# Linear conic formulations for two-party correlations and values of nonlocal games 

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#### Abstract

In this work we study the sets of two-party correlations generated from a Bell scenario involving two spatially separated systems with respect to various physical models. We show that the sets of classical, quantum, no-signaling and unrestricted correlations can be expressed as projections of affine sections of appropriate convex cones. As a by-product, we identify a spectrahedral outer approximation to the set of quantum correlations which is contained in the first level of the Navascués, Pironio and Acín (NPA) hierarchy and also a sufficient condition for the set of quantum correlations to be closed. Furthermore, by our conic formulations, the value of a nonlocal game over the sets of classical, quantum, no-signaling and unrestricted correlations can be cast as a linear conic program. This allows us to show that a semidefinite programming upper bound to the classical value of a nonlocal game introduced by Feige and Lovász is in fact an upper bound to the quantum value of the game and moreover, it is at least as strong as optimizing over the first level of the NPA hierarchy. Lastly, we show that deciding the existence of a perfect quantum (resp. classical) strategy is equivalent to deciding the feasibility of a linear conic program over the cone of completely positive semidefinite matrices (resp. completely positive matrices). By specializing the results to synchronous nonlocal games, we recover the conic formulations for various quantum and classical graph parameters that were recently derived in the literature.


Keywords. Quantum correlations, nonlocal games, completely positive semidefinite cone, conic programming, quantum graph parameters, semidefinite programming relaxations.

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## 1 Introduction

Consider two parties, Alice and Bob, who individually perform measurements on a shared physical system without communicating. A problem of fundamental importance with which we are primarily concerned in this work is to characterize the structure of the sets of correlations that can arise between Alice and Bob, with respect to various physical models.

In one of the most celebrated discoveries of modern physics John Bell showed that quantum mechanical systems can exhibit correlations that cannot be reproduced within the framework of classical physics [3]. This fact has received extensive experimental verification, see [21, 1] for examples. In addition to their theoretical significance, these correlations have been increasingly regarded as a valuable resource for distributed tasks such as unconditionally secure cryptography [19] and randomness certification [13] among others.

In order to tackle this problem, we take the viewpoint of linear conic optimization. Specifically, we introduce the notion of conic correlations and show that the sets of classical, quantum, nosignaling and unrestricted correlations can be expressed as conic correlations over appropriate convex cones. Consequently, conic correlations provide us with a unified framework where we can study the properties of many interesting families of correlations. Furthermore, using our conic characterizations, we can express the classical, quantum, no-signaling and unrestricted values of a nonlocal game as linear conic programs. This allows one to use the arsenal of linear conic programming theory in order to study how the various values of a nonlocal game relate to each other and to better understand their properties.

There exists a significant body of work addressing these questions from a mathematical optimization perspective. In the celebrated work [32] Navascués, Pironio and Acín constructed a hierarchy of spectrahedral outer approximations to the set of quantum correlations. Another fundamental result is that the quantum value of an XOR nonlocal game is given by a semidefinite program [38, 11]. Furthermore, the quantum value of a unique nonlocal game can be tightly approximated using semidefinite programming [25]. Lastly, mathematical optimization has also proven to be extremely useful for (classical and quantum) parallel repetition results [20, 12, 25, 15, 16].

Two-party correlations. Consider the following thought experiment: Two spatially separated parties, Alice and Bob, perform measurements on some shared physical system. Alice has a set of possible measurements at her disposal, where each measurement is labeled by some element of a finite set $S$. The set of possible outcomes of each of Alice's measurements is labeled by the elements of some finite set $A$. Similarly, Bob has a set $T$ of possible measurements at his disposal each with possible outcomes labeled by the elements of some finite set $B$. Note that we use the term "measurement" very loosely at this point as the details depend on the underlying physical theory. We refer to a thought experiment as described above as a Bell scenario.

At each run of the experiment Alice and Bob without communicating choose measurements $s \in S$ and $t \in T$ respectively which they use to measure their individual systems. Following the measurement they get $a \in A$ and $b \in B$ as outcomes. Since the measurement process is probabilistic, each time the experiment is conducted Alice and Bob might generate different outcomes. The Bell scenario is completely described by the joint conditional probability distribution $p=\left(p(a, b \mid s, t)_{a, b, s, t}\right.$, where $p(a, b \mid s, t)$ denotes the conditional probability that upon performing measurements $s \in S$ and $t \in T$, Alice and Bob get outcomes $a \in A$ and $b \in B$, respectively.

For any Bell scenario, the set of all joint probability distributions, denoted by $\mathcal{P}$, consists of all vectors $p=(p(a, b \mid s, t)) \in \mathbb{R}^{|A \times B \times S \times T|}$ that satisfy $p(a, b \mid s, t) \geq 0$, for all $a, b, s, t$, and $\sum_{a \in A, b \in B} p(a, b \mid s, t)=1$, for all $s, t$. The elements of $\mathcal{P}$ are called correlation vectors or simply corre-
lations.
A question of fundamental theoretical interest is to describe the correlations that can arise within a Bell scenario as described above with respect to various physical models. We now briefly introduce the models and the corresponding sets of correlations that are relevant to this work. For additional details the reader is referred to the extensive survey [7] and references therein.

Classical correlations. A classical strategy allows Alice and Bob to determine their outputs by employing both private and shared randomness. Formally, a classical strategy is given by:
(i) A shared random variable $i$ with domain $[n]$, each sample occurring with probability $k_{i}$.
(ii) For each $i \in[n]$ and $s \in S$ a probability distribution $\left\{x_{a}^{s, i}: a \in A\right\}$.
(iii) For each $i \in[n]$ and $t \in T$ a probability distribution $\left\{y_{b}^{t, i}: b \in B\right\}$.

Given that the value of the shared randomness is $i \in[n]$, if Alice chooses measurement $s \in S$ she determines her output $a \in A$ by sampling from the distribution $\left\{x_{a}^{s, i}\right\}_{a \in A}$. Bob acts analogously and determines his output by sampling from the distribution $\left\{y_{b}^{t, i}\right\}_{b \in B}$. Formally, a correlation $p \in$ $\mathcal{P}$ is called classical if there exist nonnegative scalars $\left\{k_{i}\right\}_{i,},\left\{x_{a}^{s, i}\right\}_{a, s, i},\left\{y_{b}^{t, i}\right\}_{b, t, i}$ satisfying $\sum_{i \in[n]} k_{i}=$ 1, $\sum_{a \in A} x_{a}^{s, i}=\sum_{b \in B} y_{b}^{t, i}=1$, for all $s, t, i$ and

$$
\begin{equation*}
p(a, b \mid s, t)=\sum_{i=1}^{n} k_{i} x_{a}^{s, i} y_{b}^{t, i}, \text { for all } a, b, s, t \tag{1}
\end{equation*}
$$

We denote the set of classical correlations by $\mathcal{C}$. Note that in the literature, classical correlations are also referred to as "local" and denoted by $\mathcal{L}$.

The set of classical correlations forms a convex polytope in $\mathbb{R}^{|A \times B \times S \times T|}$. Its vertices correspond to deterministic strategies, i.e., correlations of the form $p(a, b \mid s, t)=\delta_{a, \alpha(s)} \delta_{b, \beta(t)}$ for some pair of functions $\alpha: S \rightarrow A$ and $\beta: T \rightarrow B$, where $\delta_{i, j}$ denotes the Kronecker delta function.

Quantum correlations. A quantum strategy for a Bell scenario allows Alice and Bob to determine their outputs by performing measurements on a shared quantum state (the reader is referred to Section 2 for background on quantum information and the context behind the mathematical formalism in the following discussion.) A correlation $p \in \mathcal{P}$ is called quantum if there exists finite dimensional complex Euclidean spaces $\mathcal{X}$ and $\mathcal{Y}$, a unit vector $\psi \in \mathcal{X} \otimes \mathcal{Y}$, Hermitian positive semidefinite (psd) operators $\left\{X_{a}^{s}\right\}_{a \in A}$ satisfying $\sum_{a} X_{a}^{s}=\mathbb{I}_{\mathcal{X}}$, for each $s \in S$ and Hermitian psd operators $\left\{Y_{b}^{t}\right\}_{b \in B}$ satisfying $\sum_{b} Y_{b}^{t}=\mathbb{I}_{\mathcal{Y}}$, for each $t \in T$, where

$$
\begin{equation*}
p(a, b \mid s, t)=\psi^{*}\left(X_{a}^{s} \otimes Y_{b}^{t}\right) \psi, \text { for all } a, b, s, t \tag{2}
\end{equation*}
$$

We denote the set of quantum correlations by $\mathcal{Q}$.
The set of quantum correlations is a non-polyhedral set whose structure has been extensively studied but is nevertheless not well understood (e.g. see [7]). In particular, it is not even known whether $\mathcal{Q}$ is closed. On the positive side, Navascués, Pironio, and Acín (NPA) in [32] identified a hierarchy of spectrahedral outer approximations to the set of quantum correlations. Although the NPA hierarchy converges, it is not known whether it converges to the set of quantum correlations.

No-signaling correlations. A correlation $p \in \mathcal{P}$ is no-signaling if Alice's local marginal probabilities are independent of Bob's choice of measurement and, symmetrically, Bob's local marginal probabilities are independent of Alice's choice of measurement. Algebraically, $p \in \mathcal{P}$ is no-signaling if it satisfies:

$$
\begin{gather*}
\sum_{b \in B} p(a, b \mid s, t)=\sum_{b \in B} p\left(a, b \mid s, t^{\prime}\right), \text { for all } s \in S, t \neq t^{\prime} \in T, \text { and }  \tag{3}\\
\sum_{a \in A} p(a, b \mid s, t)=\sum_{a \in A} p\left(a, b \mid s^{\prime}, t\right), \text { for all } t \in T, s \neq s^{\prime} \in S \tag{4}
\end{gather*}
$$

We denote the set of no-signaling correlations by $\mathcal{N S}$.
The no-signaling conditions (3) and (4) are a natural physical requirement since if they are violated at least one party can receive information about the other party's input instantaneously, contradicting the fact that information cannot travel faster than the speed of light.

It is immediate from physical context that every classical correlation is also quantum (cf. Theorem 3.5). Furthermore, it is easy to verify that every quantum correlation is no-signaling (cf. Theorem 3.8). On the other hand, it is well-known that there exist quantum correlations that are not classical and no-signaling correlations that are not quantum. In other words, we have that

$$
\begin{equation*}
\mathcal{C} \subsetneq \mathcal{Q} \subsetneq \mathcal{N S} \subsetneq \mathcal{P}, \tag{5}
\end{equation*}
$$

and in this paper we give (alternative) algebraic proofs of these containments.
Two-player one-round nonlocal games. As we mentioned, the set of quantum correlations is a strict superset of the set of classical correlations. How can we identify quantum correlations that are not classical? One approach is via the framework of nonlocal games which we now introduce.

A nonlocal game is a thought experiment between two spatially separated parties, Alice and Bob, who can only communicate with a third party, a referee, who decides whether they win or lose. Formally, a (two-player one-round) nonlocal game is specified by four finite sets $A, B, S, T$, a probability distribution $\pi$ on $S \times T$ and a Boolean predicate $V: A \times B \times S \times T \rightarrow\{0,1\}$. We denote the nonlocal game by $\mathcal{G}(\pi, V)$ or simply $\mathcal{G}$ when there is no need to specify $\pi$ and $V$.

The nonlocal game $\mathcal{G}(\pi, V)$ proceeds as follows: The referee using the distribution $\pi$ samples a pair of questions $(s, t) \in S \times T$ and sends $s$ to Alice and $t$ to Bob. After receiving their questions, Alice and Bob use some strategy to determine their answers $a \in A$ and $b \in B$ which they send back to the referee. The players win the game if $V(a, b \mid s, t)=1$ and they lose otherwise.

The objective of the players is to maximize their probability of winning the game. To do this the players are not allowed to communicate after they receive their questions but they can agree on some common strategy before the start of the game using their knowledge of $V$ and $\pi$.

Fix a particular strategy for the game that gives rise to the correlation $p=(p(a, b \mid s, t)) \in \mathcal{P}$. The probability that Alice and Bob win the game using this strategy is given by

$$
\sum_{s \in S} \sum_{t \in T} \pi(s, t) \sum_{a \in A} \sum_{b \in B} V(a, b \mid s, t) p(a, b \mid s, t) .
$$

For a fixed set of correlations $\mathcal{S} \subseteq \mathcal{P}$ we denote by $\omega_{\mathcal{S}}(\mathcal{G})$ the maximum probability Alice and Bob can win the game $\mathcal{G}$ when they use strategies that generate correlations that lie in $\mathcal{S}$. Formally:

$$
\begin{equation*}
\omega_{\mathcal{S}}(\mathcal{G}):=\sup \left\{\sum_{s \in S} \sum_{t \in T} \pi(s, t) \sum_{a \in A} \sum_{b \in B} V(a, b \mid s, t) p(a, b \mid s, t): p \in \mathcal{S}\right\} \tag{6}
\end{equation*}
$$

In this paper we restrict our attention to (i) the classical value denoted $\omega_{\mathcal{C}}(\mathcal{G})$, (ii) the quantum value denoted $\omega_{\mathcal{Q}}(\mathcal{G})$, (iii) the no-signaling value denoted $\omega_{\mathcal{N S}}(\mathcal{G})$ and (iv) the unrestricted value denoted $\omega_{\mathcal{P}}(\mathcal{G})$. As an immediate consequence of the set inclusions given in Equation (5) we have

$$
\omega_{\mathcal{C}}(\mathcal{G}) \leq \omega_{\mathcal{Q}}(\mathcal{G}) \leq \omega_{\mathcal{N S}}(\mathcal{G}) \leq \omega_{\mathcal{P}}(\mathcal{G})
$$

for any nonlocal game $\mathcal{G}$.
As a concrete example of the above definitions we now describe the CHSH game [10]. This game has $A=B=S=T=\{0,1\}, \pi$ is uniform, and $V(a, b \mid s, t)=1$ if and only if $a \oplus b=s \cdot t$, where $\oplus$ denotes addition modulo 2. Informally, the referee sends a random bit $s$ to Alice and an independently random bit $t$ to Bob. The players respond with single bits $a$ and $b$, respectively. Alice and Bob win if $V(a, b \mid s, t)=1$, i.e., if $a \oplus b$ is equal to the logical AND of their questions. It is well-known that the no-signaling value of the CHSH game is 1 , the quantum value is $\cos ^{2}(\pi / 8) \approx$ 0.85 and the classical value is $3 / 4$.

Convex cones of interest. Consider a vector space $\mathcal{V}$ endowed with an inner product $\langle\cdot, \cdot\rangle$. The Gram matrix of the vectors $\left\{x_{i}\right\}_{i=1}^{n} \subseteq \mathcal{V}$, denoted by

$$
\begin{equation*}
\operatorname{Gram}\left(\left\{x_{i}\right\}_{i=1}^{n}\right), \tag{7}
\end{equation*}
$$

is the $n \times n$ matrix whose $(i, j)$ entry is given by $\left\langle x_{i}, x_{j}\right\rangle$. We say that the vectors $\left\{x_{i}\right\}_{i=1}^{n}$ form a Gram representation of $X=\operatorname{Gram}\left(\left\{x_{i}\right\}_{i=1}^{n}\right)$.

We denote by $\mathcal{S}^{n}$ the set of $n \times n$ real symmetric matrices which we equip with the HilbertSchmidt inner product $\langle X, Y\rangle:=\operatorname{Tr}(X Y)$. A matrix $X \in \mathcal{S}^{n}$ is called positive semidefinite (psd) if $X=\operatorname{Gram}\left(\left\{x_{i}\right\}_{i=1}^{n}\right)$ for some family of real vectors $\left\{x_{i}\right\}_{i=1}^{n} \subseteq \mathbb{R}^{d}$ (for some $d \geq 1$ ). Equivalently, a matrix is psd if and only if its eigenvalues are nonnegative. A nonsingular psd matrix is called positive definite. We denote by $\mathcal{S}_{+}^{n}$ (resp. $\mathcal{S}_{++}^{n}$ ) the set of $n \times n$ psd matrices (resp. positive definite matrices). The set $\mathcal{S}_{+}^{n}$ forms a closed, convex, self-dual cone whose structure is well understood (e.g. see [2] and references therein). Linear optimization over $\mathcal{S}_{+}^{n}$ is called semidefinite programming (SDP) and its optimal value can be approximated within arbitrary precision in polynomial time using the ellipsoid method, under reasonable assumptions (e.g. see [4]).

The nonnegative cone, denoted by $\mathcal{N}^{n}$, consists of the $n \times n$ entrywise nonnegative matrices in $\mathcal{S}^{n}$. It is easy to verify that $\mathcal{N}^{n}$ is a self-dual cone.

A matrix is called doubly nonnegative if its psd and entrywise nonnegative. We denote by $\mathcal{D N} \mathcal{N}^{n}$ the set of $n \times n$ doubly nonnegative matrices, i.e.,

$$
\begin{equation*}
\mathcal{D N N}^{n}:=\left\{X \in \mathcal{S}_{+}^{n}: X_{i, j} \geq 0, \text { for all } 1 \leq i, j \leq n\right\}, \tag{8}
\end{equation*}
$$

which is known to form a full-dimensional closed convex cone.
A matrix $X \in \mathcal{S}^{n}$ is called completely positive if

$$
\begin{equation*}
X=\operatorname{Gram}\left(\left\{x_{i}\right\}_{i=1}^{n}\right), \text { where }\left\{x_{i}\right\}_{i=1}^{n} \subseteq \mathbb{R}_{+}^{d}(\text { for some } d \geq 1) . \tag{9}
\end{equation*}
$$

The set of $n \times n$ completely positive matrices forms a full-dimensional closed convex cone known as the completely positive cone, and is denoted by $\mathcal{C P}{ }^{n}$. The structure of the $\mathcal{C P}$ cone has been extensively studied, e.g. see [5]. Optimization over $\mathcal{C P}$ is intractable since there exist NP-hard combinatorial optimization problems that can be formulated as linear optimization problems over $\mathcal{C P}$ [14] (see also Section (4). On the positive side, there exist SDP hierarchies that can be used to approximate $\mathcal{C P}$ from the interior [26] and from the exterior [34].

Thinking of nonnegative vectors as diagonal psd matrices suggests a natural generalization of the completely positive cone. A matrix $X \in \mathcal{S}^{n}$ is called completely positive semidefinite (cpsd) if

$$
\begin{equation*}
X=\operatorname{Gram}\left(\left\{X_{i}\right\}_{i=1}^{n}\right), \text { where }\left\{X_{i}\right\}_{i=1}^{n} \subseteq \mathcal{S}_{+}^{d}(\text { for some } d \geq 1) . \tag{10}
\end{equation*}
$$

The set of $n \times n$ cpsd matrices forms a full-dimensional convex cone denoted by $\mathcal{C} \mathcal{S}_{+}^{n}$. The $\mathcal{C} \mathcal{S}_{+}$ cone was introduced recently as a tool to provide conic programming formulations for the quantum chromatic number of a graph [27] and quantum graph homomorphisms [36] (cf. Section 5). Nevertheless, its structure appears to be very complicated. In particular it is not known whether $\mathcal{C} \mathcal{S}_{+}$forms a closed set [8]. Furthermore, given a matrix $X \in \mathcal{C} \mathcal{S}_{+}^{n}$, no upper bound is known on the size of the psd matrices in a Gram representation for $X$. This is in contrast to the completely positive cone, where we can always find a Gram representation satisfying (9) using nonnegative vectors whose dimension is at most quadratic in the size of the matrix. Lastly, combining results from [27] and [24] it follows that linear optimization over $\mathcal{C \mathcal { S } _ { + }}$ is NP-hard.

It is immediate from the definitions given above that for all $n \geq 1$ we have

$$
\begin{equation*}
\mathcal{C} \mathcal{P}^{n} \subseteq \mathcal{C} \mathcal{S}_{+}^{n} \subseteq \mathcal{D N} \mathcal{N}^{n} \tag{11}
\end{equation*}
$$

For $n \leq 4$ it is known that $\mathcal{C P}{ }^{n}=\mathcal{D N N}^{n}$ [31]. On the other hand, for $n \geq 5$ all the inclusions in (11) are known to be strict [27].

As the mathematical formulation of quantum mechanics is stated in terms of psd matrices with complex entries, in some parts of this work we consider matrices with complex entries. We denote by $\mathcal{H}^{n}$ the set of $n \times n$ Hermitian matrices. A matrix $X \in \mathcal{H}^{n}$ is called Hermitian positive semidefinite if $z^{*} X z \geq 0$, for all $z \in \mathbb{C}^{n}$. We denote by $\mathcal{H}_{+}^{n}$ (resp. $\mathcal{H}_{++}^{n}$ ) the set of $n \times n$ Hermitian positive semidefinite matrices (resp. $n \times n$ Hermitian positive definite matrices). Occasionally, we also use the notation $\mathcal{H}_{+}(\mathcal{X})$ to denote the positive operators acting on a finite dimensional complex Euclidean space $\mathcal{X}$. For a matrix $X \in \mathcal{H}^{n}$ we write $X=\mathcal{R}(X)+i \mathcal{I}(X)$, where $\mathcal{R}(X)$ is the real part and $\mathcal{I}(X)$ is the imaginary part of $X$. If $X$ is Hermitian we get that $\mathcal{R}(X)$ is real symmetric and $\mathcal{I}(X)$ is real skew-symmetric. Moreover, for $X, Y \in \mathcal{H}^{n}$ we have $\langle X, Y\rangle=$ $\operatorname{Tr}(\mathcal{R}(X) \mathcal{R}(Y)-\mathcal{I}(X) \mathcal{I}(Y))$. For a matrix $X \in \mathbb{C}^{n \times n}$, set

$$
T(X):=\frac{1}{\sqrt{2}}\left(\begin{array}{cc}
\mathcal{R}(X) & -\mathcal{I}(X)  \tag{12}\\
\mathcal{I}(X) & \mathcal{R}(X)
\end{array}\right),
$$

and notice that $T$ is a bijection between complex $n \times n$ matrices and real $2 n \times 2 n$ matrices. More importantly, we have that $X \in \mathcal{H}_{+}^{n}$ if and only if $T(X) \in \mathcal{S}_{+}^{2 n}$ and moreover $\langle X, Y\rangle=\langle T(X), T(Y)\rangle$ for all $X, Y \in \mathcal{H}_{+}^{n}$. This shows that the set of cpsd matrices does not change if we allow the psd matrices in the Gram decompositions to be Hermitian psd instead of just real psd.

Lastly, a symmetric matrix $X \in \mathcal{N}^{N}$ (where $N:=|S||A|+|T||B|$ ) is called no-signaling, denoted by $\mathcal{N S O}$, if it satisfies

$$
\begin{gathered}
\sum_{a \in A} X[(s, a),(t, b)]=\sum_{a \in A} X\left[\left(s^{\prime}, a\right),(t, b)\right], \forall b \in B, t \in T, s \neq s^{\prime} \in S, \text { and } \\
\sum_{b \in B} X[(s, a),(t, b)]=\sum_{b \in B} X\left[(s, a),\left(t^{\prime}, b\right)\right], \forall a \in A, s \in S, t \neq t^{\prime} \in T
\end{gathered}
$$

Contributions. Consider a Bell scenario with question sets $S, T$ and answer sets $A, B$ and set $N:=|S||A|+|T||B|$. In this work we mostly consider symmetric $N \times N$ matrices. The rows and
columns of such a matrix are each indexed by $S \times A$ and $T \times B$ and it is useful to think of $X$ as being partitioned into blocks $X_{i, j}$, where each block is indexed by a pair of questions $i, j \in S \cup T$. The size of each block is $(i)|A| \times|A|$ if $i, j \in S,(i i)|A| \times|B|$ if $i \in S, j \in T,($ iii $)|B| \times|A|$ if $i \in T, j \in S$ and $(i v)|B| \times|B|$ if $i, j \in T$.

For $i, j \in S \cup T$ we define $J_{i, j} \in \mathcal{S}^{N}$ to be the matrix which acts on a matrix $X \in \mathcal{S}^{N}$ by summing all entries in block $X_{i, j}$, i.e.,

$$
\begin{equation*}
\left\langle J_{i, j}, X\right\rangle=\sum_{k, l} X[(i, k),(j, l)] . \tag{13}
\end{equation*}
$$

At times, we also consider symmetric $(N+1) \times(N+1)$ matrices, which have an extra row and column indexed by " 0 ". We also extend the operator $J_{i, j}$ defined in (13) to act on $\mathcal{S}^{N+1}$, where we define

$$
\begin{equation*}
\left\langle J_{0, i}, X\right\rangle=\sum_{k} X[0,(i, k)] \text {, for all } i \in S \cup T \text { and }\left\langle J_{0,0}, X\right\rangle=X[0,0] . \tag{14}
\end{equation*}
$$

Lastly, in the final part of this work we also consider matrices in $\mathcal{S}^{|S \times A|}$. In this case, for any $s, s^{\prime} \in S$ we denote by $J_{s, s^{\prime}}$ the operator that acts on $X \in \mathcal{S}^{|S \times A|}$ by summing the entries in block $X_{s, s^{\prime}}$.

For brevity, we do not specify in the notation whether the operator $J_{i, j}$ acts on $\mathcal{S}^{N}, \mathcal{S}^{1+N}$ or $\mathcal{S}^{|S \times A|}$ as this is always clear from context.
Correlation sets. In Section 3 we express the sets of classical, quantum, no-signaling and unrestricted correlations as projections of affine slices of appropriate convex cones. To achieve this we use the following definition.

Definition 1.1. For a convex cone $\mathcal{K} \subseteq \mathcal{N}^{N}$ the set of $\mathcal{K}$-correlations, denoted $\operatorname{Corr}(\mathcal{K})$, is defined as the set of vectors $p=(p(a, b \mid s, t)) \in \mathbb{R}^{|A \times B \times S \times T|}$ for which there exists a matrix $X \in \mathcal{K}$ satisfying:

$$
\begin{align*}
\left\langle J_{i, j}, X\right\rangle & =1, \text { for all } i, j \in S \cup T, \text { and }  \tag{15}\\
X[(s, a),(t, b)] & =p(a, b \mid s, t), \text { for all } a, b, s, t .
\end{align*}
$$

In Theorem 3.7we show there exist appropriate choices of convex cones $\mathcal{K}$ for which the sets of $\mathcal{K}$-correlations capture the sets of classical, quantum, no-signaling and unrestricted correlations. Specifically:
Result 1. Consider an arbitrary vector $p=(p(a, b \mid s, t)) \in \mathbb{R}^{|A \times B \times S \times T|}$. Then,
(i) $p$ is a classical correlation (i.e., $p \in \mathcal{C}$ ) if and only if $p \in \operatorname{Corr}(\mathcal{C P})$.
(ii) $p$ is quantum correlation (i.e., $p \in \mathcal{Q}$ ) if and only if $p \in \operatorname{Corr}\left(\mathcal{C S}_{+}\right)$.
(iii) $p$ is a no-signaling correlation (i.e., $p \in \mathcal{N S}$ ) if and only if $p \in \operatorname{Corr}(\mathcal{N S O})$.
(iv) $p$ is a correlation (i.e., $p \in \mathcal{P}$ ) if and only if $p \in \operatorname{Corr}(\mathcal{N})$.

We note that upon completion of this work we found that a result similar to Result $\prod_{\text {(ii) has }}$ been derived independently in the unpublished note [28].

The use of convex cones to characterize the sets of quantum, classical, and no-signaling correlations was also an essential ingredient in [22].

As suggested by Result 1 the notion of conic correlations provides a general framework allowing us to phrase and study the properties of many interesting sets of correlations. Notice that
whenever $\mathcal{K}_{1} \subseteq \mathcal{K}_{2}$ we have that $\operatorname{Corr}\left(\mathcal{K}_{1}\right) \subseteq \operatorname{Corr}\left(\mathcal{K}_{2}\right)$. Consequently, the inclusions from (11) combined with Result 1 imply that $\mathcal{C} \subseteq \mathcal{Q} \subseteq \mathcal{P}$. As already mentioned in the introduction these inclusions are well-known but the notion of conic correlations allows us to recover them within a purely mathematical framework.

Note that when $\mathcal{K}$ is a closed convex cone the set of $\mathcal{K}$-correlations is also closed. Furthermore, recall that the set of quantum correlations is not known to be closed. As an immediate consequence of Result 1 (ii) it follows that if the $\mathcal{C} \mathcal{S}_{+}$cone is closed then the set of quantum correlations is also closed (cf. Proposition 3.9). The same observation was made independently in the unpublished note [28]. To the best of our knowledge, the first work where the structure of the closure of the set of quantum correlations was studied is [23].

In Theorem 3.8 we show that for any $\mathcal{K} \subseteq \mathcal{D N} \mathcal{N}$ we have $\operatorname{Corr}(\mathcal{K}) \subseteq \mathcal{N S}$. This fact combined with Result 1 and the inclusion $\mathcal{C} \mathcal{S}_{+} \subseteq \mathcal{D N \mathcal { N }}$ implies that

$$
\begin{equation*}
\mathcal{Q} \subseteq \operatorname{Corr}(\mathcal{D N \mathcal { N }}) \subseteq \mathcal{N S} \tag{16}
\end{equation*}
$$

i.e., $\operatorname{Corr}(\mathcal{D N N})$ forms a spectrahedral outer approximation for the set of quantum correlations which is contained in the set of no-signaling correlations. In Theorem 3.16 we compare $\operatorname{Corr}(\mathcal{D N} \mathcal{N})$ with the first level of the NPA hierarchy, denoted by NPA ${ }^{(1)}$. We are able to show the following:

Result 2. For any Bell scenario we have that $\operatorname{Corr}(\mathcal{D N N}) \subseteq \operatorname{NPA}^{(1)}$.
Game values. In Section 4 we study the value of a nonlocal game when the players use strategies that generate classical, quantum, no-signaling or unrestricted correlations. To state our results in a succinct manner we introduce some notation that is used throughout the paper. The cost matrix of a game $\mathcal{G}(\pi, V)$ is the $|S \times A|$ by $|T \times B|$ matrix $C$ whose entries are given by

$$
\begin{equation*}
C[(s, a),(t, b)]:=\pi(s, t) V(a, b \mid s, t), \text { for all } a \in A, b \in B, s \in S, t \in T . \tag{17}
\end{equation*}
$$

The symmetric cost matrix of the game $\mathcal{G}$ is the $N \times N$ matrix

$$
\hat{C}:=\frac{1}{2}\left(\begin{array}{cc}
0 & C  \tag{18}\\
C^{\top} & 0
\end{array}\right) .
$$

For a convex cone $\mathcal{K} \subseteq \mathcal{N}^{N}$ we denote by $\omega(\mathcal{K}, \mathcal{G})$ the maximum success probability of winning $\mathcal{G}$ when the players use strategies that generate $\mathcal{K}$-correlations, i.e.,

$$
\begin{equation*}
\omega(\mathcal{K}, \mathcal{G}):=\sup \left\{\langle\hat{C}, X\rangle:\left\langle J_{i, j}, X\right\rangle=1, \text { for all } i, j \in S \cup T, X \in \mathcal{K}\right\} . \tag{K}
\end{equation*}
$$

Note that $\left(\overline{\mathcal{P K}_{\mathcal{K}}}\right)$ is an instance of a linear conic program over the convex cone $\mathcal{K}$. As an immediate consequence of Result 1 the classical, quantum, no-signaling and unrestricted values of a nonlocal game can all be expressed as linear conic programs over appropriate convex cones.

Having established conic formulations for $\omega_{\mathcal{C}}(\mathcal{G})$ and $\omega_{\mathcal{Q}}(\mathcal{G})$ we also study the corresponding dual conic programs and their properties in Section 4 Furthermore, we use our formulations to compare the various values of a nonlocal game. For this, let $\operatorname{SDP}^{(1)}(\mathcal{G})$ denote the value of the SDP obtained by optimizing over NPA ${ }^{(1)}$, i.e., the first level of the NPA hierarchy. In Proposition 4.5 we show:

Result 3. For any game $\mathcal{G}$ we have $\omega_{\mathcal{Q}}(\mathcal{G}) \leq \omega(\mathcal{D N N}, \mathcal{G}) \leq \operatorname{SDP}^{(1)}(\mathcal{G})$.
Interestingly, $\omega(\mathcal{D N \mathcal { N }}, \mathcal{G})$ was already introduced by Feige and Lovász as an SDP upper bound to the classical value of a nonlocal game [20].

In a very recent and independent work, a similar observation was also made by creating a new SDP hierarchy approximating the quantum value of a nonlocal game [6]. The first level of that hierarchy corresponds to $\omega(\mathcal{D N N}, \mathcal{G})$.

Lastly, we use our conic formulations to study the problem of deciding the existence of a strategy that wins a nonlocal game with certainty.

Definition 1.2. Consider a nonlocal game $\mathcal{G}(\pi, V)$ and a convex cone $\mathcal{K} \subseteq \mathcal{N}$. We say that $\mathcal{G}$ admits a perfect $\mathcal{K}$-strategy if $\omega(\mathcal{K}, \mathcal{G})=1$ and moreover, this value is achieved by some correlation in $\operatorname{Corr}(\mathcal{K})$.

We show that deciding the existence of a perfect $\mathcal{K}$-strategy is equivalent to the feasibility of a linear conic program over $\mathcal{K}$. This fact combined with Result 1 implies that deciding the existence of a perfect classical, quantum, no-signaling and unrestricted strategy is equivalent to the feasibility of a conic program over the cones $\mathcal{C P}, \mathcal{C} \mathcal{S}_{+}, \mathcal{N S O}$ and $\mathcal{N}$, respectively (cf. Corollary 4.7).

It is well-known that deciding the existence of a perfect classical strategy is NP-hard (see also Section (5). Furthermore, it was recently shown that deciding the existence of a perfect quantum strategy is also NP-hard [24]. Nevertheless, this problem is currently not known to be decidable. Note that our reformulation as a conic feasibility program does not render the problem decidable as no algorithms are known for determining the feasibility of a $\mathcal{C} \mathcal{S}_{+-}$program.

In Section 5we restrict to Bell scenarios where $S=T$ and $A=B$. We first specialize our conic characterizations from Section 3 to synchronous correlations, i.e., correlations with the property that whenever the players receive the same question they need to respond with the same answer.

Recently there has been interest in the study of synchronous correlations as they correspond to perfect strategies for graph homomorphism games and more generally, synchronous nonlocal games (e.g. [35, 30, 18]). Another characterization of the set of synchronous quantum correlations in terms of the existence of a $C^{*}$-algebra with certain properties was given in [35]. Furthermore, it was shown in [18] that Connes' embedding conjecture is equivalent to showing that two families of sets of quantum synchronous correlations coincide.

Furthermore, in Section 5 we study synchronous nonlocal games, i.e., games where both players share the same question and answer sets and in order to win, whenever they receive the same question they have to respond with the same answer (e.g. [9, 27, 36, 29]). The notion of synchronous games was implicit in [35] and was formally defined in [30] and [18]. We focus on the problem of deciding whether a synchronous game admits a perfect classical or quantum strategy. Synchronous games have the property that perfect strategies generate synchronous correlations (e.g. see [30]). In Theorem 5.10 we show that this problem is equivalent to the feasibility of a conic program with matrix variables of size $|S \times A|$. Specializing this to graph homomorphism and graph coloring games we recover in a uniform manner the conic formulations for quantum graph homomorphisms, the quantum chromatic number and the quantum independence number that were recently derived in the literature [27, 36].

Paper organization. In Section 2 we introduce the notation and background on linear algebra, quantum mechanics, and linear conic programming needed for this work. In Section 3 we discuss how correlations corresponding to various physical models can be represented as projections of affine slices of appropriate convex cones and identify a spectrahedral outer approximation for the set of quantum correlations. In Section4 we show that values of nonlocal games can be formulated as conic programming problems and we further discuss the Feige-Lovász SDP relaxation for the value of a nonlocal game. Additionally, we show that deciding the existence of a perfect strategy is equivalent to a conic feasibility problem. Finally, in Section 5we specialize our characterizations to synchronous correlations and synchronous game values.

## 2 Notation and background

Linear algebra. A finite dimensional complex Euclidean space refers to the vector space $\mathbb{C}^{n}$ (for some $n \geq 1$ ) equipped with the canonical inner product on $\mathbb{C}^{n}$. We denote by $\left\{e_{i}\right\}_{i=1}^{n}$ the standard orthonormal basis of $\mathbb{C}^{n}$, and by $e$ the vector of all 1's of appropriate dimension. Given two complex Euclidean spaces $\mathcal{X}, \mathcal{Y}$ we denote by $\mathrm{L}(\mathcal{X}, \mathcal{Y})$ the space of linear operators from $\mathcal{X}$ to $\mathcal{Y}$ which we endow with the Hilbert-Schmidt inner product $\langle X, Y\rangle:=\operatorname{Tr}\left(X^{*} Y\right)$ for $X, Y \in \mathrm{~L}(\mathcal{X}, \mathcal{Y})$. For an operator $X \in \mathrm{~L}(\mathcal{X}, \mathcal{Y})$ we denote its adjoint operator by $X^{*} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and its transpose by $X^{\top} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$. We use the correspondence between $\mathrm{L}(\mathcal{X}, \mathcal{Y})$ and $\mathcal{Y} \otimes \mathcal{X}$ given by the map vec : $\mathrm{L}(\mathcal{X}, \mathcal{Y}) \rightarrow \mathcal{Y} \otimes \mathcal{X}$, which is given by $\operatorname{vec}\left(e_{i} e_{j}^{*}\right)=e_{i} \otimes e_{j}$, on basis vectors and is extended linearly. The vec(•) map is a linear bijection between $\mathrm{L}(\mathcal{X}, \mathcal{Y})$ and $\mathcal{Y} \otimes \mathcal{X}$ and furthermore it is an isometry, i.e., $\langle X, Y\rangle=\langle\operatorname{vec}(Y), \operatorname{vec}(X)\rangle$ for all $X, Y \in \mathrm{~L}(\mathcal{X}, \mathcal{Y})$. We make repeated use of the fact

$$
\begin{equation*}
\operatorname{vec}(W)^{*}(X \otimes Y) \operatorname{vec}(Z)=\operatorname{vec}(W)^{*} \operatorname{vec}\left(X Z Y^{\top}\right)=\left\langle W, X Z Y^{\top}\right\rangle, \tag{19}
\end{equation*}
$$

for Hermitian operators $X, Y, Z, W$ of the appropriate size (e.g. see [40]).
Any vector $\psi \in \mathcal{Y} \otimes \mathcal{X}$ can be uniquely expressed as $\psi=\sum_{i=1}^{d} \lambda_{i} y_{i} \otimes x_{i}$ for some integer $d \geq 1$, positive scalars $\left\{\lambda_{i}\right\}_{i=1}^{d}$, and orthonormal sets $\left\{y_{i}\right\}_{i=1}^{d} \subseteq \mathcal{Y}$ and $\left\{x_{i}\right\}_{i=1}^{d} \subseteq \mathcal{X}$. An expression of this form is known as a Schmidt decomposition for $\psi$ and is derived by the singular value decomposition of $\operatorname{vec}^{-1}(\psi)$. The scalars $\left\{\lambda_{i}\right\}_{i=1}^{d}$ and the integer $d$ are uniquely defined and are called the Schmidt coefficients and the Schmidt rank of $\psi$, respectively. Suppose $\psi=\sum_{i=1}^{d} \lambda_{i} y_{i} \otimes x_{i}$ is a Schmidt decomposition for $\psi$, then we have that $\|\psi\|_{2}^{2}=\sum_{i=1}^{d} \lambda_{i}^{2}$. Lastly, given $x \in \mathbb{C}^{n}$ we define $\operatorname{Diag}(x):=\sum_{i=1}^{n} x_{i} e_{i} e_{i}^{*}$.

Quantum mechanics. In this section, we give a brief overview of the mathematical formulation of quantum mechanics. The reader is referred to [33] and [40] for a more thorough introduction.

According to the axioms of quantum mechanics, associated to any physical system X is a finite dimensional complex Euclidean space $\mathcal{X}$. The state space of X is identified with the set of unit vectors in $\mathcal{X}$. A measurement on a system X is specified by a family of Hermitian psd operators $\left\{X_{i}: i \in \mathcal{I}\right\} \subseteq \mathcal{H}_{+}(\mathcal{X})$ with the property that $\sum_{i \in \mathcal{I}} X_{i}=\mathbb{I}_{\mathcal{X}}$. The set $\mathcal{I}$ labels the set of possible outcomes of the measurement. According to the axioms of quantum mechanics, when the measurement $\left\{X_{i}: i \in \mathcal{I}\right\}$ is performed on a system X which is in state $\psi \in \mathcal{X}$ the outcome $i \in \mathcal{I}$ occurs with probability $p(i)=\psi^{*} X_{i} \psi$. Notice that $\{p(i): i \in \mathcal{I}\}$ forms a valid probability distribution since by the definition of a measurement we have that $p(i) \geq 0$ for all $i \in \mathcal{I}$ and $\sum_{i \in \mathcal{I}} p(i)=1$.

Consider two quantum systems X and Y with corresponding state spaces $\mathcal{X}$ and $\mathcal{Y}$ respectively. According to the axioms of quantum mechanics the Euclidean space that corresponds to the joint system $(\mathrm{X}, \mathrm{Y})$ is given by the tensor product $\mathcal{X} \otimes \mathcal{Y}$ of the individual spaces. Furthermore, if the systems X and Y are independently prepared in states $\psi_{1} \in \mathcal{X}$ and $\psi_{2} \in \mathcal{Y}$ then the state of the joint system is given by $\psi_{1} \otimes \psi_{2} \in \mathcal{X} \otimes \mathcal{Y}$. A state in $\mathcal{X} \otimes \mathcal{Y}$ of the form $\psi_{1} \otimes \psi_{2}$ for some $\psi_{1} \in \mathcal{X}$, $\psi_{2} \in \mathcal{Y}$ is called a product state. Quantum states that cannot be written as convex combinations of product states are called entangled. Lastly, any two measurements $\left\{X_{i}: i \in \mathcal{I}\right\} \subseteq \mathcal{H}_{+}(\mathcal{X})$ and $\left\{Y_{j}: j \in \mathcal{J}\right\} \subseteq \mathcal{H}_{+}(\mathcal{Y})$ on the individual systems X and Y define a product measurement on the joint system with outcomes $\{(i, j): i \in \mathcal{I}, j \in \mathcal{J}\}$. The corresponding measurement operators are given by $\left\{X_{i} \otimes Y_{j}: i \in \mathcal{I}, j \in \mathcal{J}\right\} \subseteq \mathcal{H}_{+}(\mathcal{X} \otimes \mathcal{Y})$ and the probability of outcome $(i, j) \in \mathcal{I} \times \mathcal{J}$, when measuring the quantum state $\psi$, is equal to $\psi^{*}\left(X_{i} \otimes Y_{j}\right) \psi$.

Convex analysis and linear conic programming. In this section, we introduce conic programming and state the duality results that are relevant to this work. For additional details, the reader is referred to [4].

Let $\mathcal{V}$ be a finite dimensional vector space equipped with inner product $\langle\cdot, \cdot\rangle$. Given a subset $A \subseteq \mathcal{V}$ we denote by $\operatorname{cl}(A)$ the closure of $A$ and by $\operatorname{int}(A)$ the interior of $A$ with respect to the topology induced by the inner product. A subset $\mathcal{K} \subseteq \mathcal{V}$ is called a cone if $X \in \mathcal{K}$ implies that $\lambda X \in \mathcal{K}$ for all $\lambda \geq 0$. A cone $\mathcal{K}$ is convex if $X, Y \in \mathcal{K}$ implies that $X+Y \in \mathcal{K}$. For any cone $\mathcal{K}$ we can define its dual cone, denoted by $\mathcal{K}^{*}$, given by

$$
\mathcal{K}^{*}:=\{S \in \mathcal{V}:\langle X, S\rangle \geq 0 \text { for all } X \in \mathcal{K}\} .
$$

The dual cone $\mathcal{K}^{*}$ is always closed. A cone $\mathcal{K}$ is called self-dual if $\mathcal{K}=\mathcal{K}^{*}$. For every convex cone $\mathcal{K}$ we have that $\left(\mathcal{K}^{*}\right)^{*}=\operatorname{cl}(\mathcal{K})$. As a consequence a cone $\mathcal{K}$ is closed if and only if $\mathcal{K}=\left(\mathcal{K}^{*}\right)^{*}$.

Consider two finite dimensional inner-product spaces $\mathcal{V}$ and $\mathcal{W}$ and a convex cone $\mathcal{K} \subseteq \mathcal{V}$. A linear conic program (over the cone $\mathcal{K}$ ) is specified by a triple $(C, \mathcal{L}, B)$ where $C \in \mathcal{V}, B \in \mathcal{W}$ and $\mathcal{L}: \mathcal{V} \rightarrow \mathcal{W}$ is a linear transformation. To such a triple we associate two optimization problems:

$$
\begin{aligned}
\text { Primal problem (P) } & p:=\sup \{\langle C, X\rangle: \mathcal{L}(X)=B, X \in \mathcal{K}\} \\
\text { Dual problem (D) } & d:=\inf \left\{\langle B, Y\rangle: \mathcal{L}^{*}(Y)-C \in \mathcal{K}^{*}, Y \in \mathcal{W}\right\},
\end{aligned}
$$

referred to as the primal and the dual, respectively. For brevity, sometimes we drop the "linear" and just refer to them as conic programs. We call $p$ the primal value and $d$ the dual value of $(C, \mathcal{L}, B)$.

Linear conic programming constitutes a wide generalization of several well-studied models of mathematical optimization. For example, setting $\mathcal{V}=\mathbb{R}^{n}, \mathcal{W}=\mathbb{R}^{m}$ (equipped with the canonical inner-product) and $\mathcal{K}=\mathbb{R}_{+}^{n}$ then ( P ) and ( D ) form a pair of primal-dual linear programs. Furthermore, setting $\mathcal{V}=S^{n}, \mathcal{W}=\mathcal{S}^{m}$ (equipped with the Hilbert-Schmidt inner product) and $\mathcal{K}=\mathcal{S}_{+}^{n}$ then $(\mathrm{P})$ and $(\mathrm{D})$ form a pair of primal-dual semidefinite programs.

A conic program $(C, \mathcal{L}, B)$ is primal feasible if $\{X \in \mathcal{V}: \mathcal{L}(X)=B\} \cap \mathcal{K} \neq \emptyset$ and primal strictly feasible if $\{X \in \mathcal{V}: \mathcal{L}(X)=B\} \cap \operatorname{int}(\mathcal{K}) \neq \emptyset$. Analogously, the conic program $(C, \mathcal{L}, B)$ is called dual feasible if there exists $Y \in \mathcal{W}$ such that $\mathcal{L}^{*}(Y)-C \in \mathcal{K}^{*}$ and dual strictly feasible if there exists $Y \in \mathcal{W}$ such that $\mathcal{L}^{*}(Y)-C \in \operatorname{int}\left(\mathcal{K}^{*}\right)$. The set of feasible solutions of a linear programming problem is called a polyhedron and the set of feasible solutions of a semidefinite programming problem is called a spectrahedron.

Conic programs share some of the duality theory available for linear and semidefinite programs. In particular, the dual value is always an upper bound on the primal value and, moreover, equality and attainment hold assuming appropriate constraint qualifications.

Theorem 2.1. Let $(C, \mathcal{L}, B)$ be a linear conic program over a convex cone $\mathcal{K}$.
(i) (Weak duality) If $X$ (resp. Y) is primal (dual) feasible then $\langle C, X\rangle \leq\langle B, Y\rangle$.
(ii) (Strong duality) Suppose $\mathcal{K}$ is a closed convex cone. If the primal is strictly feasible and $p<+\infty$ we have that $p=d$ and moreover the dual value is attained. Symmetrically, if the dual program is strictly feasible and $d>-\infty$ then $p=d$ and the primal value is attained.

Strong duality results in the conic programming setting are stated for closed convex cones. For a closed convex cone $\mathcal{K}$ we have $\mathcal{K}=\left(\mathcal{K}^{*}\right)^{*}$ so the duality results are symmetric with respect to the primal and the dual problem. Since the $\mathcal{C} \mathcal{S}_{+}$cone is not known to be closed we cannot apply Theorem 2.1] (ii) for $\mathcal{K}=\mathcal{C} \mathcal{S}_{+}$. In Section4 we apply Theorem 2.1(ii) to $\operatorname{cl}\left(\mathcal{C S}_{+}\right)$and $\mathcal{C} \mathcal{S}_{+}^{*}$.

## 3 Correlations as projections of affine slices of convex cones

In this section we study the sets of classical, quantum, no-signaling and unrestricted correlations and express them in a uniform manner as projections of affine slices of appropriate convex cones. Using these characterizations we identify a spectrahedral outer approximation to the set of quantum correlations which is contained in the first level of the NPA hierarchy and a sufficient condition for showing that the set of quantum correlations is closed.

An algebraic characterization of quantum correlations. We start by investigating the structure of the quantum states that can be used to generate a quantum correlation and show they can be taken to have a specific form. We make use the fact that if $X \in \mathcal{H}_{+}^{n}$ then $Y X Y^{*} \in \mathcal{H}_{+}^{n}$, for any $n \times n$ matrix $Y$.

Lemma 3.1. Any quantum correlation $p=(p(a, b \mid s, t)) \in \mathcal{Q}$ can be generated by a quantum state of the form $\psi=\sum_{i=1}^{d} \sqrt{\lambda_{i}} e_{i} \otimes e_{i} \in \mathbb{C}^{d} \otimes \mathbb{C}^{d}$ (for some $d \geq 1$ ), where $\sum_{i=1}^{d} \lambda_{i}=1$ and $\left\{e_{i}: i \in[d]\right\}$ is the standard basis for $\mathbb{C}^{d}$.
Proof. Since $p \in \mathcal{Q}$ there exist a quantum state $\psi \in \mathcal{X} \otimes \mathcal{Y}$ and quantum measurement operators $\left\{X_{a}^{s}\right\}_{a \in A} \subseteq \mathcal{H}_{+}(\mathcal{X})$ and $\left\{Y_{b}^{t}\right\}_{b \in B} \subseteq \mathcal{H}_{+}(\mathcal{Y})$ satisfying $p(a, b \mid s, t)=\psi^{*}\left(X_{a}^{s} \otimes Y_{b}^{t}\right) \psi$, for all $a \in$ $A, b \in B, s \in S, t \in T$. By the Schmidt decomposition, the vector $\psi \in \mathcal{X} \otimes \mathcal{Y}$ can be expressed as $\psi=\sum_{i=1}^{d} \sqrt{\lambda_{i}} x_{i} \otimes y_{i}$, where $\sum_{i=1}^{d} \lambda_{i}=1$ and $\left\{x_{i}\right\}_{i=1}^{d} \subseteq \mathcal{X},\left\{y_{i}\right\}_{i=1}^{d} \subseteq \mathcal{Y}$ are orthonormal sets of vectors. Set $U:=\sum_{i=1}^{d} e_{i} x_{i}^{*}$ and $U^{\prime}:=\sum_{i=1}^{d} e_{i} y_{i}{ }^{*}$ and note that

- $\tilde{\psi}:=\left(U \otimes U^{\prime}\right) \psi=\sum_{i=1}^{d} \sqrt{\lambda_{i}} e_{i} \otimes e_{i} \in \mathbb{C}^{d} \otimes \mathbb{C}^{d}$ is a valid quantum state.
- For all $s$, the matrices $\left\{\tilde{X}_{a}^{s}:=U X_{a}^{s} U^{*}: a \in A\right\}$ are a measurement on $\mathbb{C}^{d}$.
- For all $t$, the matrices $\left\{\tilde{Y}_{b}^{t}:=U^{\prime} Y_{b}^{t}\left(U^{\prime}\right)^{*}: b \in B\right\}$ are a measurement on $\mathbb{C}^{d}$.
- $\tilde{\psi}^{*}\left(\tilde{X}_{a}^{s} \otimes \tilde{Y}_{b}^{t}\right) \tilde{\psi}=\psi^{*}\left(X_{a}^{s} \otimes Y_{b}^{t}\right) \psi$, for all $a \in A, b \in B, s \in S, t \in T$.

Thus, the strategy given by the quantum state $\tilde{\psi}$ and the quantum measurements $\left\{\tilde{X}_{a}^{s}\right\}_{a \in A}$ and $\left\{\tilde{Y}_{b}^{t}\right\}_{b \in B}$ also generates $p$ and has the desired properties.

Based on Lemma 3.1] we arrive at a new algebraic characterization of the set of quantum correlations that is of central importance in Section 3 ,
Theorem 3.2. For any $p \in \mathbb{R}^{|A \times B \times S \times T|}$, the following are equivalent:
(i) $p$ is a quantum correlation.
(ii) There exist operators $K,\left\{X_{a}^{s}\right\}_{s, a},\left\{Y_{b}^{t}\right\}_{t, b} \in \mathcal{H}_{+}^{d}$ (for some $d \geq 1$ ) such that

$$
\begin{align*}
& \langle K, K\rangle=1 \\
& \sum_{a} X_{a}^{s}=\sum_{b} Y_{b}^{t}=K, \forall s, t  \tag{20}\\
& p(a, b \mid s, t)=\left\langle X_{a}^{s}, Y_{b}^{t}\right\rangle, \forall a, b, s, t
\end{align*}
$$

Proof. Let $p=(p(a, b \mid s, t)) \in \mathcal{Q}$. By (2) there exist a quantum state $\psi \in \mathcal{X} \otimes \mathcal{Y}$ and measurements $\left\{\tilde{X}_{a}^{s}\right\}_{a \in A} \subseteq \mathcal{H}_{+}(\mathcal{X})$ and $\left\{\tilde{Y}_{b}^{t}\right\}_{b \in B} \subseteq \mathcal{H}_{+}(\mathcal{Y})$ with

$$
\begin{equation*}
p(a, b \mid s, t)=\psi^{*}\left(\tilde{X}_{a}^{s} \otimes \tilde{Y}_{b}^{t}\right) \psi, \text { for all } a, b, s, t . \tag{21}
\end{equation*}
$$

By Lemma3.1 we may assume $\mathcal{X}=\mathcal{Y}=\mathbb{C}^{d}$, for some integer $d \geq 1$ and that $\psi=\sum_{i=1}^{d} \sqrt{\lambda_{i}} e_{i} \otimes e_{i}$, where $\sum_{i=1}^{d} \lambda_{i}=1$. Define $K:=\sum_{i=1}^{d} \sqrt{\lambda_{i}} e_{i} e_{i}^{*} \in \mathcal{H}_{+}^{d}$ and notice that $\operatorname{vec}(K)=\psi$ and $\langle K, K\rangle=1$. Set $X_{a}^{s}:=K^{1 / 2}\left(\tilde{X}_{a}^{s}\right) K^{1 / 2}$ for all $a, s$ and $Y_{b}^{t}:=K^{1 / 2}\left(\tilde{Y}_{b}^{t}\right)^{\top} K^{1 / 2}$, for all $b, t$. These operators satisfy $\sum_{a \in A} X_{a}^{s}=\sum_{b \in B} Y_{b}^{t}=K$, for all $s \in S, t \in T$.

Using the definitions above and properties of the vec map (cf. (19)) we have

$$
\begin{equation*}
\left\langle X_{a}^{s}, Y_{b}^{t}\right\rangle=\operatorname{vec}(K)^{*}\left(\tilde{X}_{a}^{s} \otimes \tilde{Y}_{b}^{t}\right) \operatorname{vec}(K)=\psi^{*}\left(\tilde{X}_{a}^{s} \otimes \tilde{Y}_{b}^{t}\right) \psi=p(a, b \mid s, t), \tag{22}
\end{equation*}
$$

for all $a \in A, b \in B, s \in S, t \in T$ and thus (20) is feasible.
Conversely let $K,\left\{X_{a}^{s}\right\}_{s \in S, a \in A},\left\{Y_{b}^{t}\right\}_{t \in T, b \in B}$ be feasible for (20). Without loss of generality, we may assume $K$ has full rank. Define $\psi:=\operatorname{vec}(K)$ and notice that $\|\psi\|_{2}=1$. For all $a, s$ set $\tilde{X}_{a}^{s}:=K^{-1 / 2} X_{a}^{s} K^{-1 / 2}$ and for all $b, t$ set $\tilde{Y}_{b}^{t}:=\left(K^{-1 / 2} Y_{b}^{t} K^{-1 / 2}\right)^{\top}$. Since $K^{-1 / 2} \in \mathcal{H}_{+}^{d}$ we have that $\tilde{X}_{a}^{s}, \tilde{Y}_{b}^{t} \in \mathcal{H}_{+}^{d}$, for all $a, b, s, t$ (where we also use the fact that $X \in \mathcal{H}_{+}^{n}$ if and only if $X^{\top} \in \mathcal{H}_{+}^{n}$ ). Clearly, these operators satisfy $\sum_{a \in A} \tilde{X}_{a}^{s}=\mathbb{I}_{d}=\sum_{b \in B} \tilde{Y}_{b}^{t}=\mathbb{I}_{d}$, for all $s \in S, t \in T$. Reversing the calculation in (22) we get that $p(a, b \mid s, t)=\left\langle X_{a}^{s}, Y_{b}^{t}\right\rangle=\psi^{*}\left(\tilde{X}_{a}^{s} \otimes \tilde{Y}_{b}^{t}\right) \psi$, for all $a, b, s, t$ which shows that $p \in \mathcal{Q}$.

Remark 3.3. A close inspection of the proof of Theorem 3.2 allows us to explicitly work out the dependency of the parameter $d$ on the dimension of the underlying quantum system. Specifically, for a correlation $p \in \mathcal{Q}$ that is generated by a state $\psi \in \mathcal{X} \otimes \mathcal{Y}$ there exist Hermitian psd matrices of size $d \leq \min \{\operatorname{dim}(\mathcal{X}), \operatorname{dim}(\mathcal{Y})\}$ that satisfy (20). Conversely, if (20) has a feasible solution with matrices (real or complex) of size $d \geq 1$ then the correlation $p=(p(a, b \mid s, t))$ can be generated by a state in $\mathbb{C}^{d} \otimes \mathbb{C}^{d}$. This observation can be used to derive a lower bound on the dimension of a Hilbert space needed to generate an arbitrary quantum correlation [37].

Remark 3.4. As another by-product of Theorem 3.2 we characterize the quantum correlations that can be generated using a maximally entangled state. Specifically, it follows easily from the proof of Theorem 3.2 that a quantum correlation $p \in \mathcal{Q}$ can be generated using the $d$-dimensional maximally entangled state $\psi_{d}:=\operatorname{vec}\left(\frac{1}{\sqrt{d}} \mathbb{I}_{d}\right)$ if and only if (20) is feasible with $K=\frac{1}{\sqrt{d}} \mathbb{I}_{d}$.

An algebraic characterization of classical correlations. As every classical correlation is also quantum, any classical correlation admits a representation as a quantum correlation for some appropriate choice of quantum state and measurement operators. In the next result we show that a correlation is classical if and only if (20) admits a solution with diagonal psd matrices.

Theorem 3.5. For any $p \in \mathbb{R}^{|A \times B \times S \times T|}$, the following are equivalent:
(i) $p$ is a classical correlation.
(ii) $p$ can be generated by diagonal measurement operators and a state of the form $\psi=\sum_{i=1}^{n} \sqrt{\lambda} e_{i} \otimes e_{i}$, where $\sum_{i=1}^{n} \lambda_{i}=1, \lambda_{i} \geq 0$ for $i \in[n]$.
(iii) There exists a solution to (20) with diagonal matrices.

Proof. (i) $\Longrightarrow$ (ii): By definition, for any classical correlation $p \in \mathcal{P}$ there exist nonnegative scalars $k_{i} \geq 0, x_{a}^{s, i} \geq 0$, and $y_{b}^{t, i} \geq 0$ satisfying $p(a, b \mid s, t)=\sum_{i=1}^{n} k_{i} x_{a}^{s, i} y_{b}^{t, i}$, for all $a \in A, b \in B, s \in S, t \in$ $T$, where $\sum_{i=1}^{n} k_{i}=1, \sum_{a \in A} x_{a}^{s, i}=1$ for all $i \in[n], s \in S$ and $\sum_{b \in B} y_{b}^{t, i}=1$ for all $i \in[n], t \in T$. Set

- $\psi:=\sum_{i=1}^{n} \sqrt{k_{i}} e_{i} \otimes e_{i}$, which is a quantum state of the form in Lemma 3.1.
- $X_{a}^{s}:=\sum_{i=1}^{n} x_{a}^{s, i} e_{i} e_{i}^{*}$, for $s \in S, a \in A$, which are diagonal,
- $Y_{b}^{t}:=\sum_{i=1}^{n} y_{b}^{t, i} e_{i} e_{i}^{*}$, for $t \in T, b \in B$, which are diagonal.

A straightforward calculation shows that the state $\psi$ and the measurements $\left\{X_{a}^{s}\right\}_{s \in S, a \in A}$ and $\left\{Y_{b}^{t}\right\}_{t \in T, b \in B}$ generate $p$.
$($ ii $) \Longrightarrow(i)$ : Suppose $p$ can be generated by $\psi=\sum_{i=1}^{n} \sqrt{\lambda_{i}} e_{i} \otimes e_{i}$ and diagonal measurement operators $\left\{X_{a}^{s}\right\}_{s, a}$ and $\left\{Y_{b}^{t}\right\}_{t, b}$. Set

- $k_{i}:=\lambda_{i}$, for all $i \in[n]$,
- $x_{a}^{s, i}:=X_{a}^{s}[i, i]$, for all $i \in[n], s \in S$ and $a \in A$,
- $y_{b}^{t, i}:=Y_{b}^{t}[i, i]$, for all $i \in[n], t \in T$ and $b \in B$,
and notice that this defines a classical strategy that generates the correlation $p$.
(ii) $\Longleftrightarrow$ (iii): This is clear from the proof of Theorem 3.2 noting that $K$ being diagonal and satisfying $\langle K, K\rangle=1$, implies that the quantum state $\psi:=\operatorname{vec}(K)$ is of the required form.

Conic characterization of correlations. Recall that for a convex cone $\mathcal{K} \subseteq \mathcal{N}$, we say that $p=$ $(p(a, b \mid s, t))$ is a $\mathcal{K}$-correlation, denoted by $p \in \operatorname{Corr}(\mathcal{K})$, if and only if there exists $X \in \mathcal{K}^{N}$ such that $\left\langle J_{i, j}, X\right\rangle=1$, for all $i, j \in S \cup T$, and $X[(s, a),(t, b)]=p(a, b \mid s, t)$, for all $a, b, s, t$. (cf. Definition 1.1). In this section we show that the sets of classical, quantum, no-signaling and unrestricted correlations can be expressed as the sets of conic correlations for appropriate convex cones. The characterizations for the quantum and classical case rely on Theorem 3.2 and Theorem [3.5, respectively.

Recall that for $i, j \in S \cup T$ (resp. $i, j \in\{0\} \cup S \cup T$ ) we set $J_{i, j}$ to be the matrix which acts on a matrix $X \in \mathcal{S}^{N}$ (resp. $X \in \mathcal{S}^{N+1}$ ) by summing all entries in block $X_{i, j}$ (cf. (13) and (14)). We start with a geometric lemma of central importance in this section. For the definition of a Gram matrix of a family of vectors recall (7).

Lemma 3.6. Consider vectors $\left\{x_{a}^{s}\right\}_{s \in S, a \in A}$ and $\left\{y_{b}^{t}\right\}_{t \in T, b \in B}$ in some Euclidean space $\mathcal{X}$.
(a) For $X:=\operatorname{Gram}\left(\left\{x_{a}^{s}\right\}_{s \in S, a \in A},\left\{y_{b}^{t}\right\}_{t \in T, b \in B}\right)$ the following are equivalent:
(i) $\exists k \in \mathcal{X}$ such that $\langle k, k\rangle=1$, and $\sum_{a} x_{a}^{s}=\sum_{b} y_{b}^{t}=k$, for all $s, t$.
(ii) $\left\langle J_{i, j}, X\right\rangle=1$, for all $i, j \in S \cup T$.
(b) Set $\tilde{X}:=\operatorname{Gram}\left(k,\left\{x_{a}^{s}\right\}_{s \in S, a \in A},\left\{y_{b}^{t}\right\}_{t \in T, b \in B}\right)$ where $k \in \mathcal{X}$ with $\langle k, k\rangle=1$. The following are equivalent:
(i) $\sum_{a \in A} x_{a}^{s}=\sum_{b \in B} y_{b}^{t}=k$, for all $s, t$.
(ii) $\left\langle J_{i, j}, \tilde{X}\right\rangle=1$, for all $i, j \in\{0\} \cup S \cup T$.

Proof. We start with part (a). To show (i) implies (ii), consider $i \in S$ and $j \in T$ and notice that

$$
\left\langle J_{i, j}, X\right\rangle=\sum_{a \in A} \sum_{b \in B} X[(i, a),(j, b)]=\sum_{a \in A} \sum_{b \in B}\left\langle x_{a}^{i}, y_{b}^{j}\right\rangle=\langle k, k\rangle=1 .
$$

For the other direction, define $x_{i}:=\sum_{a \in A} x_{a}^{i}$ for all $i \in S$ and $y_{i}:=\sum_{b \in B} y_{b}^{i}$ for all $i \in T$. Notice that for any $i, j \in S$ the equation $\left\langle J_{i, j}, X\right\rangle=1$ is equivalent to $\left\langle x_{i}, x_{j}\right\rangle=1$. This implies that $\left\langle x_{i}-x_{j}, x_{i}-x_{j}\right\rangle=0$ for all $i, j \in S$ and thus $x_{i}=x_{j}$ for all $i, j \in S$. Similarly we have that
$y_{i}=y_{j}$ for all $i, j \in T$. Lastly, fix any $i \in S$ and $j \in T$ and notice that $\left\langle J_{i, j}, X\right\rangle=1$ implies that $\left\langle x_{i}, y_{j}\right\rangle=1$. As before this shows that $x_{i}=y_{j}$.

We proceed with part (b). It is easy to see that (i) implies (ii). For the other direction we have from part (a) that there exists $k^{\prime} \in \mathcal{X}$ such that $\sum_{a \in A} x_{a}^{s}=\sum_{b \in B} y_{b}^{t}=k^{\prime}$, for all $s \in S, t \in T$, and $\left\langle k^{\prime}, k^{\prime}\right\rangle=1$. It suffices to show that $k=k^{\prime}$. For this, notice that $\left\langle k-k^{\prime}, k-k^{\prime}\right\rangle=2-2\left\langle k, k^{\prime}\right\rangle=$ $2-2 \sum_{a \in A} \tilde{X}[0,(s, a)]=0$, and the proof is concluded.

We now state and prove our main result in this section.
Theorem 3.7. For any Bell scenario we have that

$$
\mathcal{C}=\operatorname{Corr}(\mathcal{C P}), \mathcal{Q}=\operatorname{Corr}\left(\mathcal{C} \mathcal{S}_{+}\right), \mathcal{N S}=\operatorname{Corr}(\mathcal{N S O}), \mathcal{P}=\operatorname{Corr}(\mathcal{N})
$$

Proof. We first show $\mathcal{Q}=\operatorname{Corr}\left(\mathcal{C S}_{+}\right)$. By Theorem 3.2]we have that $p=(p(a, b \mid s, t)) \in \mathcal{Q}$ if and only if there exist operators $\left\{X_{a}^{s}\right\}_{s, a},\left\{Y_{b}^{t}\right\}_{t, b}, K$ satisfying (20). Setting $X:=\operatorname{Gram}\left(\left\{X_{a}^{s}\right\}_{s, a},\left\{Y_{b}^{t}\right\}_{t, b}, K\right)$ it follows by Lemma 3.6 (a) that $\left\langle J_{i, j}, X\right\rangle=1$, for all $i, j \in S \cup T$. As $X \in \mathcal{C} \mathcal{S}_{+}$, this gives $p \in$ $\operatorname{Corr}\left(\mathcal{C S}_{+}\right)$.

Conversely, fix $p \in \operatorname{Corr}\left(\mathcal{C} \mathcal{S}_{+}\right)$and let $X=\operatorname{Gram}\left(\left\{X_{a}^{s}\right\}_{s, a},\left\{Y_{b}^{t}\right\}_{t, b}\right) \in \mathcal{C} \mathcal{S}_{+}^{N}$ satisfying (15). By Lemma 3.6 (a) there exists a Hermitian psd matrix $K$ with $\langle K, K\rangle=1$ and $\sum_{a} X_{a}^{s}=\sum_{b} Y_{b}^{t}=$ $K, \forall s, t$. By Theorem 3.2 we get $p \in \mathcal{Q}$.

The case $\mathcal{C}=\operatorname{Corr}(\mathcal{C P})$ follows similarly by Theorem 3.5. Lastly, $\mathcal{N S}=\operatorname{Corr}(\mathcal{N S O})$ and $\mathcal{P}=\operatorname{Corr}(\mathcal{N})$ follow from the definitions of $\mathcal{N S O}$ and $\mathcal{N}$.

As exemplified by Theorem 3.7 the notion of $\mathcal{K}$-correlations has significant expressive power as it captures many correlation sets of physical significance. In our next result we continue the study of conic correlations and identify a sufficient condition in terms of the cone $\mathcal{K}$ so that the corresponding set of correlations $\operatorname{Corr}(\mathcal{K})$ satisfies the no-signaling conditions.

Theorem 3.8. For any convex cone $\mathcal{K} \subseteq \mathcal{D N} \mathcal{N}$ we have that $\operatorname{Corr}(\mathcal{K}) \subseteq \mathcal{N S}$.
Proof. For any $p=(p(a, b \mid s, t)) \in \operatorname{Corr}(\mathcal{K})$ there exists $X \in \mathcal{D N N}^{N}$ such that $\left\langle J_{i, j}, X\right\rangle=1$ for all $i, j \in S \cup T$ and $p(a, b \mid s, t)=X[(s, a),(t, b)]$ for all $a, b, s, t$. If $X=\operatorname{Gram}\left(\left\{x_{a}^{s}\right\}_{s \in S, a \in A},\left\{y_{b}^{t}\right\}_{t \in T, b \in B}\right)$ it follows from Lemma 3.6 (a) that there exists a vector $k$ such that

$$
\begin{equation*}
\sum_{a \in A} x_{a}^{s}=\sum_{b \in B} y_{b}^{t}=k, \text { for all } s \in S, t \in T, \text { and }\langle k, k\rangle=1 \tag{23}
\end{equation*}
$$

For $s \neq s^{\prime} \in S$ and $t \in T$ we get from (23) that

$$
\sum_{a \in A} X[(s, a),(t, b)]=\sum_{a \in A}\left\langle x_{a}^{s}, y_{b}^{t}\right\rangle=\left\langle k, y_{b}^{t}\right\rangle=\sum_{a \in A}\left\langle x_{a}^{s^{\prime}}, y_{b}^{t}\right\rangle=\sum_{a \in A} X\left[\left(s^{\prime}, a\right),(t, b)\right],
$$

and thus $\sum_{a \in A} p(a, b \mid s, t)=\sum_{a \in A} p\left(a, b \mid s^{\prime}, t\right)$ for all $t \in T$ and $s \neq s^{\prime} \in S$. Symmetrically, we have that $\sum_{b \in B} p(a, b \mid s, t)=\sum_{b \in B} p\left(a, b \mid s, t^{\prime}\right)$ for all $s \in S$ and $t \neq t^{\prime} \in T$ and thus $p \in \mathcal{N S}$.

Another consequence of Theorem 3.7 is a sufficient condition for showing that $\mathcal{Q}$ is closed. Note that for every cone $\mathcal{K} \subseteq \mathcal{N}$, the set of matrices satisfying $\left\langle J_{i, j}, X\right\rangle=1$, for all $i, j \in S \cup T$ is bounded. Consequently, if $\mathcal{K}$ is closed it follows that $\operatorname{Corr}(\mathcal{K})$ is compact. This implies the following proposition.

Proposition 3.9. If the cone $\mathcal{C} \mathcal{S}_{+}$is closed then $\mathcal{Q}$ is also closed.

We conclude this section with a second formulation for the sets of $\mathcal{C P}, \mathcal{C S}{ }_{+}$and $\mathcal{D N N}$-correlations. We use these formulations to compare $\operatorname{Corr}(\mathcal{D N N})$ with the first level of the NPA hierarchy and in Section 5 where we recover the conic programming formulations for certain quantum graph parameters.

Lemma 3.10. Consider a correlation $p=(p(a, b \mid s, t)) \in \mathcal{P}$ and define the matrix

$$
P:=\sum_{a, b, s, t} p(a, b \mid s, t) e_{s} e_{t}^{\top} \otimes e_{a} e_{b}^{\top} .
$$

For any cone $\mathcal{K} \in\left\{\mathcal{C P}, \mathcal{C} \mathcal{S}_{+}, \mathcal{D N N}\right\}$ we have that $p=(p(a, b \mid s, t)) \in \operatorname{Corr}(\mathcal{K})$ if and only if there exists a matrix

$$
\tilde{X}=\left(\begin{array}{ccc}
1 & x^{\top} & y^{\top}  \tag{24}\\
x & X & P \\
y & P^{\top} & Y
\end{array}\right) \in \mathcal{K}^{1+N},
$$

such that $\left\langle J_{i, j}, \tilde{X}\right\rangle=1$, for all $i, j \in\{0\} \cup S \cup T$.
Proof. This follows from the definition of $\operatorname{Corr}(\mathcal{K})$ combined with Lemma[3.6(b).

A spectrahedral outer approximation for quantum correlations. In this section we use Theorem 3.7 to derive a new spectrahedral outer approximation for the set of quantum correlations. Furthermore, we show that our approximation is at least as strong as the first level of the NPA hierarchy.

In Theorem [3.7 we showed that $\mathcal{Q}=\operatorname{Corr}\left(\mathcal{C S}_{+}\right)$. As $\mathcal{C} \mathcal{S}_{+} \subseteq \mathcal{D N \mathcal { N }}$ we immediately get a necessary and efficiently verifiable condition for membership in the set of quantum correlations.

Proposition 3.11. For any Bell scenario we have $\mathcal{Q} \subseteq \operatorname{Corr}(\mathcal{D N \mathcal { N }}) \subseteq \mathcal{N S}$.
As already mentioned the set of quantum correlations is a non-polyhedral set whose structure is poorly understood. In [32] Navascués, Pironio and Acín constructed a hierarchy of spectrahedral outer approximations to the set of quantum correlations. The mathematical derivation of the NPA hierarchy is involved and is beyond the scope of this paper. For the precise definition and its properties the reader is referred to [32]. In this work we only consider the first level of the NPA hierarchy, denoted by NPA ${ }^{(1)}$, that we introduce below.

For this we need the following definition. For $p=(p(a, b \mid s, t)) \in \mathcal{N} \mathcal{S}$ we denote by $p_{A}(a \mid s)$ Alice's local marginal probabilities for all $a \in A, s \in S$ and by $p_{B}(b \mid t)$ Bob's local marginal probabilities for all $b \in B, t \in T$. Note that these are well-defined by the no-signaling conditions (3) and (4).

It is useful to arrange the marginal probabilities in a vector as follows:

$$
\begin{equation*}
p_{A}(s):=\sum_{a \in A} p_{A}(a \mid s) e_{a} \in \mathbb{R}_{+}^{|A|}, \text { and } p_{B}(t):=\sum_{b \in B} p_{B}(b \mid t) e_{b} \in \mathbb{R}_{+}^{|B|}, \tag{25}
\end{equation*}
$$

for all $s \in S$ and $t \in T$, respectively, and

$$
\begin{equation*}
p_{A}:=\sum_{s, a} e_{s} \otimes p_{A}(s) \in \mathbb{R}_{+}^{|S \times A|} \text { and } p_{B}:=\sum_{t, b} e_{t} \otimes p_{B}(t) \in \mathbb{R}_{+}^{|T \times B|} . \tag{26}
\end{equation*}
$$

Remark 3.12. Let $\mathcal{K} \in\left\{\mathcal{C P}, \mathcal{C S}_{+}, \mathcal{D N N}\right\}$. Given $p=(p(a, b \mid s, t)) \in \operatorname{Corr}(\mathcal{K})$ it follows from the proof of Lemma 3.10 that every feasible solution to (24) satisfies $\tilde{X}[0,(s, a)]=p_{A}(a \mid s)$, for all $a \in A, s \in S$ and $\tilde{X}[0,(t, b)]=p_{B}(b \mid t)$, for all $t \in T, b \in B$. We make use of this fact in Theorem 3.16

Using the vectors given in (26) we can now give the description of NPA ${ }^{(1)}$.
Definition 3.13. Consider a correlation $p=(p(a, b \mid s, t)) \in \mathcal{N S}$ and define

$$
P:=\sum_{a, b, s, t} p(a, b \mid s, t) e_{s} e_{t}^{\top} \otimes e_{a} e_{b}^{\top},
$$

and $p_{A}, p_{B}$ as defined in (26). Then $p \in \mathrm{NPA}^{(1)}$ if and only if there exists a matrix

$$
\tilde{X}:=\left(\begin{array}{ccc}
1 & p_{A}^{\top} & p_{B}^{\top}  \tag{27}\\
p_{A} & X & P \\
p_{B} & P^{\top} & Y
\end{array}\right) \in \mathcal{S}_{+}^{1+N} \text {, satisfying: }
$$

(i) $X\left[(s, a),\left(s, a^{\prime}\right)\right]=\delta_{a, a^{\prime}} p_{A}(a \mid s)$, for all $s \in S, a, a^{\prime} \in A$,
(ii) $Y\left[(t, b),\left(t, b^{\prime}\right)\right]=\delta_{b, b^{\prime}} p_{B}(b \mid t)$, for all $t \in T, b, b^{\prime} \in B$.

Remark 3.14. Using Lemma 3.6 it is easy to verify that $\mathrm{NPA}^{(1)}$ can be expressed as the projection (onto the blocks that are indexed by $S \times T$ ) of the set of matrices in $\mathcal{S}_{+}^{1+N}$ satisfying the following constraints
(i) $\left\langle J_{i, j}, \tilde{X}\right\rangle=1$, for all $i, j \in\{0\} \cup S \cup T$,
(ii) $\tilde{X}[(s, a),(t, b)] \geq 0$, for all $s \in S, t \in T, a \in A, b \in B$,
(iii) $\tilde{X}\left[(s, a),\left(s, a^{\prime}\right)\right]=0$, for all $s \in S, a \neq a^{\prime} \in A$,
(iv) $\tilde{X}\left[(t, b),\left(t, b^{\prime}\right)\right]=0$, for all $t \in T, b \neq b^{\prime} \in B$.

We make use of this fact in Section 4
Our last result in this section is that the set of $\mathcal{D N} \mathcal{N}$-correlations is contained in $\mathrm{NPA}^{(1)}$. We start with a simple lemma that we use in the proof.
Lemma 3.15. Consider $x, y \in \mathbb{R}_{+}^{n}$ with $\langle e, x\rangle=\langle e, y\rangle=1$. If the matrix $\left(\begin{array}{cc}1 & x^{\top} \\ x & \operatorname{Diag}(y)\end{array}\right)$ is positive semidefinite then we have that $x=y$.
Proof. By Schur complements (e.g. see [4]) we have $\left(\begin{array}{cc}1 & x^{\top} \\ x & \operatorname{Diag}(y)\end{array}\right) \in \mathcal{S}_{+}^{n+1}$ if and only if $\operatorname{Diag}(y)-$ $x x^{\boldsymbol{\top}} \in \mathcal{S}_{+}^{n}$. Note that $\left\langle e e^{\top}, \operatorname{Diag}(y)-x x^{\boldsymbol{\top}}\right\rangle=0$. Since $\operatorname{Diag}(y)-x x^{\top}$ is psd we get $\left(\operatorname{Diag}(y)-x x^{\boldsymbol{\top}}\right) e=$ 0 (where we use the well-known fact that for $X \in \mathcal{S}_{+}^{n}$ we have $x^{\top} X x=0$ if and only if $X x=0$ ). Lastly, as $\langle e, x\rangle=1$, it follows from the preceding equality that $x=y$.

We are now ready to prove the last result in this section.
Theorem 3.16. For any Bell scenario we have that $\operatorname{Corr}(\mathcal{D N N}) \subseteq \mathrm{NPA}^{(1)}$.

Proof. Consider a correlation $p \in \operatorname{Corr}(\mathcal{D N} \mathcal{N})$. By Theorem 3.8 we have that $p \in \mathcal{N S}$ and thus the marginal probability distributions $p_{A}$ and $p_{B}$ are well-defined. By Remark 3.12 there exists

$$
\tilde{X}:=\left(\begin{array}{ccc}
1 & p_{A}^{\top} & p_{B}^{\top}  \tag{28}\\
p_{A} & X & P \\
p_{B} & P^{\top} & Y
\end{array}\right) \in \mathcal{D N}^{1+N},
$$

satisfying $\left\langle J_{i, j}, \tilde{X}\right\rangle=1$, for all $i, j \in\{0\} \cup S \cup T$. Fix $s \in S$ and $a \neq a^{\prime} \in A$ and set $E_{a, a^{\prime}}^{s} \in \mathcal{S}_{+}^{1+N}$ to be the matrix with entries $E_{a, a^{\prime}}^{s}[(s, a),(s, a)]=1, E_{a, a^{\prime}}^{s}\left[\left(s, a^{\prime}\right),\left(s, a^{\prime}\right)\right]=1, E_{a, a^{\prime}}^{s}\left[(s, a),\left(s, a^{\prime}\right)\right]=-1$, $E_{a, a^{\prime}}^{s}\left[\left(s, a^{\prime}\right),(s, a)\right]=-1$ and 0 otherwise. Furthermore, define

$$
\begin{equation*}
X^{\prime}:=\tilde{X}+\tilde{X}\left[(s, a),\left(s, a^{\prime}\right)\right] E_{a, a^{\prime}}^{s}, \tag{29}
\end{equation*}
$$

and notice that $X^{\prime}\left[(s, a),\left(s, a^{\prime}\right)\right]=0$. Moreover, since $\tilde{X} \in \mathcal{D N} \mathcal{N}^{1+N}$ we have that $X^{\prime} \in \mathcal{S}_{+}^{1+N}$ and since $\left\langle J_{i, j}, E_{a, a^{\prime}}^{s}\right\rangle=0$ it follows from (29) that $\left\langle J_{i, j}, X^{\prime}\right\rangle=1$ for all $i, j \in\{0\} \cup S \cup T$. Clearly, this argument can be repeated for all $s \in S, a \neq a^{\prime} \in A$ and symmetrically for all $t \in T$ and $b \neq b^{\prime} \in B$. In this way we construct a matrix

$$
Z:=\left(\begin{array}{ccc}
1 & p_{A}^{\top} & p_{B}^{\top}  \tag{30}\\
p_{A} & Z_{1} & P \\
p_{B} & P^{\top} & Z_{2}
\end{array}\right) \in \mathcal{S}_{+}^{1+N},
$$

satisfying $\left\langle J_{i, j}, Z\right\rangle=1$ for all $i, j \in\{0\} \cup S \cup T, Z_{1}\left[(s, a)\left(s, a^{\prime}\right)\right]=0$ for every $s \in S, a \neq a^{\prime} \in A$, and $Z_{2}\left[(t, b),\left(t, b^{\prime}\right)\right]=0$ for every $t \in T, b \neq b^{\prime} \in B$. It remains to show that $Z[(s, a),(s, a)]=p_{A}(a \mid s)$ for all $a \in A, s \in S$ and $Z[(t, b),(t, b)]=p_{B}(b \mid t)$ for all $t \in T$ and $b \in B$. For this, fix $s \in S$ and notice that the principal submatrix of $Z$ indexed by $\{[0,0]\} \cup\{[0,(s, a)]: a \in A\}$ is given by $\left(\begin{array}{cc}1 & p_{A}(s)^{\top} \\ p_{A}(s) & \operatorname{Diag}(y)\end{array}\right) \in \mathcal{S}_{+}^{1+|A|}$. Since $\langle y, e\rangle=1$ it follows by Lemma 3.15 that $y=p_{A}(s)$. Since the same argument can be repeated for all other diagonal blocks of $Z$, the proof is concluded.

We do not know if the containment given in Theorem 3.16 is strict. One difficulty in proving the converse inclusion is that for any matrix feasible for (27) we do not have control of the signs of the entries in the off-diagonal blocks.

## 4 Conic programming formulations for game values

In this section we study the value of a nonlocal game when the players use strategies that generate classical, quantum, no-signaling or unrestricted correlations. By Theorem 3.7, the classical, quantum, no-signaling and unrestricted values can be formulated as linear conic programs over appropriate convex cones. Any conic program has an associated dual which we derive in our setting and investigate its properties. This allows us to identify a sufficient condition for showing that the $\mathcal{C} \mathcal{S}_{+}$cone is not closed. Furthermore, we identify a new SDP upper bound to the quantum value of an arbitrary nonlocal game which we show is at most the value of the SDP obtained when we optimize over the first level of the NPA hierarchy. Lastly, we show that the problem of deciding whether a nonlocal game admits a perfect $\mathcal{K}$-strategy is equivalent to deciding the feasibility of a linear conic program over $\mathcal{K}$. In particular, deciding whether a nonlocal game admits a perfect quantum strategy is equivalent to the feasibility of a linear conic program over the $\mathcal{C} \mathcal{S}_{+}$cone.

Primal formulations. Recall that the maximum probability of winning a game $\mathcal{G}$ using $\mathcal{K}$-correlations is given by
$\left(\mathcal{P}_{\mathcal{K}}\right) \quad \omega(\mathcal{K}, \mathcal{G}):=\sup \left\{\langle\hat{C}, X\rangle:\left\langle J_{i, j}, X\right\rangle=1\right.$, for all $\left.i, j \in S \cup T, X \in \mathcal{K}\right\}$,
where $\hat{C}$ is the symmetric cost matrix defined in (17) and the matrices $J_{i, j}$ are defined in (13).
By Theorem 3.7 the sets of classical, quantum, no-signaling and unrestricted correlations can be expressed as the set of $\mathcal{K}$-correlations over some appropriate convex cone $\mathcal{K} \subseteq \mathcal{N}^{N}$. Specifically, we have:

Theorem 4.1. For any nonlocal game $\mathcal{G}(\pi, V)$ we have:
(i) The classical value $\omega_{\mathcal{C}}(\mathcal{G})$ equal to $\omega(\mathcal{C P}, \mathcal{G})$.
(ii) The quantum value $\omega_{\mathcal{Q}}(\mathcal{G})$ equal to $\omega\left(\mathcal{C S}_{+}, \mathcal{G}\right)$.
(iii) The no-signaling value $\omega_{\mathcal{N S}}(\mathcal{G})$ equal to $\omega(\mathcal{N S O}, \mathcal{G})$.
(iv) The unrestricted value $\omega_{\mathcal{P}}(\mathcal{G})$ equal to $\omega(\mathcal{N}, \mathcal{G})$.

Note that $\left(\overline{\mathcal{P}_{\mathcal{K}}}\right)$ is a linear conic program over the convex cone $\mathcal{K}$. Our next goal is to apply the theory of linear conic optimization to $\left(\overline{\mathcal{P}_{\mathcal{K}}}\right.$ to understand how the various values relate to each other and to study their properties.

Dual formulations. The dual conic program associated to $\left(\overline{\mathcal{P}_{\mathcal{K}}}\right)$ is given by:

$$
\begin{equation*}
\xi(\mathcal{K}, \mathcal{G}):=\inf \left\{\sum_{i, j \in S \cup T} v_{i, j}: \sum_{i, j \in S \cup T} v_{i, j} J_{i, j}-\hat{C} \in \mathcal{K}^{*}\right\} . \tag{K}
\end{equation*}
$$

We start by analyzing the primal-dual pair of conic programs $\left(\mathcal{P}_{\mathcal{K}}\right)$ and $\left(\mathcal{D}_{\mathcal{K}}\right)$. By weak duality (cf. Theorem 2.1 (i)) the optimal value of the dual program upper bounds the optimal value of the primal, i.e., for any game $\mathcal{G}$ we have $\omega(\mathcal{K}, \mathcal{G}) \leq \xi(\mathcal{K}, \mathcal{G})$. For this to hold with equality, it suffices to determine whether strong duality holds for the primal or the dual (cf. Theorem[2.1(ii)). Note that for any cone $\mathcal{K} \subseteq \mathcal{S}_{+}^{N}$ the primal program ( $\mathcal{P}_{\mathcal{K}}$ ) is not strictly feasible. To see this, fix indices $i \in S$, $j \in T$, and define the (nonzero) psd matrix

$$
M:=J_{i, i}+J_{j, j}-2 J_{i, j} \in \mathcal{S}_{+}^{N} .
$$

Any matrix $X$ feasible for $\left(\mathcal{P}_{\mathcal{K}}\right)$ satisfies $\langle M, X\rangle=0$ and so $X \notin \operatorname{int}(\mathcal{K}) \subseteq \mathcal{S}_{+++}^{N}$.
Also, notice that if the cone $\mathcal{K}$ is not closed then we cannot apply strong duality directly to the primal-dual pair. However, as we now show, under the additional assumption that $\mathcal{K}$ is a closed convex cone, strong duality holds for the primal-dual pair of conic programs ( $\mathcal{P}_{\mathcal{K}}$ ) and ( $\mathcal{D}_{\mathcal{K}}$ ).

Proposition 4.2. Consider a game $\mathcal{G}$ and let $\mathcal{K} \subseteq \mathcal{N}$ be a closed convex cone such that ( $\mathcal{P}_{\mathcal{K}}$ ) is primal feasible. Then we have that $\omega(\mathcal{K}, \mathcal{G})=\xi(\mathcal{K}, \mathcal{G})$ and moreover there exists an optimal solution for $\left(\mathcal{P}_{\mathcal{K}}\right)$.

Proof. Since $\left(\mathcal{P}_{\mathcal{K}}\right)$ is feasible, the dual value $\xi(\mathcal{K}, \mathcal{G})$ is bounded below by 0 . It remains to show that the dual program is strictly feasible for the range of cones we consider. Notice that the program ( $\mathcal{D}_{\mathcal{K}}$ is strictly feasible for $\mathcal{K}=\mathcal{N}$ since $\operatorname{int}(\mathcal{N})=\{X: X[i, j]>0$ for all $i, j\}$ and $\sum_{i, j \in S \cup T} v_{i, j} J_{i, j}-\hat{C} \in \operatorname{int}(\mathcal{N})$ by setting each $v_{i, j}$ to be a very large positive constant. Furthermore, for $\mathcal{K} \subseteq \mathcal{N}$ we have that $\mathcal{N}=\mathcal{N}^{*} \subseteq \mathcal{K}^{*}$ implying $\operatorname{int}(\mathcal{N}) \subseteq \operatorname{int}\left(\mathcal{K}^{*}\right)$. Thus $(\overline{\mathcal{L}})$ is strictly feasible for all cones $\mathcal{K} \subseteq \mathcal{N}$. The proof is concluded by Theorem[2.1(ii).

Since $\mathcal{C} \mathcal{S}_{+}^{*}=\left(\operatorname{cl}\left(\mathcal{C S}{ }_{+}\right)\right)^{*}$, we get the following corollary.
Corollary 4.3. For any nonlocal game $\mathcal{G}$ we have $\omega\left(\operatorname{cl}\left(\mathcal{C S}_{+}\right), \mathcal{G}\right)=\xi\left(\mathcal{C} \mathcal{S}_{+}, \mathcal{G}\right)$.
Recall that the $\mathcal{C} \mathcal{S}_{+}$cone is not known to be closed [27, 8]. It follows from Corollary 4.3 that a sufficient condition for showing that the cone $\mathcal{C} \mathcal{S}_{+}$is not closed is to identify a game $\mathcal{G}$ for which $\omega\left(\mathcal{C} \mathcal{S}_{+}, \mathcal{G}\right)<\xi\left(\mathcal{C} \mathcal{S}_{+}, \mathcal{G}\right)$.

The Feige-Lovász SDP relaxation. Notice that the tractability of the conic program $\left(\overline{P_{\mathcal{K}}}\right)$ depends on the underlying cone $\mathcal{K}$. In this section we focus on the case $\mathcal{K}=\mathcal{D N} \mathcal{N}$ for which ( $\left.\overline{\mathcal{P}_{\mathcal{K}}}\right)$ becomes an instance of a semidefinite program. Specifically, using the definition of $\operatorname{Corr}(\mathcal{D N N})$ we have:

$$
\omega(\mathcal{D N N}, \mathcal{G})=\max \left\{\langle\hat{C}, X\rangle:\left\langle J_{i, j}, X\right\rangle=1, \forall i, j \in S \cup T, X \in \mathcal{D N}^{N}\right\}
$$

which we define as a maximization as the feasible region is compact.
Note that whenever $\mathcal{K}_{1} \subseteq \mathcal{K}_{2}$ we have that $\omega\left(\mathcal{K}_{1}, \mathcal{G}\right) \leq \omega\left(\mathcal{K}_{2}, \mathcal{G}\right)$. By Proposition 3.11 it follows that $\omega(\mathcal{D N N}, \mathcal{G})$ is an SDP upper bound to the quantum value of a nonlocal game, that never exceeds the no-signaling value.

Proposition 4.4. For any game $\mathcal{G}$ we have $\omega_{\mathcal{Q}}(\mathcal{G}) \leq \omega(\mathcal{D N N}, \mathcal{G}) \leq \omega_{\mathcal{N S}}(\mathcal{G})$.
As it turns out, this SDP was already studied by Feige and Lovász as an upper bound to the classical value of an arbitrary nonlocal game (cf. Equations (5)-(9) in [20]). On the other hand, Proposition 4.4 yields a much stronger result, namely that $\omega(\mathcal{D N} \mathcal{N}, \mathcal{G})$ is in fact an upper bound to the quantum value, so in particular it also upper bounds $\omega_{\mathcal{C}}(\mathcal{G})$. To the best of our knowledge, prior to this work, the only known result relating $\omega(\mathcal{D N \mathcal { N }}, \mathcal{G})$ with the quantum value is that they are equal for XOR games [39, Theorem 22].

We conclude this section by comparing $\omega(\mathcal{D N \mathcal { N }}, \mathcal{G})$ with the maximum probability of winning the game $\mathcal{G}$ when the players use strategies that generate correlations in the first level of the NPA hierarchy, denoted by $\operatorname{SDP}^{(1)}(\mathcal{G})$. As an immediate consequence of Theorem 3.16 we have that:

Proposition 4.5. For any game $\mathcal{G}$ we have that $\omega(\mathcal{D N N}, \mathcal{G}) \leq \operatorname{SDP}^{(1)}(\mathcal{G})$.
At present, we have not been able to identify a game for which this inequality is strict. Lastly, by Remark 3.14 it is easy to see that $\operatorname{SDP}^{(1)}$ is equal to:

$$
\begin{array}{cl}
\operatorname{maximize} & \langle\hat{C}, X\rangle \\
\text { subject to } & \left\langle J_{i, j}, X\right\rangle=1, \text { for all } i, j \in S \cup T, \\
& X[(s, a),(t, b)] \geq 0, \text { for all } s \in S, t \in T, a \in A, b \in B, \\
& X\left[(s, a),\left(s, a^{\prime}\right)\right]=0, \text { for all } s \in S, a \neq a^{\prime} \in A,  \tag{31}\\
& X\left[(t, b),\left(t, b^{\prime}\right)\right]=0, \text { for all } t \in T, b \neq b^{\prime} \in B, \\
& X \in \mathcal{S}_{+}^{N} .
\end{array}
$$

The SDP given in (31) is the "canonical" SDP relaxation for $\omega_{\mathcal{Q}}(\mathcal{G})$ that is usually considered in the quantum information literature (e.g. see [25]).

Perfect strategies. Recall that for a convex cone $\mathcal{K} \subseteq \mathcal{N}$ we say that the game $\mathcal{G}(\pi, V)$ admits a perfect $\mathcal{K}$-strategy if $\omega(\mathcal{K}, \mathcal{G})=1$ and moreover, this value is achieved by some correlation in $\operatorname{Corr}(\mathcal{K})$ (cf. Definition 1.2). Using our conic formulations we show that deciding the existence of a perfect $\mathcal{K}$-strategy for an arbitrary nonlocal game can be cast as a conic program over the cone $\mathcal{K}$.

Lemma 4.6. Let $\mathcal{G}(\pi, V)$ be a game with question sets $S, T$ and answer sets $A, B$ and let $\mathcal{K} \subseteq \mathcal{N}^{N}$. The game $\mathcal{G}$ admits a perfect $\mathcal{K}$-strategy if and only if the following conic program is feasible:

$$
\begin{align*}
& X \in \mathcal{K}, \quad\left\langle J_{i, j}, X\right\rangle=1, \forall i, j \in S \cup T  \tag{K}\\
& X[(s, a),(t, b)]=0, \forall a, b, s, t \text { with } \pi(s, t)>0 \text { and } V(a, b \mid s, t)=0
\end{align*}
$$

Proof. For any $p=(p(a, b \mid s, t)) \in \operatorname{Corr}(\mathcal{K})$ we have that

$$
\begin{equation*}
\sum_{s, t} \pi(s, t) \sum_{a, b} V(a, b \mid s, t) p(a, b \mid s, t) \leq \sum_{s, t} \pi(s, t) \sum_{a, b} p(a, b \mid s, t)=1 \tag{32}
\end{equation*}
$$

Therefore $\mathcal{G}$ admits a perfect $\mathcal{K}$-strategy if and only if (32) holds throughout with equality for some $p=(p(a, b \mid s, t)) \in \operatorname{Corr}(\mathcal{K})$. This is equivalent to

$$
\pi(s, t)(V(a, b \mid s, t)-1) p(a, b \mid s, t)=0, \text { for all } a \in A, b \in B, s \in S, t \in T
$$

which shows that $p(a, b \mid s, t)=0$ when $\pi(s, t)>0$ and $V(a, b \mid s, t)=0$.
Lemma 4.6, combined with Theorem 3.7 implies the following.
Corollary 4.7. For any nonlocal game $\mathcal{G}$ we have that:
(i) $\mathcal{G}$ admits a perfect classical strategy if and only if $\left(\mathcal{F}_{\mathcal{C P}}\right)$ is feasible.
(ii) $\mathcal{G}$ admits a perfect quantum strategy if and only if $\left(\mathcal{F}_{\mathcal{C} \mathcal{S}_{+}}\right)$is feasible.
(iii) $\mathcal{G}$ admits a perfect no-signaling strategy if and only if $\left(\mathcal{F}_{\mathcal{N S O}}\right)$ is feasible.
(iv) $\mathcal{G}$ admits a perfect unrestricted strategy if and only if $\left(\mathcal{F}_{\mathcal{N}}\right)$ is feasible.

We conclude this section with an equivalent form of Corollary 4.7 which is used in Section 5 , This follows easily using Lemma 3.10 .
 if and only if there exists $\tilde{X} \in \mathcal{K}^{1+N}$ satisfying:

- $\left\langle J_{i, j}, \tilde{X}\right\rangle=1$, for all $i, j \in\{0\} \cup S \cup T$, and
- $\tilde{X}[(s, a),(t, b)]=0, \forall a, b, s, t$ with $\pi(s, t)>0$ and $V(a, b \mid s, t)=0$.


## 5 Synchronous correlations and game values

Throughout this section we only consider Bell scenarios where $A=B$ and $S=T$. Given such a scenario we study synchronous correlations, i.e., correlations with the property that the players respond with the same answer whenever they receive the same question. First, we specialize Theorem 3.7 to synchronous correlations and show that our conic characterizations assume a particularly simple form. Based on these simplified characterizations we study the maximum probability
of winning a nonlocal game using strategies that generate synchronous correlations. This allows us to derive conic programming formulations for deciding the existence of perfect strategies for synchronous nonlocal games. As a corollary, we recover in a uniform manner the conic programming formulations for deciding the existence of a classical and quantum graph homomorphisms [36] and also the conic programming formulations for the quantum chromatic and the quantum independence number [27].

Synchronous correlations. We start this section with a central definition.
Definition 5.1. A correlation $p \in \mathcal{P}$ is called synchronous if the players always respond with the same answer upon receiving the same question, i.e.,

$$
\begin{equation*}
p\left(a, a^{\prime} \mid s, s\right)=0, \text { for all } s \in S \text { and } a \neq a^{\prime} \in A . \tag{33}
\end{equation*}
$$

Our first result is a geometric lemma that is essential in obtaining simplified conic characterizations for quantum and classical synchronous correlations.

Lemma 5.2. Let $\mathcal{X}$ be a Euclidean space and consider two families of psd matrices $\left\{X_{i}: i \in[n]\right\} \subseteq \mathcal{H}_{+}(\mathcal{X})$ and $\left\{Y_{i}: i \in[n]\right\} \subseteq \mathcal{H}_{+}(\mathcal{X})$ satisfying:
(i) $\sum_{i=1}^{n} X_{i}=\sum_{i=1}^{n} Y_{i}$, and
(ii) $\left\langle X_{i}, Y_{j}\right\rangle=0$, for all $i \neq j \in[n]$.

Then we have that $X_{i}=Y_{i}$ for all $i \in[n]$.
Proof. Fix $i \in[n]$ and let $\lambda$ be the largest eigenvalue of $X_{i}$ with corresponding (normalized) eigenvector $v$ and let $\mu$ be the largest eigenvalue of $Y_{i}$. By condition (ii), we know that $Y_{j} v=0$ for all $j \neq i \in[n]$. Using this, we have

$$
\begin{equation*}
\lambda=v^{*} X_{i} v \leq v^{*}\left(\sum_{j=1}^{n} X_{j}\right) v=v^{*}\left(\sum_{j=1}^{n} Y_{j}\right) v=v^{*} Y_{i} v \leq \mu, \tag{34}
\end{equation*}
$$

proving that $\mu \geq \lambda$. By symmetry, we also have $\lambda \geq \mu$ proving that $\lambda=\mu$. This shows that (34) holds throughout with equality and thus $v$ is also an eigenvector of $Y_{i}$ corresponding to eigenvalue $\lambda=\mu$ as well. Lastly, define $X_{i}^{\prime}:=X_{i}-\lambda v v^{*}, Y_{i}^{\prime}:=Y_{i}-\lambda v v^{*}$ and for $j \neq i$ set $X_{j}^{\prime}:=X_{j}$ and $Y_{j}^{\prime}:=Y_{j}$. Notice that the matrices $\left\{X_{i}^{\prime}\right\}_{i=1}^{n} \subseteq \mathcal{H}_{+}(\mathcal{X})$ and $\left\{Y_{i}^{\prime}\right\}_{i=1}^{n} \subseteq \mathcal{H}_{+}(\mathcal{X})$ satisfy conditions (i) and (ii) and the proof is concluded by an inductive argument.

Based on Lemma5.2 we now derive a second result that we use in Theorem 5.4 below and in our study of perfect strategies.

Lemma 5.3. Consider a family of vectors $\left\{x_{a}^{s}\right\}_{s \in S, a \in A}$ in some Euclidean space $\mathcal{X}$.
(a) For $X:=\operatorname{Gram}\left(\left\{x_{a}^{s}\right\}_{s \in S, a \in A}\right)$ the following are equivalent:
(i) There exists $k \in \mathcal{X}$ satisfying $\sum_{a \in A} x_{a}^{s}=k$ for all $s \in S$ and $\langle k, k\rangle=1$.
(ii) $\left\langle J_{s, s^{\prime}}, X\right\rangle=1$, for all $s, s^{\prime} \in S$.
(b) Set $\tilde{X}:=\operatorname{Gram}\left(k,\left\{x_{a}^{s}\right\}_{s \in S, a \in A}\right)$ where $k \in \mathcal{X}$ with $\langle k, k\rangle=1$. The following are equivalent:
(i) $\sum_{a \in A} x_{a}^{s}=k$, for all $s \in S$.
(ii) $\left\langle J_{s, s}, \tilde{X}\right\rangle=\left\langle J_{0, s}, \tilde{X}\right\rangle=1$, for all $s \in\{0\} \cup S$.

Proof. The proof is similar to the proof of Lemma 3.6 and we omit most cases. We only consider case (b) and show that (ii) implies (i). Notice that

$$
\begin{equation*}
\left\langle k-\sum_{a \in A} x_{a}^{s}, k-\sum_{a \in A} x_{a}^{s}\right\rangle=\langle k, k\rangle-2 \sum_{a \in A}\left\langle k, x_{a}^{s}\right\rangle+\left\langle\sum_{a \in A} x_{a}^{s}, \sum_{a \in A} x_{a}^{s}\right\rangle . \tag{35}
\end{equation*}
$$

By assumption we have that $\langle k, k\rangle=1,\left\langle J_{0, s}, \tilde{X}\right\rangle=\sum_{a}\left\langle k, x_{a}^{s}\right\rangle=1$ and $\left\langle\sum_{a} x_{a}^{s}, \sum_{a} x_{a}^{s}\right\rangle=$ $\left\langle J_{s, s}, \tilde{X}\right\rangle=1$. Substituting in (35) the proof is concluded.

We now arrive at the main result in this section.
Theorem 5.4. Consider a Bell scenario with question set $S$ and answer set $A$. Furthermore, consider a synchronous correlation $p=\left(p\left(a, a^{\prime} \mid s, s^{\prime}\right)\right) \in \mathcal{P}$ and set $P:=\sum_{a, a^{\prime}, s, s^{\prime}} p\left(a, a^{\prime} \mid s, s^{\prime}\right) e_{s} e_{s^{\prime}}^{\top} \otimes e_{a} e_{a^{\prime}}^{\top}$. For every cone $\mathcal{K} \in\left\{\mathcal{C P}, \mathcal{C S}_{+}\right\}$the following are equivalent:
(i) $p \in \operatorname{Corr}(\mathcal{K})$.
(ii) $P \in \mathcal{K}^{|S \times A|}$.
(iii) There exists $\tilde{X}=\left(\begin{array}{cc}1 & x^{\boldsymbol{\top}} \\ x & P\end{array}\right) \in \mathcal{K}^{1+|S \times A|}$ s.t. $\left\langle J_{0, s}, \tilde{X}\right\rangle=1, \forall s \in\{0\} \cup S$.

Proof. We only consider $\mathcal{K}=\mathcal{C} \mathcal{S}_{+}$the case $\mathcal{K}=\mathcal{C P}$ being similar.
(i) $\Longrightarrow$ (ii): Since $p=\left(p\left(a, a^{\prime} \mid s, s^{\prime}\right)\right) \in \operatorname{Corr}\left(\mathcal{C S}_{+}\right)$, by Theorem 3.2 there exist psd matrices $\left\{X_{a}^{s}\right\}_{s \in S, a \in A},\left\{Y_{a}^{s}\right\}_{s, a}, K \in \mathcal{S}_{+}^{d}$ (for some $d \geq 1$ ) such that $p\left(a, a^{\prime} \mid s, s^{\prime}\right)=\left\langle X_{a}^{s}, Y_{a^{\prime}}^{s^{\prime}}\right\rangle$ for all $a, a^{\prime}, s, s^{\prime}$, $\sum_{a} X_{a}^{s}=\sum_{a} Y_{a}^{s}=K$ for all $s \in S$ and $\langle K, K\rangle=1$. Since $p$ is synchronous it follows from Lemma 5.2] that $X_{a}^{s}=Y_{a}^{s}$ for all $s \in S$ and $a \in A$. This implies that $P \in \mathcal{C} \mathcal{S}_{+}^{|S \times A|}$.
(ii) $\Longrightarrow$ (iii): Since $P \in \mathcal{C} \mathcal{S}_{+}^{|S \times A|}$ there exist matrices $\left\{X_{a}^{s}\right\}_{s \in S, a \in A} \in \mathcal{S}_{+}^{d}$ (for some $d \geq 1$ ) such that $p\left(a, a^{\prime} \mid s, s^{\prime}\right)=\left\langle X_{a}^{s}, X_{a^{\prime}}^{s^{\prime}}\right\rangle$ for all $a, a^{\prime}, s, s^{\prime}$. By Lemma 5.3(a) there exists $K \in \mathcal{S}_{+}^{d}$ such that $\langle K, K\rangle=1$ and $\sum_{a} X_{a}^{s}=K$ for all $s \in S$. The proof is concluded by noticing that $\operatorname{Gram}\left(K,\left\{X_{a}^{s}\right\}_{s \in S, a \in A}\right)$ is feasible for (iii).
(iii) $\Longrightarrow(i)$ : Let $\tilde{X} \in \mathcal{C} \mathcal{S}_{+}^{1+|S \times A|}$ be feasible for (iii) and consider $K,\left\{X_{a}^{s}\right\}_{s \in S, a \in A} \in \mathcal{S}_{+}^{d}$ such that $\tilde{X}=\operatorname{Gram}\left(K,\left\{X_{a}^{s}\right\}_{s \in S, a \in A}\right)$. Since $p \in \mathcal{P}$ we have that $\left\langle J_{s, s}, \tilde{X}\right\rangle=1$, for all $s \in S$. Thus, by Lemma 5.3(b) we have that $\sum_{a} X_{a}^{s}=K$, for all $s \in S$. Lastly, Theorem 3.2 implies $p \in \operatorname{Corr}\left(\mathcal{C S} \mathcal{S}_{+}\right)$.

As a consequence of Theorem 5.4 we arrive at the following conic characterization of the sets of synchronous quantum and classical correlations:

Corollary 5.5. Consider a Bell scenario with question set $S$ and answer set $A$ and let $\mathcal{K} \in\left\{\mathcal{C P}, \mathcal{C} \mathcal{S}_{+}\right\}$. The set of synchronous $\mathcal{K}$-correlations is given by

$$
\left\{X \in \mathcal{K}^{|S \times A|}:\left\langle J_{s, s^{\prime}}, X\right\rangle=1, \text { for } s, s^{\prime} \text { and } X\left[(s, a),\left(s, a^{\prime}\right)\right]=0, \text { for } a \neq a^{\prime}, s\right\},
$$

where we identify a correlation vector $p=\left(p\left(a, a^{\prime} \mid s, s^{\prime}\right)\right)$ with the square matrix

$$
P:=\sum_{a, a^{\prime}, s, s^{\prime}} p\left(a, a^{\prime} \mid s, s^{\prime}\right) e_{s} e_{s^{\prime}}^{\top} \otimes e_{a} e_{a^{\prime}}^{\top} .
$$

Synchronous value. In this section we study the value of a nonlocal game when the players use strategies that generate synchronous correlations.
Definition 5.6. For any convex cone $\mathcal{K} \subseteq \mathcal{N}^{N}$, the $\mathcal{K}$-synchronous value of a nonlocal game $\mathcal{G}$, denoted $\omega_{\text {syn }}(\mathcal{K}, \mathcal{G})$, is defined as the maximum probability of winning the game when the players are only allowed to use strategies that generate synchronous $\mathcal{K}$-correlations.

As we now show by Corollary 5.5 we get a conic programming formulation for the classical and quantum synchronous value of a nonlocal game with matrix variables of size $|S \times A|$.
Proposition 5.7. Consider a game $\mathcal{G}$ with question set $S$ and answer set $A$. For a cone $\mathcal{K} \subseteq \mathcal{D N}^{|S \times A|}$ define

$$
\begin{aligned}
\nu(\mathcal{K}, \mathcal{G}):=\text { supremum } & \frac{1}{2}\left\langle C+C^{\top}, X\right\rangle \\
\text { subject to } & \left\langle J_{s, s^{\prime}}, X\right\rangle=1, \text { for all } s, s^{\prime} \in S, \\
& X\left[(s, a),\left(s, a^{\prime}\right)\right]=0, \text { for } s \in S, a \neq a^{\prime} \in A, \\
& X \in \mathcal{K}^{|S \times A|} .
\end{aligned}
$$

Then $\omega_{\text {syn }}(\mathcal{C P}, \mathcal{G})=\nu(\mathcal{C P}, \mathcal{G})$ and $\omega_{\text {syn }}\left(\mathcal{C S}_{+}, \mathcal{G}\right)=\nu\left(\mathcal{C} \mathcal{S}_{+}, \mathcal{G}\right)$.
Note that in the proposition above we use $\frac{1}{2}\left\langle C+C^{\top}, X\right\rangle$ as the objective function. This is because this is equal to $\sum_{a, a^{\prime}, s, s^{\prime}} \pi(s, t) V\left(a, a^{\prime} \mid s, s^{\prime}\right) p\left(a, a^{\prime} \mid s, s^{\prime}\right)$, which is exactly the probability Alice and Bob win the game.

There are many examples of games for which the optimal classical strategy generates a synchronous correlation but optimal quantum correlations are not synchronous (e.g. the CHSH game). This raises the following question: What is the optimal value for such games when one restricts to synchronous quantum strategies? Perhaps the interesting thing to see is if the power of quantum strategies comes from the fact that the optimal quantum strategies do not need to be synchronous. The following corollary of Proposition 5.7 gives a partial answer to the above question. It is a consequence of the fact that $\mathcal{C P}^{n}=\mathcal{C} \mathcal{S}_{+}^{n}=\mathcal{D N \mathcal { N } ^ { n }}$ for any $n \leq 4$ [27].

Corollary 5.8. For any nonlocal game $\mathcal{G}$ with identical binary question sets (i.e., $S=T$ and $|S|=2$ ) and identical binary answer sets (i.e., $A=B$ and $|A|=2$ ), we have that the synchronous classical and synchronous quantum values coincide and are expressible as a semidefinite program, i.e.,

$$
\begin{equation*}
\omega_{\text {syn }}(\mathcal{C P}, \mathcal{G})=\omega_{\text {syn }}\left(\mathcal{C S}_{+}, \mathcal{G}\right)=\nu(\mathcal{D N \mathcal { N }}, \mathcal{G}) . \tag{36}
\end{equation*}
$$

As an example, consider the CHSH game for which there exists an optimal classical strategy which is synchronous (Alice and Bob just output 0 ) with success probability $3 / 4$. Then, it follows from Corollary 5.8 that the synchronous quantum value is also $3 / 4$. That is, quantum strategies cannot be synchronous to win CHSH with greater probability than classical strategies. For games with large question or answer sets, Corollary 5.8 does not help. However, Proposition 5.7 implies that $\nu(\mathcal{D N \mathcal { N }}, \mathcal{G})$ is a tractable upper bound on the synchronous quantum value of $\mathcal{G}$.

Perfect strategies for synchronous games. In this section we focus on a class of nonlocal games for which any perfect strategy generates a synchronous correlation.
Definition 5.9. A nonlocal game $\mathcal{G}=(\pi, V)$ is called synchronous if both players share the same question set $S$ and the same answer set $A$, and

$$
V\left(a, a^{\prime} \mid s, s\right)=0, \text { for all } s \in S, a \neq a^{\prime} \in A, \text { and } \pi(s, s)>0, \text { for all } s \in S .
$$

By Corollary 4.7, deciding the existence of a perfect $\mathcal{K}$-strategy is equivalent to the feasibility of a linear conic program with matrix variables of size $|(S \times A) \times(T \times B)|$. Moreover, by Lemma 4.6 , any perfect strategy for a synchronous game generates a synchronous correlation. Thus we can use Theorem 5.4 to derive a conic program with matrix variables of size $|S \times A|$ whose feasibility is equivalent to the existence of a perfect $\mathcal{K}$-strategy.

Theorem 5.10. Let $\mathcal{G}(\pi, V)$ be a synchronous game and $\mathcal{K} \in\left\{\mathcal{C P}, \mathcal{C} \mathcal{S}_{+}\right\}$. The following are equivalent:
(i) $\mathcal{G}$ admits a perfect $\mathcal{K}$-strategy.
(ii) There exists a matrix $X \in \mathcal{K}^{|S \times A|}$ satisfying:

- $\left\langle J_{s, s^{\prime}}, X\right\rangle=1, \forall s, s^{\prime}$,
- $X\left[(s, a),\left(s, a^{\prime}\right)\right]=0, \forall s, a \neq a^{\prime}$,
- $X\left[(s, a),\left(s^{\prime}, a^{\prime}\right)\right]=0, \forall a, a^{\prime}, s, s^{\prime}$ with $\pi\left(s, s^{\prime}\right)>0$ and $V\left(a, a^{\prime} \mid s, s^{\prime}\right)=0$.
(iii) There exists a matrix $\tilde{X} \in \mathcal{K}^{1+|S \times A|}$ satisfying:
- $\left\langle J_{s, s}, \tilde{X}\right\rangle=\left\langle J_{0, s}, \tilde{X}\right\rangle=1, \forall s \in\{0\} \cup S$,
- $\tilde{X}\left[(s, a),\left(s, a^{\prime}\right)\right]=0, \forall s, a \neq a^{\prime}$,
- $X\left[(s, a),\left(s^{\prime}, a^{\prime}\right)\right]=0, \forall a, a^{\prime}, s, s^{\prime}$ with $\pi\left(s, s^{\prime}\right)>0$ and $V\left(a, a^{\prime} \mid s, s^{\prime}\right)=0$.

The proof is omitted as it is an easy consequence of Theorem 5.4. In the next two sections we specialize Theorem 5.10 to graph coloring and more generally, graph homomorphism games and derive conic formulations for the existence of perfect strategies for these classes of games.

Graph Homomorphisms. Given two undirected graphs $H$ and $G$, a graph homomorphism from $H$ to $G$, denoted $H \rightarrow G$, is an adjacency preserving map from the vertex set of $H$ to the vertex set of $G$, i.e., a function $f: V(H) \rightarrow V(G)$ with the property that $f(h) \sim_{G} f\left(h^{\prime}\right)$ whenever $h \sim_{H} h^{\prime}$. Here we study the $(H, G)$-homomorphism game where Alice and Bob are trying to convince a referee that there exists a graph homomorphism from $H$ to $G$. To verify their claim the referee sends each player a vertex of $H$. Each player responds with a vertex of $G$. The player's answers model a homomorphism $f: H \rightarrow G$, i.e., the answer to a question $h \in V(H)$ should be $f(h) \in V(G)$.

Formally, in the $(H, G)$-homomorphism game the players share the same question set $S:=V(H)$ and the same answer set $A:=V(G)$. The distribution over the question set is the uniform distribution on $\{(h, h): h \in V(H)\} \cup\left\{\left(h, h^{\prime}\right): h \sim_{H} h^{\prime}\right\}$. Lastly, the verification predicate is given by

$$
V\left(g, g^{\prime} \mid h, h^{\prime}\right)= \begin{cases}0, & \text { if } h=h^{\prime} \text { and } g \neq g, \\ 0, & \text { if } h \sim_{H} h^{\prime} \text { and }\left(g \not \chi_{G} g \text { or } g=g^{\prime}\right), \\ 1, & \text { otherwise. }\end{cases}
$$

Notice that the existence of a homomorphism $H \rightarrow G$ can be understood via the $(H, G)$ homomorphism game. Specifically, there exists a graph homomorphism from $H$ to $G$ if and only if the $(H, G)$-homomorphism game admits a perfect classical strategy. This is easy to see using the fact that in the classical setting we may assume without loss of generality that both players are using deterministic strategies. This motivates the following definition.

Definition 5.11. For graphs $G$ and $H$, we say there exists a quantum graph homomorphism from $H$ to $G$, denoted $H \xrightarrow{q} G$, if the $(H, G)$-homomorphism game admits a perfect quantum strategy.

Quantum graph homomorphisms were introduced recently in [28]. Using our conic characterizations for the sets of quantum and classical correlations we arrive at a natural conic generalization of the notion of graph homomorphism.

Definition 5.12. For a convex cone $\mathcal{K} \subseteq \mathcal{N}$ we say that there exists a $\mathcal{K}$-homomorphism from $H$ to $G$ if and only if the $(H, G)$-homomorphism game admits a perfect $\mathcal{K}$-strategy.

By Lemma 4.6 the existence of a $\mathcal{K}$-homomorphism from $H$ to $G$ is equivalent to the feasibility of a linear conic program over $\mathcal{K}$. The similar notion of strong $\mathcal{K}$-homomorphism was introduced recently in [36]. Since the $(H, G)$-homomorphism game is synchronous we can use the conic formulations from Theorem 5.10 to show that the two notions of conic homomorphisms coincide for $\mathcal{K} \in\left\{\mathcal{C P}, \mathcal{C} \mathcal{S}_{+}\right\}$. We note that strong $\mathcal{K}$-homomorphisms are only defined for a certain class of convex cones called frabjous. Working over frabjous cones ensures that the $\mathcal{K}$-homomorphism relation is reflexive and transitive, mimicking classical graph homomorphisms.

As an immediate consequence of Theorem 5.10 (ii) it follows that deciding the existence of a classical (resp. quantum) graph homomorphism can be formulated as a feasibility conic program over the cone of completely positive (resp. completely positive semidefinite) matrices.

Corollary 5.13. Consider two graphs $H$ and $G$ and let $\mathcal{K} \in\left\{\mathcal{C P}, \mathcal{C} \mathcal{S}_{+}\right\}$. The $(H, G)$-homomorphism game admits a perfect $\mathcal{K}$-strategy if and only if there exists $X \in \mathcal{K}^{|V(H) \times V(G)|}$ such that

- $\sum_{g \in V(G)} \sum_{g^{\prime} \in V(G)} X\left[(h, g),\left(h^{\prime}, g^{\prime}\right)\right]=1$, for all $h, h^{\prime} \in V(H)$, and
- $X\left[(h, g),\left(h^{\prime}, g^{\prime}\right)\right]=0$, when $\left(h=h^{\prime}\right.$ and $\left.g \neq g^{\prime}\right)$ or $\left(h \sim_{H} h^{\prime}\right.$ and $\left.g \not \chi_{G} g^{\prime}\right)$.

We note that the case $\mathcal{K}=\mathcal{C P}$ (resp. $\mathcal{K}=\mathcal{C} \mathcal{S}_{+}$) corresponds to Theorem 4.1 in [36] (resp. Theorem 4.3 in [36]).

Chromatic and independence number. A $k$-coloring for a graph $G$ corresponds to an assignment of one out of $k$ possible colors to its vertices so that adjacent vertices receive different colors. The chromatic number of a graph $G$, denoted $\chi(G)$, is equal to the smallest integer $k \geq 1$ for which $G$ admits a $k$-coloring. Notice that $G$ admits a $k$-coloring if and only if there exists a homomorphism from $G$ into $K_{k}$, i.e., the complete graph on $k$ vertices. Thus, $\chi(G)$ may be equivalently defined as the smallest $k \geq 1$ for which the ( $G, K_{k}$ ) -homomorphism game admits a perfect classical strategy. The quantum chromatic number of a graph $G$, denoted $\chi_{q}(G)$, is equal to the smallest $k \geq 1$ for which the $\left(G, K_{k}\right)$-homomorphism game admits a perfect quantum strategy.

The independence number of a graph $G$, denoted $\alpha(G)$, is equal to the largest number of pairwise nonadjacent vertices of $G$. Notice that $G$ contains $k$ pairwise nonadjacent vertices if and only if there exists a homomorphism from $K_{k}$ into $\bar{G}$, where $\bar{G}$ denotes the complement of the graph $G$. As a result, $\alpha(G)$ can be equivalently defined as the largest integer $k \geq 1$ for which the ( $K_{k}, \bar{G}$ )homomorphism game admits a perfect classical strategy. Analogously, the quantum independence number of $G$, denoted $\alpha_{q}(G)$, is the largest $k \geq 1$ for which the ( $K_{k}, \bar{G}$ )-homomorphism game admits a perfect quantum strategy.

The quantum chromatic number was introduced and studied in [9] and the quantum independence number in [29]. It was recently shown that deciding whether the quantum chromatic number of a graph is at most 3 is NP-hard [24].

Using Theorem 5.10 (iii) we immediately get conic programming formulations for the quantum chromatic number and the quantum independence number of a graph. We note that these formulations were also identified in Proposition 4.10 and Proposition 4.1 in [27], respectively.

Corollary 5.14. The quantum chromatic number of a graph $G$ is equal to the smallest integer $k \geq 1$ for which there exists $X \in \mathcal{C} \mathcal{S}_{+}^{|V(G)| k+1}$ satisfying:

- $X[0,0]=1$;
- $\sum_{i, i^{\prime} \in[k]} X\left[(g, i),\left(g, i^{\prime}\right)\right]=\sum_{i \in[k]} X[0,(g, i)]=1$, for all $g \in V(G)$;
- $X\left[(g, i),\left(g^{\prime}, i^{\prime}\right)\right]=0$, when $\left(g=g^{\prime}\right.$ and $\left.i \neq i^{\prime}\right)$ or $\left(g \sim g^{\prime}\right.$ and $\left.i=i^{\prime}\right)$.

The quantum independence number of a graph $G$ is equal to the largest integer $k \geq 1$ for which there exists a matrix $X \in \mathcal{C S}_{+}^{k|V(G)|+1}$ satisfying:

- $X[0,0]=1$;
- $\sum_{g, g^{\prime} \in V(G)} X\left[(i, g),\left(i, g^{\prime}\right)\right]=\sum_{g \in V(G)} X[0,(i, g)]=1$, for all $i \in[k]$;
- $X\left[(i, g),\left(i^{\prime}, g^{\prime}\right)\right]=0$, when $\left(i=i^{\prime}\right.$ and $\left.g \neq g^{\prime}\right)$ or $\left(i \neq i^{\prime}\right.$ and $\left.g \simeq g^{\prime}\right)$.

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## References

[1] A. Aspect, P. Grangier, and G. Roger. Experimental realization of Einstein-Podolsky-RosenBohm Gedankenexperiment: A new violation of Bell's inequalities. Physical Review Letters, 49:91-94, Jul 1982.
[2] A. Barvinok. A Course in Convexity. American Mathematical Society, 2002.
[3] J. S. Bell. On the Einstein Podolsky Rosen paradox. Physics, 1(3):195-200, 1964.
[4] A. Ben-Tal and A. Nemirovski. Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications. MOS-SIAM Series on Optimization. 2001.
[5] A. Berman and N. Shaked-Monderer. Completely Positive Matrices. World Scientific, 2003.
[6] M. Berta, O. Fawzi, and V. B. Scholz. Quantum bilinear optimization. arXiv:1506.08810, 2015.
[7] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner. Bell nonlocality. Reviews of Modern Physics, 86(2):419, 2014.
[8] S. Burgdorf, M. Laurent, and T. Piovesan. On the closure of the completely positive semidefinite cone and linear approximations to quantum colorings. arXiv:1502.02842, 2015.
[9] P. J. Cameron, A. Montanaro, M. W. Newman, S. Severini, and A. Winter. On the quantum chromatic number of a graph. Electronic Journal of Combinatorics, 14(1), 2007.
[10] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt. Proposed experiment to test local hidden-variable theories. Physical Review Letters, 23(15):880-884, 1969.
[11] R. Cleve, P. Høyer, B. Toner, and J. Watrous. Consequences and limits of nonlocal strategies. In Proceedings of the 19th Annual IEEE Conference on Computational Complexity, pages 236-249, 2004.
[12] R. Cleve, W. Slofstra, F. Unger, and S. Upadhyay. Perfect parallel repetition theorem for quantum XOR proof systems. Computational Complexity, 17(2):282-299, 2008.
[13] R. Colbeck. Quantum and relativistic protocols for secure multi-party computation. PhD thesis, Trinity College, University of Cambridge, 2006.
[14] E. de Klerk and D. V. Pasechnik. Approximation of the stability number of a graph via copositive programming. SIAM Journal on Optimization, 12(4):875-892, 2002.
[15] I. Dinur and D. Steurer. Analytical approach to parallel repetition. In Proceedings of the 46 th ACM Symposium on Theory of Computing, 2014.
[16] I. Dinur, D. Steurer, and T. Vidick. A parallel repetition theorem for entangled projection games. In Proceedings of the 29th IEEE Conference on Computational Complexity, pages 197-208, 2014.
[17] M. Dür. Copositive programming - a survey. In Recent advances in optimization and its applications in engineering, pages 3-20. Springer, 2010.
[18] K. J. Dykema and V. Paulsen. Synchronous correlation matrices and Connes' embedding conjecture. Journal of Mathematical Physics, 57:015214, 2016.
[19] A. K. Ekert. Quantum cryptography based on Bell's theorem. Physical Review Letters, 67(6):661-663, 1991.
[20] U. Feige and L. Lovász. Two-prover one-round proof systems: Their power and their problems. In Proceedings of the twenty-fourth annual ACM symposium on theory of computing, pages 733-744. ACM, 1992.
[21] S. J. Freedman and J. F. Clauser. Experimental test of local hidden-variable theories. Physical Review Letters, 28:938-941, 1972.
[22] T. Fritz. Polyhedral duality in Bell scenarios with two binary observables. Journal of Mathematical Physics, 53:072202, 2012.
[23] T. Fritz. Tsirelson's problem and Kirchberg's conjecture. Reviews in Mathematical Physics, 24(5):1250012, 2012.
[24] Z. Ji. Binary constraint system games and locally commutative reductions. arXiv:1310.3794, 2013.
[25] J. Kempe, O. Regev, and B. Toner. Unique games with entangled provers are easy. SIAM Journal on Computing, 39(7):3207-3229, 2010.
[26] J. B. Lasserre. New approximations for the cone of copositive matrices and its dual. Mathematical Programming, 144(1-2):265-276, 2013.
[27] M. Laurent and T. Piovesan. Conic approach to quantum graph parameters using linear optimization over the completely positive semidefinite cone. SIAM Journal on Optimization, 25(4):2461-2493, 2015.
[28] L. Mančinska and D. E. Roberson. Note on the correspondence between quantum correlations and the completely positive semidefinite cone. Unpublished manuscript, available at https://sites.google.com/site/davideroberson/, 2014.
[29] L. Mančinska and D. E. Roberson. Graph homomorphisms for quantum players. Journal of Combinatorial Theory, Series B, to appear, 2015.
[30] L. Mančinska, D. E. Roberson, and A. Varvitsiotis. On deciding the existence of perfect entangled strategies for nonlocal games. Chicago Journal of Theoretical computer science, to appear, 2016.
[31] J. E. Maxfield and H. Minc. On the matrix equation $X^{\prime} X=A$. Proceedings of the Edinburgh Mathematical Society (Series 2), 13(02):125-129, 1962.
[32] M. Navascués, S. Pironio, and A. Acín. A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations. New Journal of Physics, 10(7):073013, 2008.
[33] M. A. Nielsen and I. L. Chuang. Quantum computation and quantum information. Cambridge University Press, 2000.
[34] P. A. Parrilo. Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization. PhD thesis, California Institute of Technology, 2000.
[35] V. I. Paulsen, S. Severini, D. Stahlke, I. G. Todorov, and A. Winter. Estimating quantum chromatic numbers. Journal of Functional Analysis, 270(6):2188-2222, 2016.
[36] D. E. Roberson. Conic formulations of graph homomorphisms. Journal of Algebraic Combinatorics, pages 1-37, 2016.
[37] J. Sikora, A. Varvitsiotis, and Z. Wei. On the minimum dimension of a Hilbert space needed to generate a quantum correlation. arXiv:1507.00213, 2015.
[38] B. S. Tsirelson. Quantum analogues of the Bell inequalities: The case of two spatially separated domains. Journal of Soviet Mathematics, 36:557-570, 1987.
[39] S. Upadhyay. Quantum Information and Variants of Interactive Proof Systems. PhD thesis, University of Waterloo, 2011.
[40] J. Watrous. Theory of quantum information, lecture notes. Available at https://cs.uwaterloo.ca/~watrous/LectureNotes.html, 2011.

## A Constraint satisfaction problems

It is natural to ask what kinds of combinatorial problems admit linear conic formulations of a similar form. In this appendix we show that all examples considered in this work can be cast in the common framework of binary constraint satisfaction problems.

An instance of a constraint satisfaction problem (CSP) is specified by a triple ( $\mathcal{V}, \mathcal{D}, \mathcal{C})$ where the elements of $\mathcal{V}=\left\{x_{1}, . ., x_{n}\right\}$ are called the variables of the CSP, the elements of $\mathcal{D}=\left\{D_{1}, \ldots, D_{n}\right\}$
are the domains of the corresponding variables and the elements of $\mathcal{C}=\left\{C_{1}, \ldots, C_{m}\right\}$ are called the constraints of the CSP. Each constraint $C_{i}$ involves a subset of variables $\left\{x_{i_{1}}, \ldots, x_{i_{i}}\right\} \subseteq \mathcal{V}$ and is defined as some $t_{i}$-ary relation on $D_{i_{1}} \times \cdots \times D_{i_{i}}$. The number of variables $t_{i}$ is called the arity of the constraint $C_{i}$. We say that a CSP is satisfiable if there exists an assignment of values to each variable from its corresponding domain so that every constraint is satisfied. A CSP that only involves constraints of arity 2 is called a binary CSP.

Deciding the existence of a homomorphism from a graph $H$ to a graph $G$ can be formulated as an instance of a binary CSP. Specifically, we have one variable for each vertex of $H$ and the domain of each variable is the vertex set of $G$. Lastly, for every edge $e=\left(h, h^{\prime}\right) \in E(H)$ we have a constraint $C_{e}$ of arity 2 involving the variables $h$ and $h^{\prime}$; the constraint is given by $C_{e}=\{E(G)\}$.

To any binary constraint satisfaction problem $\mathcal{P}:=(\mathcal{V}, \mathcal{D}, \mathcal{C})$ we may associate a two-player nonlocal game, denoted $\mathcal{G}(\mathcal{P})$, having the property that the CSP is satisfiable if and only if the game admits a perfect classical strategy. The game is defined as follows: The referee selects uniformly at random a pair of variables $\left(x_{i}, x_{j}\right) \in \mathcal{V} \times \mathcal{V}$ and sends $x_{i}$ to Alice and $x_{j}$ to Bob. For the players to win they need to respond to the referee with an element of $D_{i}$ and $D_{j}$, respectively. Furthermore, if there exists some constraint $C_{k}$ that involves the variables $x_{i}$ and $x_{j}$ then the answers of the players must satisfy the constraint. Lastly, if the players receive the same variables as questions they have to provide identical answers ensuring that the game is synchronous.

Definition A.1. A binary constraint satisfaction problem $\mathcal{P}$ is called quantumly satisfiable if the nonlocal game $\mathcal{G}(\mathcal{P})$ admits a perfect quantum strategy.

Notice that the notion of quantum satisfiability of binary CSP's generalizes the concept of quantum graph homomorphisms. Indeed, it is immediate from the definitions that there exists a quantum graph homomorphism from $H$ to $G$ if and only if the nonlocal game corresponding to the homomorphism CSP admits a perfect quantum strategy.

The majority of the literature concerning CSP's usually focuses on binary CSP's. The reason for this is that any non-binary CSP $\mathcal{P}$ can be converted to a binary CSP $\mathcal{P}^{\prime}$ such that $\mathcal{P}$ is satisfiable if and only if $\mathcal{P}^{\prime}$ is satisfiable. The transformation is straightforward: For each constraint $C_{i}$ of $\mathcal{P}$ we introduce one variable $c_{i}$ in $\mathcal{P}^{\prime}$. The domain of the variable $c_{i}$ is given by all assignments that satisfy the constraint $C_{i}$ in $\mathcal{P}$. Lastly, for every two constraints $C_{i}, C_{j}$ of $\mathcal{P}$ that share a variable $x_{k}$ we add a binary constraint between the variables $c_{i}, c_{j}$ in $\mathcal{P}^{\prime}$. This constraint excludes those satisfying assignments for $C_{i}$ and $C_{j}$ where the common variable $x_{k}$ receives different values.

The discussion above allows us to generalize the notion of quantum satisfiability from binary CSP's to arbitrary ones. Combining this fact with Theorem 5.10 we get the following corollary.

Corollary A.2. Deciding whether an arbitrary constraint satisfaction problem is satisfiable (resp. quantumly satisfiable) is equivalent to deciding the feasibility of a linear conic program over $\mathcal{C P}$ (resp. $\mathcal{C} \mathcal{S}_{+}$).


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