalised affine function $a: X \rightarrow[-\infty, \infty]$ with $-\iota_{B} \leq a \leq \iota_{A}$. Choosing

$$
C=\{x \in X \mid a(x)<0\}, D=\{x \in X \mid a(x) \geq 0\}
$$

concludes the proof.
Theorem 4.2 also implies the following.
Corollary 4.3. Assume that $X$ is semidivisible monoid and $f: X \rightarrow[-\infty, \infty]$ is convex. Then $f$ is the supremum over its generalised affine minorants.

Proof. Clearly we have

$$
\begin{equation*}
f(x) \geq \sup \{a(x) \mid a \leq f, a \text { is affine }\} . \tag{4.11}
\end{equation*}
$$

To show that equality holds, assume to the contrary that we have a strict inequality in (4.11). The function $g$ which equals the supremum at $x$ and $-\infty$ everywhere else is concave. By Theorem 4.2, there exists an affine function $a$ such that

$$
\sup \{a(x) \mid a \leq f, a \text { is affine }\}<a(x)<f(x)
$$

which is a contradiction.
Example 4.1 (Non separation). In the non-divisible setting, Theorem 4.2 fails even for everywhere finite functions. Take for example $X=\mathbb{Z}^{2}$. Let $A=\operatorname{conv}_{\mathbb{R}^{2}}(\{(0,2),(1,0)\})$ and $B=\operatorname{conv}_{\mathbb{R}^{2}}(\{(0,1),(2,0)\})$, and

$$
\begin{aligned}
& f(x)=2 \sqrt{5} d_{A}(x)-1 \\
& g(x)=-2 \sqrt{5} d_{B}(x)+1 .
\end{aligned}
$$

where $d_{A}(x)=\inf _{a \in A}\|x-a\|_{\mathbb{R}^{2}}$. Note that for every $x \in \mathbb{Z}^{2}$ such that $x \notin A$, we have $d_{A}(x) \geq \frac{1}{\sqrt{5}}$. Similarly, if $x \in \mathbb{Z}^{2}$ and $x \notin B$, we have $d_{B}(x) \geq \frac{1}{\sqrt{5}}$. For every $x \in \mathbb{Z}^{2}$, either $x \notin A$ or $x \notin B$ and so $d_{A}(x)+d_{B}(x) \geq \frac{1}{\sqrt{5}}$. Hence,

$$
f(x)-g(x)=2 \sqrt{5}\left(d_{A}(x)+d_{B}(x)\right)-2 \geq 0
$$

and so $g \leq f$ on $\mathbb{Z}^{2}$. Also, $f$ and $-g$ are convex, since they are convex on all of $\mathbb{R}^{2}$ (the distance to a convex set in a vector space is a convex function). Assume that $a$ is affine and satisfies $g \leq a \leq f$. By the choice of $f$ and $g, a$ has to be finite everywhere. Since $a$ is affine, we can write $a\left(m_{1}, m_{2}\right)=c+\alpha_{1} m_{1}+\alpha_{2} m_{2}$, where $c, \alpha_{1}, \alpha_{2} \in \mathbb{R}$. Since $a \leq f$, we can choose $x=(0,2)$ and $x=(1,0)$ and obtain

$$
c+2 \alpha_{2} \leq-1, c+\alpha_{1} \leq-1
$$

Similarly, since $a \geq g$ we get

$$
c+2 \alpha_{1} \geq 1, c+\alpha_{2} \geq 1
$$

Altogether, we get both $c \leq-3$ and $c \geq 3$.


Figure 3. Convex separation in a lattice
Example 4.2. Let $(X, \wedge)$ be a semimodule induced by a semilattice $X$. This is divisible since $x \wedge x=x$. Thus, $\operatorname{conv}(S)$ is the sub semilattice generated by $S$. In this case convex and subadditive functions coincide, and so Theorems 4.1 and 4.2 both assert the un-obvious result that disjoint sub meet-lattices lie in partitioning sublattices. See Figure 3. Note that since $X$ contains nontrivial idempotent elements, it cannot be embedded in a group (see [Ham05]). See also [Pon14] for a study of convexity in semilattices.

## Part II: Convex operators on groups

## 5. Analysis of convex operators on groups

We turn now to results for operators on groups. By Example 3.2 and Remark 2.2, we could derive many of these results using $\mathbb{Q}$-modules but we prefer to highlight the use of only monoidal structure.
5.1. Subdifferential calculus of operators. Here we assume that $X, Y$ are groups, and $f: X \rightarrow(Y \cup\{\infty\}, \leq)$, where $\infty$ is a maximal element with respect to the partial order $\leq$ on $Y$. Assume also that $\leq$ is compatible with the group operation, i.e., if $x \geq y$ iff $x-y \geq 0$. We also assume that the order is at least inductive, i.e., that every countable chain has an upper bound. In Subsection 5.3, we will need to further assume that $\leq$ is a complete order, i.e., that every order bounded set has an infimum and supremum. Of course $Y$ may be $\mathbb{R}$ as before.

Remark 5.1. A partial order in a Banach space is order complete if and only it is latticial. Moreover, order completeness of the range characterises the Hahn-Banach extension theorem holding. By contrast if the cone has a bounded complete base, the order is inductive. Thus, in Euclidean space all pointed closed convex cones induce inductive orders. (See [BV10, Bor82, BPT84, BT92] for much more on these technicalities in the vector space setting.) $\diamond$

As in Definition 2.12, $f$ is said to be subadditive if $f(x+y) \leq f(x)+f(y)$. We can similarly define $\mathbb{N}$-sublinear and convex functions.
Definition 5.1 (Domain of convex function). Let $X, Y$ be groups and $f: X \rightarrow Y \cup\{\infty\}$ be convex. Define the domain of $f$ to be the set

$$
\operatorname{dom}(f)=\{x \in X \mid f(x)<\infty\} .
$$

It is easily shown that the domain of a convex function of a convex subset of $X$. The core of the domain is then:

Definition 5.2 (Core of domain). Let $X, Y$ be groups and let $f: X \rightarrow Y \cup\{\infty\}$ be a convex function. Define the core of the domain of $f$ to be

$$
\operatorname{core}(\operatorname{dom}(f))=\{x \in X \mid \forall h \in X, \exists n \in \mathbb{N}, \exists g \in X, n g=h, f(x+g)<\infty\} .
$$

By choosing $h=0$, it follows that core $(\operatorname{dom}(f)) \subseteq \operatorname{dom}(f)$. More generally, we can define the core of a convex function.

Definition 5.3 (Core of convex set). Let $X$ be a group and $C \subseteq X$ a convex set. Define the core of $C$ to be the set

$$
\operatorname{core}(C)=\{x \in X \mid \forall h \in X, \exists n \in \mathbb{N}, \exists g \in X, n g=h, x+g \in C\}
$$

Again, we have core $(C) \subseteq C$. Now we define the directional derivative.
Definition 5.4 (Directional derivative). Let $X$ be a group, $(Y \cup\{\infty\})$ a group with an inductive order, and $f: X \rightarrow Y \cup\{\infty\}$ a convex function. For $x \in \operatorname{core}(\operatorname{dom}(f))$, define

$$
f_{x}(h)=\inf \{n(f(x+g)-f(x)) \mid n g=h, f(x+g)<\infty\} .
$$

Before proceed to the study of directional derivatives, we need the following technical proposition.

Proposition 5.1. Assume that $\left\{a_{n}\right\}_{n \in \mathbb{N}}$ and $\left\{b_{n}\right\}_{n \in \mathbb{N}}$ are two decreasing sequences in an inductive and compatible cone. Then

$$
\inf _{n \in \mathbb{N}}\left\{a_{n}+b_{n}\right\}=\inf _{n \in \mathbb{N}} a_{n}+\inf _{n \in \mathbb{N}} b_{b}
$$

Proof. Let $n, m \in N$ with $n>m$. Then since $\left\{b_{n}\right\}_{n \in \mathbb{N}}$ is decreasing, we have $a_{n}+b_{n} \leq a_{n}+b_{m}$. Thus, we have

$$
\inf _{n \in \mathbb{N}}\left\{a_{n}+b_{n}\right\} \leq\left[\inf _{n \in \mathbb{N}} a_{n}\right]+b_{m}
$$

Taking the infimum over $m$ gives $\inf _{n \in \mathbb{N}}\left\{a_{n}+b_{n}\right\} \leq \inf _{n \in \mathbb{N}} a_{n}+\inf _{m \in \mathbb{N}} b_{m}$. The converse inequality is clear. This completes the proof.

We have the following.
Proposition 5.2 (One-sided derivatives, I). Assume that a group $X$ is a p-semidivisible group, and $(Y, \leq)$ is a group with an inductive order. Assume also that $f: X \rightarrow Y \cup\{\infty\}$ is convex and $x \in \operatorname{core}(\operatorname{dom}(f))$. Then $f_{x}$ is an everywhere finite, $\mathbb{N}$-sublinear function.

Proof. For arbitrarily large $n, n^{\prime} \in \mathbb{N}$ with $n<n^{\prime}$ we can find $g, g^{\prime} \in X$ such that $n g=$ $n^{\prime} g^{\prime}=h$ and $f(x+g)<\infty, f\left(x+g^{\prime}\right)<\infty$. We have $n^{\prime}\left(x+g^{\prime}\right)=n(x+g)+\left(n^{\prime}-n\right) x$, and so by convexity $n^{\prime} f\left(x+g^{\prime}\right) \leq n f(x+g)+\left(n^{\prime}-n\right) f(x)$. Therefore, we have

$$
n^{\prime}\left(f\left(x+g^{\prime}\right)-f(x)\right) \leq n(f(x+g)-f(x))
$$

Also, if $g, g^{\prime} \in X$ are such that $n g=n^{\prime} g^{\prime}=h$, then $\left(n+n^{\prime}\right) x=n(x-g)+n^{\prime}(x+g)$ and so again by convexity, we have

$$
n(f(x)-f(x-g)) \leq n^{\prime}\left(f\left(x+g^{\prime}\right)-f(x)\right)
$$

Thus, the sequence $\{n(f(x+g)-f(x)) \mid n g=h, f(x+g)<\infty\}$ is decreasing and bounded from below. Since $\leq$ is an inductive order on $Y, f_{x}(h)$ exists and is finite. To show that $f_{x}(0) \leq 0$, note that we can choose $g=0$ in Definition 5.4 and obtain $f_{x}(0) \leq 0$. To prove
the positive homogeneity of $f_{x}$, choose, $g, g^{\prime} \in X$ such that $p^{l} g=p h$ and $p^{l} g^{\prime}=h$. Then we have $p^{l+1}\left(x+g^{\prime}\right)=p^{l+1} x+p h=p^{l+1} x+p^{l} g=\left(p^{l+1}-p^{l}\right) x+p^{l}(x+g)$. Thus, since $f$ is convex, we have

$$
p^{l+1} f\left(x+g^{\prime}\right) \leq\left(p^{l+1}-p^{l}\right) f(x)+p^{l} f(x+g)
$$

or in other words,

$$
p^{l+1}\left(f\left(x+g^{\prime}\right)-f(x)\right) \leq p^{l}(f(x+g)-f(x))
$$

Taking the limit as $l \rightarrow \infty$ and using the fact that the sequence in Definition 5.4 is decreasing, we get $p f_{x}(h) \leq f_{x}(p h)$. On the other hand, we have,

$$
\begin{aligned}
p f_{x}(h) & =\inf \{p n(f(x+g)-f(x) \mid n g=h\} \\
& \stackrel{(*)}{\geq} \inf \{p n(f(x+g)-f(x) \mid p n g=p h\} \\
& \stackrel{(* *)}{=} f_{x}(p h) .
\end{aligned}
$$

In $(*)$ we used the fact that if $p g=h$ then $p n g=p h$ (but we might have a bigger set on which we take the infimum). In $(* *)$ we used the fact in Definition 5.4 the infimum is taken over a decreasing sequence. This shows that $p_{x}(p h)=p f_{x}(h)$. Finally, to show subadditivity, note that $p\left(x+g_{1}+\cdots+g_{p}\right)=\left(x+p g_{1}\right)+\cdots+\left(x+p g_{p}\right)$, and so by convexity of $f$,

$$
\begin{align*}
p\left(f\left(x+g_{1}+\cdots+g_{p}\right)-f(x)\right) & \\
& \leq\left(f\left(x+p g_{1}\right)-f(x)\right)+\cdots+\left(f\left(x+p g_{p}\right)-f(x)\right) . \tag{5.1}
\end{align*}
$$

Multiply (5.1) by $n$ and then choose $g_{1}, \ldots, g_{p}$ such that $n g_{1}=h_{1}, \ldots, n g_{p}=h_{p}$. This is possible since we may assume without loss of generality that $n=p^{l}$ for some $l \in \mathbb{N}$, and this is because the sequence $\{n(f(x+g)-f(x)) \mid n g=h, f(x+g)<\infty\}$ is decreasing. We get

$$
\begin{equation*}
\left.p\left(n f\left(x+g_{1}+\cdots+g_{p}\right)-f(x)\right)\right) \leq \sum_{j=1}^{p} n\left(f\left(x+g_{j}\right)-f(x)\right) \tag{5.2}
\end{equation*}
$$

By Definition 5.4, we have

$$
\begin{equation*}
\left.p\left(n f\left(x+g_{1}+\cdots+g_{p}\right)-f(x)\right)\right) \geq p f_{x}\left(h_{1}+\cdots+h_{p}\right) \tag{5.3}
\end{equation*}
$$

To evaluate the right side of (5.2), note that for each $1 \leq j \leq p$, the sequence

$$
\left\{n\left(f\left(x+g_{j}\right)-f(x)\right) \mid n g_{j}=h_{j}, f\left(x+g_{j}\right)<\infty\right\}
$$

is decreasing. Thus, using Proposition 5.1 and taking the infimum over the right side of (5.2), we get,

$$
\begin{align*}
& \inf \left\{\sum_{j=1}^{p} n\left(f\left(x+g_{j}\right)-f(x)\right) \mid n g_{j}=p h_{j}, f\left(x+g_{j}\right)<\infty, 1 \leq j \leq p\right\} \\
& =\sum_{j=1}^{p} \inf \left\{n\left(f\left(x+g_{j}\right)-f(x)\right) \mid n g_{j}=p h_{j}, f\left(x+g_{j}\right)<\infty, 1 \leq j \leq p\right\} \\
& =\sum_{j=1}^{p} f_{x}\left(p h_{j}\right) \tag{5.4}
\end{align*}
$$

Combining (5.3) and (5.4), we get

$$
p f_{x}\left(h_{1}+\ldots h_{p}\right) \leq f_{x}\left(p h_{1}\right)+\cdots+f_{x}\left(p h_{p}\right)
$$

and so, since $f_{x}\left(p h_{j}\right)=p f_{x}\left(h_{j}\right), 1 \leq j \leq p$, we get

$$
f_{x}\left(h_{1}+\ldots h_{p}\right) \leq f_{x}\left(h_{1}\right)+\cdots+f_{x}\left(h_{p}\right)
$$

Note that here we used the fact that $\leq$ is compatible with the group operations on $Y$, and therefore we have $p y_{1} \leq p y_{2} \Longrightarrow y_{1} \leq y_{2}$. Next, note that since is assumed to be $p$ prime, $p \geq 2$. Choosing $h_{3}=\cdots=h_{p}=0$, we get

$$
\begin{aligned}
f\left(h_{1}+h_{2}\right) & \leq f_{x}\left(h_{1}\right)+f_{x}\left(h_{2}\right)+f_{x}\left(h_{3}\right)+\cdots+f_{x}\left(h_{p}\right) \\
& \stackrel{(*)}{\leq} f_{x}\left(h_{1}\right)+f_{x}\left(h_{2}\right)+0 \\
& =f_{x}\left(h_{1}\right)+f_{x}\left(h_{2}\right) .
\end{aligned}
$$

where in $(*)$ we used the fact that $f_{x}(0) \leq 0$. Altogether we have that $f_{x}$ is subadditive and $f(p x)=p f(x)$. Now apply Proposition 2.7 to deduce that $f$ is $\mathbb{N}$-sublinear, and the proof is complete.
Remark 5.2. The proof of Proposition 5.2 shows that the sequence $n(f(x+g)-f(x))$, $n g=h, f(x+g)<\infty$, is decreasing. If we assume that we have both $p X=X$ and $q X=X$, then in (5.4), we can choose $n=p^{l}$ or $n=q^{l}$ for every $l \in \mathbb{N}$ and the infimum would be the same in both cases.

In the case when $f$ is not only convex, but actually $\mathbb{N}$-sublinear, we have the following stronger result.

Proposition 5.3 (One-sided derivatives, II). Assume that $X$ is a group, $(Y, \leq)$ is a group with an inductive order, and $f: X \rightarrow Y \cup\{\infty\}$ is $\mathbb{N}$-sublinear map, and $x \in \operatorname{core}(\operatorname{dom}(f))$. Then $f_{x}$ is an everywhere finite $\mathbb{N}$-sublinear map, that satisfies in addition $f_{x}(0)=0, f_{x}(x)=$ $-f_{x}(-x)=f(x)$.

Proof. When $f$ is $\mathbb{N}$-sublinear, (5.4) becomes

$$
f_{x}(h)=\inf \{f(n x+h)-n f(x) \mid f(n x+h)<\infty\} .
$$

Since $f$ is positively homogeneous, it is easy to see that $f_{x}(x)=-f_{x}(-x)=f(x)$ and $f_{x}(0)=0$. To show the positive homogeneity of $f_{x}$, use the fact that, as in the proof of Proposition 5.2, the sequence $\{f(n x+h)-n f(x)\}$ is decreasing, and so we have for all $m \in \mathbb{N}$,

$$
\begin{aligned}
f_{x}(m h) & =\inf \{f(n x+m h)-n f(x) \mid f(n x+m h)<\infty\} \\
& =\inf \{f(m k x+m h)-m k f(x) \mid f(m k x+m h)<\infty\} \\
& =m \inf \{f(k x+h)-k f(x) \mid f(k x+h)<\infty\} \\
& =m f_{x}(h) .
\end{aligned}
$$

To show the subadditivity, take $n_{1}, n_{2} \in \mathbb{N}$. Since $f$ is subadditive, we have,

$$
\begin{aligned}
f_{x}\left(h_{1}+h_{2}\right) & \leq f\left(\left(n_{1}+n_{2}\right) x+h_{1}+h_{2}\right)-\left(n_{1}+n_{2}\right) f(x) \\
& \leq\left(f\left(n_{1} x+h_{1}\right)-n_{1} f(x)\right)+\left(f\left(n_{2} x+h_{2}\right)-n_{2} f(x)\right)
\end{aligned}
$$

Taking the infimum over all $n_{1}, n_{2} \in \mathbb{N}$ such that $f\left(n_{1} x+h_{1}\right)<\infty, f\left(n_{2} x+h_{2}\right)<\infty$, the subadditivity follows.

Given two monoids $X$ and $Y$, let $\mathcal{L}(X, Y)$ be the collection of all additive maps between $X$ and $Y$. As in the vector space setting, define the following:

$$
\partial f\left(x_{0}\right)=\left\{a \in \mathcal{L}(X, Y) \mid f\left(x_{0}\right)+a(h) \leq f\left(x_{0}+h\right)\right\} .
$$

In the vector space setting it is usually required that $a\left(x-x_{0}\right) \leq f(x)-f\left(x_{0}\right)$. However, in order to avoid taking differences, we use the above definition. Let $\mathcal{L}(X, Y)$ be the space of all additive maps between $X$ and $Y$. Then it follows that $\partial f\left(x_{0}\right) \subseteq \mathcal{L}(X, Y)$.

Proposition 5.4. Assume that $X$ is a p-semidivisible group, $(Y, \leq)$ is a group with an inductive order, and $f: X \rightarrow Y \cup\{\infty\}$ is subadditive and satisfies $f(p x)=p f(x)$ for all $x \in X$. If $x \in \operatorname{core}(\operatorname{dom}(f))$, then $f_{x} \leq f$ and

$$
f_{x}(x)+f_{x}(-x) \leq 0
$$

Proof. To prove the first assertion, note that

$$
\begin{aligned}
f_{x}(h) & \stackrel{(*)}{=} \inf \{n(f(x+g)-f(x)) \mid n g=h, f(x+g)<\infty\} \\
& =\inf \left\{p^{l}(f(x+g)-f(x)) \mid p^{l} g=h, f(x+g)<\infty\right\} \\
& \leq \inf \left\{p^{l} f(g) \mid p^{l} g=h, f(x+g)<\infty\right\} \\
& \stackrel{(* *)}{=} f(h),
\end{aligned}
$$

where in $(*)$ we used the fact that $\{n(f(x+g)-f(x)) \mid n g=h, f(x+g)<\infty\}$ is a decreasing sequence and in $(* *)$ we used the fact that $f(p x)=p f(x)$. To prove the second assertion, choose $g$ such that $p g=x$ and note that

$$
\begin{aligned}
f_{x}(x)+f_{x}(-x) & \leq p(f(x+g)-f(x))+m(f(x-g)-f(x)) \\
& =f((p+1) x)+f((p-1) x)-2 p f(x) \\
& \leq 0
\end{aligned}
$$

where in the last inequality we used the subadditivity of $f$.
Proposition 5.5. If $(Y, \leq)$ satisfies that for every $m \in \mathbb{N} m y_{1} \leq m y_{2} \Longrightarrow y_{1} \leq y_{2}$ then $\partial p\left(x_{0}\right)$ is convex in $\mathcal{L}(X, Y)$.

Proof. For $a_{1}, \ldots, a_{n}, a \in \mathcal{L}(X, Y)$, assume that $m a=\sum_{i=1}^{n} m_{i} a_{i}, m=\sum_{i=1}^{n} m_{i}$. Then we have

$$
m\left(f\left(x_{0}\right)+a(x)\right)=\sum_{i=1}^{n}\left(f\left(x_{0}\right)+a_{i}(x)\right) \leq \sum_{i=1}^{n} m_{i} f(x)=m f(x)
$$

By the assumption on $Y$, it follows that $f\left(x_{0}\right)+a(x) \leq f(x)$.
5.2. The maximum or max formula. We show that the well known max formula [BV10, BL06] holds in this generality.

Theorem 5.1 (Max formula). Assume that $X$ is a p-semidivisible group, that $(Y, \leq)$ is an additive group with an inductive order, and $f: X \rightarrow Y \cup\{\infty\}$ is convex. Assume also that for some $x_{0} \in \operatorname{core}(\operatorname{dom}(f))$, we have

$$
\begin{equation*}
f_{x_{0}}\left(x_{0}\right)+f_{x_{0}}\left(-x_{0}\right) \leq 0 \tag{5.5}
\end{equation*}
$$

Then we have

$$
\begin{equation*}
f_{x_{0}}(h)=\max \left\{a(h) \mid a \in \partial f\left(x_{0}\right)\right\} \tag{5.6}
\end{equation*}
$$

In particular, $f$ admits additive minorants, and $\partial f\left(x_{0}\right) \neq \emptyset$. The maximal element in (5.6) is bounded.

Proof. Define $\mathcal{C}$ to be the set of all pairs $(\varphi, S)$, where $S \subseteq X$, and $\varphi: X \rightarrow Y \cup\{\infty\}$ is $\mathbb{N}$-sublinear and satisfies $\varphi \leq f_{x_{0}}$, and $\sup _{s \in S}(\varphi(s)+\varphi(-s)) \leq 0$. Define a partial order on $\mathcal{C}$ by

$$
\left(\varphi_{1}, S_{1}\right) \leq\left(\varphi_{2}, S_{2}\right) \Longleftrightarrow \varphi_{1} \geq \varphi_{2}, S_{1} \subseteq S_{2}
$$

$(\mathcal{C}, \leq)$ is inductive, as both $\leq$ and $\subseteq$ are inductive orders. By Proposition 5.2, we have $f_{x_{0}}(0)=0$, implying that $\left(f_{x_{0}},\{0\}\right) \in \mathcal{C}$ and so $\mathcal{C} \neq \emptyset$. Therefore, $\mathcal{C}$ has a maximal element $(\bar{\varphi}, \bar{S})$. We claim that we must have $\bar{S}=X$. Otherwise, choose $y \in X \backslash \bar{S}$. Since $(\bar{\varphi}, \bar{S}) \in$ $\mathcal{C}$, in particular it follows that the function $\bar{\varphi}$ satisfies the hypotheses of Proposition 5.4. Also, $y \in \operatorname{core}(\operatorname{dom}(f))$ since $\bar{\varphi} \leq f_{x_{0}}$ and $f_{x_{0}}$ is everywhere finite (by Proposition 5.2). Therefore, Proposition 5.4 implies that $\bar{\varphi}_{y} \leq \bar{\varphi}$ and $\bar{\varphi}_{y}(y)+\bar{\varphi}_{y}(-y) \leq 0$. This means that $\left(\bar{\varphi}_{y}, \bar{S} \cup\{y\}\right) \in \mathcal{C}$, which is a contradiction to the maximality of $(\bar{\varphi}, \bar{S})$. Thus, we have $\bar{S}=X$. Next, we claim that $\bar{\varphi}$ is additive on $X$. If not, then since $\bar{\varphi}$ is subadditive, there must exist $x, h \in X$ such that $\bar{\varphi}(x+h)-\bar{\varphi}(h)<\bar{\varphi}(x)$. But then $\bar{\varphi}_{x} \leq \bar{\varphi}$ which is again a contradiction to the maximality of $(\bar{\varphi}, \bar{S})$. Since $\bar{\varphi} \leq f_{x_{0}}$ and by (5.5) we have $\bar{\varphi}\left(-x_{0}\right) \leq f_{x_{0}}\left(-x_{0}\right) \leq-f_{x_{0}}\left(x_{0}\right)$, it follows that $\bar{\varphi}\left(x_{0}\right)=f_{x_{0}}\left(x_{0}\right)$ and $\bar{\varphi}$ is bounded. Choosing $a=\bar{\varphi}$ proves (5.6). Since $x \in \operatorname{core}(\operatorname{dom}(f))$, Definition 5.4 implies that the maximal element in (5.6) is indeed bounded. This completes the proof.

An instructive setting is when $Y$ is the symmetric matrices endowed with the (non-lattical) semidefinite order.

Remark 5.3 (Well posedness). If $f$ is $\mathbb{N}$-sublinear and $n g=x$, then by positive homogeneity, we have $n(f(x-g)-f(x))=f((n-1) x)-f(n x)=-f(x)$ and $n(f(x+g)-f(x))=$ $f((n+1) x)-f(n x)=f(x)$. In particular, $f_{x}(x)+f_{x}(-x) \leq 0$ for every $x \in \operatorname{core}(\operatorname{dom}(f))$. Thus, every $\mathbb{N}$-sublinear function satisfies the assumptions of Theorem 5.1.

Remark 5.4. Using Proposition 5.3, we have that Theorem 5.1 holds if $f$ is $\mathbb{N}$-sublinear, even if we omit the subdivisibility assumption.
5.3. Fenchel-Rockafellar duality. As in vector spaces, define the additive dual group of a group $X$ to be

$$
X^{*}=\{\varphi: X \rightarrow \mathbb{R} \mid \varphi \text { is additive }\}
$$

Then $X^{*}$ is an additive group with the addition being point-wise addition. We emphasise that $X^{*}$ is not the group of homomorphisms of $X$. How rich a notion this is depends on the given group.

Consider now $(Z, \leq)$ which is order complete. We still require that $\leq$ is compatible with the group operation. Define the conjugate function $f^{\star}: X^{*} \rightarrow Z \cup\{\infty\}$ to be

$$
\begin{equation*}
f^{\star}(\varphi)=\sup _{x \in X}\{\varphi(x)-f(x)\} \tag{5.7}
\end{equation*}
$$

The conjugate function has been studied extensively in the vector space setting. See for example [BL06, BV10, Roc97]. Note that $f^{*}(\varphi)=+\infty$ will happen if (5.7) has no upper bound. Before proving the Fenchel duality theorem for groups, we need the following proposition.

Proposition 5.6. Assume that $X_{1}, X_{2}, Z$ are groups, where $X_{1}$ is semidivisible and $(Z, \leq)$ is an order complete group. Let $T: X_{1} \rightarrow X_{2}$ be additive, and assume that $f: X_{1} \rightarrow Z \cup\{\infty\}$ and $g: X_{2} \rightarrow Z \cup\{\infty\}$ are convex. If we define $h: X_{2} \rightarrow Z \cup\{\infty\}$ by

$$
h(u)=\inf _{x \in X_{1}}[f(x)+g(T x+u)],
$$

then $h$ is convex, and it domain is given by

$$
\begin{equation*}
\operatorname{dom}(h)=\operatorname{dom}(g)-T \operatorname{dom}(g) \tag{5.8}
\end{equation*}
$$

Proof. First, note that since $g$ is convex and $T$ is additive, it follows that $g \circ T: X_{1} \rightarrow Z \cup\{\infty\}$ is convex. Next, to show the convexity of $h$, let $m_{1}, \ldots, m_{n} \in \mathbb{N}, u_{1}, \ldots, u_{n}, u \in X_{2}$ such that $m u=\sum_{i=1}^{n} m_{i} u_{i}, m=\sum_{i=1}^{n} m_{i}$. Let $x_{1}, \ldots, x_{n} \in X_{1}$. By Proposition 2.6, we may assume that $m=p^{l}$, where $p$ is a prime satisfying $p X=X$. Hence, there exists $x \in X_{1}$ such that $m x=\sum_{i=1}^{n} m_{i} x_{i}$. We have

$$
\begin{aligned}
m h(u) & \leq m(f(x)+g(T x+u)) \\
& \leq \sum_{i=1}^{n} m_{i}\left(f\left(x_{i}\right)+g\left(T x_{i}+u_{i}\right)\right)
\end{aligned}
$$

Taking the infimum over $x_{1}, \ldots, x_{n} \in X$, we get

$$
m h(u) \leq \sum_{i=1}^{n} m_{i} h\left(u_{i}\right)
$$

The proof of (5.8) is immediate. This completes the proof.
Theorem 5.2 (Fenchel-Young inequality for groups). Suppose that $X, Z$, are groups, $Z$ is order complete, and $f: X \rightarrow Z \cup\{\infty\}$. Then for every $x \in X$ and every $\varphi \in X^{*}$,

$$
f(x)+f^{\star}(\varphi) \geq \varphi(x)
$$

Equality holds if and only if $\varphi \in \partial f(x)$.
Proof. By definition (5.7), $\varphi(x)-f(x) \leq f^{\star}(\varphi)$ which implies $f(x)+f^{\star}(\varphi) \geq \varphi(x)$. If $\varphi \in \partial f(x)$, then $f(x)+\varphi(y-x) \leq f(y)$ and so $f(x)-\varphi(x) \leq f(y)-\varphi(y)$. Taking the infimum over the right side gives $f(x)-\varphi(x) \leq-f^{\star}(\varphi)$ which then gives $f(x)+f^{\star}(\varphi)=\varphi(x)$. Conversely, by the definition of $f^{\star}$, if $f(x)+f^{\star}(\varphi)=\varphi(x)$ then $\varphi(y-x) \leq f(y)-f(x)$, and so $\varphi \in \partial f(x)$ as required.

Example 5.1. If $X$ is a meet lattice then additive functions are identically 0 , since for every $m \in \mathbb{N}$ we have

$$
f(x)=f(\overbrace{x \wedge \cdots \wedge x}^{m \text { times }})=m f(x) .
$$

Hence $X^{*}=\{0\}$ and Theorem 5.2 simply gives $f(x) \geq \inf _{x \in X} f(x)$.
For an additive map $T: X_{1} \rightarrow X_{2}$ define the adjoint $T^{*}: X_{2}^{*} \rightarrow X_{1}^{*}$ in the usual way

$$
\left(T^{*} x_{2}^{*}\right)\left(x_{1}\right)=x_{2}^{*}\left(T x_{1}\right), x_{1} \in X_{1}, x_{2}^{*} \in X_{2}^{*} .
$$

We are now in a position to state and prove the Fenchel duality theorem.
Theorem 5.3 (Weak and strong Fenchel duality). Let $X_{1}, X_{2}, Z$, be groups, and $(Z, \leq)$ an order complete group. Given $f: X_{1} \rightarrow Z \cup\{\infty\}, g: X_{2} \rightarrow Z \cup\{\infty\}$ and an additive map $T: X_{1} \rightarrow X_{2}$, define

$$
\begin{aligned}
& P=\inf _{x \in X_{1}}\{f(x)+g(T x)\} \\
& D=\sup _{\varphi^{*} \in X_{2}^{*}}\left\{-f^{\star}\left(T^{*} \varphi\right)-g^{\star}(\varphi)\right\}
\end{aligned}
$$

Then $P \geq D$ (weak duality). In particular, if $P=-\infty$ then $D=-\infty$. If, in addition, $X_{1}$ is semidivisible, $f$ and $g$ are convex and we assume

$$
0 \in \operatorname{core}(\operatorname{dom}(g)-T \operatorname{dom}(f))
$$

then $P=D$ (strong duality) and $D$ is attained when finite.
Proof. To prove weak duality, note that $P \geq D$ is equivalent to

$$
\inf _{\substack{x \in X_{1} \\ \varphi \in X_{2}^{*}}}\left[f(x)+f^{\star}\left(T^{*} \varphi\right)+g(T x)+g^{\star}(-\varphi)\right] \geq 0
$$

By Theorem 5.2, we have $f(x)+f^{\star}\left(T^{*} \varphi\right) \geq\left(T^{*} \varphi\right)(x)$ and $g(T x)+g^{\star}(-\varphi) \geq-\varphi(T x)$. Then by the definition of $T^{*}$ we have $\left(T^{*} \varphi\right)(x)-\varphi(T x)=0$.

To prove strong duality, define $h: X_{2} \rightarrow Z \cup\{\infty\}$,

$$
h(u)=\inf _{x \in X_{1}}\{f(x)+g(T x+u)\}
$$

By Proposition 5.6, $h$ is convex and $\operatorname{dom}(h)=\operatorname{dom}(g)-T \operatorname{dom}(f)$ is a convex set. Since we assume that $0 \in \operatorname{core}(\operatorname{dom}(g)-T \operatorname{dom}(f))$, applying Theorem 5.1 for $h$ and $x_{0}=0$ implies that there exists $\varphi: X_{2} \rightarrow Z \cup\{\infty\}$ additive such that $\varphi(u) \leq h(u)-h(0)$ (note that since we choose $x_{0}=0$ in Theorem 5.1, the condition $h_{x_{0}}\left(x_{0}\right)+h_{x_{0}}\left(-x_{0}\right) \leq 0$ holds, as $h_{x}(0)=0$ always). Hence,

$$
\begin{aligned}
h(0) & \leq h(u)-\varphi(u) \leq f(x)+g(T x+u)-\varphi(u) \\
& =\left[f(x)-\left(T^{*} \varphi\right)(x)\right]+[g(T x+u)-(-\varphi(T x+u))] .
\end{aligned}
$$

Taking the infimum over $x \in X_{1}, u \in X_{2}$ implies

$$
h(0) \leq-f^{\star}\left(T^{*} \varphi\right)-g^{\star}(-\varphi) \leq D
$$

Since $h(0)=P$, strong duality follows. Again the dual supremum is attained when finite.

Example 5.2. If $X_{2}$ is a meet lattice, then $X_{2}^{*}=\{0\}$ and

$$
D=-f^{\star}(0)-g^{\star}(0)=\inf _{x \in X_{1}} f(x)+\inf _{x \in X_{2}} g(x)
$$

which is clearly smaller than $P$.
Remark 5.5. Assume that in Theorem 5.3 we have $\mathbb{N}$-sublinear functions rather than convex functions. Then if we use Proposition 5.3, Theorem 5.3 still holds even if we omit the subdivisibility assumption.

Next we discuss applications of Theorem 5.3. One of the classical applications, is a representation for the subdifferential of a sum of convex functions. We show that such a result holds for groups as well.

Theorem 5.4 (Sum rule for subdifferentials). Suppose $f: X_{1} \rightarrow Z \cup\{\infty\}, g: X_{2} \rightarrow Z \cup\{\infty\}$, for $(Z, \leq)$ an order complete group and $T: X_{1} \rightarrow X_{2}$ is additive. Then

$$
\partial(f+g \circ T)\left(x_{0}\right) \supseteq \partial f\left(x_{0}\right)+T^{*} \partial g\left(x_{0}\right)
$$

If, in addition, $X_{1}$ is semidivisible, $0 \in \operatorname{core}(\operatorname{dom}(g)-T \operatorname{dom}(f))$, while $f$ and $g$ are convex, then equality holds.

Proof. The first inclusion follows immediately. To prove the equality case, let $\phi \in \partial(f+$ $g \circ T)\left(x_{0}\right)$. Then the function $(f-\phi)+g \circ T$ is minimised at $x_{0}$. Assume without loss of generality that the minimum is 0 . By the strong Fenchel duality result with $P=D=0$, there exists $\varphi \in X_{2}^{*}$ such that

$$
0=-(f-\phi)^{\star}\left(T^{*} \varphi\right)-g^{\star}(-\varphi)=-f^{\star}\left(T^{*} \varphi+\phi\right)-g^{\star}(-\varphi)
$$

Hence, for every $x_{1} \in X_{1}$ and $x_{2} \in X_{2}$, we have

$$
\begin{equation*}
0 \leq(f-\phi)\left(x_{1}\right)-T^{*} \varphi\left(x_{1}\right)+g\left(x_{2}\right)+\varphi\left(x_{2}\right) \tag{5.9}
\end{equation*}
$$

In particular, choosing $x_{1}=x_{0}$, we have for all $x_{2} \in X_{2}$,

$$
-\varphi\left(x-T x_{0}\right) \leq(f-\phi)\left(x_{0}\right)+g\left(x_{2}\right)=g\left(x_{2}\right)-g\left(T x_{0}\right)
$$

where in the last equality we used our assumption that $(f-\phi)\left(x_{0}\right)+g\left(T x_{0}\right)=0$. Thus, we have $-\varphi \in \partial g\left(T x_{0}\right)$. Also, by (5.9), we have

$$
\sup _{x_{1} \in X_{1}}\left(-g\left(T x_{1}\right)-T^{*} \varphi\left(x_{1}\right)\right) \leq \inf _{x_{1} \in X_{1}}\left((f-\phi)\left(x_{1}\right)-T^{*} \varphi\left(x_{1}\right)\right)
$$

Thus there exists $z_{0} \in Z$ such that for all $x_{1} \in X_{1}$,

$$
-g\left(T x_{1}\right) \leq\left(T^{*} \varphi\right)\left(x_{1}\right)+z_{0} \leq(f-\phi)\left(x_{1}\right)
$$

and equality holds when $x_{1}=x_{0}$. Hence $z_{0}=0$ and $T^{*} \varphi+\phi \in \partial f\left(x_{0}\right)$, which completes the proof of the theorem.

Another application of Theorem 5.3 is a Hahn-Banach theorem for groups.
Theorem 5.5 (Hahn-Banach theorem for groups). Let $X$ be a group, $X^{\prime} \subseteq X$ a subgroup, and $(Z, \leq)$ an order complete group. Assume that $f: X \rightarrow Z$ is $\mathbb{N}$-sublinear and $h: X^{\prime} \rightarrow Z$ is additive such that $h \leq f$ on $X^{\prime}$. Then there exists $\bar{h}: X \rightarrow Z$ additive such that $\bar{h} \leq f$ and $\bar{h}=h$ on $X^{\prime}$.


Figure 4. Single minorant in $\mathbb{R}$
Proof. Choose $X_{1}=X_{2}=X$ and let $T: X \rightarrow X$ be the identity map. Choose $g: X^{\prime} \rightarrow$ $Z \cup\{\infty\}$ to be $g=-h+\iota_{X^{\prime}}$, where

$$
\iota_{X^{\prime}}(x)= \begin{cases}0 & x \in X^{\prime}, \\ \infty & x \notin X^{\prime} .\end{cases}
$$

Since $f: X \rightarrow Z$, $\operatorname{dom}(f)=X$. Also, $\operatorname{dom}(g)=X^{\prime}$. Thus $0 \in \operatorname{core}(\operatorname{dom}(f)-T \operatorname{dom}(g))$ and we can thus use Theorem 5.3. Note that by Remark 5.5 we do not need to assume subdivisibility as we are dealing with $\mathbb{N}$-sublinear functions. Now, by Theorem 5.3, we have

$$
\begin{align*}
0 & \leq \inf _{x \in X}\left\{f(x)-h(x)+\iota_{X^{\prime}}(x)\right\} \\
& =\inf _{x \in X}\{f(x)+g(x)\} \\
& =\sup _{\varphi \in X^{*}}\left\{-f^{\star}(\varphi)-g^{\star}(-\varphi)\right\} . \tag{5.10}
\end{align*}
$$

Thus, there exists $\varphi \in X^{*}$ such that for all $x \in X^{\prime}, f^{\star}(\varphi) \leq \varphi(x)-h(x)$. Since $f$ is sublinear, $f(0)=0$ and so it follows that $f^{\star}(\varphi) \geq 0$ or in other words $h(x) \leq \varphi(x), x \in X^{\prime}$. Since $X^{\prime}$ is a subgroup and $\varphi$ is additive, we have $h(x)=\varphi(x)$ on $X^{\prime}$ and $g^{\star}(-\varphi)=0$. Now (5.10) implies that $f^{\star}(\varphi)=0$, which implies that $\varphi(x) \leq f(x)$ for all $x \in X$.

Remark 5.6. If $X$ and $Z$ are groups and $f, g: X \rightarrow Z \cup\{\infty\}$ are additive with $g \leq f$, then $f=g$. However, if $X$ is only a semigroup, this is no longer always true. As a result, we cannot expect strong Hahn-Banach type theorems on arbitrary semigroups.

Theorem 5.6 (Sandwich theorem for groups). Assume that $X_{1}$ is a semidivisible group, $X_{2}$ a group, and $(Z \cup\{\infty\}, \leq)$ a group with complete order. Let $f: X_{1} \rightarrow Z \cup\{\infty\}$, $-g: X_{2} \rightarrow Z \cup\{\infty\}$ be convex and $T: X_{1} \rightarrow X_{2}$ be additive, such that $g \circ T \leq f$. Assume that $0 \in \operatorname{core}(\operatorname{dom}(g)-T \operatorname{dom}(f))$. Then there exists an additive function $a: X \rightarrow Z$ such that $g \circ T \leq a \leq f$.

Proof. Using Theorem 5.3, we have $P \leq 0$ and so there exists $\varphi \in X_{2}^{*}$ such that $(-g)^{\star}(-\varphi) \leq$ $-f^{\star}\left(T^{*} \varphi\right)$. This implies that

$$
\begin{equation*}
\sup _{x \in X_{1}}\left(-g(T x)-T^{*} \varphi(x)\right) \leq \inf _{x \in X_{1}}\left(f(x)-T^{*} \varphi(x)\right) . \tag{5.11}
\end{equation*}
$$

$T^{*} \varphi$ is the required additive function. If $P=-\infty$ in Theorem 5.3, then $P<-\alpha<0$ for every $\alpha>0$ and so inequality (5.11) still holds.

Remark 5.7. By Proposition 5.3, Theorem 5.6 holds if we replace convex functions by $\mathbb{N}$-sublinear, even if we omit the subdivisibility assumption.

Remark 5.8. Even for $X=\mathbb{R}$, the only additive minorant may be $a=0$. Consider the subadditive (non-convex) function $f(x)=\sqrt{|x|}$. See Figure 4.

## 6. Subadditive optimisation

Let $f, g_{1}, \ldots, g_{k}: X \rightarrow[-\infty, \infty]$ and $b \in \mathbb{R}$. Define $v: \mathbb{R}^{k} \rightarrow[-\infty, \infty]$ by

$$
\begin{equation*}
v(b)=v\left(b_{1}, \ldots, b_{k}\right)=\inf \left\{f(x) \mid x \in X, g_{1}(x) \leq b_{1}, \ldots, g_{k}(x) \leq b_{k}\right\} \tag{6.1}
\end{equation*}
$$

$v$ is also known as the value function. We have the following.
Proposition 6.1 (Subadditive and sublinear value functions). Assume that $X$ is a monoid and $f, g_{1}, \ldots, g_{k}: X \rightarrow[-\infty, \infty]$ are subadditive. Then the function $v: \mathbb{R}^{k} \rightarrow[-\infty, \infty]$ defined by (6.1) is subadditive. If, in addition, $X$ is assumed to be $p$-semidivisible and $f, g_{1}, \ldots, g_{k}$ satisfy $f(p x)=p f(x), g_{i}(p x)=p g_{i}(x), 1 \leq i \leq k$ then $v$ satisfies $v(p x)=p v(x)$. In particular, by Proposition 2.7, $v$ is convex.

Proof. Let $x_{1}, x_{2} \in X$ be such that $g_{i}\left(x_{1}\right) \leq b_{i}, g\left(x_{2}\right) \leq c_{i}, 1 \leq i \leq k$. Since $g_{1}, \ldots, g_{k}$ are subadditive, $g_{i}\left(x_{1}+x_{2}\right) \leq g_{i}\left(x_{1}\right)+g_{i}\left(x_{2}\right) \leq b_{i}+c_{i}, 1 \leq i \leq k$. Thus, of $b=\left(b_{1}, \ldots, b_{k}\right)$, $c=\left(c_{1}, \ldots, c_{k}\right)$, then

$$
v(b+c) \leq f\left(x_{1}+x_{2}\right) \leq f\left(x_{1}\right)+f\left(x_{2}\right),
$$

where we used the subadditivity of $f$. Taking the infimum over the right side, the first assertion follows. To prove the second assertion, we only need to prove positive homogeneity. Indeed, for every $x \in X$, since $X$ is $p$-semidivisible, there exists $y \in X$ satisfying $x=p y$. As a result,

$$
\begin{aligned}
v(p b) & =\inf \left\{f(x) \mid x \in X, g_{1},(x) \leq p b_{1}, \ldots, g_{k}(x) \leq p b_{k}\right\} \\
& =\inf \left\{f(p y) \mid y \in X, g_{1}(p y) \leq p b_{1}, \ldots, g_{k}(p y) \leq p b_{k}\right\} \\
& =\inf \left\{p f(y) \mid y \in X, p g_{1}(y) \leq p b_{1}, \ldots, p g_{k}(y) \leq p b_{k}\right\} \\
& =p \inf \left\{f(y) \mid y \in X, g_{1}(y) \leq b_{1}, \ldots, g_{k}(y) \leq b_{k}\right\} \\
& =p v(b),
\end{aligned}
$$

and we are done.
Remark 6.1. The result holds if the module is over a semidivisible semiring $R$ and $f$ and $g$ are subadditive functions.

In the sublinear case, we may now apply Theorem 5.1 to the function $h$ of Proposition 6.1 to describe $h$ in terms of additive minorants.

Example 6.1. Let $b \in \mathbb{R}$, and let

$$
\inf \{-x \mid 2 x \leq b, x \in \mathbb{Z}\}=-\left\lceil\frac{b}{2}\right\rceil
$$

Thus, in the nondivisible setting, even if $k=1$ and $f$ and $g_{1}$ are additive, $v$ need not be homogeneous.

In general integer programming [Wil97, AV95] adding the sub additive, but not $\mathbb{N}$-homogeneous, ceiling function $\lceil\cdot\rceil$ allows one to reconstruct integer value functions but the additive minorants do not suffice. This is discussed in [TW81, BJ82]. It is interesting to ask what class of groups allows an analogue of the ceiling?

We note also that methods that were originally developed to study linear programming results in vector spaces, such as the cutting-plane method [Kel60], can also be used to study integer linear programming problems. See also [AV95, LL02] and the survey [BV] for more information on the cutting-planes method, and [BJ82, Gom58, GB60, LL02] for more information on integer programming.
6.1. Lagrange multipliers in action. Suppose now that we have an optimisation problem with $m$ constraints:

$$
\inf \left\{f(x) \mid g_{1}(x) \leq 0, \ldots, g_{k}(x) \leq 0\right\}
$$

Let $g(x)=\left(g_{1}(x), \ldots, g_{k}(x)\right) \in \mathbb{R}^{m}$. Define the Lagrangian function $L: X \times \mathbb{R}^{k} \rightarrow(-\infty, \infty]$ to be

$$
L(x, \lambda)=f(x)+\lambda \cdot g(x)
$$

Here, $\lambda \cdot g(x)$ is the standard inner product in $\mathbb{R}^{k}$. We say that $\bar{\lambda} \in \mathbb{R}^{k}$ is a Lagrange multiplier if the Lagrangian function $L(\cdot, \bar{\lambda})$ has the same infimum as $f$ on $X$. We will now show that Lagrange multipliers can be used to compute the subdifferential of the maximum of convex function. In the vector space case, this fact has several different proofs. We chose this particular version to show the use of Lagrange multipliers in the group setting.

Theorem 6.1. Let $X$ be a semidivisible group and $f_{i}: X \rightarrow(-\infty, \infty]$ be convex functions, where $i \in I$, I being a finite index set. Let $f=\max _{1 \leq i \leq k} f_{i}$. For $x_{0} \in \bigcap_{i \in I\left(x_{0}\right)} \operatorname{core}\left(\operatorname{dom}\left(f_{i}\right)\right)$, where $I\left(x_{0}\right)=\left\{1 \leq i \leq k \mid f_{i}\left(x_{0}\right)=f\left(x_{0}\right)\right\}$. Then we have

$$
\partial f\left(x_{0}\right)=\operatorname{conv}\left(\bigcup_{i \in I\left(x_{0}\right)} \partial f_{i}\left(x_{0}\right)\right)
$$

Proof. The inclusion $\supseteq$ follows immediately from the fact the subdifferential is convex (Proposition 5.5 with $Y=\mathbb{R}$ ). To prove the other inclusion, consider the constrained minimisation problem

$$
\begin{equation*}
\inf \left\{t \mid t \in \mathbb{R}, x \in X, f_{1}(x) \leq t, \ldots, f_{k}(x) \leq t\right\} \tag{6.2}
\end{equation*}
$$

Note that this infimum equals $\inf _{x \in X} f(x)$. Assume first that $0 \in \partial f\left(x_{0}\right)$, which means that the infimum in (6.2) is attained at $x_{0}$. Define the following auxiliary value function $v: \mathbb{R}^{I\left(x_{0}\right)} \rightarrow[-\infty, \infty]$,

$$
v(b)=\inf \left\{t \mid f_{i}(x)-t \leq b_{i}, i \in I\left(x_{0}\right)\right\}
$$

We have $v(b) \geq f\left(x_{0}\right)-\max _{i \in I\left(x_{0}\right)}\left|b_{i}\right|>-\infty$. Also, since we assumed that

$$
x_{0} \in \bigcap_{i \in I\left(x_{0}\right)} \operatorname{core}\left(\operatorname{dom}\left(f_{i}\right)\right)
$$

it follows that $0 \in \operatorname{core}(\operatorname{dom}(v))$. By Proposition 6.1, $v$ is convex. Thus, by Theorem 5.1, there exists $\bar{\lambda} \in \partial v(0)$ (again we are allowed to use the max formula because we are at
$x_{0}=0$ ). We note also that if $b \in \mathbb{R}_{+}^{I\left(x_{0}\right)}$ then we also have $v(b) \leq f\left(x_{0}\right)$ (infimum over a larger set) and also $v(0)=f\left(x_{0}\right)$. Thus, we have

$$
f\left(x_{0}\right)=v(0) \leq v(b)+\bar{\lambda} \cdot b \leq f\left(x_{0}\right)+\bar{\lambda} \cdot b,
$$

which means that $\bar{\lambda} \in \mathbb{R}_{+}^{I\left(x_{0}\right)}$. Hence,

$$
\begin{aligned}
t & \geq v\left(\left(f_{i}(x)-t\right)_{i \in I\left(x_{0}\right)}\right) \\
& \geq v(0)-\bar{\lambda} \cdot\left(f_{i}(x)-t\right)_{i \in I\left(x_{0}\right)} \\
& =f\left(x_{0}\right)-\bar{\lambda} \cdot\left(f_{i}(x)-t\right)_{i \in I\left(x_{0}\right)}
\end{aligned}
$$

and so

$$
t+\bar{\lambda} \cdot\left(f_{i}(x)-t\right)_{i \in I\left(x_{0}\right)} \geq f\left(x_{0}\right)
$$

which means that $\bar{\lambda}$ is a minimiser for the Lagrangian function. In other words, we can find $\bar{\lambda} \in \mathbb{R}^{I\left(x_{0}\right)}$ that minimises

$$
\begin{equation*}
t+\sum_{i \in I\left(x_{0}\right)} \lambda_{i}\left(f_{i}(x)-t\right)=t\left(1-\sum_{i \in I\left(x_{0}\right)} \lambda_{i}\right)+\sum_{i \in I\left(x_{0}\right)} \lambda_{i} f_{i}(x) \tag{6.3}
\end{equation*}
$$

We must have $\sum_{i \in I\left(x_{0}\right)} \bar{\lambda}_{i}=1$. If not, then we can choose $t$ that would make (6.3) go to $-\infty$. Thus, we have

$$
\sum_{i \in I\left(x_{0}\right)} \bar{\lambda}_{i} f_{i}\left(x_{0}\right) \leq \sum_{i \in I\left(x_{0}\right)} \bar{\lambda}_{i} f_{i}(x),
$$

and so $0 \in \partial\left(\sum_{i \in I\left(x_{0}\right)} \bar{\lambda}_{i} f_{i}\right)\left(x_{0}\right)$. If, in general, we have that $\phi \in \partial f\left(x_{0}\right)$, then $0 \in \partial(f-$ $\phi)\left(x_{0}\right)$ and then we repeat the same argument to conclude that $\phi \in \partial\left(\sum_{i \in I\left(x_{0}\right)} \bar{\lambda}_{i} f_{i}\right)\left(x_{0}\right)$. Altogether, we get

$$
\partial f\left(x_{0}\right) \subseteq \bigcup\left\{\partial\left(\sum_{i \in I\left(x_{0}\right)} \lambda_{i} f_{i}\right)\left(x_{0}\right) \mid \lambda_{i} \geq 0, \sum_{i \in I\left(x_{0}\right)} \lambda_{i}=1\right\} .
$$

Now, Theorem 5.4 implies that the right side is equal to

$$
\operatorname{conv}\left(\bigcup_{i \in I\left(x_{0}\right)} \partial f_{i}\left(x_{0}\right)\right)
$$

and so we have

$$
\partial f\left(x_{0}\right) \subseteq \operatorname{conv}\left(\bigcup_{i \in I\left(x_{0}\right)} \partial f_{i}\left(x_{0}\right)\right)
$$

which proves the other inclusion and concludes the proof.
Remark 6.2. Combining Theorem 6.1 with Proposition 2.5 allows us to consider subadditive optimisation problems with finitely many constraints.

## 7. Conclusion

This paper grew out of a lecture that the first author gave in 1983 and then put aside until 2015 when the second author joined him in recreating and extending the original results. One original intention was to better understand the difficulty of integer programming as that of programming over a non-divisible group. See also [ $\left.\mathrm{BEE}^{+} 14, \mathrm{FGL} 05\right]$. In so doing we have uncovered many interesting connections but as of now made little progress directly for integer programming.

Surely there are many other classical results for which one can find elegant and even useful generalisations. Hopefully this paper will serve as an invitation to others to join the pursuit.

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