Generalized monotone operators and their averaged resolvents

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Abstract

The correspondence between the monotonicity of a (possibly) set-valued operator and the firm nonexpansiveness of its resolvent is a key ingredient in the convergence analysis of many optimization algorithms. Firmly nonexpansive operators form a proper subclass of the more general – but still pleasant from an algorithmic perspective – class of averaged operators. In this paper, we introduce the new notion of conically nonexpansive operators which generalize nonexpansive mappings. We characterize averaged operators as being resolvents of comonotone operators under appropriate scaling. As a consequence, we characterize the proximal point mappings associated with hypoconvex functions as cocoercive operators, or equivalently; as displacement mappings of conically nonexpansive operators. Several examples illustrate our analysis and demonstrate tightness of our results.

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1 Introduction

In this paper, we assume that

X is a real Hilbert space,

with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Monotone operators form a beautiful class of operators that play a crucial role in modern optimization. This class includes subdifferential operators of proper lower semicontinuous convex functions as well as matrices with positive semidefinite symmetric part. (For detailed discussions on monotone operators and the connection to optimization problems, we refer the reader to [2], [5], [6], [7], [10], [11], [20], [25], [26], [27], [31], [32], and the references therein.)

The correspondence between the maximal monotonicity of an operator and the firm nonexpansiveness of its *resolvent* is of central importance from an algorithmic perspective: to find a critical point of the former, iterate the later!

Indeed, firmly nonexpansive operators belong to the more general and pleasant class of *averaged* operators. Let $x_0 \in X$ and let $T: X \to X$ be averaged. Thanks to the Krasnosel'skiĭ–Mann iteration (see [17], [18] and also [2, Theorem 5.14]), the sequence $(T^n x_0)_{n \in \mathbb{N}}$ converges weakly to a fixed point of T. When T is the *proximal mapping* associated with a proper lower semicontinuous convex function f, the set of fixed points of T is the set of critical point of f; equivalently the set of minimizers of f. In fact, iterating T is this case produces the famous proximal point algorithm, see [24]. The main goal of this paper is to answer the question: Can we explore a new correspondence between a set-valued operator and its resolvent which generalizes the fundamental correspondence between monotone operators and firmly nonexpansive mappings (see Fact 2.1)? Our approach relies on the new notion of conically nonexpansive operators as well as the notions of ρ -monotonicity (respectively ρ -comonotonicity) which, depending on the value of ρ , reduce to strong monotonicity, monotonicity or hypomonotonicity (respectively).

Although some correspondences between a monotone operator ($\rho \ge 0$) and its resolvent have been established in [3], our analysis here not only provides more quantifications and but also goes beyond monotone operators. We now summarize the three main results of this paper:

R1 We show that, when $\rho > -1$, the resolvent of a ρ -monotone operator as well as the resolvent of its inverse are single-valued and have full domain. This allows us to extend the classical theorem by Minty (see Fact 2.2) to this class of operators (see Theorem 2.16).

- **R2** We characterize conically nonexpansive operators (respectively averaged operators and nonexpansive operators) to be resolvents of ρ -comonotone operators with $\rho > -1$ (respectively $\rho > -\frac{1}{2}$ and $\rho \ge -\frac{1}{2}$) (see Corollary 3.10 and also Table 1).
- **R3** As a consequence of **R2**, we obtain a novel characterization of the proximal point mapping associated with a *hypoconvex* function¹ (under appropriate scaling of the function) to be a conically nonexpansive mapping, or equivalently, the displacement mapping of a coccercive operator (see Theorem 6.4).

The remainder of this paper is organized as follows. Section 2 is devoted to the study of the properties of ρ -monotone and ρ -comonotone operators. In Section 3, we provide a characterization of averaged operators as resolvents of ρ -comonotone operators. Section 4 provides useful correspondences between an operators and its resolvent as well as its reflected resolvent. In Section 5, we focus on ρ -monotone and ρ -comonotone linear operators. In the final Section 6, we establish the connection to hypoconvex functions.

The notation we use is standard and follows, e.g., [2] or [25].

2 ρ -monotone and ρ -comonotone operators

Let $A: X \Rightarrow X$. Recall that the *resolvent* of A is $J_A = (\mathrm{Id} + A)^{-1}$ and the *reflected resolvent* of A is $R_A = 2J_A - \mathrm{Id}$, where $\mathrm{Id}: X \to X: x \mapsto x$. The *graph* of A is $\mathrm{gra} A = \{(x, u) \in X \times X \mid u \in Ax\}$. Let $T: X \to X$ and let $\alpha \in]0, 1[$. Recall that

- (i) *T* is *nonexpansive* if $(\forall (x, y) \in X \times X) ||Tx Ty|| \le ||x y||$.
- (ii) *T* is α -averaged if there exists a nonexpansive operator $N: X \to X$ such that $T = (1 \alpha) \operatorname{Id} + \alpha N$; equivalently, $(\forall (x, y) \in X \times X)$ we have

$$(1-\alpha)\|(\mathrm{Id}-T)x - (\mathrm{Id}-Ty)\|^2 \le \alpha(\|x-y\|^2 - \|Tx-Ty\|^2).$$
(1)

(iii) *T* is *firmly nonexpansive* if *T* is $\frac{1}{2}$ -averaged. Equivalently, if $(\forall (x, y) \in X \times X)$ $||Tx - Ty||^2 + ||(Id - T)x - (Id - T)y||^2 \le ||x - y||^2$.

We begin this section by stating the following two useful facts.

Fact 2.1. (see, e.g., [13, Theorem 2]) Let *D* be a nonempty subset of *X*, let $T: D \to X$, and set $A = T^{-1} - \text{Id}$. Then $T = J_A$. Moreover, the following hold:

¹This is also known as *weakly convex* function.

- (i) *T* is firmly nonexpansive if and only if *A* is monotone.
- (ii) *T* is firmly nonexpansive and D = X if and only if *A* is maximally monotone.

Fact 2.2 (Minty's Theorem). [19] (see also [2, Theorem 21.1]) *Let* $A : X \Rightarrow X$ *be monotone. Then*

$$\operatorname{gra} A = \{ (J_A x, (\operatorname{Id} - J_A) x) \mid x \in \operatorname{ran}(\operatorname{Id} + A) \}.$$
(2)

Moreover,

A is maximally monotone
$$\Leftrightarrow$$
 ran $(\mathrm{Id} + A) = X.$ (3)

Definition 2.3. *Let* $A : X \rightrightarrows X$ *and let* $\rho \in \mathbb{R}$ *. Then*

(i) A is ρ -monotone if $(\forall (x, u) \in \operatorname{gra} A)$ $(\forall (y, v) \in \operatorname{gra} A)$ we have

$$\langle x - y, u - v \rangle \ge \rho \|x - y\|^2.$$
(4)

(ii) A is maximally ρ -monotone if A is ρ -monotone and there is no ρ -monotone operator B: X \Rightarrow X such that gra B properly contains gra A, i.e., for every $(x, u) \in X \times X$,

$$(x,u) \in \operatorname{gra} A \iff (\forall (y,v) \in \operatorname{gra} A) \langle x-y,u-v \rangle \ge \rho ||x-y||^2.$$
 (5)

(iii) A is ρ -comonotone if $(\forall (x, u) \in \operatorname{gra} A)$ $(\forall (y, v) \in \operatorname{gra} A)$ we have

$$\langle x - y, u - v \rangle \ge \rho \|u - v\|^2.$$
(6)

(iv) A is maximally ρ -comonotone if A is ρ -comonotone and there is no ρ -comonotone operator B: X \Rightarrow X such that gra B properly contains gra A, i.e., for every $(x, u) \in X \times X$,

$$(x,u) \in \operatorname{gra} A \Leftrightarrow (\forall (y,v) \in \operatorname{gra} A) \langle x-y,u-v \rangle \ge \rho \|u-v\|^2.$$
 (7)

Some comments are in order.

Remark 2.4.

- (i) When $\rho = 0$, both ρ -monotonicity of A and ρ -comonotonicity of A reduce to the monotonicity of A; equivalently to the monotonicity of A^{-1} .
- (ii) When $\rho < 0$, ρ -monotonicity is known as ρ -hypomonotonicity, see [25, Example 12.28] and [7, Definition 6.9.1]. In this case, the ρ -comonotonicity is also known as ρ -cohypomonotonicity (see [12, Definition 2.2]).
- (iii) In passing, we point out that when $\rho > 0$, ρ -monotonicity of A reduces to ρ -strong monotonicity of A, while ρ -comonotonicity of A reduces to ρ -cocoercivity² of A.

²Let $\beta > 0$ and let $T: X \to X$. Recall that T is β -cocoercive if βT is firmly nonexpansive, i.e., $(\forall (x,y) \in X \times X) \ \langle x - y, Tx - Ty \rangle \ge \beta \|Tx - Ty\|^2$.

Unlike classical monotonicity, ρ -comonotonicity of A is *not* equivalent to ρ -comonotonicity of A^{-1} . Instead, we have the following correspondences.

Lemma 2.5. Let $A: X \rightrightarrows X$ and let $\rho \in \mathbb{R}$. The following are equivalent:

- (i) A is ρ -comonotone.
- (ii) $A^{-1} \rho$ Id is monotone.
- (iii) A^{-1} is ρ -monotone, i.e., $(\forall (x, u) \in \operatorname{gra} A^{-1}) (\forall (y, v) \in \operatorname{gra} A^{-1}) \langle x y, u v \rangle \ge \rho \|x y\|^2$.

Proof. "(i) \Rightarrow (ii)": Let {(x, u), (y, v)} $\subseteq X \times X$. Then {(x, u), (y, v)} \subseteq gra($A^{-1} - \rho$ Id) \Leftrightarrow [$u \in A^{-1}x - \rho x$ and $v \in A^{-1}y - \rho y$] \Leftrightarrow {($x, u + \rho x$), ($y, v + \rho y$)} \subseteq gra A^{-1} \Leftrightarrow {($u + \rho x, x$), ($v + \rho y, y$)} \subseteq gra $A \Rightarrow \langle x - y, u - v + \rho(x - y) \rangle \ge \rho ||x - y||^2 \Leftrightarrow$ $\rho ||x - y||^2 + \langle x - y, u - v \rangle \ge \rho ||x - y||^2 \Leftrightarrow \langle u - v, x - y \rangle \ge 0$.

"(ii) \Rightarrow (iii)": Let $\{(x, u), (y, v)\} \subseteq \text{gra } A^{-1}$. Then $\{(x, u - \rho x), (y, v - \rho y)\} \subseteq \text{gra}(A^{-1} - \rho \operatorname{Id})$. Hence $\langle x - y, u - v - \rho(x - y)\rangle \ge 0$; equivalently $\langle x - y, u - v \rangle \ge \rho \|x - y\|^2$.

 $\begin{array}{ll} \text{``(iii)} \Rightarrow \text{(i)'': Let } \{(x,u),(y,v)\} &\subseteq X \times X. \text{ Then } \{(x,u),(y,v)\} &\subseteq \operatorname{gra} A \Leftrightarrow \{(u,x),(v,y)\} \subseteq \operatorname{gra} A^{-1} \Rightarrow \langle x-y,u-v \rangle \geq \rho \|u-v\|^2. \end{array}$

Lemma 2.6. Let $A: X \rightrightarrows X$ and let $\rho \in \mathbb{R}$. Then the following hold:

- (i) gra $A = \{(u + \rho x, x) \mid (x, u) \in \operatorname{gra}(A^{-1} \rho \operatorname{Id})\}.$
- (ii) $\operatorname{gra}(A^{-1} \rho \operatorname{Id}) = \{(u, x \rho u) \mid (x, u) \in \operatorname{gra} A\}.$

Proof. (i): Let $(x, u) \in X \times X$. Then $(x, u) \in \text{gra}(A^{-1} - \rho \operatorname{Id}) \Leftrightarrow u \in A^{-1}x - \rho x \Leftrightarrow u + \rho x \in A^{-1}x \Leftrightarrow x \in A(u + \rho x) \Leftrightarrow (u + \rho x, x) \in \operatorname{gra} A$. This proves " \supseteq " in (i). The opposite inclusion can be proved similarly. (ii): The proof proceeds similar to that of (i).

Lemma 2.7. Let $A: X \rightrightarrows X$ and let $\rho \in \mathbb{R}$. The following are equivalent:

- (i) A is maximally ρ -comonotone.
- (ii) $A^{-1} \rho$ Id is maximally monotone.

Proof. Note that Lemma 2.5 implies that *A* is ρ -comonotone $\Leftrightarrow A^{-1} - \rho \operatorname{Id}$ is monotone. "(i) \Rightarrow (ii)": Let $(y,v) \in X \times X$. Then (y,v) is monotonically related to $\operatorname{gra}(A^{-1} - \rho \operatorname{Id}) \Leftrightarrow (\forall (x,u) \in \operatorname{gra}(A^{-1} - \rho \operatorname{Id})) \langle x - y, u - v \rangle \geq 0 \Leftrightarrow$

 $(\forall (x, u) \in \operatorname{gra}(A^{-1} - \rho \operatorname{Id})) \langle x - y, u - v \rangle + \rho ||x - y||^2 \ge \rho ||x - y||^2 \Leftrightarrow (\forall (x, u) \in \operatorname{gra}(A^{-1} - \rho \operatorname{Id})) \langle x - y, u + \rho x - (v + \rho y) \rangle \ge \rho ||x - y||^2$. Because the last inequality holds for all $(x, u) \in \operatorname{gra}(A^{-1} - \rho \operatorname{Id})$, the parametrization of gra *A* given in Lemma 2.6(i) and the *maximal* ρ -component of *A* imply that $(v + \rho y, y) \in \operatorname{gra} A$. Therefore, by Lemma 2.6(ii), $(y, v) \in \operatorname{gra}(A^{-1} - \rho \operatorname{Id})$.

"(ii) \Rightarrow (i)": Let $(y,v) \in X \times X$. Then (y,v) is ρ -comonotonically related to gra A $\Leftrightarrow (\forall (x,u) \in \text{gra } A) \langle x - y, u - v \rangle \geq \rho ||u - v||^2 \Leftrightarrow (\forall (x,u) \in \text{gra } A) \langle x - \rho u - (y - \rho v), u - v \rangle \geq 0$. It follows from Lemma 2.6(ii) and the *maximal* monotonicity of $A^{-1} - \rho$ Id that $(v, y - \rho v) \in \text{gra}(A^{-1} - \rho$ Id), equivalently, using Lemma 2.6(i), $(y,v) \in \text{gra } A$.

Remark 2.8. Note that when $\rho < 0$, the (maximal) monotonicity of $A^{-1} - \rho$ Id is equivalent to the (maximal) monotonicity of the Yosida approximation $(A^{-1} - \rho \operatorname{Id})^{-1}$. Such a characterization is presented in [7, Proposition 6.9.3].

Proposition 2.9. Let $A: X \Rightarrow X$ be maximally ρ -comonotone where $\rho > -1$. Then $ran(Id + A^{-1}) = X$.

Proof. By Lemma 2.7, $A^{-1} - \rho \operatorname{Id}$ is maximally monotone. Consequently, because $1 + \rho > 0$, the operator $\frac{1}{1+\rho}(A^{-1} - \rho \operatorname{Id})$ is maximally monotone. Applying (3) to $\frac{1}{1+\rho}(A^{-1} - \rho \operatorname{Id})$ we have $\operatorname{ran}(\operatorname{Id} + A^{-1}) = \operatorname{ran}((1+\rho)\operatorname{Id} + (A^{-1} - \rho \operatorname{Id})) = (1 + \rho)\operatorname{ran}(\operatorname{Id} + \frac{1}{1+\rho}(A^{-1} - \rho \operatorname{Id})) = (1 + \rho)X = X.$

Proposition 2.10. *Let* $A : X \rightrightarrows X$ *. Then the following hold:*

- (i) $J_{A^{-1}} = \text{Id} J_A$.
- (ii) $\operatorname{ran}(\operatorname{Id} + A^{-1}) = \operatorname{dom}(\operatorname{Id} J_A) = \operatorname{ran}(\operatorname{Id} + A).$

Proof. (i): This follows from [2, Proposition 23.7(ii) and Definition 23.1]. (ii): Using (i), we have $\operatorname{ran}(\operatorname{Id} + A^{-1}) = \operatorname{dom}(\operatorname{Id} + A^{-1})^{-1} = \operatorname{dom} J_{A^{-1}} = \operatorname{dom}(\operatorname{Id} - J_A) = (\operatorname{dom} \operatorname{Id}) \cap (\operatorname{dom} J_A) = \operatorname{dom} J_A = \operatorname{ran}(\operatorname{Id} + A).$

Corollary 2.11 (surjectivity of Id + A and Id + A⁻¹). Let $A: X \Rightarrow X$ be maximally ρ -comonotone where $\rho > -1$. Then

$$\operatorname{dom} J_A = \operatorname{ran}(\operatorname{Id} + A) = X,\tag{8}$$

and

$$\operatorname{dom}(\operatorname{Id} - J_A) = \operatorname{ran}(\operatorname{Id} + A^{-1}) = X.$$
(9)

Proof. Combine Proposition 2.9 and Proposition 2.10(i)&(ii).

Proposition 2.12 (single-valuedness of the resolvent). Let $A: X \Rightarrow X$ be ρ comonotone where $\rho > -1$. Then $J_A = (Id + A)^{-1}$ and $J_{A^{-1}} = Id - J_A$ are at most
single-valued.

Proof. Let $x \in \text{dom } J_A = \text{ran}(\text{Id} + A)$ and let $(u, v) \in X \times X$. Then $\{u, v\} \subseteq J_A x$ $\Leftrightarrow [x - u \in Au \text{ and } x - v \in Av] \Rightarrow \langle (x - u) - (x - v), u - v \rangle \ge \rho ||u - v||^2 \Leftrightarrow -||u - v||^2 \ge \rho ||u - v||^2$. Since $\rho > -1$, the last inequality implies that u = v. Now combine with Proposition 2.10(i).

Corollary 2.13 (See also [23, Proposition 3.4]). Let $A: X \Rightarrow X$ be maximally ρ comonotone where $\rho > -1$. Then $J_A = (\mathrm{Id} + A)^{-1}$ and $J_{A^{-1}} = \mathrm{Id} - J_A$ are single-valued
and dom $J_A = \mathrm{dom} J_{A^{-1}} = X$.

In Example 2.14 below, we illustrate that the assumption that $\rho > -1$ is critical in the conclusion of Corollary 2.11 and Proposition 2.12.

Example 2.14. Suppose that $X \neq \{0\}$. Let C be a nonempty closed convex subset of X, let $r \in \mathbb{R}_+$, set $B = -\operatorname{Id} - rP_C$, set $A = B^{-1}$ and set $\rho = -(1+r) \leq -1$. Then the following hold:

- (i) $B \rho$ Id is maximally monotone.
- (ii) A is maximally ρ -comonotone.
- (iii) $\operatorname{ran}(\operatorname{Id} + A) = \operatorname{ran}(\operatorname{Id} + A^{-1}) = (\rho + 1)C = -rC.$
- (iv) Id +A is surjective $\Leftrightarrow [C = X \text{ and } r > 0].$

(v) J_A is at most single-valued $\Leftrightarrow J_{A^{-1}}$ is at most single-valued $\Leftrightarrow [C = X \text{ and } r > 0]$.

Proof. (i): Indeed, $B - \rho \operatorname{Id} = -\operatorname{Id} - rP_C + (1 + r) \operatorname{Id} = r(\operatorname{Id} - P_C)$. It follows from [2, Example 23.4 & Proposition 23.11(i)] that $\operatorname{Id} - P_C$ is maximally monotone. Because $r \ge 0$, the operator $B - \rho \operatorname{Id} = r(\operatorname{Id} - P_C)$ is maximally monotone as well.

(ii): Combine (i) and Lemma 2.7.

(iii): The first identity is Proposition 2.10(ii). Now $ran(Id + A^{-1}) = ran(Id + B) = ran(-rP_C) = -r ran P_C = -rC = (\rho + 1)C$.

(iv): This is a direct consequence of (iii).

(v): The first equivalence follows from Proposition 2.10(i). Note that $[r = 0 \text{ or } C = \{0\}] \Leftrightarrow rC = \{0\} \Leftrightarrow rP_C \equiv 0 \Leftrightarrow B = -\operatorname{Id} \Leftrightarrow \operatorname{gra} J_{A^{-1}} = \operatorname{gra} J_B = \{0\} \times X$. Now suppose that r > 0. Then $J_{A^{-1}} = J_B = (\operatorname{Id} + B)^{-1} = (-rP_C)^{-1} = (\operatorname{Id} + N_C) \circ (-r^{-1}\operatorname{Id})$ which is at most single-valued $\Leftrightarrow C = X$, by e.g., [2, Theorem 7.4].

Proposition 2.15. Let $A: X \implies X$ be ρ -comonotone, where $\rho > -1$, and such that ran(Id + A) = X. Then A is maximally ρ -comonotone.

Proof. Let $(x, u) \in X \times X$ such that $(\forall (y, v) \in \operatorname{gra} A)$

$$\langle x - y, u - v \rangle \ge \rho \|u - v\|^2.$$
⁽¹⁰⁾

It follows from the surjectivity of Id +*A* that there exists $(y, v) \in X \times X$ such that $v \in Ay$ and $x + u = y + v \in (\text{Id} + A)y$. Consequently, (10) implies that $\rho ||u - v||^2 \le \langle x - y, u - v \rangle = \langle -(u - v), u - v \rangle = -||u - v||^2$. Hence, because $\rho > -1$, we have u = v and thus x = y which proves the maximality of *A*.

Theorem 2.16 (Minty parametrization). *Let* $A : X \Rightarrow X$ *be* ρ *-comonotone where* $\rho > -1$ *. Then*

$$\operatorname{gra} A = \left\{ (J_A x, (\operatorname{Id} - J_A) x) \mid x \in \operatorname{ran}(\operatorname{Id} + A) \right\}.$$
(11)

Moreover, A is maximally ρ -comonotone \Leftrightarrow ran(Id + A) = X, in which case

$$\operatorname{gra} A = \left\{ (J_A x, (\operatorname{Id} - J_A) x) \mid x \in X \right\}.$$
(12)

Proof. Let $(x, u) \in X \times X$. In view of Proposition 2.12 we have $(x, u) \in \text{gra } A \Leftrightarrow u \in Ax \Leftrightarrow x + u \in x + Ax = (\text{Id} + A)x \Leftrightarrow x = J_A(x + u) \Leftrightarrow [z := x + u \in \text{ran}(\text{Id} + A), x = J_Az \text{ and } u = x + u - x = x + u - J_A(x + u) = (\text{Id} - J_A)z]$. The equivalence of maximal ρ -comonotonicity of A and the surjectivity of Id + A follows from combining Corollary 2.11 and Proposition 2.15.

Corollary 2.17. *Suppose that* $A: X \rightrightarrows X$ *is maximally* ρ *-comonotone where* $\rho > -1$ *and let* $(x, u) \in X \times X$ *. Then the following hold:*

- (i) $(x, u) \in \operatorname{gra} J_A \Leftrightarrow (u, x u) \in \operatorname{gra} A.$
- (ii) $(x, u) \in \operatorname{gra} R_A \Leftrightarrow \left(\frac{1}{2}(x+u), \frac{1}{2}(x-u)\right) \in \operatorname{gra} A.$

Proof. Let $(x, u) \in X \times X$ and note that in view of Proposition 2.12 and Theorem 2.16 $J_A: X \to X$ and consequently $R_A: X \to X$ are single-valued.

(i): We have $(x, u) \in \text{gra } J_A \Leftrightarrow u = J_A x \Leftrightarrow x - u = (\text{Id} - J_A)x$. Now use Theorem 2.16.

(ii): We have $(x, u) \in \text{gra } R_A \Leftrightarrow u = R_A x = 2J_A x - x \Leftrightarrow x + u = 2J_A x \Leftrightarrow J_A x = \frac{1}{2}(x+u) \Leftrightarrow x - J_A x = x - \frac{1}{2}(x+u) = \frac{1}{2}(x-u) \Leftrightarrow (\frac{1}{2}(x+u), \frac{1}{2}(x-u)) \in \text{gra } A$, where the last equivalence follows from Theorem 2.16.

3 *ρ*-comonotonicity and averagedness

We start this section with the following definition.

Definition 3.1. Let $T: X \to X$ and let $\alpha \in]0, +\infty[$. Then T is α -conically nonexpansive *if there exists a nonexpansive operator* $N: X \to X$ such that $T = (1 - \alpha) \operatorname{Id} + \alpha N$.

Remark 3.2. In view of Definition 3.1, it is clear that T is α -averaged if and only if [T α -conically nonexpansive and $\alpha \in [0, 1[]$. Similarly, T is nonexpansive if and only if T 1-conically nonexpansive.

The proofs of the next two results are straightforward and hence omitted.

Lemma 3.3. Let $T: X \to X$ and let $\alpha \in [0, +\infty[$. Then

T is
$$\alpha$$
-conically nonexpansive \Leftrightarrow Id $-T$ is $\frac{1}{2\alpha}$ -cocoercive. (13)

Lemma 3.4. Let *D* be a nonempty subset of *X*, let $T: D \to X$, let $N: D \to X$, let $\alpha \in [1, +\infty[$ and set $T = (1 - \alpha) \operatorname{Id} + \alpha N$. Suppose that $N: D \to X$ is nonexpansive. Then $(\forall (x, y) \in D \times D)$ we have

$$||Tx - Ty|| \le (2\alpha - 1)||x - y||, \tag{14}$$

i.e., *T* is Lipschitz with constant $2\alpha - 1$.

One can directly verify the following result.

Lemma 3.5. *Let* $(x, y) \in X \times X$ *and let* $\alpha \in \mathbb{R}$ *. Then*

$$\alpha^{2} \|x\|^{2} - \|(\alpha - 1)x + y\|^{2} = 2\alpha \langle x - y, y \rangle - (1 - 2\alpha) \|x - y\|^{2}.$$
 (15)

Lemma 3.6. Let D be a nonempty subset of X, let $N: D \to X$, let $\alpha \in \mathbb{R}$ and set $T = (1 - \alpha) \operatorname{Id} + \alpha N$. Then N is nonexpansive if and only if $(\forall (x, y) \in D \times D)$ we have

$$2\alpha \langle Tx - Ty, (\mathrm{Id} - T)x - (\mathrm{Id} - T)y \rangle \ge (1 - 2\alpha) \| (\mathrm{Id} - T)x - (\mathrm{Id} - T)y \|^2.$$
(16)

Proof. Let $(x, y) \in D \times D$. Applying Lemma 3.5 with (x, y) replaced by (x - y, Tx - Ty), we learn that

$$2\alpha \langle Tx - Ty, (\mathrm{Id} - T)x - (\mathrm{Id} - T)y \rangle - (1 - 2\alpha) \| (\mathrm{Id} - T)x - (\mathrm{Id} - T)y \|^2 \quad (17a)$$

$$= \alpha^{2} \|x - y\|^{2} - \|(\alpha - 1)(x - y) + (1 - \alpha)(x - y) + \alpha(Nx - Ny)\|^{2}$$
(17b)

$$= \alpha^{2} (\|x - y\|^{2} - \|Nx - Ny\|^{2}).$$
(17c)

Now *N* is nonexpansive $\Leftrightarrow ||x - y||^2 - ||Nx - Ny||^2 \ge 0$ and the conclusion directly follows.

We now provide new characterizations of averaged and nonexpansive operators.

Corollary 3.7. *Let* D *be a nonempty subset of* X*, let* $T: D \to X$ *, let* $\alpha \in [0, +\infty[$ *and let* $(x, y) \in D \times D$ *. Then the following hold:*

- (i) T is nonexpansive $\Leftrightarrow 2\langle Tx Ty, (Id T)x (Id T)y \rangle \ge \|(Id T)x (Id T)y\|^2$.
- (ii) *T* is α -conically nonexpansive $\Leftrightarrow 2\alpha \langle Tx Ty, (\mathrm{Id} T)x (\mathrm{Id} T)y \rangle \geq (1 2\alpha) \| (\mathrm{Id} T)x (\mathrm{Id} T)y \|^2$.

Proof. (i): Apply Lemma 3.6 with $\alpha = 1$.

(ii): A direct consequence of Lemma 3.6.

Proposition 3.8. Let D be a nonempty subset of X, let $T: D \to X$, let $\alpha \in [0, +\infty[$, set $A = T^{-1} - \text{Id}$ and set $N = \frac{1}{\alpha}T - \frac{1-\alpha}{\alpha}\text{Id}$, *i.e.*, $T = J_A = (1-\alpha)\text{Id} + \alpha N$. Then the following hold:

- (i) *T* is α -conically nonexpansive $\Leftrightarrow N$ is nonexpansive $\Leftrightarrow A$ is $(\frac{1}{2\alpha} 1)$ -comonotone.
- (ii) [*T* is α -conically nonexpansive and D = X] \Leftrightarrow [*N* is nonexpansive and D = X] \Leftrightarrow *A* is maximally $(\frac{1}{2\alpha} - 1)$ -comonotone.

Proof. (i): The first equivalence is Definition 3.1. We now turn to the second equivalence. " \Rightarrow ": Let $\{(x,u), (y,v)\} \subseteq \text{gra } A$. Then $(x,u) = (T(x + u), (\mathrm{Id} - T)(x + u))$ and likewise $(y,v) = (T(y + v), (\mathrm{Id} - T)(y + v))$. It follows from Lemma 3.6 applied with (x, y) replaced by (x + u, y + v) that $2\alpha\langle x - y, u - v\rangle \geq (1 - 2\alpha) ||u - v||^2$. Since $\alpha > 0$, the conclusion follows by dividing both sides of the last inequality by 2α . " \Leftarrow ": Using Theorem 2.16, we learn that $(\forall (x, y) \in D \times D) \{(Tx, (\mathrm{Id} - T)x), (Ty, (\mathrm{Id} - T)y)\} \subseteq \mathrm{gra} A$ and hence $\langle Tx - Ty, (\mathrm{Id} - T)x - (\mathrm{Id} - T)y \rangle \geq (1 - 2\alpha) ||(\mathrm{Id} - T)x - (\mathrm{Id} - T)y||^2$. Thus $2\alpha\langle Tx - Ty, (\mathrm{Id} - T)x - (\mathrm{Id} - T)y \rangle \geq (1 - 2\alpha) ||(\mathrm{Id} - T)x - (\mathrm{Id} - T)y||^2$. Now use Lemma 3.6.

(ii): Note that dom $N = \text{dom } T = \text{ran} T^{-1} = \text{ran}(\text{Id} + A)$. Now combine (i) and Theorem 2.16.

Proposition 3.9. Let *D* be a nonempty subset of *X*, let $T: D \to X$, let $\alpha \in [0, +\infty[$, set $A = T^{-1} - \text{Id}$, *i.e.*, $T = J_A$, and set $\rho = \frac{1}{2\alpha} - 1 > -1$. Then the following equivalences hold:

- (i) *T* is α -conically nonexpansive $\Leftrightarrow A$ is ρ -comonotone.
- (ii) [*T* is α -conically nonexpansive and D = X] $\Leftrightarrow A$ is maximally ρ -comonotone.

- (iii) *T* is nonexpansive $\Leftrightarrow A$ is $\left(-\frac{1}{2}\right)$ -comonotone.
- (iv) [*T* is nonexpansive and D = X] $\Leftrightarrow A$ is maximally $\left(-\frac{1}{2}\right)$ -comonotone.

If we assume that $\alpha \in [0, 1[$ *, equivalently,* $\rho > -\frac{1}{2}$ *, then we additionally have:*

- (v) *T* is α -averaged \Leftrightarrow *A* is ρ -comonotone.
- (vi) [*T* is α -averaged and D = X] $\Leftrightarrow A$ is maximally ρ -monotone.

Proof. (i)&(ii): This follows from Proposition 3.8(i)&(ii). (iii)–(vi): Combine (i) and (ii) with Remark 3.2. ■

Corollary 3.10. (The characterization corollary). Let $T: X \to X$. Then the following *hold:*

- (i) *T* is nonexpansive if and only if it is the resolvent of a maximally $\left(-\frac{1}{2}\right)$ -comonotone operator *A*: *X* \Rightarrow *X*.
- (ii) Let $\alpha \in [0, +\infty[$. Then *T* is α -conically nonexpansive if and only if it is the resolvent of a ρ -comonotone operator $A: X \Longrightarrow X$, where $\rho = \frac{1}{2\alpha} 1 > -1$ (i.e., $\alpha = \frac{1}{2(\rho+1)}$).
- (iii) Let $\alpha \in [0, 1[$. Then T is α -averaged if and only if it is the resolvent of a ρ -comonotone operator $A: X \rightrightarrows X$ where $\rho = \frac{1}{2\alpha} 1 > -\frac{1}{2}$ (i.e., $\alpha = \frac{1}{2(\rho+1)}$).

Example 3.11. Suppose that U is a closed linear subspace of X and set $N = 2P_U - \text{Id}$. Let $\alpha \in [0, +\infty[$, set $T_{\alpha} = (1 - \alpha) \text{Id} + \alpha N$, and set $A_{\alpha} = (T_{\alpha})^{-1} - \text{Id}$. Then for every $\alpha \in [0, +\infty[$, T_{α} is α -conically nonexpansive and

$$A_{\alpha} = \begin{cases} N_{U}, & \text{if } \alpha = \frac{1}{2};\\ \frac{2\alpha}{1-2\alpha} P_{U^{\perp}}, & \text{otherwise.} \end{cases}$$
(18)

Moreover, A_{α} is $(\frac{1}{2\alpha} - 1)$ -comonotone.

Proof. First note that $T_{\alpha} = (1 - \alpha) \operatorname{Id} + \alpha (2P_U - \operatorname{Id}) = (1 - 2\alpha) \operatorname{Id} + 2\alpha P_U$. The case $\alpha = \frac{1}{2}$ is clear by, e.g., [2, Example 23.4]. Now suppose that $\alpha \in [0, +\infty[\setminus \{\frac{1}{2}\}, \alpha]$ and let $y \in X$. Then $y \in A_{\alpha}x \Leftrightarrow x + y \in (\operatorname{Id} + A_{\alpha})x \Leftrightarrow x = T_{\alpha}(x + y) = (1 - 2\alpha)(x + y) + 2\alpha P_U(x + y) \Leftrightarrow x = x + y - 2\alpha(\operatorname{Id} - P_U)(x + y) \Leftrightarrow y = 2\alpha P_{U^{\perp}}(x + y) = 2\alpha P_{U^{\perp}}x + 2\alpha P_{U^{\perp}}x + 2\alpha P_{U^{\perp}}x + 2\alpha P_U(x + y)$. Therefore, $y = \frac{2\alpha}{1 - 2\alpha}P_{U^{\perp}}x$, and the conclusion follows in view of Corollary 3.10(ii).

Proposition 3.12. Let $A: X \Rightarrow X$ be such that dom $A \neq \emptyset$, let $\rho \in [-1, +\infty[$, set $D = \operatorname{ran}(\operatorname{Id} + A)$, set $T = J_A$, i.e., $A = T^{-1} - \operatorname{Id}$, and set $N = 2(\rho + 1)T - (2\rho + 1)\operatorname{Id}$, *i.e.*, $T = \frac{2\rho+1}{2(\rho+1)}\operatorname{Id} + \frac{1}{2(\rho+1)}N$. Then the following equivalences hold:

- (i) A is ρ -comonotone \Leftrightarrow N is nonexpansive.
- (ii) A is maximally ρ -comonotone \Leftrightarrow N is nonexpansive and D = X.

Proof. (i): Set $\alpha = \frac{1}{2(\rho+1)}$ and note that $\alpha > 0$. It follows from Proposition 2.12 that $T = J_A$ is single-valued. Now use Proposition 3.8(i). (ii): Combine (i) and Proposition 3.8(ii).

Proposition 3.13. Let $A: X \Rightarrow X$ be such that dom $A \neq \emptyset$, let $\rho \in [-1, +\infty[$, set $D = \operatorname{ran}(\operatorname{Id} + A)$, set $T = J_A$, i.e., $A = T^{-1} - \operatorname{Id}$, and set $\alpha = \frac{1}{2(\rho+1)}$. Then we have the following equivalences:

- (i) A is ρ -comonotone \Leftrightarrow T is $\frac{1}{2(\rho+1)}$ -conically nonexpansive.
- (ii) A is maximally ρ -comonotone \Leftrightarrow T is α -conically nonexpansive and D = X.
- (iii) A is $\left(-\frac{1}{2}\right)$ -comonotone \Leftrightarrow T is nonexpansive.
- (iv) A is maximally $\left(-\frac{1}{2}\right)$ -comonotone \Leftrightarrow T is nonexpansive and D = X.
- (v) [A is ρ -comonotone and $\rho > -\frac{1}{2}$] \Leftrightarrow T is α -averaged.
- (vi) [A is maximally ρ -monotone and $\rho > -\frac{1}{2}$] \Leftrightarrow [T is α -averaged and D = X].

Proof. (i)–(vi): Use Proposition 3.9.

Corollary 3.14. Let $A: X \rightrightarrows X$ be maximally ρ -comonotone and $\rho > -\frac{1}{2}$. Then J_A is $\frac{1}{2(\rho+1)}$ -averaged.

The following corollary provides an alternative proof to [7, Proposition 6.9.6].

Corollary 3.15. Let $A: X \rightrightarrows X$ be maximally ρ -comonotone and $\rho \ge -\frac{1}{2}$. Then zer A is closed and convex.

Proof. It is clear that zer $A = \text{Fix } J_A$. The conclusion now follows from combining [2, Corollary 4.14] and Proposition 3.13(iv).

Table 1 below summarizes the main results of this section.

ρ	Α	A^{-1}	J_A	$J_{A^{-1}}$
$\bullet \longrightarrow 0$	ρ -cocoercive \Leftrightarrow	ρ -strongly \Leftrightarrow monotone	$\frac{1}{2(\rho+1)}$ - \Leftrightarrow conically nonexpansive	$(\rho + 1)$ -cocoercive
← 0	monotone \Leftrightarrow	monotone \Leftrightarrow	firmly nonexpan- ⇔ sive	firmly nonexpan- sive
-0.5 0	ρ - \Leftrightarrow comonotone	ρ - \Leftrightarrow monotone	$\frac{1}{2(\rho+1)}$ -averaged \Leftrightarrow	$(\rho + 1)$ -cocoercive
-0.5	ρ - \Leftrightarrow comonotone		nonexpansive \Leftrightarrow	$\frac{1}{2}$ -cocoercive
← -1 -0.5	ρ - \Leftrightarrow comonotone	ρ - \Leftrightarrow monotone	$\frac{1}{2(\rho+1)}$ \Leftrightarrow conically nonexpansive	$(\rho + 1)$ -cocoercive
-1	ρ - \Leftrightarrow comonotone	$ \begin{array}{cc} \rho - & \Rightarrow \\ \text{monotone} \end{array} $	may fail to be at \Leftrightarrow most single-valued	may fail to be at most single- valued

Table 1: Properties of an operator *A* and its inverse A^{-1} along with the corresponding resolvents J_A and $J_{A^{-1}}$ respectively, for different values of $\rho \in \mathbb{R}$. Here, *A* satisfies the implication: $\{(x, u), (y, v)\} \subseteq \text{gra } A \Rightarrow \langle x - y, u - v \rangle \ge \rho ||u - v||^2$.

4 Further properties of the resolvent J_A and the reflected resolvent R_A

We start this section with the following useful lemma.

Lemma 4.1. Let $T: X \to X$, let $\alpha \in [0, 1[$. Then the following hold:

- (i) *T* is α -averaged $\Leftrightarrow 2T \text{Id} = (1 2\alpha) \text{Id} + 2\alpha N$ for some nonexpansive $N: X \to X$.
- (ii) $[T = \frac{\alpha}{2}(\mathrm{Id} + N) \text{ and } N \text{ is nonexpansive}] \Leftrightarrow -(2T \mathrm{Id}) \text{ is } \alpha \text{-averaged}^3$, in which case T is a Banach contraction with Lipschitz constant $\alpha < 1$.
- (iii) *T* is $\frac{1}{2}$ -strongly monotone $\Leftrightarrow 2T \text{Id is monotone}$.

Proof. (i): We have: *T* is α -averaged \Leftrightarrow [$T = (1 - \alpha)$ Id $+\alpha N$ and *N* is nonexpansive] \Leftrightarrow [$2T - \text{Id} = (2 - 2\alpha)$ Id $+2\alpha N - \text{Id} = (1 - 2\alpha)$ Id $+2\alpha N$ and *N* is nonexpansive].

(ii): Indeed, $[T = \frac{\alpha}{2}(\mathrm{Id} + N) \text{ and } N \text{ is nonexpansive}] \Leftrightarrow 2T - \mathrm{Id} = (\alpha - 1) \mathrm{Id} + \alpha N = -((1 - \alpha) \mathrm{Id} + \alpha(-N))$, equivalently $2T - \mathrm{Id}$ is α -negatively averaged.

(iii): We have: *T* is $\frac{1}{2}$ -strongly monotone $\Leftrightarrow T - \frac{1}{2}$ Id is monotone $\Leftrightarrow 2T -$ Id is monotone.

³This is also known as α -negatively averaged (see [14, Definition 3.7]).

Before we proceed, we recall the following useful fact (see, e.g., [2, Proposition 4.35]).

Fact 4.2. Let $T: X \to X$, let $(x, y) \in X \times X$ and let $\alpha \in]0, 1[$. Then

$$T \text{ is } \alpha \text{- averaged} \Leftrightarrow \|Tx - Ty\|^2 + (1 - 2\alpha)\|x - y\|^2 \le 2(1 - \alpha)\langle x - y, Tx - Ty\rangle.$$
(19)

Proposition 4.3. Let $\alpha \in [0, 1[$, let $\beta \in]-\frac{1}{2}$, $+\infty[$, let $A: X \rightrightarrows X$ and suppose that A is β -comonotone. Then the following hold:

- (i) A is β -comonotone $\Leftrightarrow J_A$ is $\frac{1}{2(1+\beta)}$ -averaged $\Leftrightarrow R_A = (1 \frac{1}{1+\beta}) \operatorname{Id} + \frac{1}{1+\beta} N$ for some nonexpansive $N: X \to X$.
- (ii) A is β -strongly monotone $\Leftrightarrow [J_A = \frac{1}{2(\beta+1)}(\mathrm{Id} + N) \text{ and } N \text{ is nonexpansive}] \Leftrightarrow -R_A$ is $\frac{1}{\beta+1}$ -averaged, in which case J_A is a Banach contraction with Lipschitz constant $\frac{1}{\beta+1} < 1$.
- (iii) A is nonexpansive $\Leftrightarrow J_A$ is $\frac{1}{2}$ -strongly monotone $\Leftrightarrow R_A$ is monotone.
- (iv) A is α -averaged $\Leftrightarrow R_A$ is $\frac{1-\alpha}{\alpha}$ -cocoercive.
- (v) A is firmly nonexpansive $\Leftrightarrow R_A$ is firmly nonexpansive.

Proof. Let $\{(x, u), (y, v)\} \subseteq X \times X$. Using Corollary 2.17(i), we have $\{(x, u), (y, v)\} \subseteq$ gra $J_A \Leftrightarrow \{(u, x - u), (v, y - v)\} \subseteq$ gra A, which we shall use repeatedly.

(i): Let $\{(x, u), (y, v)\} \subseteq \operatorname{gra} J_A$. We have

A is β -comonotone

$$\Leftrightarrow \beta \| (x-y) - (u-v) \|^2 \le \langle (x-y) - (u-v), u-v \rangle$$
(20a)

$$\Leftrightarrow \beta \|x - y\|^2 + \beta \|u - v\|^2 - 2\beta \langle x - y, u - v \rangle \le \langle x - y, u - v \rangle - \|u - v\|^2$$
(20b)
$$\Leftrightarrow \beta \|x - y\|^2 + (\beta + 1)\|u - v\|^2 \le (2\beta + 1)\langle x - y, u - v \rangle$$
(20c)

$$\Leftrightarrow \|u - v\|^2 + \frac{\beta}{\beta + 1} \|x - y\|^2 \le \frac{2\beta + 1}{\beta + 1} \langle x - y, u - v \rangle$$
(20d)

$$\Leftrightarrow \|u - v\|^{2} + \left(1 - \frac{1}{\beta + 1}\right) \|x - y\|^{2} \le 2\left(1 - \frac{1}{2(\beta + 1)}\right) \langle x - y, u - v \rangle$$
(20e)

$$\Leftrightarrow J_A \text{ is } \frac{1}{2(\beta+1)} \text{-averaged}, \tag{20f}$$

$$\Leftrightarrow R_A = \left(1 - \frac{1}{1+\beta}\right) \operatorname{Id} + \frac{1}{1+\beta} N \text{ for some nonexpansive } N \colon X \to X,$$
 (20g)

where the last two equivalences follow from Fact 4.2 and Lemma 4.1(i), respectively.

(ii): We start by proving the equivalence of the first and third statement. (see [14, Proposition 5.4] for " \Rightarrow " and also [22, Proposition 2.1(iii)]). Let {(*x*, *u*), (*y*, *v*)} \subseteq

gra $(-R_A)$, i.e., $\{(x, -u), (y, -v)\} \subseteq$ gra R_A . In view of Corollary 2.17(ii), this is equivalent to $\{(\frac{1}{2}(x-u), \frac{1}{2}(x+u)), (\frac{1}{2}(y-v), \frac{1}{2}(y+v))\} \subseteq$ gra A. We have

A is β -strongly monotone

$$\Leftrightarrow \langle (x-y) + (u-v), (x-y) - (u-v) \rangle \ge \beta \| (x-y) - (u-v) \|^2$$
(21a)

$$\Leftrightarrow ||x - y||^{2} - ||u - v||^{2} \ge \beta ||x - y||^{2} + \beta ||u - v||^{2} - 2\beta \langle x - y, u - v \rangle$$
(21b)

$$\Leftrightarrow 2\beta \langle x - y, u - v \rangle \ge (\beta - 1) \|x - y\|^2 + (\beta + 1) \|u - v\|^2$$
(21c)

$$\Leftrightarrow \frac{2\beta}{\beta+1} \langle x - y, u - v \rangle \ge \frac{\beta-1}{\beta+1} \|x - y\|^2 + \|u - v\|^2$$
(21d)

$$\Leftrightarrow 2\left(1-\frac{1}{\beta+1}\right)\langle x-y,u-v\rangle \ge \left(1-\frac{2}{\beta+1}\right)\|x-y\|^2 + \|u-v\|^2 \tag{21e}$$

$$\Leftrightarrow -R_A \text{ is } \frac{1}{\beta+1} \text{-averaged}, \tag{21f}$$

where the last equivalence follows from Fact 4.2. Now apply Lemma 4.1(ii) to prove the equivalence of the second and third statements in (ii).

(iii): Let $\{(x, u), (y, v)\} \subseteq \text{gra } J_A$ and note that Corollary 2.17(i) implies that $x - u \in Au, y - v \in Av, 2u - x \in (\text{Id} - A)u$ and $2v - y \in (\text{Id} - A)v$. It follows from Corollary 3.7(i) applied with (T, x, y) replaced by (A, u, v) that

A is nonexpansive
$$\Leftrightarrow \langle (x-y) - (u-v), 2(u-v) - (x-y) \rangle$$

$$\geq -\frac{1}{2} \|2(u-v) - (x-y)\|^2 \qquad (22a)$$

$$\Leftrightarrow - \|x - y\|^2 - 2\|u - v\|^2 + 3\langle x - y, u - v \rangle$$

$$\geq 2\|u - v\|^2 - 1\|x - y\|^2 + 2\langle x - y, u - v \rangle$$
(22b)

$$\geq -2\|u-v\|^{2} - \frac{1}{2}\|x-y\|^{2} + 2\langle x-y, u-v\rangle$$
(22b)

$$\Leftrightarrow \langle x - y, u - v \rangle \ge \frac{1}{2} ||x - y||^2$$
(22c)

$$\Leftrightarrow J_A \text{ is } \frac{1}{2} \text{-strongly monotone}$$
(22d)

$$\Leftrightarrow R_A \text{ is monotone,} \tag{22e}$$

where the last equivalence follows from Lemma 4.1(iii).

(iv): Let $\{(x, u), (y, v)\} \subseteq X \times X$. Using Corollary 2.17 we have $\{(x, u), (y, v)\} \subseteq$ gra $R_A \Leftrightarrow \{(\frac{1}{2}(x+u), \frac{1}{2}(x-u)), (\frac{1}{2}(y+v), \frac{1}{2}(y-v))\} \subseteq$ gra A. Let $\{(x, u), (y, v)\} \subseteq$ gra R_A . Applying Corollary 3.7(ii) with (T, x, y) replaced by $(A, \frac{1}{2}(x+u), \frac{1}{2}(y+v))$ and Remark 3.2, we learn that

A is
$$\alpha$$
-averaged $\Leftrightarrow 2\alpha \langle \frac{1}{2}((x-y) - (u-v)), u-v \rangle \geq (1-2\alpha) ||u-v||^2$ (23a)

$$\Leftrightarrow \ \alpha \langle x - y, u - v \rangle - \alpha \|u - v\|^2 \ge (1 - 2\alpha) \|u - v\|^2$$
(23b)

$$\Leftrightarrow \ \frac{\alpha}{1-\alpha} \langle x - y, u - v \rangle \ge \|u - v\|^2,$$
(23c)

equivalently R_A is $\frac{1-\alpha}{\alpha}$ -coccoercive. (v): Apply (iv) with $\alpha = \frac{1}{2}$.

Remark 4.4. *Proposition 4.3(i)* generalizes the conclusion of [14, Proposition 5.3]. Indeed, if $\beta > 0$ we have A is β -cocoercive, equivalently R_A is $\frac{1}{\beta+1}$ -averaged.

5 ρ -monotone and ρ -comonotone linear operators

Let $A \in \mathbb{R}^{n \times n}$ and set $A_s = \frac{A+A^{\mathsf{T}}}{2}$. In the following we use $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ to denote the smallest and largest eigenvalue of A, respectively, provided all eigenvalues of A are real.

Proposition 5.1. *Suppose that* $A \in \mathbb{R}^{n \times n}$ *. Then the following hold:*

- (i) A is ρ -monotone $\Leftrightarrow \lambda_{\min}(A_s) \ge \rho$.
- (ii) A is ρ -comonotone $\Leftrightarrow \lambda_{\min}(A_s \rho A^{\mathsf{T}} A) \geq 0$.

Proof. Let $x \in \mathbb{R}^n$. (i): A is ρ -monotone $\Leftrightarrow \langle x, Ax \rangle \ge \rho ||x||^2 \Leftrightarrow \langle x, (A - \rho \operatorname{Id})x \rangle \ge 0$ $\Leftrightarrow \langle x, (A - \rho \operatorname{Id})_s x \rangle \ge 0 \Leftrightarrow \langle x, (A_s - \rho \operatorname{Id})x \rangle \ge 0 \Leftrightarrow A_s - \rho \operatorname{Id} \succeq 0 \Leftrightarrow A_s \succeq \rho \operatorname{Id} \Leftrightarrow \lambda_{\min}(A_s) \ge \rho$. (ii): A is ρ -comonotone $\Leftrightarrow \langle x, Ax \rangle \ge \rho ||Ax||^2 \Leftrightarrow \langle x, (A_s - \rho A^{\mathsf{T}}A)x \rangle \ge 0 \Leftrightarrow A_s - \rho A^{\mathsf{T}}A \succeq 0 \Leftrightarrow \lambda_{\min}(A_s - \rho A^{\mathsf{T}}A) \ge 0$.

Example 5.2. Suppose that $N: X \to X$ is continuous and linear such that $N^* = -N$ and $N^2 = -\text{Id.}$ Then N is nonexpansive. Moreover, let $\lambda \in [0, 1[$, set $T_{\lambda} = (1 - \lambda) \text{Id} + \lambda N$ and set $A_{\lambda} = (T_{\lambda})^{-1} - \text{Id.}$ Then the following hold:

(i) We have

$$A_{\lambda} = \frac{\lambda}{(1-\lambda)^2 + \lambda^2} \big((1-2\lambda) \operatorname{Id} - N \big).$$
(24)

- (ii) A_{λ} is ρ -monotone with optimal $\rho = \frac{\lambda(1-2\lambda)}{\lambda^2+(1-\lambda)^2}$.
- (iii) A_{λ} is ρ -comonotone with optimal $\rho = \frac{1-2\lambda}{2\lambda}$.

Proof. Let $x \in X$. Then $||Nx||^2 = \langle Nx, Nx \rangle = \langle x, N^*Nx \rangle = \langle x, -N^2x \rangle = \langle x, x \rangle = ||x||^2$. Hence *N* is nonexpansive; in fact, *N* is an isometry. Now set

$$B_{\lambda} = \frac{\lambda}{(1-\lambda)^2 + \lambda^2} ((1-2\lambda) \operatorname{Id} - N).$$
(25)

(i): We have

$$(\mathrm{Id} + B_{\lambda})T_{\lambda} = \left(\mathrm{Id} + \frac{\lambda}{(1-\lambda)^{2} + \lambda^{2}} \left((1-2\lambda) \,\mathrm{Id} - N \right) \right) \left((1-\lambda) \,\mathrm{Id} + \lambda N \right)$$
(26a)

$$= \frac{1}{(1-\lambda)^2 + \lambda^2} \left((1-\lambda) \operatorname{Id} - \lambda N \right) \left((1-\lambda) \operatorname{Id} + \lambda N \right)$$
(26b)

$$= \frac{1}{(1-\lambda)^2 + \lambda^2} \left((1-\lambda)^2 \operatorname{Id} - \lambda^2 N^2 \right) = \operatorname{Id}.$$
(26c)

Similarly, one can show that $T_{\lambda}(\text{Id} + B_{\lambda}) = \text{Id}$ and the conclusion follows.

(ii): Using (i), we have

$$\langle x, A_{\lambda}x \rangle = \frac{\lambda}{(1-\lambda)^2 + \lambda^2} \big((1-2\lambda) \|x\|^2 - \langle Nx, x \rangle \big)$$
(27a)

$$= \frac{\lambda(1-2\lambda)}{(1-\lambda)^2 + \lambda^2} ||x||^2.$$
 (27b)

(iii): Using (i), we have

$$\|A_{\lambda}x\|^{2} = \frac{\lambda^{2}}{((1-\lambda)^{2}+\lambda^{2})^{2}} \left((1-2\lambda)^{2}\|x\|^{2}+\|Nx\|^{2}\right)$$
(28a)

$$= \frac{\lambda^2}{((1-\lambda)^2 + \lambda^2)^2} ((1-2\lambda)^2 + 1) ||x||^2.$$
 (28b)

Therefore, combining with (27b) we obtain

$$\langle x, A_{\lambda}x \rangle = \frac{(1-2\lambda)((1-\lambda)^2 + \lambda^2)}{\lambda((1-2\lambda)^2 + 1)} \cdot \frac{\lambda^2((1-2\lambda)^2 + 1)}{((1-\lambda)^2 + \lambda^2)^2} \|x\|^2$$
(29a)

$$=\frac{(1-2\lambda)((1-\lambda)^2+\lambda^2)}{\lambda((1-2\lambda)^2+1)}\|A_{\lambda}x\|^2,$$
(29b)

$$=\frac{1-2\lambda}{2\lambda}\|A_{\lambda}x\|^{2},$$
(29c)

and the conclusion follows.

6 Hypoconvex functions

In this section, we apply results in the previous sections to characterize proximal mappings of hypoconvex functions. We shall assume that $f: X \to]-\infty, +\infty]$ is a proper lower semicontinuous function minorized by a concave quadratic function: $\exists \nu \in \mathbb{R}, \beta \in \mathbb{R}, \alpha \ge 0$ such that

$$(\forall x \in X) \quad f(x) \ge -\alpha \|x\|^2 - \beta \|x\| + \nu.$$

For $\mu > 0$, the Moreau envelope of *f* is defined by

$$e_{\mu}f(x) = \inf_{y \in X} \left(f(y) + \frac{1}{2\mu} ||x - y||^2 \right),$$

and the associated proximal mapping $Prox_{\mu f}$ by

$$\operatorname{Prox}_{\mu f}(x) = \underset{y \in X}{\operatorname{argmin}} \left(f(y) + \frac{1}{2\mu} \|x - y\|^2 \right), \tag{30}$$

where $x \in X$. We shall use ∂f for the subdifferential mapping from convex analysis.

Definition 6.1. An abstract subdifferential $\partial_{\#}$ associates a subset $\partial_{\#}f(x)$ of X to f at $x \in X$, and it satisfies the following properties:

- (i) $\partial_{\#}f = \partial f$ if f is a proper lower semicontinuous convex function;
- (ii) $\partial_{\#}f = \nabla f$ if f is continuously differentiable;
- (iii) $0 \in \partial_{\#} f(x)$ if f attains a local minimum at $x \in \text{dom } f$;
- (iv) for every $\beta \in \mathbb{R}$,

$$\partial_{\#}\left(f+\beta\frac{\|\cdot-x\|^2}{2}\right) = \partial_{\#}f+\beta(\operatorname{Id}-x).$$

The Clarke–Rockafellar subdifferential, Mordukhovich subdifferential, and Fréchet subdifferential all satisfy Definition 6.1(i)–(iv), see, e.g., [8], [21, 20], so they are $\partial_{\#}$. Related but different abstract subdifferentials have been used in [1, 15, 29].

Recall that *f* is $\frac{1}{\lambda}$ -hypoconvex (see [25, 30]) if

$$f((1-\tau)x + \tau y) \le (1-\tau)f(x) + \tau f(y) + \frac{1}{2\lambda}\tau(1-\tau)\|x-y\|^2,$$
(31)

for all $(x, y) \in X \times X$ and $\tau \in]0, 1[$.

Proposition 6.2. If $f: X \to]-\infty, +\infty]$ is a proper lower semicontinuous $\frac{1}{\lambda}$ -hypoconvex *function, then*

$$\partial_{\#}f = \partial \left(f + \frac{1}{2\lambda} \|\cdot\|^2\right) - \frac{1}{\lambda} \operatorname{Id}.$$
 (32)

Consequently, for a hypoconvex function the Clarke–Rockafellar, Mordukhovich, and Fréchet subdifferential operators all coincide.

Proof. For the convex function $f + \frac{1}{2\lambda} \| \cdot \|^2$, apply Definition 6.1(i) and (iv) to obtain

$$\partial \left(f + \frac{1}{2\lambda} \| \cdot \|^2 \right) = \partial_\# \left(f + \frac{1}{2\lambda} \| \cdot \|^2 \right) = \partial_\# f + \frac{1}{\lambda} \operatorname{Id}$$

from which (32) follows.

Let f^* denote the Fenchel conjugate of f. The following result is well known in \mathbb{R}^n , see, e.g., [25, Exercise 12.61(b)(c), Example 11.26(d) and Proposition 12.19], and [30]. In fact, it also holds in a Hilbert space.

Proposition 6.3. *The following are equivalent:*

(i) f is $\frac{1}{\lambda}$ -hypoconvex.

- (ii) $f + \frac{1}{2\lambda} \|\cdot\|^2$ is convex.
- (iii) Id $+\lambda \partial_{\#} f$ is maximally monotone.
- (iv) $(\forall \mu \in]0, \lambda[) \operatorname{Prox}_{\mu f}$ is $\lambda / (\lambda \mu)$ -Lipschitz continuous with

$$\operatorname{Prox}_{\mu f} = J_{\mu \partial_{\#} f} = (\operatorname{Id} + \mu \partial_{\#} f)^{-1}.$$
(33)

(v) $(\forall \mu \in]0, \lambda[) \operatorname{Prox}_{\mu f}$ is single-valued and continuous.

Proof. "(i) \Leftrightarrow (ii)": Simple algebraic manipulations.

"(ii)⇒(iii)": As

$$\partial\left(f + \frac{1}{2\mu} \|\cdot\|^2\right) = \partial_{\#}\left(f + \frac{1}{2\mu} \|\cdot\|^2\right) = \partial_{\#}f + \frac{1}{\mu}\operatorname{Id}$$

is maximally monotone, $\text{Id} + \mu \partial_{\#} f$ is maximally monotone.

"(iii) \Rightarrow (iv)": By Definition 6.1(iii) and (iv), $y \in Prox_{\mu f}(x)$ implies that

$$0 \in \partial_{\#} \left(f(y) + \frac{1}{2\mu} \|y - x\|^2 \right) = \partial_{\#} f(y) + \frac{1}{\mu} (y - x).$$

Thus, one has

$$(\forall x \in X) \operatorname{Prox}_{\mu f}(x) \subseteq (\operatorname{Id} + \mu \partial_{\#} f)^{-1}(x).$$
 (34)

Using

$$\mathrm{Id} + \mu \partial_{\#} f = \frac{\lambda - \mu}{\lambda} \Big(\mathrm{Id} + \frac{\mu}{\lambda - \mu} (\mathrm{Id} + \lambda \partial_{\#} f) \Big)$$

yields

$$(\mathrm{Id} + \mu \partial_{\#} f)^{-1} = J_A \circ \left(\frac{\lambda}{\lambda - \mu} \mathrm{Id}\right),$$

where $A = \frac{\mu}{\lambda - \mu} (\text{Id} + \lambda \partial_{\#} f)$ is maximally monotone by the assumption. Since J_A is nonexpansive on X, $(\text{Id} + \mu \partial_{\#} f)^{-1}$ is $\lambda / (\lambda - \mu)$ -Lipschitz. Together with (34), we obtain $\text{Prox}_{\mu f} = (\text{Id} + \mu \partial_{\#} f)^{-1}$.

"(iv) \Rightarrow (v)": Clear.

"(v) \Rightarrow (ii)": Let $x \in X$ and let $\mu \in [0, \lambda[$. We have

$$e_{\mu}f(x) = \frac{1}{2\mu} \|x\|^2 - \left(f + \frac{1}{2\mu} \|\cdot\|^2\right)^* \left(\frac{x}{\mu}\right),\tag{35}$$

and e_{μ} is locally Lipschitz, see, e.g., [16, Proposition 3.3(b)]. By [4, Proposition 5.1], (v) implies that $e_{\mu}f$ is Fréchet differentiable with $\nabla e_{\mu}f = \mu^{-1}(\text{Id} - \text{Prox}_{\mu}f)$. Then

 $(f + \frac{1}{2\mu} \| \cdot \|^2)^*$ is Fréchet differentiable by (35). It follows from [28, Theorem 1] that $f + \frac{1}{2\mu} \| \cdot \|^2$ is convex. Since this hold for every $\mu \in]0, \lambda[$, (ii) follows.

We now provide a new refined characterization of hypoconvex functions in terms of the cocoercivity of their proximal operators; equivalently, of the conical nonexpansiveness of the displacement mapping of their proximal operators.

Theorem 6.4. *Let* $\mu \in]0, \lambda[$ *. Then the following are equivalent.*

- (i) f is $\frac{1}{\lambda}$ -hypoconvex.
- (ii) Id Prox_{μf} is $\frac{\lambda}{2(\lambda-\mu)}$ -conically nonexpansive.
- (iii) $\operatorname{Prox}_{\mu f}$ is $\frac{\lambda-\mu}{\lambda}$ -cocoercive.

Proof. "(i) \Leftrightarrow (ii)": Using $0 < \frac{\mu}{\lambda} < 1$ we have

$$f \text{ is } \frac{1}{\lambda} \text{-hypoconvex}$$

$$\Leftrightarrow \text{ Id } +\lambda\partial_{\#}f \text{ is maximally monotone} \qquad (by \text{ Proposition 6.3})$$

$$\Leftrightarrow \frac{\mu}{\lambda} \text{ Id } +\mu\partial_{\#}f \text{ is maximally monotone}$$

$$\Leftrightarrow \mu\partial_{\#}f \text{ is maximally } (-\frac{\mu}{\lambda}) \text{-monotone}$$

$$\Leftrightarrow (\mu\partial_{\#}f)^{-1} \text{ is maximally } (-\frac{\mu}{\lambda}) \text{-comonotone} \qquad (by \text{ Lemma 2.7})$$

$$\Leftrightarrow J_{(\mu\partial_{\#}f)^{-1}} \text{ is } \frac{\lambda}{2(\lambda-\mu)} \text{-conically nonexpansive} \qquad (by \text{ Corollary 3.10(ii)})$$

$$\Leftrightarrow \text{ Id } -J_{\mu\partial_{\#}f} \text{ is } \frac{\lambda}{2(\lambda-\mu)} \text{-conically nonexpansive} \qquad (by \text{ Proposition 2.10(i)})$$

$$\Leftrightarrow \text{ Id } -\text{Prox}_{\mu f} \text{ is } \frac{\lambda}{2(\lambda-\mu)} \text{-conically nonexpansive} \qquad (by (33))$$

"(ii) \Leftrightarrow (iii)": Use Lemma 3.3.

Corollary 6.5. Suppose that $f: X \to \mathbb{R}$ is Fréchet differentiable such that ∇f is Lipschitz with a constant $1/\lambda$. Then the following hold:

- (i) Id $+\lambda \nabla f$ is maximally monotone.
- (ii) f is $\frac{1}{\lambda}$ -hypoconvex.
- (iii) $f + \frac{1}{2\lambda} \|\cdot\|^2$ is convex.
- (iv) $(\forall \mu \in]0, \lambda[)$ Prox_{*uf*} is single-valued.
- (v) $(\forall \mu \in]0, \lambda[) \operatorname{Prox}_{\mu f} is \frac{\lambda \mu}{\lambda}$ -cocoercive.

(vi) $(\forall \mu \in]0, \lambda[) \operatorname{Prox}_{\mu f} = J_{\mu \partial_{\#} f} = (\operatorname{Id} + \mu \nabla f)^{-1}.$

(vii)
$$(\forall \mu \in]0, \lambda[)$$
 Id $- \operatorname{Prox}_{\mu f}$ is $\frac{\lambda}{2(\lambda - \mu)}$ -conically nonexpansive.

Proof. Definition 6.1(ii) implies that $(\forall x \in X) \ \partial_{\#} f(x) = \{\nabla f(x)\}$. (i): Indeed, $\lambda \nabla f$ is nonexpansive. Now the conclusion follows from [2, Example 20.29]. (ii)–(vii): Combine (i) with Proposition 6.3 and Theorem 6.4.

Finally, we give two examples to illustrate our results.

Example 6.6. Suppose that $X = \mathbb{R}$. Let $\lambda > 0$ and set, for every λ , $f_{\lambda} : x \mapsto \exp(x) - \frac{1}{2\lambda}x^2$. Then f is $\frac{1}{\lambda}$ -hypoconvex by Proposition 6.3, $f'_{\lambda} : x \mapsto \exp(x) - \frac{x}{\lambda}$, and we have $\operatorname{Id} + \lambda f'_{\lambda} = \lambda \exp$ is maximally monotone. Moreover, for every $\mu \in [0, \lambda]$ we have

$$\operatorname{Prox}_{\mu f_{\lambda}}(x) = \left(\operatorname{Id} + \mu f_{\lambda}'\right)^{-1}(x) = \left(\left(1 - \frac{\mu}{\lambda}\right)\operatorname{Id} + \mu \exp\right)^{-1}(x)$$
(36a)

$$=\begin{cases} \ln(\frac{\lambda}{\mu}), & \text{if } \mu = \lambda;\\ \frac{\lambda x}{\lambda - \mu} - \text{Lambert } W(\frac{\lambda \mu \exp(\lambda x / (\lambda - \mu))}{\lambda - \mu}), & \text{if } \mu \in]0, \lambda[, \end{cases}$$
(36b)

where the first identity in (36a) follows from Corollary 6.5(vi).

Example 6.7. Let *D* be a nonempty closed convex subset of *X*, let $\lambda > 0$ and set, for every λ , $f_{\lambda} = \iota_D - \frac{1}{2\lambda} \|\cdot\|^2$. Then *f* is $\frac{1}{\lambda}$ -hypoconvex by Proposition 6.3, and $\partial_{\#}f_{\lambda} = N_D - \frac{1}{\lambda}$ Id by Proposition 6.2. Moreover, for every $\lambda > 0$, we have Id $+\lambda \partial_{\#}f_{\lambda} = N_D$ is maximally monotone. Finally, using (33) and [2, Example 23.4] we have for every $\mu \in]0, \lambda[$

$$\operatorname{Prox}_{\mu f_{\lambda}} = \left(\operatorname{Id} + \mu \partial_{\#} f_{\lambda}\right)^{-1} = \left(\left(1 - \frac{\mu}{\lambda}\right)\operatorname{Id} + \mu N_{D}\right)^{-1}$$
(37a)

$$= \left((1 - \frac{\mu}{\lambda}) (\operatorname{Id} + N_D) \right)^{-1} = P_D \circ \left(\frac{\lambda}{\lambda - \mu} \operatorname{Id} \right).$$
(37b)

In particular, if D is a closed convex cone we learn that $\operatorname{Prox}_{\mu f_{\lambda}} = \frac{\lambda}{\lambda - \mu} P_{D}$.

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