# Decreasing Minimization on M-convex Sets: Background and Structures

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#### Abstract

The present work is the first member of a pair of papers concerning decreasingly-minimal (dec-min) elements of a set of integral vectors, where a vector is dec-min if its largest component is as small as possible, within this, the next largest component is as small as possible, and so on. This discrete notion, along with its fractional counterpart, showed up earlier in the literature under various names.

The domain we consider is an M-convex set, that is, the set of integral elements of an integral base-polyhedron. A fundamental difference between the fractional and the discrete case is that a base-polyhedron has always a unique dec-min element, while the set of dec-min elements of an M-convex set admits a rich structure, described here with the help of a 'canonical chain'. As a consequence, we prove that this set arises from a matroid by translating the characteristic vectors of its bases with an integral vector.

By relying on these characterizations, we prove that an element is dec-min if and only if the square-sum of its components is minimum, a property resulting in a new type of min-max theorems. The characterizations also give rise, as shown in the companion paper, to a strongly polynomial algorithm, and to several applications in the areas of resource allocation, network flow, matroid, and graph orientation problems, which actually provided a major motivation to the present investigations. In particular, we prove a conjecture on graph orientation.

**Keywords**: Submodular optimization, Matroid, Base-polyhedron, M-convex set, Lexicographic minimization.

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# Contents

1	Intro	oduction	3
	1.1	Background problems	3
		1.1.1 Orientations of graphs	3
		1.1.2 A resource allocation problem	4
		1.1.3 Network flows	5
		1.1.4 Matroid bases	5
	1.2	Main goals	5
	1.3	Notation	7
2	Base	e-polyhedra and M-convex sets	7
3	Cha	racterizing a decreasingly minimal element	9
	3.1	Decreasing minimality	9
	3.2	Characterizing a dec-min element	10
	3.3	Minimizing the sum of the $k$ largest components	13
4	Decomposition by pre-decreasingly minimal elements		14
	4.1	Max-minimizers and pre-dec-min elements	15
	4.2	The peak-set $S_1$	16
	4.3	Separating along $S_1$	17
	4.4	The matroid $M_1$ on $S_1$	18
	4.5	Value-fixed elements of $S_1$	20
5	Desc	cription of the set of all decreasingly minimal elements	21
	5.1	Canonical partition and canonical chain	21
	5.2	Obtaining the canonical chain and value-sequence from a dec-min element	23
	5.3	Matroidal description of the set of dec-min elements	25
6	Integral square-sum and difference-sum minimization		26
	6.1	Symmetric convex minimization	26
	6.2	Min-max theorem for integral square-sum	29
	6.3	The set of optimal duals to integral square-sum minimization	32
7	Con	tinuous versus discrete	34
8	Con	clusion	36

# **1** Introduction

We investigate a problem which we call "discrete decreasing minimization." An element of a set of vectors is called decreasingly minimal (dec-min) if its largest component is as small as possible, within this, its second largest component is as small as possible, and so on. The term discrete decreasing minimization refers to the problem of finding a dec-min element (or even a cheapest dec-min element with respect to a given weighting) of a set of integral vectors. In the present work, this set is an M-convex set, which is nothing but the set of integral elements of an integral base-polyhedron. Note that one may consider the analogous term "increasing maximization" (inc-max), as well. This dichotomy is the reason why we avoid the usage of term "lexicographic optimal" used in the literature.

The goal of this paper is to develop structural characterizations of the set of dec-min elements of an M-convex set. These form the bases, in [13], for developing a strongly polynomial algorithm, as well as for exploring and exhibiting various applications. Actually, earlier special cases played a major motivating role for our investigations, which are described in Section 1.1. The main results will be described in Section 1.2. The research was strongly motivated by the theory of Discrete Convex Analysis (DCA), but the paper is self-contained and does not rely on any prerequisite from DCA.

#### 1.1 Background problems

There are several different sources underlying the study of discrete decreasing minimization.

#### 1.1.1 Orientations of graphs

Let G = (V, E) be an undirected graph. Orienting an edge e = uv means the operation that replaces e by one of the two oppositely directed edges (sometimes called arcs) uv or vu. A directed graph arising from G by orienting all of its edges is called an orientation of G (see Figure 1). A graph orientation problem consists of finding an orientation of G meeting some specified properties such as in-degree constraints (lower and upper bounds) and/or various connectivity prescriptions. One goal is to characterize undirected graphs for which the requested orientation exists, and another related goal is to design an algorithm for finding the orientation.

The literature is quite rich in orientation results, for a relatively wide overview, see the book [10]. While upper and lower bounds are often imposed on the in-degree of each node in an orientation of *G*, there are other type of requirements for an orientation concerning the global distribution of the in-degrees of nodes. That is, the goal is to find orientations (with possible connectivity expectations) whose in-degree vector (on the node-set) is felt intuitively evenly distributed: 'fair', 'equitable', 'egalitarian', 'uniform'. For example, how can one determine the minimum value  $\beta_1$  of the largest in-degree of a (*k*-edge-connected) orientation? Even further, after determining  $\beta_1$ , it might be interesting to minimize the number of nodes with in-degree  $\beta_1$  among orientations of *G* with largest in-degree  $\beta_1$ . Or, a more global equitability feeling is captured if we minimize the sum of squares of the in-degrees. For example, the in-degree vector  $m_3 = (1, 3, 0)$  in Figure 1 (with square-sum 10) is felt less 'fair' than  $m_2 = (2, 2, 0)$  (with square-sum 8) and  $m_1 = (2, 1, 1)$  (with square-sum 6).

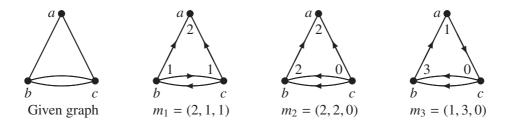


Figure 1: Orientations of a graph

A formally different definition was recently suggested and investigated by Borradaile et al. [4] who called an orientation of G = (V, E) egalitarian if the highest in-degree of the nodes is as small as possible, and within this, the second highest (but not necessarily distinct) in-degree is as small as possible, and within this, the third highest in-degree is as small as possible, and so on. In other words, if we rearrange the in-degrees of nodes in a decreasing order, then the sequence is lexicographically minimal. In order to emphasize that the in-degrees are considered in a decreasing order, we prefer the use of the more expressive term **decreasingly minimal** (dec-min, for short) for such an orientation, rather than egalitarian.

This change of terminology is reasonable since one may also consider the mirror problem of finding an **increasingly maximal** (or **inc-max**, for short) orientation that is an orientation of G in which the smallest in-degree is as large as possible, within this, the second smallest in-degree is as large as possible, and so on. Intuitively, such an orientation may equally be felt 'egalitarian' in the informal meaning of the word.

Borradaile et al. [4], however, proved that an orientation of a graph is decreasingly minimal (egalitarian in their original term) if and only if there is no 'small' improvement, where a small improvement means the reorientation of a dipath from some node *s* to another node *t* with in-degrees  $\rho(t) \ge \rho(s) + 2$ . This theorem immediately implies that an orientation is decreasingly minimal if and only if it is increasingly maximal, and therefore we could retain the original terminology "egalitarian orientation" used in [4].

However, when orientations are considered with specific requirements such as strong (or, more generally, k-edge-) connectivity and/or in-degree bounds on the nodes, the possible equivalence of decreasingly minimal and increasingly maximal orientations had not yet been investigated. Actually, Borradaile et al. [4] conjectured that a strong orientation of a graph is decreasingly minimal (among strong orientations) if and only if there is no small improvement preserving strong connectivity. This, if true, would imply immediately that decreasing minimality and increasing maximality do coincide for strong orientations, as well. In [13] we shall prove this conjecture even in its extended form concerning k-edge-connected and in-degree constrained orientations.

#### 1.1.2 A resource allocation problem

Another source of our investigations is due to Harvey et al. [23] who solved the problem of minimizing  $\sum [d_F(s)(d_F(s)+1): s \in S]$  over the semi-matchings F of a simple bipartite graph G = (S, T; E). Here a semi-matching is a subset F of edges for which  $d_F(t) = 1$  holds for every node  $t \in T$ . Harada et al. [22] solved the minimum edge-cost version of this problem. The framework of Harvey et al. was extended by Bokal et al. [3] to quasi-matchings, and, even further, to degree-bounded quasimatchings by Katrenič and Semanišin [30]. It turns out that these problems are strongly related to minimization of a separable convex function over (integral elements of) a base-polyhedron which has been investigated in the literature under the name of "resource allocation problems under submodular constraints" ([7], [26], [25], [29], [27], [28]). Ghodsi et al. [20] considered the problem of finding a semi-matching F of G = (S, T; E) whose degree-vector restricted to S is increasingly maximal. This problem "constrained max-min fairness" originated from modelling a fair sharing problem for datacenter jobs. Here T corresponds to the set of available computers while S to the set of users. An edge st exists if user s can run her program on computer t. Ghodsi et al. also consider the fractional version, which finds a real vector  $x : E \to \mathbf{R}_+$  so that  $d_x(t) = 1$  for every  $t \in T$  and the vector  $(d_x(s): s \in S)$  is increasingly maximal. (Here  $d_x(v) := \sum [x(uv): uv \in E]$ ). When x is requested to be  $\{0, 1\}$ -valued, we are back at the subgraph version. It should be emphasized that, unlike the well-known situation with ordinary bipartite matchings, the optima for the subgraph version and for the fractional version may be different.

#### 1.1.3 Network flows

There is a much earlier, strongly related problem concerning network flows, due to Megiddo [35], [36]. We are given a digraph D = (V, A) with a source-set  $S \subset V$  and a single sink-node  $t \in V - S$ . Let  $g : A \to \mathbf{R}_+$  be a capacity function. A flow means a function  $x : A \to \mathbf{R}_+$  for which the net out-flow  $\delta_x(v) - \varrho_x(v) = 0$  if  $v \in V - (S \cup \{t\})$  and  $\delta_x(v) - \varrho_x(v) \ge 0$  if  $v \in S$ . (Here  $\varrho_x(v) := \sum [x(uv) : uv \in A]$ and  $\delta_x(v) := \sum [x(vu) : vu \in A]$ .) A flow *x* is feasible if  $x \le g$ . The **flow amount** of *x* is defined by  $\varrho_x(t) - \delta_x(t)$ . Megiddo solved the problem of finding a feasible flow of maximum flow amount which is, in his term, "source-optimal" at *S*. Source-optimality is the same as requiring that the net out-flow vector on *S* is increasingly maximal. It must be emphasized that the flow in Megiddo's problem is not requested to be integer-valued.

The integrality property is a fundamental feature of ordinary network flows. It states that in case of an integer-valued capacity function g there always exists a maximum flow which is integer-valued. In this light, it is quite surprising that the integer-valued (or discrete) version of Megiddo's inc-max problem (source-optimal in his term), where the capacity function g is integer-valued and the max flow is required to be integer-valued, has not been investigated in the literature. We consider the present work the first such attempt.

#### 1.1.4 Matroid bases

The fourth source of discrete decreasing minimization problems is due to Levin and Onn [31] who used the term "shifted optimization." They considered the following matroid optimization problem. For a specified integer k, find k bases  $Z_1, Z_2, ..., Z_k$  of a matroid M on S in such a way that the vector  $\sum_i \chi_{Z_i}$  be, in our term, decreasingly minimal, where  $\chi_Z$  is the incidence (or characteristic) vector of a subset Z. They apply the following natural approach to reduce the problem to classic results of matroid theory. First replace each element s of S by k copies to be parallel in the resulting matroid M' on the new ground-set  $S' = S_1 \cup S_2 \cup \cdots \cup S_k$  where  $S_1, \ldots, S_k$  are the k copies of S. Assign then a 'rapidly increasing' cost function to the copies. (The paper [31] explicitly describes what rapidly increasing means). Then a minimum cost basis of the matroid  $M_0$  obtained by multiplying M' k-times will be a solution to the problem. (By definition, a basis of  $M_0$  is the union of k disjoint bases of M').

#### 1.2 Main goals

Each of the problems in Section 1.1 may be viewed as a special case of a single discrete optimization problem: *Characterize decreasingly minimal elements of an M-convex set* [37, 38] (or, in other words, dec-min integral elements of an integral base-polyhedron). By one of its equivalent definitions, an M-convex set is nothing but the set of integral elements of an integral base-polyhedron.

We characterize dec-min elements of an M-convex set as those admitting no local improvement, and prove that the set of dec-min elements is itself an M-convex set arising by translating a matroid base-polyhedron with an integral vector. This result implies that decreasing minimality and increasing maximality coincide for M-convex sets. We shall also show that an element of an M-convex set is dec-min precisely if it is a square-sum minimizer. Using the characterization of dec-min elements, we shall derive a novel min-max theorem for the minimum square-sum of elements of an integral member of a base-polyhedron.

The structural description of the set of dec-min elements of an M-convex set in terms of a matroid makes it possible to solve the algorithmic problem of finding a minimum cost dec-min element. (In the continuous counterpart of decreasing minimization, this problem simply does not exist due to the uniqueness of the fractional dec-min element of a base-polyhedron.) In the companion paper [13], we shall describe a polynomial algorithm for finding a minimum cost (in-degree constrained) dec-min orientation. Furthermore, we shall outline an algorithm to solve the minimum cost version of the resource allocation problem of Harvey et al. [23] mentioned in Section 1.1.2. Moreover, as an essential extension of the algorithm of Harada et al. [22], we describe a strongly polynomial algorithm

to solve a minimum cost version of the decreasingly minimal degree-bounded subgraph problem in a bipartite graph G = (S, T, E). One may consider two versions here. In the simpler one, we have a cost-function on the node-set of G, that is, on the ground-set of the corresponding M-convex set. Due to the matroidal description of the set of dec-min elements of an M-convex set, this min-cost version becomes rather easy since the matroid greedy algorithm can be applied. Significantly more complicated is the other min-cost version, where a cost-function is given on the set of edges of G. The latter version is also solved in [13].

The topic of our investigations may be interpreted as a discrete counterpart of the work by Fujishige [17] from 1980 on the lexicographically optimal base of a base-polyhedron B, where lexicographical optimality is essentially the same as decreasing minimality. He proved that there is a unique lexicographically optimal member  $x_0$  of B, and  $x_0$  is the unique minimum norm (that is, the minimum square-sum) element of B. This uniqueness result reflects a characteristic difference between the behaviour of the fractional and the discrete versions of decreasing minimization since in the latter case the set of dec-min elements (of an M-convex set) is typically not a singleton, and it actually has, as indicated above, a matroidal structure. While the present paper focuses on the unweighted case, the lexicographically optimal base of a base-polyhedron is defined and analyzed with respect to a weight vector in [17].

Fujishige also introduced the concept of principal partitions concerning the dual structure of the minimum norm point of a base-polyhedron. Actually, he introduced a special chain of the subsets of ground-set *S* and his principal partition arises by taking the difference sets of this chain. We will prove that there is an analogous concept in the discrete case, as well. As an extension of the above-mentioned elegant result of Borradaile et al. [5] concerning graphs, we show that there is a 'canonical chain' describing the structure of dec-min elements of an M-convex set. The relation between our canonical partition and Fujishige's principal partition is clarified in [12], showing that the canonical partition is an intrinsic structure of an M-convex set consistent with the principal partition of a base-polyhedron.

The paper of Fujishige is one of the early representatives of the rich literature of related topics. The section of "Survey of early papers" in [12] provides a relatively complete overview of these results, along with an outline of their relationship.

The present paper is organized, as follows. After formally introducing the basic notions, terminology, and notation in Section 2, we prove, in Section 3, two characterizations of an element m of an M-convex sets to be decreasingly minimal. The first one is a co-**NP**-characterization consisting of an easily checkable certificate for m not to be dec-min, while the second one is an **NP**-characterization consisting of an easily checkable certificate for m to be dec-min. The first characterization implies immediately that m is dec-min precisely if it is inc-max. This is a property that fails to hold for the intersection of two M-convex sets.

In Sections 4 and 5, we show how the set of all dec-min elements of an M-convex set can be obtained from a matroid, and hence it is also an M-convex set. The main device is the dual concept of canonical chains and partitions. In Section 6, we prove that an element *m* of an M-convex set is decmin if and only if its  $\ell_2$ -norm  $\|\cdot\|_2$  (or equivalently, the square sum of its components) is minimum. On one hand, this result seems surprising in the light of the fact that there is an example for two elements  $m_1$  and  $m_2$  of an M-convex set for which  $m_1$  is decreasingly smaller than  $m_2$ , but  $||m_1||_2 > ||m_2||_2$ . On the other hand, we shall use the coincidence of dec-minimality and square-sum minimality to prove a min-max theorem for the minimum square-sum of the components over the elements of an M-convex set. As a special case, this provides a min-max formula for the the minimum of  $\sum \{\rho_D(v)^2 : v \in V\}$  over all orientations *D* of an undirected graph on node-set *V*, where  $\rho_D(v)$  denotes the in-degree of *v* in *D*. To our best knowledge, no min-max formulas of similar type appeared earlier in the literature.

In Section 7, we discuss the relationship between the continuous and discrete versions of decreasing minimization on a base-polyhedron. Both dec-min elements and partitions of the ground-set are compared between the continuous and discrete versions. Finally in Section 8, we give a perspective of our series of research on discrete decreasing minimization.

#### 1.3 Notation

Throughout the paper, *S* denotes a finite non-empty ground-set. For elements  $s, t \in S$ , we say that  $X \subset S$  is an  $s\overline{i}$ -set if  $s \in X \subseteq S - t$ . For a vector  $m \in \mathbb{R}^S$  (or function  $m : S \to \mathbb{R}$ ), the restriction of m to  $X \subseteq S$  is denoted by m|X. We also use the notation  $\widetilde{m}(X) = \sum [m(s) : s \in X]$ . With a small abuse of notation, we do not distinguish between a one-element set  $\{s\}$  called a singleton and its only element s. When we work with a chain C of non-empty sets  $C_1 \subset C_2 \subset \cdots \subset C_q$ , we sometimes use  $C_0$  to denote the empty set without assuming that  $C_0$  is a member of C. The characteristic (or incidence) vector of a subset Z is denoted by  $\chi_Z$ , that is,  $\chi_Z(s) = 1$  if  $s \in Z$  and  $\chi_Z(s) = 0$  otherwise. For a polyhedron B,  $\overline{B}$  (pronounce: dotted B) denotes the set of integral members (elements, vectors, points) of B, that is,

$$\overset{\dots}{B} := B \cap \mathbf{Z}^S. \tag{1.1}$$

For a set-function *h*, we allow it to have value  $+\infty$  or  $-\infty$ , while  $h(\emptyset) = 0$  is assumed throughout. Where h(S) is finite, the **complementary function**  $\overline{h}$  is defined by  $\overline{h}(X) = h(S) - h(S - X)$ . For functions  $f : S \to \mathbb{Z} \cup \{-\infty\}$  and  $g : S \to \mathbb{Z} \cup \{+\infty\}$  with  $f \le g$ , the polyhedron  $T(f, g) = \{x \in \mathbb{R}^S :$  $f \le x \le g\}$  is called a **box**. If  $g(s) \le f(s) + 1$  holds for every  $s \in S$ , we speak of a **small box**. For example, the (0, 1)-box is small, and so is any set consisting of a single integral vector.

#### **2** Base-polyhedra and M-convex sets

Let *S* be a finite non-empty ground-set. Let *b* be a set-function for which  $b(X) = +\infty$  is allowed but  $b(X) = -\infty$  is not. The submodular inequality for subsets  $X, Y \subseteq S$  is defined by

$$b(X) + b(Y) \ge b(X \cap Y) + b(X \cup Y).$$

We say that *b* is **submodular** if the submodular inequality holds for every pair of subsets  $X, Y \subseteq S$  with finite *b*-values. A set-function *p* is **supermodular** if -p is submodular.

For a submodular integer-valued set-function b on S for which  $b(\emptyset) = 0$  and b(S) is finite, the **base-polyhedron** B in  $\mathbb{R}^S$  is defined by

$$B = B(b) = \{x \in \mathbf{R}^S : \widetilde{x}(S) = b(S), \ \widetilde{x}(Z) \le b(Z) \text{ for every } Z \subset S\},$$
(2.1)

which is possibly unbounded.

A special base-polyhedron is the one of matroids. Given a matroid M, Edmonds proved that the polytope (that is, the convex hull) of the incidence (or characteristic) vectors of the bases of M is the base-polyhedron B(r) defined by the rank function r of M, that is,  $B(r) = \{x \in \mathbb{R}^S : \tilde{x}(S) = r(S) \text{ and } \tilde{x}(Z) \le r(Z) \text{ for every subset } Z \subset S \}$ . It can be proved that a kind of converse also holds, namely, every (integral) base-polyhedron in the unit (0, 1)-cube is a matroid base-polyhedron. We call the translation of a matroid base-polyhedron a **translated matroid base-polyhedron**. It follows that the intersection of a base-polyhedron with a small box is a translated matroid base-polyhedron.

A base-polyhedron B(b) is never empty, and B(b) is known to be an integral polyhedron. (A rational polyhedron is **integral** if each of its faces contains an integral element. In particular, a pointed rational polyhedron is integral if all of its vertices are integral.) By convention, the empty set is also considered a base-polyhedron. Note that a real-valued submodular function *b* also defines a base-polyhedron B(b) but in the present work we are interested only in integer-valued submodular functions and integral base-polyhedra.

We call the set *B* of integral elements of an integral base-polyhedron *B* an **M-convex set**. Originally, this basic notion of DCA introduced by Murota [37] (see, also the book [38]), was defined as a set of integral points in  $\mathbf{R}^S$  satisfying certain exchange axioms, and it is known that the two properties are equivalent ([38, Theorem 4.15]). While an integral base-polyhedron *B* defines an M-convex set as  $\widetilde{B} = B \cap \mathbf{Z}^S$ , an M-convex set induces an integral base-polyhedron as its convex hull. This implies,

in particular, that two distinct integral base-polyhedra  $B_1$  and  $B_2$  define distinct M-convex sets  $B_1$  and  $B_2$ . The set of integral elements of a translated matroid base-polyhedron will be called a **matroidal M-convex set**.

A non-empty base-polyhedron *B* can also be defined by a supermodular function *p* for which  $p(\emptyset) = 0$  and p(S) is finite as follows:

$$B = B'(p) = \{x \in \mathbf{R}^S : \widetilde{x}(S) = p(S), \widetilde{x}(Z) \ge p(Z) \text{ for every } Z \subset S\}.$$
(2.2)

It is known that *B* uniquely determines both *p* and *b*, namely,  $b(Z) = \max{\tilde{x}(Z) : x \in B}$  and  $p(Z) = \min{\tilde{x}(Z) : x \in B}$ . The functions *p* and *b* are complementary functions, that is, b(X) = p(S) - p(S - X) or p(X) = b(S) - b(S - X) (where b(S) = p(S)).

For a set  $Z \,\subset S$ , p|Z denotes the restriction of p to Z, while p' = p/Z is the set-function on S - Z obtained from p by contracting Z, which is defined for  $X \subseteq S - Z$  by  $p'(X) = p(X \cup Z) - p(Z)$ . Note that p/Z and  $\overline{p}|(S - Z)$  are complementary set-functions. It is also known for disjoint subsets  $Z_1$  and  $Z_2$  of S that

$$(p/Z_1)/Z_2 = p/(Z_1 \cup Z_2),$$
 (2.3)

When p(Z) is finite, the base-polyhedron B'(p|Z) is called the **restriction** of B'(p) to Z.

Let  $\{S_1, \ldots, S_q\}$  be a partition of *S* and let  $p_i$  be a supermodular function on  $S_i$ . Let *p* denote the supermodular function on *S* defined by  $p(X) := \sum [p_i(S_i \cap X) : i = 1, \ldots, q]$  for  $X \subseteq S$ . The base-polyhedron B'(p) is called the **direct sum** of the *q* base-polyhedra  $B'(p_i)$ . Obviously, a vector  $x \in \mathbf{R}^S$  is in B'(p) if and only if each  $x_i$  is in  $B'(p_i)$   $(i = 1, \ldots, q)$ , where  $x_i$  denotes the restriction  $x|S_i$ of *x* to  $S_i$ .

It is known that a face *F* of a non-empty base-polyhedron *B* is also a base-polyhedron. The (special) face of B'(p) defined by the single equality  $\tilde{x}(Z) = p(Z)$  is the direct sum of the base polyhedra B'(p|Z) and B'(p|Z). More generally, any face *F* of *B* can be described with the help of a chain  $(\emptyset \subset) C_1 \subset C_2 \subset \cdots \subset C_\ell = S$  of subsets by  $F := \{z : z \in B, p(C_i) = \tilde{z}(C_i) \text{ for } i = 1, \dots, \ell\}$ . (In particular, when  $\ell = 1$ , the face *F* is *B* itself.) Let  $S_1 := C_1$  and  $S_i := C_i - C_{i-1}$  for  $i = 2, \dots, \ell$ . Then *F* is the direct sum of the base-polyhedra  $B'(p_i)$ , where  $p_i$  is a supermodular function on  $S_i$  defined by  $p_i(X) := p(X \cup C_{i-1}) - p(C_{i-1})$  for  $X \subseteq S_i$ . In other words,  $p_i$  is a set-function on  $S_i$  obtained from *p* by deleting  $C_{i-1}$  and contracting  $S - C_i$ . The unique supermodular function  $p_F$  defining the face *F* is given by  $\sum [p_i(S_i \cap X) : i = 1, \dots, \ell]$ . A face *F* is the set of elements *x* of *B* minimizing *cx* whenever  $c : S \to \mathbf{R}$  is a linear cost function such that c(s) = c(t) if  $s, t \in S_i$  for some *i* and c(s) > c(t) if  $s \in S_i$  and  $t \in S_j$  for some subscripts i < j.

The intersection of an integral base-polyhedron B = B'(p) (=  $B(\overline{p})$ ) and an integral box T(f, g) is an integral base-polyhedron. The intersection is non-empty if and only if

$$p \le \widetilde{g} \quad \text{and} \quad f \le \overline{p}.$$
 (2.4)

For an element *m* of a base-polyhedron B = B(b) defined by a submodular function *b*, we call a subset  $X \subseteq S$  *m*-tight (with respect to *b*) if  $\widetilde{m}(X) = b(X)$ . It is known (e.g., Theorem 14.2.8 in [10]) that, for a given subset  $X \subseteq S$ , the face  $B_X := \{x \in B : \widetilde{x}(X) = b(X)\}$  of *B* is a non-empty base-polyhedron. This means that a subset *X* is *m*-tight precisely if *m* is in  $B_X$ . Clearly, the empty set and *S* are *m*-tight, and *m*-tight sets are closed under taking union and intersection (see, for example, Lemma 2.4.7 in [10]). Therefore, for each subset  $Z \subseteq S$ , the intersection  $T_m(Z; b)$  of all *m*-tight sets including *Z* is the unique smallest *m*-tight set including *Z*. When  $Z = \{s\}$  is a singleton, we simply write  $T_m(s; b)$  to denote the smallest *m*-tight set containing *s*. This set admits a representation  $T_m(s; b) = \{t \in S : m + \chi_s - \chi_t \in B(b)\}$ . When the submodular function *b* is understood from the context, we abbreviate  $T_m(Z; b)$  to  $T_m(Z)$ .

Analogously, when B = B'(p) is given by a supermodular function p, we call  $X \subseteq S$  *m*-tight (with respect to p) if  $\widetilde{m}(X) = p(X)$ . In this case, we also use the analogous notation  $T_m(Z) = T_m(Z; p)$  and  $T_m(s) = T_m(s; p)$ . We have  $T_m(s; p) = \{t \in S : m - \chi_s + \chi_t \in B'(p)\}$ . Observe that for complementary functions b and p, X is *m*-tight with respect to b precisely if S - X is *m*-tight with respect to p.

**Example 2.1.** The set of in-degree vectors of orientations of a given undirected graph forms an Mconvex set. Consider the undirected graph G = (V, E) in the left-most of Figure 1, where  $V = \{a, b, c\}$ and the set *E* of edges consists of *ab*, *ac*, and a pair of parallel edges between *b* and *c*. An orientation of *G* means a directed graph *D* that is obtained from *G* by orienting the edges of *G*. The in-degree vector of *D* is the vector *m* on *V* whose component at  $v \in V$  is equal to the number of edges entering *v* in *D*. Three different orientations are depicted in Figure 1 with the corresponding in-degree vectors,  $m_1 = (2, 1, 1), m_2 = (2, 2, 0), and m_3 = (1, 3, 0)$ . The set of in-degree vectors of all orientations of *G* is known to form an M-convex set, say, *B*. In the present example, the M-convex set *B* consists of 10 members as

$$B = \{(2, 1, 1), (2, 2, 0), (2, 0, 2), (1, 3, 0), (1, 0, 3), (1, 2, 1), (1, 1, 2), (0, 2, 2), (0, 3, 1), (0, 1, 3)\},\$$

where  $m_1, m_2$ , and  $m_3$  are underlined. Note that different orientations may result in the same in-degree vector. The supermodular function p describing the M-convex set B (or the base-polyhedron B) is given as follows. For any  $X \subseteq V$ , let  $i_G(X)$  denote the number of edges of G induced by X, that is,  $i_G(X) = |\{uv \in E : \{u, v\} \subseteq X\}|$ . This function  $i_G$  is a (non-negative) integer-valued supermodular function, and we have B = B'(p) for  $p = i_G$ . For any orientation D with in-degree vector m, a subset X of V is m-tight if and only if there is no edge entering X in D. In the present example,  $X = \{b, c\}$  is m-tight for  $m = m_1, m_2$ , whereas it is not  $m_3$ -tight.

# **3** Characterizing a decreasingly minimal element

#### **3.1** Decreasing minimality

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For a vector x, let  $x \downarrow$  denote the vector obtained from x by rearranging its components in a decreasing order. For example, we call two vectors x and y (of same dimension) value-equivalent if  $x \downarrow = y \downarrow$ .

A vector x is **decreasingly smaller** than vector y, in notation  $x <_{dec} y$  if  $x \downarrow$  is lexicographically smaller than  $y \downarrow$  in the sense that they are not value-equivalent and  $x \downarrow (j) < y \downarrow (j)$  for the smallest subscript j for which  $x \downarrow (j)$  and  $y \downarrow (j)$  differ. For example, x = (2, 5, 5, 1, 4) is decreasingly smaller than y = (1, 5, 5, 5, 1) since  $x \downarrow = (5, 5, 4, 2, 1)$  is lexicographically smaller than  $y \downarrow = (5, 5, 5, 1, 1)$ . We write  $x \leq_{dec} y$  to mean that x is decreasingly smaller than or value-equivalent to y.

For a set Q of vectors,  $x \in Q$  is **decreasingly minimal** (dec-min, for short) if  $x \leq_{dec} y$  for every  $y \in Q$ . Note that the dec-min elements of Q are value-equivalent. Therefore an element m of Q is dec-min if its largest component is as small as possible, within this, its second largest component (with the same or smaller value than the largest one) is as small as possible, and so on. An element x of Q is said to be a **max-minimized** element (a **max-minimizer**, for short) if its largest component is as small as possible. A max-minimizer element x is **pre-decreasingly minimal** (**pre-dec-min**, for short) in Q if the number of its largest components is as small as possible. Obviously, a dec-min element is pre-dec-min, and a pre-dec-min element is max-minimized. In Example 2.1, for example,  $m_1 = (2, 1, 1)$  is dec-min in B,  $m_2 = (2, 2, 0)$  is a max-minimizer that is not dec-min, and  $m_3 = (1, 3, 0)$ is not a max-minimizer.

In an analogous way, for a vector x, we let  $x\uparrow$  denote the vector obtained from x by rearranging its components in an increasing order. A vector y is **increasingly larger** than vector x, in notation  $y >_{inc} x$ , if they are not value-equivalent and  $y\uparrow(j) > x\uparrow(j)$  holds for the smallest subscript j for which  $y\uparrow(j)$  and  $x\uparrow(j)$  differ. We write  $y \ge_{inc} x$  if either  $y >_{inc} x$  or x and y are value-equivalent. Furthermore, we call an element m of Q **increasingly maximal** (**inc-max** for short) if its smallest component is as large as possible over the elements of Q, within this its second smallest component is as large as possible, and so on.

It should be emphasized that a dec-min element of a base-polyhedron *B* is not necessarily integervalued. For example, if  $B = \{(x_1, x_2) : x_1 + x_2 = 1\}$ , then  $x^* = (1/2, 1/2)$  is a dec-min element of *B*. In this case, the dec-min members of  $\overline{B}$  are (0, 1) and (1, 0). Therefore, finding a dec-min element of *B* and finding a dec-min element of  $\overline{B}$  (the set of integral points of *B*) are two distinct problems, and we shall concentrate only on the second, discrete problem. In what follows, the slightly sloppy term integral dec-min element of *B* will always mean a dec-min element of  $\overline{B}$ . (The term is sloppy in the sense that an integral dec-min element of *B* is not necessarily a dec-min element of *B*).

We call an integral vector  $x \in \mathbb{Z}^S$  uniform if all of its components are the same integer  $\ell$ , and **near-uniform** if its largest and smallest components differ by at most 1, that is, if  $x(s) \in \{\ell, \ell + 1\}$  for some integer  $\ell$  for every  $s \in S$ . Note that if Q consists of integral vectors and the component-sum is the same for each member of Q, then any near-uniform member of Q is obviously both decreasingly minimal and increasingly maximal integral vector.

#### 3.2 Characterizing a dec-min element

Let B = B(b) = B'(p) be a base-polyhedron defined by an integer-valued submodular function b or supermodular function p (where b and p are complementary set-functions). Let m be an integral member of B, that is,  $m \in B$ . Recall the definition of m-tightness introduced at the end of Section 2.

The equivalences in the next claim will be used throughout.

**Claim 3.1.** Let  $m \in B$ , and let *s* and *t* be elements of *S*, and  $m' := m + \chi_s - \chi_t$ . The following properties are pairwise equivalent.

(A)  $m' \in \overline{B}$ .

(P1) There is no ts-set which is m-tight with respect to p.

(P2)  $s \in T_m(t; p)$ .

(B1) There is no st-set which is m-tight with respect to b.

(B2)  $t \in T_m(s; b)$ .

**Proof.** (A)  $\Rightarrow$  (P1) If X is a  $t\overline{s}$ -set which is *m*-tight with respect to p, then  $\widetilde{m}'(X) < \widetilde{m}(X) = p(X)$ , showing that  $m' \notin \overline{B}$ .

(P1)  $\Rightarrow$  (P2) Since  $T_m(t; p)$  is an *m*-tight set containing *t*, (P1) implies that *s* cannot be outside  $T_m(t; p)$ .

(P2)  $\Rightarrow$  (A) Suppose that *m'* is not in  $\widetilde{B}$ , that is, there is a subset  $X \subset S$  with  $\widetilde{m}'(X) < p(X)$ . Since *m* and *p* are integer-valued and  $\widetilde{m}(X) \ge p(X)$ , we get from the definition of *m'* that  $\widetilde{m}(X) = p(X)$  and *X* is a  $t\overline{s}$ -set, contradicting (P2).

(P1)  $\Leftrightarrow$  (B1) Since *p* and *b* are complementary set-functions, a subset *X* is *m*-tight with respect to *p* precisely if *S* – *X* is *m*-tight with respect to *b*.

(P2)  $\Leftrightarrow$  (B2) Suppose that  $t \notin T_m(s; b)$ , that is, there is an  $s\overline{t}$ -subset X which is *m*-tight with respect to *b*. Then S - X is *m*-tight with respect to *p*, implying that  $s \notin T_m(t; p)$ , that is, (P2) implies (B2). The reverse implication follows analogously.

A 1-tightening step for  $m \in \tilde{B}$  is an operation that replaces m by  $m' := m + \chi_s - \chi_t$  where s and t are elements of S for which  $m(t) \ge m(s) + 2$  and m' belongs to  $\tilde{B}$ . Note that m' is both decreasingly smaller and increasingly larger than m. Intuitively, a 1-tightening step may be viewed as a local improvement at m. Since the mean of the components of m does not change at a 1-tightening step while the square-sum of the components of m strictly drops, consecutive 1-tightening steps may occur only a finite number of times (even if B is unbounded).

As an example, consider the in-degree vector  $m = m_3 = (1, 3, 0)$  in Example 2.1, where  $S = \{a, b, c\}$  and m = (m(a), m(b), m(c)). For (s, t) = (a, b) we have  $m(t) \ge m(s) + 2$  and  $m' = m + \chi_s - \chi_t = (2, 2, 0) \in \overline{B}$ . (Note that m' is equal to  $m_2$  in Fig. 1.) Therefore, this is a 1-tightening step. In contrast, for (s, t) = (c, b) and m = (1, 3, 0), we do not have a 1-tightening step since m(t) = m(s) + 1, although  $m + \chi_s - \chi_t = (0, 3, 1) \in \overline{B}$ .

The next claim shows equivalent conditions for the non-existence of a 1-tightening step.

**Claim 3.2.** For an integral element m of the integral base-polyhedron B = B(b) = B'(p), the following conditions are pairwise equivalent.

(A) There is no 1-tightening step for m.

(P1)  $m(s) \ge m(t) - 1$  holds whenever  $t \in S$  and  $s \in T_m(t; p)$ .

(P2) Whenever  $m(t) \ge m(s) + 2$ , there is a ts-set X which is m-tight with respect to p.

(B1)  $m(s) \ge m(t) - 1$  holds whenever  $s \in S$  and  $t \in T_m(s; b)$ .

(B2) Whenever  $m(t) \ge m(s) + 2$ , there is an st-set Y which is m-tight with respect to b.

**Proof.** (A)  $\Rightarrow$  (P1) If we had a pair (s, t) of elements with  $t \in S$ ,  $s \in T_m(t; p)$  for which  $m(s) \le m(t)-2$ , then replacing *m* by  $m' := m + \chi_s - \chi_t$  would be a 1-tightening step.

(P1)  $\Rightarrow$  (P2) Suppose that (s, t) is a pair of elements for which  $m(t) \ge m(s) + 2$  but no *m*-tight  $t\bar{s}$ -set exists. Then  $s \in T_m(t; p)$ , contradicting (P1).

(P2)  $\Rightarrow$  (A) Suppose there is a 1-tightening step for *m*, that is, there are elements *s* and *t* for which  $m(t) \ge m(s) + 2$  and  $m' := m + \chi_s - \chi_t$  is in  $\overline{B}$ . But  $m' \in \overline{B}$  implies that no *m*-tight  $t\overline{s}$ -set can exist, contradicting (P2).

(P1)  $\Leftrightarrow$  (B1) follows from the equivalence of  $s \in T_m(t; p)$  and  $t \in T_m(t; b)$  established in Claim 3.1.

(P2)  $\Leftrightarrow$  (B2) follows from the property that a subset of *S* is *m*-tight with respect to *b* precisely if its complement is *m*-tight with respect to *p*.

For a given vector m in  $\mathbb{R}^S$ , we call a set  $X \subseteq S$  an m-top set (or a top-set with respect to m) if  $m(u) \ge m(v)$  holds whenever  $u \in X$  and  $v \in S - X$ . Both the empty set and the ground-set Sare m-top sets, and m-top sets are closed under taking union and intersection. If m(u) > m(v) holds whenever  $u \in X$  and  $v \in S - X$ , we speak of a strict m-top set. A set  $X \subseteq S$  is a strict m-top set if and only if X is represented as  $X = \{s \in S : m(s) \ge \alpha\}$  for some integer  $\alpha$ . For example, the vector m = (4, 2, 2, 1, 1), indexed by  $S = \{s_1, s_2, s_3, s_4, s_5\}$ , has four strict m-top sets: the empty set,  $\{s_1\}$ ,  $\{s_1, s_2, s_3\}$ , and S. Note that the number of strict non-empty m-top sets is at most n for every  $m \in B$ while  $m \equiv 0$  exemplifies that even all of the non-empty subsets of S can be m-top sets.

**Theorem 3.3.** Let *b* be an integer-valued submodular function and let  $p := \overline{b}$  be its complementary (supermodular) function. For an integral element *m* of the integral base-polyhedron B = B(b) = B'(p), the following four conditions are pairwise equivalent.

(A) There is no 1-tightening step for m (or any one of the four other equivalent properties holds in Claim 3.2).

(B) There is a chain C of m-top sets  $(\emptyset \subset) C_1 \subset C_2 \subset \cdots \subset C_{\ell} = S$  which are m-tight with respect to p (or equivalently, whose complements are m-tight with respect to b) such that the restriction  $m_i = m|S_i$  of m to  $S_i$  is near-uniform for each member  $S_i$  of the S-partition  $\{S_1, \ldots, S_{\ell}\}$ , where  $S_1 = C_1$  and  $S_i := C_i - C_{i-1}$   $(i = 2, \ldots, \ell)$ .

(C1) *m* is decreasingly minimal in  $\overline{B}$ .

(C2) m is increasingly maximal in B.

**Proof.** (B) $\rightarrow$ (A): If  $m(t) \ge m(s) + 2$ , then there is an *m*-tight set  $C_i$  containing *t* and not containing *s*, from which Property (A) follows from Claim 3.2.

 $(A) \rightarrow (B)$ : Let *C* be a longest chain consisting of non-empty *m*-tight and *m*-top sets  $C_1 \subset C_2 \subset \cdots \subset C_\ell = S$ . For notational convenience, let  $C_0 = \emptyset$  (but  $C_0$  is not a member of *C*). We claim that *C* meets the requirement of (B). If, indirectly, this is not the case, then there is a subscript  $i \in \{1, \ldots, \ell\}$  for which *m* is not near-uniform within  $S_i := C_i - C_{i-1}$ . This means that the max *m*-value  $\beta_i$  in  $S_i$  is at least 2 larger than the min *m*-value  $\alpha_i$  in  $S_i$ , that is,  $\beta_i \ge \alpha_i + 2$ . Let  $Z := \bigcup [T_m(t; p) : t \in S_i, m(t) = \beta_i]$ . Then *Z* is *m*-tight. Since  $C_i$  is *m*-tight,  $T_m(t; p) \subseteq C_i$  holds for  $t \in S_i$  and hence  $Z \subseteq C_i$ . Furthermore, (A) implies that  $m(v) \ge \beta_i - 1$  for every  $v \in Z \cap S_i$ .

Consider the set  $C' := C_{i-1} \cup Z$ . Then C' is *m*-tight, and  $C_{i-1} \subset C' \subset C_i$ . Moreover, we claim that C' is an *m*-top set. Indeed, if, indirectly, there is an element  $u \in C'$  and an element  $v \in S - C'$  for which m(u) < m(v), then  $u \in Z \cap S_i$  and  $v \in C_i - Z$  since both  $C_{i-1}$  and  $C_i$  are *m*-top sets. But this is impossible since the *m*-value of each element of  $Z \cap S_i$  is  $\beta_i$  or  $\beta_i - 1$  while the *m*-value of each element of  $C_i - Z$  is at most  $\beta_i - 1$ .

The existence of C' contradicts the assumption that C was a longest chain of *m*-tight and *m*-top sets, and therefore *m* must be near-uniform within each  $S_i$ , that is, C meets indeed the requirements in (B).

 $(C1) \rightarrow (A)$  and  $(C2) \rightarrow (A)$ : Property (A) must indeed hold since a 1-tightening step for *m* results in an element *m'* of  $\overline{B}$  which is both decreasingly smaller and increasingly larger than *m*.

(B) $\rightarrow$ (C1): We may assume that the elements of *S* are arranged in an *m*-decreasing order  $s_1, \ldots, s_n$  (that is,  $m(s_1) \ge m(s_2) \ge \cdots \ge m(s_n)$ ) in such a way that each  $C_i$  in (B) is a starting segment. Let *m'* be an element of *B* which is decreasingly smaller than or value-equivalent to *m*. Recall that m|X denoted the vector *m* restricted to a subset  $X \subseteq S$ .

**Lemma 3.4.** For each  $i = 0, 1, ..., \ell$ , vector  $m'|C_i$  is value-equivalent to vector  $m|C_i$ .

**Proof.** Induction on *i*. For i = 0, the statement is void so we assume that  $1 \le i \le \ell$ . By induction, we may assume that the statement holds for  $j \le i - 1$  and we want to prove it for *i*. Since  $m'|C_{i-1}$  is value-equivalent to  $m|C_{i-1}$  and  $C_{i-1}$  is *m*-tight, it follows that  $C_{i-1}$  is *m'*-tight, too.

Let  $\beta_i$  denote the max *m*-value of the elements of  $S_i = C_i - C_{i-1}$ . By the hypothesis in (B), the maximum and the minimum of the *m*-values in  $S_i$  differ by at most 1. Hence we can assume that there are  $r_i > 0$  elements in  $S_i$  with *m*-value  $\beta_i$  and  $|S_i| - r_i \ge 0$  elements with *m*-value  $\beta_i - 1$ .

As  $m|C_{i-1}$  is value-equivalent to  $m'|C_{i-1}$  and m' was assumed to be decreasingly smaller than or value-equivalent to m, we can conclude that  $m'|(S - C_{i-1})$  is decreasingly smaller than or valueequivalent to  $m|(S - C_{i-1})$ . Therefore,  $S_i$  contains at most  $r_i$  elements of m'-value  $\beta_i$  and hence

$$p(C_i) \le \widetilde{m}'(C_i) = \widetilde{m}'(C_{i-1}) + \widetilde{m}'(S_i)$$
  
$$\le \widetilde{m}'(C_{i-1}) + r_i\beta_i + (|S_i| - r_i)(\beta_i - 1)$$
  
$$= \widetilde{m}(C_{i-1}) + r_i\beta_i + (|S_i| - r_i)(\beta_i - 1)$$
  
$$= \widetilde{m}(C_{i-1}) + \widetilde{m}(S_i) = \widetilde{m}(C_i) = p(C_i),$$

from which equality follows everywhere. In particular,  $S_i$  contains exactly  $r_i$  elements of m'-value  $\beta_i$  and  $|S_i| - r_i$  elements of m'-value  $\beta_i - 1$ , proving the lemma.

By the lemma, m' is value-equivalent to m, and hence m is a decreasingly minimal element of B, that is, (C1) follows.

(B) $\rightarrow$ (C2): The property in (C1) that *m* is decreasingly minimal in  $\overline{B}$  is equivalent to the statement that -m is increasingly maximal in  $-\overline{B}$ , that is, (C2) holds with respect to -m and  $-\overline{B}$ . As we have already proved the implications (C2) $\rightarrow$ (A) $\rightarrow$ (B) $\rightarrow$ (C1), it follows that (C1) holds for -m and  $-\overline{B}$ . But (C1) for -m and  $-\overline{B}$  is just the same as (C2) for *m* and  $\overline{B}$ .

**Example 3.1.** Theorem 3.3 is illustrated here for a graph orientation problem. Consider the undirected graph G = (V, E) in the left of Figure 2, where  $V = \{a, b, c, d\}$  and the set *E* of edges consists of *bc*, *cd*, *da*, and five parallel edges between *a* and *b*. Recall that the associated M-convex set  $\overline{B}$  consists of the in-degree vectors of all orientations of *G*. Let *D* denote the orientation of *G* depicted in the right of Figure 2. We shall apply Theorem 3.3 to verify that the in-degree vector m = (3, 2, 2, 1) of *D* is a dec-min element of  $\overline{B}$ .

To verify Condition (A) for *m*, first note that  $m + \chi_s - \chi_t$  belongs to B for  $s, t \in V$  precisely if there is a dipath from *s* to *t* in *D*. For the in-degree vector m = (3, 2, 2, 1), we have  $m(t) \ge m(s) + 2$  only for

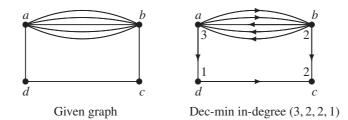


Figure 2: Orientation of a graph (Example 3.1)

(s, t) = (d, a), whereas there is no dipath from *d* to *a* in *D*. Therefore, no 1-tightening step exists for m = (3, 2, 2, 1).

To verify Condition (B) for *m*, recall (from Example 2.1) that a subset *X* of *V* is *m*-tight if and only if there is no edge entering *X* in *D*. The *m*-tight sets are  $\{a, b\}$  and  $\{a, b, d\}$  as well as the empty set and *V*. These *m*-tight sets are *m*-top sets except for  $\{a, b, d\}$ . Thus the (longest) chain of non-empty *m*-top and *m*-tight sets consists of  $C_1 \,\subset\, C_2$  where  $C_1 = \{a, b\}$  and  $C_2 = \{a, b, c, d\}$ . This chain determines the partition of *V* into two parts  $S_1 = \{a, b\}$  and  $S_2 = \{c, d\}$ , for which  $m_1 = m|S_1 = (3, 2)$  and  $m_2 = m|S_2 = (2, 1)$  are both near-uniform. Therefore, Condition (B) is satisfied by m = (3, 2, 2, 1).

To verify Conditions (C1) and (C2), we may enumerate all possible in-degree vectors. The subgraph consisting of three edges bc, cd and da admits six in-degree vectors:

$$\overline{B_0} = \{(0, 1, 1, 1), (1, 0, 1, 1), (1, 1, 1, 0), (1, 1, 0, 1), (0, 0, 2, 1), (0, 0, 1, 2)\}.$$

By adding the five parallel edges we obtain

$$\ddot{B} = \{m_0 + (k, 5 - k, 0, 0) : m_0 \in \ddot{B}_0, 0 \le k \le 5\}.$$

A straightforward inspection reveals that m = (3, 2, 2, 1) is dec-min and inc-max in B. Note that there are three further dec-min elements: (2, 3, 2, 1), (2, 3, 1, 2), and (3, 2, 1, 2).

**Remark 3.1.** The equivalence of (C1) and (C2) in Theorem 3.3 shows that an element of an M-convex set is decreasingly minimal if and only if it is increasingly maximal. In the intersection of two M-convex sets (called an  $M_2$ -convex set in [38]), however, decreasing minimality and increasing maximality do not coincide. For example, consider two M-convex sets

$$B_1 = \{(2, 0, 0, 0), (1, -1, 1, 1), (2, -1, 1, 0), (1, 0, 0, 1)\},\$$
  
$$\widetilde{B_2} = \{(2, 0, 0, 0), (1, -1, 1, 1), (2, 1, 0, 1), (1, 0, 1, 0)\}.$$

In their intersection  $\widetilde{B_1} \cap \widetilde{B_2} = \{(2,0,0,0), (1,-1,1,1)\}$ , the element x = (2,0,0,0) is increasingly maximal while y = (1,-1,1,1) is decreasingly minimal.

#### **3.3** Minimizing the sum of the *k* largest components

A decreasingly minimal element of  $\hat{B}$  has the starting property that its largest component is as small as possible. As a natural extension, one may be interested in finding a member of  $\hat{B}$  for which the sum of the *k* largest components is as small as possible. We refer to this problem as **min** *k*-**largest-sum**.

**Theorem 3.5.** Let *B* be an integral base-polyhedron and *k* an integer with  $1 \le k \le n$ . Then any dec-min element *m* of  $\ddot{B}$  is a solution to the min *k*-largest-sum problem.

**Proof.** Observe first that if  $z_1$  and  $z_2$  are dec-min elements of B, then it follows from the very definition of decreasing minimality that the sum of the first j largest components of  $z_1$  and of  $z_2$  are the same for each j = 1, ..., n.

Let *K* denote the sum of the first *k* largest components of any dec-min element, and assume indirectly that there is a member  $y \in \overline{B}$  for which the sum of its first *k* largest components is smaller than *K*. Assume that the componentwise square-sum of *y* is as small as possible. By the previous observation, *y* is not a dec-min element. Theorem 3.3 implies that there are elements *s* and *t* of *S* for which  $y(t) \ge y(s) + 2$  and  $y' := y - \chi_t + \chi_s$  is in  $\overline{B}$ . The sum of the first *k* largest components of *y'* is at most the sum of the first *k* largest components of *y*, and hence this sum is also smaller than *K*. But this contradicts the choice of *y* since the componentwise square-sum of *y'* is strictly smaller than that of *y*.

This theorem shows that M-convex sets have a striking property. Namely, any dec-min element of an M-convex set  $\ddot{B}$  is simultaneously a solution to the min k-largest-sum problem for each k = 1, 2, ..., n. We say that such an element m is a **simultaneous** k-largest-sum minimizer. This notion has been investigated in the literature of majorization [2, 33, 40] under the name of 'least majorized' element. In particular, Tamir [40] proved the existence of a least majorized integral element for integral base-polyhedra. (Actually, he proved this even for g-polymatroids, but this more general result is an easy consequence of the special case concerning base-polyhedra).

The following result shows that this property actually characterizes dec-min elements of an M-convex set.

# **Theorem 3.6.** Let *B* be an integral base-polyhedron. An element *m* of B is dec-min if and only if *m* is a simultaneous *k*-largest-sum minimizer.

**Proof.** The content of Theorem 3.5 is that a dec-min element is a simultaneous k-largest-sum minimizer. To see the converse, let  $m \in \overline{B}$  be a simultaneous k-largest-sum minimizer. Suppose indirectly that m is not dec-min. By Theorem 3.3, there is a 1-tightening step for m, that is, there are elements s and t of S with  $m(s) \ge m(t) + 2$  such that  $m' := m - \chi_s + \chi_t$  is in  $\overline{B}$ . Let k' denote the number of components of m with value at least m(s). Then the sum of the k' largest components of m' is one less than the sum of the k' largest components of m, contradicting the assumption that m is k-sum-minimizer for each k = 1, 2, ..., n.

While the *k*-largest sum is a natural objective function to conceive from the definition of decminimality, it is also possible to characterize the dec-min elements of an M-convex set as minimizers of other convex functions, which we shall discuss in depth in Section 6. We now turn to investigating the structure of the set of dec-min elements of an M-convex set. In particular, our next goal is to introduce 'dual' objects for dec-minimality, which we call the canonical chain and the canonical partition. These 'dual' objects are constructed in Sections 4 and 5 below.

# 4 Decomposition by pre-decreasingly minimal elements

We continue to assume that *p* is an integer-valued (with possible  $-\infty$  values but with finite p(S)) supermodular function, which implies that B = B'(p) is a non-empty integral base-polyhedron.

One of the main goals of this paper is to show that the set dm(B) of all dec-min elements of  $\overline{B}$  is an M-convex set, meaning that there exists an integral base-polyhedron  $B^{\bullet} \subseteq B$  (obtained by intersecting a face of B with a 'small' integral box) such that  $dm(\overline{B})$  is the set of integral elements of  $B^{\bullet}$ . In addition, we shall show that  $B^{\bullet}$  is a special base-polyhedron which is obtained from a matroid base-polyhedron by translating it with an integral vector. The base-polyhedron  $B^{\bullet}$  will be obtained with the help of a decomposition of B along a certain 'canonical' partition  $\{S_1, S_2, \ldots, S_q\}$  of S into non-empty sets. In this section, we construct the first member  $S_1$  of this partition along with a matroid on  $S_1$ . The set  $S_1$ , depending only on B, will be called the peak-set of B. The peak-set  $S_1$  allows us to decompose the problem of finding dec-min elements of  $\overline{B}$  to two independent problems on  $S_1$  and on  $S - S_1$ . In Section 5, we shall construct the other members  $S_2, S_3, \ldots$  of the partition by applying the same procedure iteratively.

#### 4.1 Max-minimizers and pre-dec-min elements

In Section 3.1 we introduced three related notions, dec-min elements, pre-dec-min elements, and max-minimizers. A dec-min element is always pre-dec-min, and a pre-dec-min element is always a max-minimizer. Recall that an element of  $\overline{B}$  is called a max-minimizer if its largest component is as small as possible, while a max-minimizer is a pre-dec-min element of  $\overline{B}$  if the number of its maximum components is smallest possible.

For a number  $\beta$ , we say that a vector is  $\beta$ -covered if each of its components is at most  $\beta$ . Throughout our discussion,  $\beta_1$  denotes the smallest integer for which  $\overline{B}$  has a  $\beta_1$ -covered element, that is,

$$\beta_1 := \min\{\max\{z(s) : s \in S\} : z \in B\}.$$
(4.1)

Note that an element *m* of  $\overline{B}$  is  $\beta_1$ -covered precisely if *m* is a max-minimizer. Moreover,  $\beta_1$  is equal to the largest component of any pre-dec-min (and hence any dec-min) element of  $\overline{B}$ . For any real number  $\alpha \in \mathbf{R}$ , let  $\lceil \alpha \rceil$  denote the smallest integer not smaller than  $\alpha$ .

**Theorem 4.1.** For the largest component  $\beta_1$  of a max-minimizer of  $\overset{\sim}{B}$ , one has

$$\beta_1 = \max\{\left\lceil \frac{p(X)}{|X|}\right\rceil : \emptyset \neq X \subseteq S\}.$$
(4.2)

**Proof.** Formula (2.4), when applied to the special case with  $f \equiv -\infty$  and  $g \equiv \beta$ , implies that *B* has a  $\beta$ -covered element if and only if

$$\beta|X| \ge p(X)$$
 whenever  $X \subseteq S$ . (4.3)

Moreover, if  $\beta$  is an integer and (4.3) holds, then *B* has an integral  $\beta$ -covered element. As  $\beta |X| \ge p(X)$  holds for an arbitrary  $\beta$  when  $X = \emptyset$ , it follows that the smallest integer  $\beta$  meeting this (4.3) is indeed max{[p(X)/|X|] :  $\emptyset \ne X \subseteq S$ }.

For a  $\beta_1$ -covered element *m* of B, let  $r_1(m)$  denote the number of  $\beta_1$ -valued components of *m*. Recall that for an element  $s \in S$  we denoted the unique smallest *m*-tight set containing *s* by  $T_m(s) = T_m(s; p)$  (that is,  $T_m(s)$  is the intersection of all *m*-tight sets containing *s*). Furthermore, let

$$S_1(m) := \bigcup \{ T_m(t) : m(t) = \beta_1 \}.$$
(4.4)

Then  $S_1(m)$  is *m*-tight and  $S_1(m)$  is actually the unique smallest *m*-tight set containing all the  $\beta_1$ -valued elements of *m*.

**Theorem 4.2.** A  $\beta_1$ -covered element m of B is pre-dec-min if and only if  $m(s) \ge \beta_1 - 1$  for each  $s \in S_1(m)$ .

**Proof.** Necessity. Let *m* be a pre-dec-min element of  $\overline{B}$ . For any  $\beta_1$ -valued element  $t \in S$  and any element  $s \in T_m(t)$ , we claim that  $m(s) \ge \beta_1 - 1$ . Indeed, if we had  $m(s) \le \beta_1 - 2$ , then the vector *m'* arising from *m* by decreasing m(t) by 1 and increasing m(s) by 1 belongs to *B* (since  $T_m(t)$  is the smallest *m*-tight set containing *t*) and has one less  $\beta_1$ -valued components than *m* has, contradicting the assumption that *m* is pre-dec-min.

Sufficiency. Let *m*' be an arbitrary  $\beta_1$ -covered integral element of *B*. Abbreviate  $S_1(m)$  by *Z* and let *h*' denote the number of elements  $z \in Z$  for which  $m'(z) = \beta_1$ . Then

$$|Z|(\beta_1 - 1) + r_1(m) = \widetilde{m}(Z) = p(Z) \le \widetilde{m}'(Z)$$
  
$$\le h'\beta_1 + (|Z| - h')(\beta_1 - 1) = |Z|(\beta_1 - 1) + h'$$
  
$$\le |Z|(\beta_1 - 1) + r_1(m'),$$

from which  $r_1(m) \leq r_1(m')$ , as required.

Define the set-function  $h_1$  on S as follows.

$$h_1(X) := p(X) - (\beta_1 - 1)|X|$$
 for  $X \subseteq S$ . (4.5)

**Theorem 4.3.** For the minimum number  $r_1$  of  $\beta_1$ -valued components of a  $\beta_1$ -covered member of B, one has

$$r_1 = \max\{h_1(X) : X \subseteq S\}.$$
 (4.6)

**Proof.** Let *m* be an element of B for which the maximum of its components is  $\beta_1$ , and let *X* be an arbitrary subset of *S*. Suppose that *X* has  $\ell \beta_1$ -valued components. Then

$$p(X) \leq \widetilde{m}(X) \leq \ell \beta_1 + (|X| - \ell)(\beta_1 - 1) = |X|(\beta_1 - 1) + \ell \leq |X|(\beta_1 - 1) + r_1(m),$$
(4.7)

from which  $r_1(m) \ge p(X) - (\beta_1 - 1)|X| = h_1(X)$ , implying that

$$r_1 = \min\{r_1(m) : m \in B, m \text{ is } \beta_1 \text{-covered }\} \ge \max\{h_1(X) : X \subseteq S\}.$$

In order to prove the reverse inequality, we have to find a  $\beta_1$ -covered integral element *m* of *B* and a subset *X* of *S* for which  $r_1(m) = h_1(X)$ , which is equivalent to requiring that each of the three inequalities in (4.7) holds with equality. That is, the following three **optimality criteria** hold: (a) *X* is *m*-tight, (b) *X* contains all  $\beta_1$ -valued components of *m*, and (c)  $m(s) \ge \beta_1 - 1$  for each  $s \in X$ .

Let *m* be a pre-dec-min element of *B*. Then  $S_1(m)$  is *m*-tight,  $S_1(m)$  contains all  $\beta_1$ -valued elements and, by Theorem 4.2,  $m(s) \ge \beta_1 - 1$  for all  $s \in S_1(m)$ , therefore *m* and  $S_1(m)$  satisfy the three optimality criteria.

Note that  $r_1$  is the number of  $\beta_1$ -valued components of any pre-dec-min element (and in particular, any dec-min element) of  $\tilde{B}$ .

#### **4.2** The peak-set $S_1$

Since the set-function  $h_1$  introduced in (4.5) is supermodular, the maximizers of  $h_1$  are closed under taking intersection and union. Let  $S_1$  denote the unique smallest subset of S maximizing  $h_1$ . In other words,  $S_1$  is the intersection of all sets maximizing  $h_1$ . We call this set  $S_1$  the **peak-set** of B (and of  $\overline{B}$ ).

**Theorem 4.4.** For every pre-dec-min (and in particular, for every dec-min) element m of  $\hat{B}$ , the set  $S_1(m)$  introduced in (4.4) is independent of the choice of m and  $S_1(m) = S_1$ , where  $S_1$  is the peak-set of B.

**Proof.** It follows from Theorem 4.3 that, given a pre-dec-min element *m* of *B*, a subset *X* is maximizing  $h_1$  precisely if the three optimality criteria mentioned in the proof hold. Since  $S_1(m)$  meets the optimality criteria, it follows that  $S_1 \subseteq S_1(m)$ . If, indirectly, there is an element  $s \in S_1(m) - S_1$ , then  $m(s) = \beta_1 - 1$  since  $S_1$  contains all the  $\beta_1$ -valued elements. By the definition of  $S_1(m)$ , there is a  $\beta_1$ -valued element  $t \in S_1(m)$  for which the smallest *m*-tight set  $T_m(t)$  contains *s*, but this is impossible since  $S_1$  is an *m*-tight set containing *t* but not *s*.

Since  $S_1 = S_1(m)$  is *m*-tight and near-uniform, we obtain that

$$\beta_1 = \left\lceil \frac{\overline{m}(S_1)}{|S_1|} \right\rceil = \left\lceil \frac{p(S_1)}{|S_1|} \right\rceil,$$

and the definitions of  $S_1$  and  $r_1$  imply that

$$r_1 = p(S_1) - (\beta_1 - 1)|S_1|.$$
(4.8)

**Proposition 4.5.**  $S_1 = \{s \in S : \text{there is a pre-dec-min element } m \in B \text{ with } m(s) = \beta_1\}$ . For every pre-dec-min element m of B,  $m(s) \ge \beta_1 - 1$  holds for every  $s \in S_1$ , and  $m(s) \le \beta_1 - 1$  holds for every  $s \in S - S_1$ .

**Proof.** If  $m(s) = \beta_1$  for some pre-dec-min m, then  $s \in S_1(m) = S_1$ . Conversely, let  $s \in S_1$  and let m be a pre-dec-min element. We are done if  $m(s) = \beta_1$ . If this is not the case, then  $m(s) = \beta_1 - 1$  by Theorem 4.2. By the definition of  $S_1(m)$ , there is an element  $t \in S_1(m)$  for which  $m(t) = \beta_1$  and  $s \in T_m(t)$ . But then  $m' := m + \chi_s - \chi_t$  is in B,  $m'(s) = \beta_1$  and m' is also pre-dec-min as it is value-equivalent to m.

#### **4.3** Separating along *S*<sub>1</sub>

Let  $S_1$  be the peak-set occurring in Theorem 4.4 and let  $S'_1 := S - S_1$ . Let  $p_1 = p|S_1$  denote the restriction of p to  $S_1$ , and let  $B_1 \subseteq \mathbf{R}^{S_1}$  denote the base-polyhedron defined by  $p_1$ , that is,  $B_1 := B'(p_1)$ . Suppose that  $S'_1 \neq \emptyset$  and let  $p'_1 := p/S_1$ , that is,  $p'_1$  is the set-function on  $S'_1$  obtained from p by contracting  $S_1$   $(p'_1(X) = p(S_1 \cup X) - p(S_1)$  for  $X \subseteq S'_1$ ).

Consider the face *F* of *B* determined by  $S_1$ , that is, *F* is the direct sum of the base-polyhedra  $B_1 = B'(p_1)$  and  $B'_1 = B'(p'_1)$ . Then the dec-min elements of  $B_1$  are exactly the integral elements of the intersection of  $B_1$  and the box given by  $\{x : \beta_1 - 1 \le x(s) \le \beta_1 \text{ for every } s\}$ . Hence the dec-min elements of  $B_1$  are near-uniform.

**Theorem 4.6.** An integral vector  $m = (m_1, m'_1)$  is a dec-min element of B if and only if  $m_1$  is a dec-min element of  $\overline{B_1}$  and  $m'_1$  is a dec-min element of  $\overline{B'_1}$ .

**Proof.** Suppose first that *m* is a dec-min element of *B*. Then  $S_1 = S_1(m)$  by Theorem 4.4 and *m* is a max-minimizer, implying that every component of *m* in  $S_1$  is of value  $\beta_1 - 1$  or value  $\beta_1$ , and *m* has exactly  $r_1$  components of value  $\beta_1$ . Therefore each of the components of  $m_1$  is  $\beta_1 - 1$  or  $\beta_1$ , that is,  $m_1$  is near-uniform. Since  $m_1$  is obviously in  $\overline{B}_1$ ,  $m_1$  is indeed dec-min in  $\overline{B}_1$ .

Since  $\widetilde{m}(S_1) = p(S_1)$ , for a set  $X \subseteq S'_1$ , we have

$$\widetilde{m}'_1(X) = \widetilde{m}(X) = \widetilde{m}(S_1 \cup X) - \widetilde{m}(S_1) = \widetilde{m}(S_1 \cup X) - p(S_1) \ge p(S_1 \cup X) - p(S_1) = p'_1(X).$$

Furthermore

$$\widetilde{m}'_{1}(S'_{1}) = \widetilde{m}(S'_{1}) = \widetilde{m}(S_{1} \cup S'_{1}) - \widetilde{m}(S_{1}) = p(S_{1} \cup S'_{1}) - p(S_{1}) = p'_{1}(S'_{1})$$

that is,  $m'_1$  is in  $B'_1$ . If, indirectly,  $m'_1$  is not dec-min, then, by applying Theorem 3.3 to  $S'_1$ ,  $m'_1$ , and  $p'_1$ , we obtain that there are elements t and s of  $S'_1$  for which  $m'_1(t) \ge m'_1(s) + 2$  and (\*) no  $t\bar{s}$ -set exists which is  $m'_1$ -tight with respect to  $p'_1$ . On the other hand, m is a dec-min element of  $\bar{B}$  for which

$$m(t) = m'_1(t) \ge m'_1(s) + 2 = m(s) + 2,$$

and hence there must be a  $t\bar{s}$ -set Y which is *m*-tight with respect to p.

Since  $S_1$  is *m*-tight with respect to *p*, the set  $S_1 \cup Y$  is also *m*-tight with respect to *p*. Let  $X := S'_1 \cap Y$ . Then

$$\widetilde{m}(X) + \widetilde{m}(S_1) = \widetilde{m}(S_1 \cup Y) = p(S_1 \cup Y) = p(S_1 \cup X),$$

and hence

$$\widetilde{m}_{1}'(X) = \widetilde{m}(X) = p(S_{1} \cup X) - \widetilde{m}(S_{1}) = p(S_{1} \cup X) - p(S_{1}) = p_{1}'(X).$$

that is, X is a  $t\bar{s}$ -set which is  $m'_1$ -tight with respect to  $p'_1$ , in contradiction with statement (\*) above that no such set exists.

To see the converse, assume that  $m_1$  is a dec-min element of  $B_1$  and  $m'_1$  is a dec-min element of  $B'_1$ . This immediately implies that m is in the face F of B determined by  $S_1$ . Suppose, indirectly, that m is not a dec-min element of B. By Theorem 3.3, there are elements t and s of S for which  $m(t) \ge m(s) + 2$  and (\*\*) no  $t\overline{s}$ -set exists which is m-tight with respect to p. If  $t \in S_1$ , then s cannot be in  $S_1$  since the m-value of each element of  $S_1$  is  $\beta_1$  or  $\beta_1 - 1$ . But  $S_1$  is  $m_1$ -tight with respect to p and hence it is m-tight with respect to p, contradicting property (\*\*). Therefore t must be in  $S'_1$ , implying, by Proposition 4.5, that s is also in  $S'_1$ .

Since  $m'_1$  is a dec-min element of  $B'_1$ , there must be a  $t\overline{s}$ -set  $Y \subset S'_1$  which is  $m'_1$ -tight with respect to  $p'_1$ . It follows that

$$\widetilde{m}(Y) = \widetilde{m}'_1(Y) = p'_1(Y) = p(S_1 \cup Y) - p(S_1) \le \widetilde{m}(S_1 \cup Y) - \widetilde{m}(S_1) = \widetilde{m}(Y),$$

from which  $\widetilde{m}(S_1 \cup Y) = p(S_1 \cup Y)$ , contradicting property (\*\*) that no *ts*-set exists which is *m*-tight with respect to *p*.

An important consequence of Theorem 4.6 is that, in order to find a dec-min element of  $\overline{B}$ , it will suffice to find separately a dec-min element of  $\overline{B_1}$  (which was shown above to be a near-uniform vector) and a dec-min element of  $\overline{B'_1}$ . The algorithmic details are discussed in [13].

**Theorem 4.7.** Let  $S_1$  be the peak-set of B. For an element  $m_1$  of  $B_1$ , the following properties are pairwise equivalent.

(A1)  $m_1 has r_1 (= p(S_1) - (\beta_1 - 1)|S_1| > 0)$  components of value  $\beta_1$  and  $|S_1| - r_1 (\ge 0)$  components of value  $\beta_1 - 1$ .

(A2)  $m_1$  is near-uniform.

(A3)  $m_1$  is dec-min in  $B_1$ .

(B1)  $m_1$  is the restriction of a dec-min element m of  $\overset{\frown}{B}$  to  $S_1$ .

(B2)  $m_1$  is the restriction of a pre-dec-min element m of  $\overline{B}$  to  $S_1$ .

**Proof.** The implications  $(A1) \rightarrow (A2) \rightarrow (A3)$  and  $(B1) \rightarrow (B2)$  are immediate from the definitions.

(A3) $\rightarrow$ (B1): Let  $m'_1$  be an arbitrary dec-min element of  $B'_1$ . By Theorem 4.6,  $m := (m_1, m'_1)$  is a dec-min element of  $\overline{B}$  and hence  $m_1$  is indeed the restriction of a dec-min element of  $\overline{B}$  to  $S_1$ .

(B2) $\rightarrow$ (A1): By Theorems 4.2 and 4.4, we have  $m_1(s) \ge \beta_1 - 1$  for each  $s \in S_1(m) = S_1$ , that is,  $\beta_1 - 1 \le m_1(s) \le \beta_1$ . By letting r' denote the number of  $\beta_1$ -valued components of  $m_1$ , we obtain by (4.8) that

$$r_1 + (\beta_1 - 1)|S_1| = p_1(S_1) = \widetilde{m}_1(S_1) = (\beta_1 - 1)|S_1| + r$$

and hence  $r' = r_1$ .

Theorem 4.6 implies that, in order to characterize the set of dec-min elements of  $\overline{B}$ , it suffices to characterize the set of dec-min elements of  $\overline{B'_1}$ .

**Theorem 4.8.** Let  $\beta_2$  denote the smallest integer for which  $B'_1$  has a  $\beta_2$ -covered element, that is,  $\beta_2 = \beta(B'_1)$ . Then

$$\beta_2 = \max\{\left\lceil \frac{p_1'(X)}{|X|} \right\rceil : \emptyset \neq X \subseteq S - S_1\},\tag{4.9}$$

where  $p'_1(X) = p(X \cup S_1) - p(S_1)$ . Furthermore,  $\beta_2$  is the largest component in  $S - S_1$  of every dec-min element of  $\tilde{B}$ , and  $\beta_2 < \beta_1$ .

**Proof.** Formula (4.9) follows by applying Theorem 4.1 to base-polyhedron  $B'_1 (= B'(p'_1))$  in place of *B*. By Theorem 4.6, the largest component in  $S - S_1$  of any dec-min element *m* of  $\overline{B}$  is  $\beta_2$ . By Theorem 4.4,  $S_1(m) = S_1$ , and the definition of  $S_1(m)$  shows that  $m(s) \le \beta_1 - 1$  holds for every  $s \in S - S_1$ , from which  $\beta_2 < \beta_1$  follows.

#### 4.4 The matroid $M_1$ on $S_1$

It is known from the theory of base-polyhedra that the intersection of an integral base-polyhedron with an integral box is a (possibly empty) integral base-polyhedron. Moreover, if the box in question is small, then the intersection is actually a translated matroid base-polyhedron (meaning that the intersection arises from a matroid base-polyhedron by translating it with an integral vector). This result is a consequence of the theorem that (\*) any integral base-polyhedron in the unit (0, 1)-cube is the convex hull of (incidence vectors of) the bases of a matroid.

Consider the special small integral box  $T_1 \subseteq \mathbb{Z}^{S_1}$  defined by

$$T_1 := \{ x : \beta_1 - 1 \le x(s) \le \beta_1 \}$$

and its intersection  $B_1^{\bullet} := B_1 \cap T_1$  with the base-polyhedron  $B_1$  investigated above. Therefore  $B_1^{\bullet}$  is a translated matroid base-polyhedron and Theorem 4.7 implies the following.

**Corollary 4.9.** The dec-min elements of  $B_1^{\circ}$  are exactly the integral elements of the translated matroid base-polyhedron  $B_1^{\circ}$ .

Our next goal is to reprove Corollary 4.9 by concretely describing the matroid in question and not relying on the background theorem (\*) mentioned above. For a dec-min element  $m_1$  of  $B_1$ , let

$$L_1(m_1) := \{ s \in S_1 : m_1(s) = \beta_1 \}.$$

We know from Theorem 4.7 that  $|L_1(m_1)| = r_1$ . Define a set-system  $\mathcal{B}_1$  as follows:

$$\mathcal{B}_1 := \{L \subseteq S_1 : L = L_1(m_1) \text{ for some dec-min element } m_1 \text{ of } B_1\}.$$
(4.10)

We need the following characterization of  $\mathcal{B}_1$ .

**Proposition 4.10.** An  $r_1$ -element subset L of  $S_1$  is in  $\mathcal{B}_1$  if and only if

$$|L \cap X| \ge p_1'(X) := p_1(X) - (\beta_1 - 1)|X| \text{ whenever } X \subseteq S_1.$$
(4.11)

**Proof.** Suppose first that  $L \in \mathcal{B}_1$ , that is, there is a dec-min element  $m_1$  of  $\overline{\mathcal{B}_1}$  for which  $L = L_1(m_1)$ . Then

$$(\beta_1 - 1)|X| + |X \cap L| = \widetilde{m}_1(X) \ge p_1(X),$$

for every subset  $X \subseteq S_1$  from which (4.11) follows.

To see the converse, let  $L \subseteq S_1$  be an  $r_1$ -element set meeting (4.11). Let

$$m_1(s) := \begin{cases} \beta_1 & \text{if } s \in L \\ \beta_1 - 1 & \text{if } s \in S - L. \end{cases}$$

$$(4.12)$$

Then obviously  $L = L_1(m_1)$ . Furthermore,

$$\widetilde{m}_1(S_1) = (\beta_1 - 1)|S_1| + |L| = (\beta_1 - 1)|S_1| + r_1 = p(S_1)$$

and

$$\widetilde{m}_1(X) = (\beta_1 - 1)|X| + |L \cap X| \ge p_1(X) \text{ whenever } X \subset S_1,$$

showing that  $m_1 \in B_1$ . Since  $m_1 \in T_1$ , we conclude that  $m_1$  is a dec-min element of  $B_1$ .

**Theorem 4.11.** The set-system  $\mathcal{B}_1$  defined in (4.10) forms the set of bases of a matroid  $M_1$  on ground-set  $S_1$ .

**Proof.** The set-system  $\mathcal{B}_1$  is clearly non-empty and all of its members are of cardinality  $r_1$ . It is widely known [6] that for an integral submodular function b on a ground-set  $S_1$  the set-system

$$\{L \subseteq S_1 : |L \cap X| \le b(X) \text{ whenever } X \subset S_1, |L| = b(S_1)\},\$$

if non-empty, satisfies the matroid basis axioms. This implies for the supermodular function  $p'_1$  that the set-system  $\{L : |L \cap X| \ge p'_1(X)$  whenever  $X \subset S_1$ ,  $|L| = p'_1(S_1)\}$ , if non-empty, forms the set of bases of a matroid. By applying this fact to the supermodular function  $p'_1$  defined by  $p'_1(X) := p_1(X) - (\beta_1 - 1)|X|$ , one obtains that  $\mathcal{B}_1$  is non-empty and forms the set of bases of a matroid.

With this matroid  $M_1$ , we can rewrite Corollary 4.9 into a more explicit form, which is convenient for our subsequent discussion.

**Corollary 4.12.** Let  $\Delta_1 : S_1 \to \mathbb{Z}$  denote the integral vector defined by  $\Delta_1(s) := \beta_1 - 1$  for  $s \in S_1$ . A member  $m_1$  of  $B_1$  is decreasingly minimal if and only if there is a basis  $B_1$  of  $M_1$  such that  $m_1 = \chi_{B_1} + \Delta_1$ .

#### **4.5 Value-fixed elements of** *S*<sub>1</sub>

We say that an element  $s \in S$  is **value-fixed** with respect to B if m(s) is the same for every dec-min element m of B. In Section 6.3, we will show a description of value-fixed elements of B. In the present section, we consider the value-fixed elements with respect to  $B_1$ , that is,  $s \in S_1$  is value-fixed if  $m_1(s)$  is the same for every dec-min element  $m_1 \in B_1$ . Recall that  $m_1 \in B_1$  was shown to be dec-min precisely if  $\beta_1 - 1 \le m_1(s) \le \beta_1$  for each  $s \in S_1$ .

A **loop** of a matroid is an element  $s \in S_1$  not belonging to any basis. (Often the singleton  $\{s\}$  is called a loop, that is,  $\{s\}$  is a one-element circuit). A **co-loop** (or cut-element or isthmus) of a matroid is an element *s* belonging to all bases.

#### **Proposition 4.13.** $M_1$ has no loops.

**Proof.** By Proposition 4.5, for every  $s \in S_1$  there is a pre-dec-min element *m* of B for which  $m(s) = \beta_1$ . Then  $m_1 := m | S_1$  is a pre-dec-min element of  $B_1$  by Theorem 4.7 from which  $s_1$  belongs to a basis of  $M_1$  by Corollary 4.12.

The proposition implies that:

**Proposition 4.14.** If  $s \in S_1$  is value-fixed (with respect to  $B_1$ ), then  $m_1(s) = \beta_1$  for every dec-min element  $m_1$  of  $B_1$ .

By Corollary 4.12, an element  $s \in S_1$  is a co-loop of  $M_1$  if and only if  $m_1(s) = \beta_1$  holds for every dec-min element  $m_1$  of  $B_1$ . This and Theorem 4.6 imply the following.

**Theorem 4.15.** For an element  $s \in S_1$ , the following properties are pairwise equivalent.

- (A) s is a co-loop of  $M_1$ .
- (B) s is value-fixed.
- (C)  $m(s) = \beta_1$  holds for every dec-min element m of B.

Our next goal is to characterize the set of value-fixed elements of  $S_1$ . Consider the family of subsets  $S_1$  defined by

$$\mathcal{F}_1 := \{ X \subseteq S_1 : \beta_1 | X | = p_1(X) \}.$$
(4.13)

The empty set belongs to  $\mathcal{F}_1$  and it is possible that  $\mathcal{F}_1$  has no other members. By standard submodularity arguments,  $\mathcal{F}_1$  is closed under taking union and intersection. Let  $F_1$  denote the unique largest member of  $\mathcal{F}_1$ . It is possible that  $F_1 = S_1$  in which case we call  $S_1$  **degenerate**.

**Theorem 4.16.** An element  $s \in S_1$  is value-fixed if and only if  $s \in F_1$ .

**Proof.** Let  $m_1$  be a dec-min member of  $B_1$ . Then

$$\beta_1|F_1| \ge \widetilde{m}_1(F_1) \ge p_1(F_1) = \beta_1|F_1|$$

and hence we must have  $\beta_1 = m_1(s)$  for every  $s \in F_1$ , that is, the elements of  $F_1$  are indeed value-fixed.

Conversely, let *s* be value-fixed, that is,  $m_1(s) = \beta_1$  for each dec-min element  $m_1$  of  $B_1$ . Let  $m_1$  be a dec-min member of  $\widetilde{B_1}$ . Let *Z* denote the unique smallest set containing *s* for which  $\widetilde{m_1}(Z) = p_1(Z)$ . (That is,  $Z = T_{m_1}(s; p_1)$ .) We claim that  $m_1(t) = \beta_1$  for every element  $t \in Z$ . For if  $m_1(t) = \beta_1 - 1$  for some *t*, then  $m'_1 := m_1 - \chi_s + \chi_t$  would also be a dec-min member of  $\widetilde{B_1}$ , contradicting the assumption that *s* is value-fixed. Therefore  $p_1(Z) = \widetilde{m_1}(Z) = \beta_1 |Z|$  from which the definition of  $F_1$  implies that  $Z \subseteq F_1$  and hence  $s \in F_1$ .

### **5** Description of the set of all decreasingly minimal elements

Let B = B'(p) denote again an integral base-polyhedron defined by the (integer-valued) supermodular function p. As in the previous section,  $\overline{B}$  continues to denote the M-convex set consisting of the integral vectors (points, elements) of B. Our present goal is to provide a complete description of the set of all decreasingly minimal (= egalitarian) elements of  $\overline{B}$  by identifying a partition of the groundset, to be named the canonical partition, inherent in this problem. As a consequence, we show that the set of dec-min elements has a matroidal structure and this feature makes it possible to solve the minimum cost dec-min problem.

#### 5.1 Canonical partition and canonical chain

In Section 4 we introduced the integer  $\beta_1$  as the minimum of the largest component of the elements of  $\overline{B}$  as well as the peak-set  $S_1$ . The peak-set  $S_1$  induces a face of B, which is the direct sum of base-polyhedra  $B_1 = B'(p_1)$  and  $B'_1 = B'(p'_1)$ , where  $p_1$  denotes the restriction of p to  $S_1$  and  $p'_1$  is obtained from p by contracting  $S_1$  (that is,  $p'_1(X) = p(S_1 \cup X) - p(S_1)$ ).

A consequence of Theorem 4.6 is that, in order to characterize the set of dec-min elements of  $\vec{B}_1$ , it suffices to characterize separately the dec-min elements of  $\vec{B}_1$  and the dec-min elements of  $\vec{B}_1$ . By Theorem 4.7, the dec-min elements of  $\vec{B}_1$  are characterized as those elements of  $\vec{B}_1$  which belong to the small box  $T_1 := \{x \in \mathbb{R}^{S_1} : \beta_1 - 1 \le x(s) \le \beta_1 \text{ for } s \in S_1\}$ . If the peak-set  $S_1$  happens to be the whole ground-set S, then the characterization of the set of dec-min elements of  $\vec{B}_1$  is complete. If  $S_1 \subset S$ , then our remaining task is to characterize the set of dec-min elements of  $\vec{B}_1$ . This can be done by repeating iteratively the separation procedure to the base-polyhedron  $B'_1 = B'(p'_1) \subseteq \mathbb{R}^{S-S_1}$  described in Section 4 for B.

In this iterative way, we are going to define a partition  $\mathcal{P}^* = \{S_1, S_2, \dots, S_q\}$  of S which determines a chain  $C^* = \{C_1, C_2, \dots, C_q\}$  where  $C_i := S_1 \cup S_2 \cup \dots \cup S_i$  (in particular  $C_q = S$ ), and the supermodular function

$$p'_i := p/C_i$$
 on set  $\overline{C_i} := S - C_i$ 

which defines the base-polyhedron  $B'_i = B'(p'_i)$  in  $\mathbf{R}^{\overline{C_i}}$ . Moreover, we define iteratively a decreasing sequence  $\beta_1 > \beta_2 > \cdots > \beta_q$  of integers, a small box

$$T_i := \{ x \in \mathbf{R}^{S_i} : \beta_i - 1 \le x(s) \le \beta_i \text{ for } s \in S_i \},$$

$$(5.1)$$

and the supermodular function  $p_i$  on  $S_i$ , where

$$p_i := p'_{i-1} | S_i \quad (= (p/C_{i-1}) | S_i), \tag{5.2}$$

that is,

 $p_i(X) = p(X \cup C_{i-1}) - p(C_{i-1}) \text{ for } X \subseteq S_i.$ 

Let  $B_i := B'(p_i) \subseteq \mathbf{R}^{S_i}$  be the base-polyhedron defined by  $p_i$ .

In the general step, suppose that the pairwise disjoint non-empty sets  $S_1, S_2, \ldots, S_{j-1}$  have already been defined, along with the decreasing sequence  $\beta_1 > \beta_2 > \cdots > \beta_{j-1}$  of integers. If  $S = S_1 \cup \cdots \cup S_{j-1}$ , then by taking q := j-1, the iterative procedure terminates. So suppose that this is not the case, that is,  $C_{j-1} \subset S$ . We assume that  $p_{j-1}$  on  $S_{j-1}$  has been defined as well as  $p'_{j-1}$  on  $\overline{C_{j-1}}$ .

Let

$$\beta_j = \max\{\left\lceil \frac{p'_{j-1}(X)}{|X|} \right\rceil : \emptyset \neq X \subseteq \overline{C_{j-1}}\},\tag{5.3}$$

that is,

$$\beta_j = \max\{\left\lceil \frac{p(X \cup C_{j-1}) - p(C_{j-1})}{|X|} \right\rceil : \emptyset \neq X \subseteq \overline{C_{j-1}}\}.$$
(5.4)

Note that, by the iterative feature of these definitions, Theorem 4.8 implies that

$$\beta_j < \beta_{j-1}$$
.

Furthermore, let  $h_j$  be a set-function on  $\overline{C_{j-1}}$  defined as follows:

$$h_j(X) := p'_{j-1}(X) - (\beta_j - 1)|X| \text{ for } X \subseteq \overline{C_{j-1}},$$
 (5.5)

and let  $S_j \subseteq \overline{C_{j-1}}$  be the peak-set of  $\overline{C_{j-1}}$  assigned to  $B'_{j-1} := B'(p'_{j-1})$ , that is,  $S_j$  is the smallest subset of  $\overline{C_{j-1}}$  maximizing  $h_j$ . Finally, let  $p_j := p'_{j-1}|S_j$  and let  $p'_j := p'_{j-1}/S_j$ . Observe by (2.3) that  $p'_j = p/C_j$ . Therefore  $p_j$  is a set-function on  $S_j$  while  $p'_j$  is defined on  $\overline{C_j}$ .

We shall refer to the partition  $\mathcal{P}^*$  and the chain  $C^*$  defined above as the **canonical partition** and **canonical chain** of *S*, respectively, assigned to *B*, while the sequence  $\beta_1 > \cdots > \beta_q$  will be called the **essential value-sequence** of  $\overline{B}$ .

**Example 5.1.** In the orientation problem in Example 3.1, the essential value-sequence is given by  $\beta_1 = 3$  and  $\beta_2 = 2$  (with q = 2) and the canonical partition is  $\mathcal{P}^* = \{S_1, S_2\}$  with  $S_1 = \{a, b\}$  and  $S_2 = \{c, d\}$ . The canonical partition for the M-convex set arising from such orientation problem coincides with the "density decomposition" of Borradaile et al. [5].

**Example 5.2.** In the proof of Theorem 3.3, we considered the longest chain consisting of non-empty *m*-tight and *m*-top sets, where *m* is a dec-min element of the M-convex set  $\overline{B}$ . One may wonder whether this longest chain is the same as the canonical chain of  $\overline{B}$ . The following example, from the area of graph orientations, demonstrates that the answer is negative.

Let G = (V, E) be an undirected graph where  $V = \{a, b, c, d, x, y\}$  and E consists of four parallel edges between a and b, four parallel edges between c and d, and the following four further edges: ax, cx, by, dy. Consider the M-convex set B consisting of the in-degree vectors of all possible orientations of G. Graph G has an orientation where the in-degree of every node is 2. This shows that the vector m := (2, 2, 2, 2, 2, 2, 2) is in B, and this uniform vector is obviously a dec-min element of B. For this m, the chain  $\{a, b\} \subset \{a, b, c, d\} \subset \{a, b, c, d, x\} \subset V$  is a longest chain consisting of four non-empty m-tight and m-top sets (and there are three other longest chains). But the canonical chain of B consists of the single member  $\{V\}$ .

Let  $B^{\oplus}$  denote the face of *B* defined by the canonical chain  $C^*$ , that is,  $B^{\oplus}$  is the direct sum of the *q* base-polyhedra  $B'(p_i)$  (i = 1, ..., q). Finally, let  $T^*$  be the direct sum of the small boxes  $T_i$  (i = 1, ..., q), that is,  $T^*$  is the integral box defined by the essential value-sequence as follows:

$$T^* := \{ x \in \mathbf{R}^S : \beta_i - 1 \le x(s) \le \beta_i \text{ whenever } s \in S_i \ (i = 1, \dots, q) \},$$
(5.6)

and let

$$B^{\bullet} := B^{\oplus} \cap T^*$$

This set  $B^{\bullet}$  is an integral base-polyhedron, since the intersection of an integral base-polyhedron with an integral box is always an integral base-polyhedron. Furthermore,  $B^{\bullet}$  is the direct sum of the qbase-polyhedra  $B_i \cap T_i$  (i = 1, ..., q), where  $B_i = B'(p_i)$ , implying that a vector m is in  $\overline{B^{\bullet}}$  if and only if each  $m_i$  is in  $\overline{B_i} \cap T_i$ , where  $m_i = m|S_i$ .

**Theorem 5.1.** Let B = B'(p) be an integral base-polyhedron on ground-set S. The set of decreasingly minimal elements of  $\overline{B}$  is (the M-convex set)  $\overline{B^{\bullet}}$ . Equivalently, an element  $m \in \overline{B}$  is decreasingly minimal if and only if its restriction  $m_i := m|S_i$  to  $S_i$  belongs to  $B_i \cap T_i$  for each i = 1, ..., q, where  $\{S_1, ..., S_q\}$  is the canonical partition of S belonging to B,  $T_i$  is the small box defined in (5.1), and  $B_i$  is the base-polyhedron  $B'(p_i)$  belonging to the supermodular set-function  $p_i$  defined in (5.2).

**Proof.** We use induction on q. Suppose first that q = 1, that is,  $S_1 = S$  and  $B_1 = B$ . If m is a dec-min element of B, then the equivalence of Properties (A1) and (A3) in Theorem 4.7 implies that m is in  $B^{\bullet}$ . If, conversely,  $m \in B^{\bullet}$ , then m is near-uniform and, by the equivalence of Properties (A1) and (A3) in Theorem 4.7 again, m is dec-min.

Suppose now that  $q \ge 2$  and consider the base-polyhedron  $B'_1 = B'(p'_1)$  appearing in Theorem 4.6. The iterative definition of the canonical partition  $\mathcal{P}^*$  implies that the canonical partition of  $S - S_1$  assigned to  $B'_1$  is  $\{S_2, \ldots, S_q\}$  and the essential value-sequence belonging to  $B'_1$  is  $\beta_2 > \beta_3 > \cdots > \beta_q$ . Also, the canonical chain  $C' := \{C'_2, \ldots, C'_q\}$  of  $B'_1$  consists of the sets  $C'_i = S_2 \cup \cdots \cup S_i = C_i - S_1$  $(i = 2, \ldots, q)$ .

By applying the inductive hypothesis to  $B'_1$ , we obtain that an integral element  $m'_1$  of  $B'_1$  is dec-min if and only if  $m'_1$  is in the face of  $B'_1$  defined by chain C' and  $m'_1$  belongs to the box  $T' := \{x \in \mathbb{R}^{S-S_1} : \beta_i - 1 \le x(s) \le \beta_i \text{ whenever } s \in S_i \ (i = 2, ..., q)\}$ . By applying Theorem 4.6, we are done in this case as well.

**Corollary 5.2.** Let B = B'(p) be an integral base-polyhedron on ground-set S. Let  $\{C_1, \ldots, C_q\}$  be the canonical chain,  $\{S_1, \ldots, S_q\}$  the canonical partition of S, and  $\beta_1 > \beta_2 > \cdots > \beta_q$  the essential value-sequence belonging to  $\overline{B}$ . Then an element  $m \in \overline{B}$  is decreasingly minimal if and only if each  $C_i$  is m-tight (that is,  $\overline{m}(C_i) = p(C_i)$ ) and  $\beta_i - 1 \le m(s) \le \beta_i$  holds for each  $s \in S_i$   $(i = 1, \ldots, q)$ .

#### 5.2 Obtaining the canonical chain and value-sequence from a dec-min element

The main goal of this section is to show that the canonical chain and value-sequence can be rather easily obtained from an arbitrary dec-min element of  $\tilde{B}$ . This approach will be crucial in developing a polynomial algorithm in [13] for computing the essential value-sequence along with the canonical chain and partition.

Let *m* be an element of *B*. We called a set  $X \subseteq S$  *m*-tight if  $\widetilde{m}(X) = p(X)$ . Recall from Section 2 that, for a subset  $Z \subseteq S$ ,  $T_m(Z) = T_m(Z; p)$  denoted the unique smallest *m*-tight set including *Z*, that is,  $T_m(Z)$  is the intersection of all the *m*-tight sets including *Z*. Obviously,

$$T_m(Z) = \cup (T_m(z) : z \in Z).$$
(5.7)

Let *m* be an arbitrary dec-min element of B. We proved that *m* is in the face  $B^{\oplus}$  of *B* defined by the canonical chain  $C^* = \{C_1, \ldots, C_q\}$  belonging to *B*. Therefore each  $C_i$  is *m*-tight with respect to *p*. Furthermore  $m_i := m | S_i$  belongs to the box  $T_i$  defined in (5.1). This implies that  $m(s) \ge \beta_i - 1$  for every  $s \in C_i$  and  $m(s') \le \beta_{i+1}$  for every  $s' \in \overline{C_i}$ . (The last inequality holds indeed since  $s' \in \overline{C_i}$  implies that  $s' \in S_j$  for some  $j \ge i + 1$  from which  $m(s') \le \beta_j \le \beta_{i+1}$ .) Since  $\beta_{i+1} \le \beta_i - 1$ , we obtain that each  $C_i$  is an *m*-top set.

Since  $m_i$  is near-uniform on  $S_i$  with values  $\beta_i$  and possibly  $\beta_i - 1$ , we obtain

$$\beta_i = \left\lceil \frac{\widetilde{m}_i(S_i)}{|S_i|} \right\rceil = \left\lceil \frac{p_i(S_i)}{|S_i|} \right\rceil = \left\lceil \frac{p(C_i) - p(C_{i-1})}{|S_i|} \right\rceil.$$

Let  $L_i := \{s \in S - C_{i-1} : m(s) = \beta_i\}$  and let  $r_i := |L_i|$ . Then  $p_i(S_i) = \widetilde{m}_i(S_i) = (\beta_i - 1)|S_i| + r_i$  and hence

$$r_i = p(C_i) - p(C_{i-1}) - (\beta_i - 1)|S_i|.$$
(5.8)

The content of the next lemma is that, once  $C_{i-1}$  is given, the next member  $C_i$  of the canonical chain (and hence  $S_i$ , as well) can be expressed with the help of m. Recall that  $T_m(L_i) = T_m(L_i; p)$  denoted the smallest *m*-tight set including  $L_i$ .

**Lemma 5.3.**  $C_i = C_{i-1} \cup T_m(L_i; p)$ .

**Proof.** Recall the definition of function  $h_i$  given in (5.5). We have

$$h_i(S_i) = r_i \tag{5.9}$$

since  $h_i(S_i) = p'_{i-1}(S_i) - (\beta_i - 1)|S_i| = p(S_i \cup C_{i-1}) - p(C_{i-1}) - (\beta_i - 1)|S_i| = \widetilde{m}(C_i) - \widetilde{m}(C_{i-1}) - (\beta_i - 1)|S_i| = \widetilde{m}(S_i) - (\beta_i - 1)|S_i| = r_i.$ 

Since  $L_i \subseteq C_i$  and each of  $C_{i-1}$ ,  $C_i$ , and  $T_m(L_i)$  are *m*-tight, we have  $C_{i-1} \cup T_m(L_i; p) \subseteq C_i$ . For  $X' := T_m(L_i) \cap \overline{C_{i-1}}$  we have

$$h_{i}(X') = p(C_{i-1} \cup T_{m}(L_{i})) - p(C_{i-1}) - (\beta_{i} - 1)|X'_{i}|$$
  
$$= \widetilde{m}(C_{i-1} \cup T_{m}(L_{i})) - \widetilde{m}(C_{i-1}) - (\beta_{i} - 1)|X'_{i}|$$
  
$$= \widetilde{m}(X') - (\beta_{i} - 1)|X'_{i}| = |L_{i}| = r_{i} = h_{i}(S_{i}),$$

that is, X' is also a maximizer of  $h_i(X)$ . Since  $S_i$  was the smallest maximizer of  $h_i$ , we conclude that  $C_{i-1} \cup T_m(L_i; p) \supseteq C_i$ .

The lemma implies that both the essential value-sequence  $\beta_1 > \cdots > \beta_q$  and the canonical chain  $C^*$  belonging to  $\overline{B}$  can be directly obtained from *m*.

**Corollary 5.4.** Let *m* be an arbitrary dec-min element of *B*. The essential value-sequence and the canonical chain belonging to B can be described as follows. Value  $\beta_1$  is the largest *m*-value and  $C_1$  is the smallest *m*-tight set containing all  $\beta_1$ -valued elements. Moreover, for i = 2, ..., q,  $\beta_i$  is the largest value of  $m|\overline{C_{i-1}}$  and  $C_i$  is the smallest *m*-tight set (with respect to *p*) containing each element of *m*-value at least  $\beta_i$ .

A detailed algorithm based on this corollary will be described in [13]. Note that a dec-min element m of B may have more than q distinct values. For example, if q = 1 and  $L_1 \subset C_1 = S$ , then m has two distinct values, namely  $\beta_1$  on the elements of  $L_1$  and  $\beta_1 - 1$  on the elements of  $S - L_1$ , while its essential value-sequence consists of the single member  $\beta_1$ .

A direct proof Corollary 5.4 implies that the chain of subsets and value-sequence assigned to a decmin element m of  $\overline{B}$  in the corollary do not depend on the choice of m. Here we describe an alternative, direct proof of this consequence.

**Theorem 5.5.** Let *m* be an arbitrary dec-min element of *B*. Let  $\beta_1$  denote the largest value of *m* and let  $C_1$  denote the smallest *m*-tight set (with respect to *p*) containing all  $\beta_1$ -valued elements. Moreover, for i = 2, 3, ..., q, let  $\beta_i$  denote the largest value of  $m|\overline{C_{i-1}}$  and let  $C_i$  denote the smallest *m*-tight set containing each element of *m*-value at least  $\beta_i$ . Then the chain  $C_1 \subset C_2 \subset \cdots \subset C_q$  and the sequence  $\beta_1 > \beta_2 > \cdots > \beta_q$  do not depend on the choice of *m*.

**Proof.** Let *z* be dec-min element of *B*. We use induction on the number of elements *t* of *S* for which m(t) > z(t). If no such an element *t* exists, then m = z and there is nothing to prove. So assume that  $z \neq m$ .

Let  $L_i := \{t \in S_i : m(t) = \beta_i\}$ . As *m* is dec-min, the definition of  $C_i$  implies that  $m(s) = \beta_i - 1$ holds for every element  $s \in S_i - L_i$ . Let  $t \in L_i$  and let  $s \in T_m(t) - L_i$ . Then  $m' := m + \chi_s - \chi_t$  is also a dec-min element of  $\overline{B}$ , and we say that m' is obtained from *m* by an elementary step. Observe that  $T_m(t) = T_{m'}(s)$  and hence the chain and the value-sequence assigned to m' is the same as those assigned to *m*.

Let *i* denote the smallest subscript for which  $m|S_i$  and  $z|S_i$  differ. Since *z* is dec-min,  $z(s) \le \beta_i$  holds for every  $s \in S_i$ . Let  $L'_i := \{t \in S_i : z(t) = \beta_i\}$ . Then  $z(v) \le \beta_i - 1$  for every  $v \in S_i - L'_i$ , and  $|L'_i| \le |L_i|$  as *z* is dec-min. Therefore

$$\widetilde{z}(S_i) \leq \beta_i |L'_i| + (\beta_i - 1)(|S_i - L'_i|) = (\beta_i - 1)|S_i| + |L'_i| \leq (\beta_i - 1)|S_i| + |L_i|.$$

On the other hand,

$$\widetilde{z}(S_i) = \widetilde{z}(C_i) - \widetilde{z}(C_{i-1}) = \widetilde{z}(C_i) - \widetilde{m}(C_{i-1})$$
  

$$\geq p(C_i) - \widetilde{m}(C_{i-1}) = \widetilde{m}(C_i) - \widetilde{m}(C_{i-1}) = \widetilde{m}(S_i) = (\beta_i - 1)|S_i| + |L_i|.$$

Therefore we have equality throughout, in particular,  $\tilde{z}(C_i) = p(C_i)$ ,  $|L'_i| = |L_i|$ , and  $z(v) = \beta_i - 1$  for every  $v \in S_i - L'_i$ .

Let  $t \in L_i$  be an element for which m(t) > z(t). Then  $m(t) = \beta_i$  and  $z(t) = \beta_i - 1$ . It follows that  $T_m(t)$  contains an element *s* for which z(s) > m(s), implying that  $m(s) = \beta_i - 1$  and  $z(s) = \beta_i$ . Now m(t) > m'(t) = z(t) holds for the dec-min element  $m' := m + \chi_s - \chi_t$  obtained from *m* by an elementary step, and therefore we are done by induction.

#### 5.3 Matroidal description of the set of dec-min elements

In Section 4.4, we introduced a matroid  $M_1$  on  $S_1$  and proved in Corollary 4.9 that the dec-min elements of  $B_1$  are exactly the integral elements of the translated base-polyhedron of  $M_1$ , where the translation means the addition of the constant vector  $(\beta_1 - 1, \dots, \beta_1 - 1)$  of dimension  $|S_1|$ . The same notions and results can be applied to each subscript  $i = 2, \dots, q$ . Furthermore, by formulating Lemma 4.11 for subscript *i* in place of 1, we obtain the following.

**Proposition 5.6.** The set-system  $\mathcal{B}_i := \{L \subseteq S_i : L = L_i(m_i) \text{ for some dec-min element } m_i \text{ of } \widetilde{B_i}\}$ forms the set of bases of a matroid  $M_i$  on ground-set  $S_i$ . An  $r_i$ -element subset L of  $S_i$  is a basis of  $M_i$ if and only if

$$|L \cap X| \ge p_i'(X) := p_i(X) - (\beta_i - 1)|X|$$
(5.10)

holds for every  $X \subseteq S_i$ .

It follows that a vector  $m_i$  on  $S_i$  is a dec-min element of  $B_i$  if and only if  $\beta_i - 1 \le m_i(s) \le \beta_i$  for each  $s \in S_i$  and the set  $L_i := \{s \in S_i : m_i(s) = \beta_i\}$  is a basis of  $M_i$ . Let  $M^*$  denote the direct sum of matroids  $M_1, \ldots, M_q$  and let  $\Delta^* \in \mathbb{Z}^S$  denote the translation vector defined by

$$\Delta^*(s) := \beta_i - 1 \text{ whenever } s \in S_i, i = 1, \dots, q.$$

By integrating these results, we obtain the following characterization.

**Theorem 5.7.** Let *B* be an integral base-polyhedron. An element *m* of (the *M*-convex set)  $\overline{B}$  is decreasingly minimal if and only if *m* can be obtained in the form  $m = \chi_L + \Delta^*$  where *L* is a basis of the matroid *M*<sup>\*</sup>. The base-polyhedron *B*<sup>•</sup> arises from the base-polyhedron of *M*<sup>\*</sup> by adding the translation vector  $\Delta^*$ . Concisely, the set of dec-min elements of *B* is a matroidal *M*-convex set.

**Cheapest dec-min element** An important algorithmic consequence of Theorems 5.1 and 5.7 is that they help solve the **cheapest dec-min element** problem, which is as follows. Let  $c : S \to \mathbf{R}$  be a cost function and consider the problem of computing a dec-min element *m* of an M-convex set  $\overline{B}$  for which *cm* is as small as possible.

By Theorem 5.7 the set  $B^{\bullet}$  of dec-min elements of B can be obtained from a matroid  $M^*$  by translation. Namely, there is a vector  $\Delta^* \in \mathbb{Z}^S$  such that m is in  $B^{\bullet}$  if and only if there is a basis L of  $M^*$  for which  $m = \chi_L + \Delta^*$ . Note that the matroid  $M^*$  arises as the direct sum of matroids  $M_i$  defined on the members  $S_i$  of the canonical partition.  $M_1$  is described in Proposition 4.10 and the other matroids  $M_i$  may be determined analogously in an iterative way. To realize this algorithmically, we must have a strongly polynomial algorithm to compute the canonical partition as well as the essential value-sequence. Such an algorithm will be described in [13].

Therefore, in order to find a minimum *c*-cost dec-min element of B, it suffices to find a minimum *c*-cost basis of  $M^*$ . Note that, in applying the greedy algorithm to the matroids  $M_i$  in question, we

need a rank oracle, which can be realized with the help of a submodular function minimization oracle by relying on the definition of bases in (4.11).

Recall that for integral bounds  $f \leq g$ , the intersection  $B_1$  of a base-polyhedron B and the box T(f,g), if non-empty, is itself a base-polyhedron. Therefore the algorithm above can be applied to the M-convex set  $B_1$ , that is, we can compute a cheapest dec-min element of the intersection  $B_1 = B \cap T(f,g)$ .

# 6 Integral square-sum and difference-sum minimization

For a vector  $z \in \mathbb{Z}^S$ , we can conceive several natural functions to measure the uniformity of its component values z(s) for  $s \in S$ . Here are two examples:

square-sum : 
$$W(z) := \sum_{n=1}^{\infty} [z(s)^2 : s \in S],$$
 (6.1)

**difference-sum** : 
$$\Delta(z) := \sum [|z(s) - z(t)| : s \neq t, s, t \in S].$$
 (6.2)

For vectors  $z_1$  and  $z_2$  with  $\tilde{z}_1(S) = \tilde{z}_2(S)$ ,  $z_1$  may be felt more uniform than  $z_2$  if  $W(z_1) < W(z_2)$ , and  $z_1$  may also be felt more uniform if  $\Delta(z_1) < \Delta(z_2)$ . The first goal of this section is to show, by establishing a fairly general theorem, that a dec-min element of an M-convex set  $\tilde{B}$  is simultaneously a minimizer of these two functions. The second goal of this section is to derive a min-max formula for the minimum integral square-sum of an element of an M-convex set  $\tilde{B}$ , along with characterizations of (integral) square-sum minimizers and dual optimal solutions.

#### 6.1 Symmetric convex minimization

Let *S* be a non-empty ground-set of *n* elements:  $S = \{1, 2, ..., n\}$ . We say that a function  $\Phi : \mathbb{Z}^S \to \mathbb{R}$  is symmetric if

$$\Phi(z(1), z(2), \dots, z(n)) = \Phi(z(\sigma(1)), z(\sigma(2)), \dots, z(\sigma(n)))$$
(6.3)

for all permutations  $\sigma$  of (1, 2, ..., n). We call a function  $\Phi : \mathbb{Z}^S \to \mathbb{R}$  convex if

$$\lambda \Phi(x) + (1 - \lambda)\Phi(y) \ge \Phi(\lambda x + (1 - \lambda)y) \tag{6.4}$$

whenever  $x, y \in \mathbb{Z}^S$ ,  $0 < \lambda < 1$ , and  $\lambda x + (1 - \lambda)y$  is an integral vector; and strictly convex if

$$\lambda \Phi(x) + (1 - \lambda)\Phi(y) > \Phi(\lambda x + (1 - \lambda)y)$$
(6.5)

whenever  $x, y \in \mathbb{Z}^S$ ,  $0 < \lambda < 1$ , and  $\lambda x + (1 - \lambda)y$  is an integral vector.

In the special case of a function in one variable, it can easily be shown that the convexity of  $\varphi : \mathbf{Z} \to \mathbf{R}$  is equivalent to the weaker requirement that the inequality

$$2\varphi(k) \le \varphi(k-1) + \varphi(k+1) \tag{6.6}$$

holds for every integer k. Such function  $\varphi$  is often called a (univariate) discrete convex function. It is strictly convex in the sense of (6.5) if and only if  $2\varphi(k) < \varphi(k-1) + \varphi(k+1)$  holds for every integer k. For example,  $\varphi(k) = k^2$  is strictly convex while  $\varphi(k) = |k|$  is convex but not strictly. Given a function  $\varphi$  in one variable satisfying (6.6), define  $\Phi$  by

$$\Phi(z) := \sum \left[ \varphi(z(s)) : s \in S \right]$$
(6.7)

for  $z \in \mathbb{Z}^S$ . Such a function  $\Phi$  is called a **symmetric separable convex function**; note that  $\Phi$  is indeed convex in the sense of (6.4). When  $\varphi$  is strictly convex,  $\Phi$  is also strictly convex.

**Example 6.1.** The square-sum W(z) in (6.1) is a symmetric convex function which is separable and strictly convex.

**Example 6.2.** The difference-sum  $\Delta(z)$  in (6.2) is a symmetric convex function which is neither separable nor strictly convex. More generally, for a nonnegative integer *K*, the function defined by

$$\Delta_K(z) := \sum [(|z(s) - z(t)| - K)^+ : s \neq t, s, t \in S]$$

is a symmetric convex function, where  $(x)^+ = \max\{x, 0\}$ .

The following statements show a close relationship between decreasing minimality and the minimization of symmetric convex  $\Phi$  over an M-convex set  $\tilde{B}$ .

**Proposition 6.1.** Let *B* be an integral base-polyhedron and  $\Phi$  a symmetric convex function. Then each dec-min element of  $\overline{B}$  is a minimizer of  $\Phi$  over  $\overline{B}$ .

**Proof.** Since the dec-min elements of  $\overline{B}$  are value-equivalent and  $\Phi$  is symmetric, the  $\Phi$ -value of each dec-min element is the same value  $\mu$ . We claim that  $\Phi(m) \ge \mu$  for each  $m \in \overline{B}$ . Suppose indirectly that there is an element *m* of  $\overline{B}$  for which  $\Phi(m) < \mu$ . Then *m* is not dec-min in  $\overline{B}$  and Property (A) in Theorem 3.3 implies that there is a 1-tightening step for *m* resulting in decreasingly smaller member of  $\overline{B}$ , that is, there exist  $s, t \in S$  such that  $m(t) \ge m(s) + 2$  and  $m' := m + \chi_s - \chi_t \in \overline{B}$ .

Let  $\alpha = m(t) - m(s)$ , where  $\alpha \ge 2$ , and define  $z = m + \alpha(\chi_s - \chi_t)$ . Since z is obtained from m by interchanging the components at s and t, the symmetry of  $\Phi$  formulated in (6.3) implies that  $\Phi(m) = \Phi(z)$ . Note that the vector z may not be a member of B. For  $\lambda = 1 - 1/\alpha$  we have

$$\lambda m + (1 - \lambda)z = \left(1 - \frac{1}{\alpha}\right)m + \frac{1}{\alpha}\left(m + \alpha(\chi_s - \chi_t)\right) = m + \chi_s - \chi_t = m' \in \overset{\dots}{B} (\subseteq \mathbb{Z}^S), \tag{6.8}$$

from which  $\lambda \Phi(m) + (1 - \lambda)\Phi(z) \ge \Phi(m')$  by convexity (6.4). Since  $\Phi(m) = \Phi(z)$ , this implies  $\Phi(m) \ge \Phi(m')$ . After a finite number of such 1-tightening steps, we arrive at a dec-min element  $m_0$  of  $\overset{\dots}{B}$ , for which  $\mu = \Phi(m_0) \le \Phi(m) < \mu$ , a contradiction.

Note that if  $\Phi$  is convex but not strictly convex, then  $\Phi$  may have minimizers that are not dec-min elements. This is exemplified by the identically zero function  $\Phi$  for which every member of  $\overline{B}$  is a minimizer. However, for strictly convex functions we have the following characterization.

**Theorem 6.2.** Given an integral base-polyhedron B and a symmetric strictly convex function  $\Phi$ , an element m of  $\overline{B}$  is a minimizer of  $\Phi$  if and only if m is a dec-min element of  $\overline{B}$ .

**Proof.** If *m* is a dec-min element, then *m* is a  $\Phi$ -minimizer by Proposition 6.1. To see the converse, let *m* be a  $\Phi$ -minimizer of  $\overline{B}$ . If, indirectly, *m* is not a dec-min element, then Property (A) in Theorem 3.3 implies that there is a 1-tightening step for *m*, that is, there exist  $s, t \in S$  such that  $m(t) \ge m(s) + 2$  and  $m' := m + \chi_s - \chi_t \in \overline{B}$ . For  $\alpha = m(t) - m(s)$ ,  $\lambda = 1 - 1/\alpha$ , and  $z = m + \alpha(\chi_s - \chi_t)$ , we have (6.8), from which we obtain  $\lambda \Phi(m) + (1 - \lambda)\Phi(z) > \Phi(m')$  by strict convexity (6.5). Since *z* is obtained from *m* by interchanging the components at *s* and *t*, the symmetry of  $\Phi$  formulated in (6.3) implies that  $\Phi(m) = \Phi(z)$ . But then  $\Phi(m) > \Phi(m')$  would follow, in contradiction to the assumption that *m* is a minimizer of  $\Phi$ .

We obtain the following as corollaries of this theorem.

**Corollary 6.3.** Let B be an integral base-polyhedron and  $\Phi$  a symmetric separable convex function. Then each dec-min element of  $\overline{B}$  is a minimizer of  $\Phi$  over  $\overline{B}$ , and the converse is also true if, in addition,  $\Phi$  is strictly convex.

**Corollary 6.4.** For an *M*-convex set  $\overline{B}$ , an element *m* of  $\overline{B}$  is a square-sum minimizer if and only if *m* is a dec-min element of  $\overline{B}$ .

An immediate consequence of Corollary 6.3 is that a square-sum minimizer of  $\hat{B}$  minimizes an arbitrary symmetric separable convex function  $\Phi$ . Note, however, that this consequence immediately follows from a much earlier result of Groenevelt [21] below, which deals with the minimization of a (not-necessarily symmetric) separable convex function.

**Theorem 6.5** (Groenevelt [21]; cf. [18, Theorem 8.1]). Let *B* be an integral base-polyhedron, *B* be the set of its integral elements, and  $\Phi(z) = \sum [\varphi_s(z(s)) : s \in S]$  for  $z \in \mathbb{Z}^S$ , where  $\varphi_s : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$  is a discrete convex function for each  $s \in S$ . An element *m* of *B* is a minimizer of  $\Phi(z)$  if and only if  $\varphi_s(m(s) + 1) + \varphi_t(m(t) - 1) \ge \varphi_s(m(s)) + \varphi_t(m(t))$  whenever  $m + \chi_s - \chi_t \in \overline{B}$ .

A dec-min element is also characterized as a difference-sum minimizer.

**Theorem 6.6.** For an *M*-convex set  $\overline{B}$ , an element *m* of  $\overline{B}$  is a difference-sum minimizer if and only if *m* is a dec-min element of  $\overline{B}$ .

**Proof.** By Proposition 6.1 every dec-min element is a difference-sum minimizer. To show the converse, suppose indirectly that there is difference-sum minimizer *m* that is not dec-min in  $\overline{B}$ . Property (A) in Theorem 3.3 implies that there is a 1-tightening step for *m*, that is, there exist  $s, t \in S$  such that  $m(t) \ge m(s) + 2$  and  $m' := m + \chi_s - \chi_t \in \overline{B}$ . Here we observe that |m'(s) - m'(t)| = |m(s) - m(t)| - 2 and

$$(|m'(v) - m'(s)| + |m'(v) - m'(t)|) - (|m(v) - m(s)| + |m(v) - m(t)|) = \begin{cases} -2 & \text{if } m(s) < m(v) < m(t) \\ 0 & \text{otherwise.} \end{cases}$$

This shows  $\Delta(m') \leq \Delta(m) - 2$ , a contradiction.

**Remark 6.1.** Corollary 6.4 says that an element *m* of an M-convex set B is dec-min precisely if *m* is a square-sum minimizer. One may feel that it would have been a more natural approach to derive this equivalence by showing that  $x \leq_{dec} y$  holds precisely if  $W(x) \leq W(y)$ . Perhaps surprisingly, however, this equivalence fails to hold, that is, the square-sum is not order-preserving with respect to the quasi-order  $\leq_{dec}$ . To see this, consider the following four vectors in increasing order:

 $m_1 = (2, 3, 3, 1) <_{\text{dec}} m_2 = (3, 3, 3, 0) <_{\text{dec}} m_3 = (2, 2, 4, 1) <_{\text{dec}} m_4 = (3, 2, 4, 0).$ 

Their square-sums admit a different order:

$$W(m_1) = 23$$
,  $W(m_2) = 27$ ,  $W(m_3) = 25$ ,  $W(m_4) = 29$ .

The four vectors  $m_i$  (i = 1, 2, 3, 4) form an M-convex set. Among these four elements,  $m_1$  is the unique dec-min element and the unique square-sum minimizer but the decreasing-order and the square-sum order of the other three elements are different. We remark that if  $\varphi$  in (6.7) is not only strictly convex but 'rapidly' increasing as well, then  $x <_{dec} y$  can be proved to be equivalent to  $\Phi(x) < \Phi(y)$ . This intuitive notion of rapid increase is formalized in [12].

**Remark 6.2.** For the intersection of two M-convex sets, dec-min elements and square-sum minimizers may not coincide. Here is an example. Let  $\vec{B_1} = \{(3, 3, 3, 0), (2, 2, 4, 1), (2, 3, 3, 1), (3, 2, 4, 0)\}$  and  $\vec{B_2} = \{(3, 3, 3, 0), (2, 2, 4, 1), (3, 2, 3, 1), (2, 3, 4, 0)\}$ , which are both M-convex. In their intersection  $\vec{B_1} \cap \vec{B_2} = \{(3, 3, 3, 0), (2, 2, 4, 1)\}$ , the vector (3, 3, 3, 0) is the unique dec-min element while (2, 2, 4, 1) is the unique square-sum minimizer. This demonstrates that the two notions of optima may differ for the intersection of two M-convex sets.

**Remark 6.3.** In Section 3.3, we considered the minimum *k*-largest-sum problem that aimed at finding an element of  $\overline{B}$  for which the sum of the *k* largest components is as small as possible. For each *k*, the sum of the *k* largest components is a symmetric convex function, and hence Theorem 3.5 is a special case of Proposition 6.1. It is noted, however, that Theorem 6.2 is not applicable to the *k*-largest-sum problem, as this function is not strictly convex. Nevertheless, a dec-min element can be characterized in terms of the *k*-largest-sum if we simultaneously consider the functions for all *k* (Theorem 3.6).

**Remark 6.4.** For  $a, b, c \ge 0$ , the function defined by

$$\Phi(z) = a \sum_{s \in S} |z(s)| + b \sum_{s \neq t} |z(s) - z(t)| + c \sum_{s \neq t} |z(s) + z(t)|$$

is a symmetric convex function. More generally, a function of the form

$$\Phi(z) = \sum_{s \in S} \varphi_1(z(s)) + \sum_{s \neq t} \varphi_2(|z(s) - z(t)|) + \sum_{s \neq t} \varphi_3(z(s) + z(t)),$$

where  $\varphi_1, \varphi_2, \varphi_3 : \mathbb{Z} \to \mathbb{R}$  are (discrete) convex functions (as defined in (6.6)), is a symmetric convex function which is not separable. Such a function is an example of the so-called 2-separable convex functions. By Theorem 6.2, a dec-min element of  $\overline{B}$  is a minimizer of function  $\Phi$  over  $\overline{B}$ . The minimization of 2-separable convex functions is investigated in depth by Hochbaum and others [1, 24, 25] using network flow techniques.

**Remark 6.5.** A min-max formula can be derived for the square-sum (see Section 6.2) and, more generally, for separable convex functions from the Fenchel-type duality theorem in DCA [37, 38]. However, we cannot use the Fenchel-type duality theorem to obtain a min-max formula for non-separable symmetric convex functions, since non-separable symmetric convex functions are not necessarily M-convex.

#### 6.2 Min-max theorem for integral square-sum

Recall the notation  $W(z) = \sum [z(s)^2 : s \in S]$  for the square-sum of  $z \in \mathbb{Z}^S$ . Given a polyhedron *B*, we say that an element  $m \in \overline{B}$  is a **square-sum minimizer** (over  $\overline{B}$ ) or that *m* is an **integral square-sum minimizer** of *B* if  $W(m) \leq W(z)$  holds for each  $z \in \overline{B}$ . The main goal of this section is to derive a min-max formula for the minimum integral square-sum of an element of an M-convex set  $\overline{B}$ , along with a characterization of (integral) square-sum minimizers.

A set-function p on S can be considered as a function defined on (0, 1)-vectors. It is known that p can be extended in a natural way to every vector  $\pi$  in  $\mathbb{R}^S$ , as follows. For the sake of this definition, we may assume that the elements of S are indexed in a decreasing order of the components of  $\pi$ , that is,  $\pi(s_1) \ge \cdots \ge \pi(s_n)$  (where the order of the components of  $\pi$  with the same value is arbitrary). For  $j = 1, \ldots, n$ , let  $I_j := \{s_1, \ldots, s_j\}$  and let

$$\hat{p}(\pi) := p(I_n)\pi(s_n) + \sum_{j=1}^{n-1} p(I_j)[\pi(s_j) - \pi(s_{j+1})].$$
(6.9)

Obviously,  $p(Z) = \hat{p}(\chi_Z)$ . The function  $\hat{p}$  is called [10, Section 14.5.1] the **linear extension** of p, where  $\hat{p}$  is a piecewise-linear function in  $\pi$ .

**Remark 6.6.** The linear extension was first considered by Edmonds [6] who proved for a polymatroid P = P(b) defined by a monotone, non-decreasing submodular function *b* that  $\max\{\pi x : x \in \tilde{P}\} = \hat{b}(\pi)$  when  $\pi$  is non-negative. The same approach shows for a base-polyhedron B = B'(p) defined by a supermodular function *p* that  $\min\{\pi x : x \in \tilde{B}\} = \hat{p}(\pi)$ . Another basic result is due to Lovász [32] who proved that *p* is supermodular if and only if  $\hat{p}$  is concave. We do not, however, explicitly need these results, and only remark that in the literature the linear extension is often called Lovász extension.

Our approach is as follows. First, we consider an arbitrary set-function p on S (supermodular or not) along with the polyhedron

$$B = B'(p) := \{x : x \in \mathbf{R}^S, \ \widetilde{x}(Z) \ge p(Z) \text{ for every } Z \subset S \text{ and } \widetilde{x}(S) = p(S) \}$$

and develop an easily checkable lower bound for the minimum square-sum over the integral elements of *B*. If this lower bound is attained by an element *m* of  $\overline{B}$ , then *m* is certainly a square-sum minimizer independently of any particular property of *p*. For general *p*, the lower bound (not surprisingly) is not always attainable. We shall prove, however, that it is attainable when *p* is supermodular. That is, we will have a min-max theorem for the minimum square-sum over an M-convex set  $\overline{B}$ , or in other words, we will have an easily checkable certificate for an element *m* of  $\overline{B}$  to be a minimizer of the square-sum.

We shall need the following two claims. For any real number  $\alpha \in \mathbf{R}$ , let  $\lfloor \alpha \rfloor$  denote the largest integer not larger than  $\alpha$ , and  $\lceil \alpha \rceil$  the smallest integer not smaller than  $\alpha$ .

**Claim 6.7.** For  $m, \pi \in \mathbb{Z}^S$ , one has

$$\sum_{s \in S} \left\lfloor \frac{\pi(s)}{2} \right\rfloor \left\lceil \frac{\pi(s)}{2} \right\rceil \ge \sum_{s \in S} m(s) [\pi(s) - m(s)].$$
(6.10)

Moreover, equality holds if and only if

$$m(s) \in \left\{ \left\lfloor \frac{\pi(s)}{2} \right\rfloor, \left\lceil \frac{\pi(s)}{2} \right\rceil \right\}$$
 for every  $s \in S$ . (6.11)

**Proof.** The claim follows by observing that  $\lfloor a/2 \rfloor \lceil a/2 \rceil \ge b(a-b)$  holds for any pair of integers *a* and *b*, where equality holds precisely if  $b \in \{\lfloor a/2 \rfloor, \lceil a/2 \rceil\}$ .

Let *p* be an arbitrary set-function on *S* with  $p(\emptyset) = 0$  and consider an integral element *m* of the polyhedron B = B'(p). Recall that a non-empty subset  $X \subseteq S$  was called a strict  $\pi$ -top set if  $\pi(u) > \pi(v)$  held whenever  $u \in X$  and  $v \in S - X$ . In what follows, for an  $m \in \overline{B}$ , *m*-tightness of a subset  $Z \subseteq S$  means  $\widetilde{m}(Z) = p(Z)$ .

**Claim 6.8.** For  $m \in \overset{\smile}{B}$  and  $\pi \in \mathbb{Z}^S$ , one has

$$\hat{p}(\pi) \le \sum_{s \in S} m(s)\pi(s).$$
(6.12)

Moreover, equality holds if and only if each (of the at most n) strict  $\pi$ -top set is m-tight.

**Proof.** Suppose that the elements of *S* are indexed in such a way that  $\pi(s_1) \ge \pi(s_2) \ge \cdots \ge \pi(s_n)$ . For  $j = 1, \dots, n$ , let  $I_j := \{s_1, \dots, s_j\}$ . Then

$$\begin{split} \hat{p}(\pi) &= p(I_n)\pi(s_n) + \sum_{j=1}^{n-1} p(I_j)[\pi(s_j) - \pi(s_{j+1})] \\ &\leq \widetilde{m}(I_n)\pi(s_n) + \sum_{j=1}^{n-1} \widetilde{m}(I_j)[\pi(s_j) - \pi(s_{j+1})] \\ &= \sum_{1 \le i \le j \le n} m(s_i)\pi(s_j) - \sum_{1 \le i \le j \le n-1} m(s_i)\pi(s_{j+1}) \\ &= \sum_{1 \le i \le j \le n} m(s_i)\pi(s_j) - \sum_{1 \le i < j' \le n} m(s_i)\pi(s_{j'}) \\ &= \sum_{j=1}^n m(s_j)\pi(s_j), \end{split}$$

from which (6.12) follows. Furthermore, we have equality in (6.12) precisely if  $\tilde{m}(I_j) = p(I_j)$  holds whenever  $\pi(s_j) - \pi(s_{j+1}) > 0$ . But this latter condition is equivalent to requiring that each strict  $\pi$ -top set is *m*-tight.

**Proposition 6.9.** Let p be an arbitrary set-function on S with  $p(\emptyset) = 0$  and let m be an integral element of the polyhedron B = B'(p). Then

$$\sum_{s \in S} m(s)^2 \ge \hat{p}(\pi) - \sum_{s \in S} \left\lfloor \frac{\pi(s)}{2} \right\rfloor \left\lceil \frac{\pi(s)}{2} \right\rceil$$
(6.13)

whenever  $\pi \in \mathbb{Z}^{S}$  is an integral vector. Furthermore, equality holds for m and  $\pi$  if and only if the following **optimality criteria** hold:

(O1) (6.11) holds: 
$$m(s) \in \left\{ \left\lfloor \frac{\pi(s)}{2} \right\rfloor, \left\lceil \frac{\pi(s)}{2} \right\rceil \right\}$$
 for every  $s \in S$ , (6.14)

(O2) each strict  $\pi$ -top-set is *m*-tight with respect to *p*. (6.15)

**Proof.** Let  $\pi \in \mathbb{Z}^S$ . By the two preceding claims,

$$\sum_{s \in S} m(s)^2 = \sum_{s \in S} m(s)\pi(s) - \sum_{s \in S} m(s)[\pi(s) - m(s)] \ge \hat{p}(\pi) - \sum_{s \in S} \left\lfloor \frac{\pi(s)}{2} \right\rfloor \left\lceil \frac{\pi(s)}{2} \right\rceil, \tag{6.16}$$

from which (6.13) follows. The claims also immediately imply that we have equality in (6.13) precisely if the optimality criteria (O1) and (O2) hold.

The min-max formula in the next theorem concerning min square-sum over the integral elements of an integral base-polyhedron can be derived from the more general Fenchel-type duality theorem in DCA (see [37] and also Theorem 8.21, page 222, in the book [38]), or from a recent framework [15] of separable discrete convex function minimization over the integer points in an integral box-TDI polyhedron. However, our proof relies only on the relatively simple characterization of dec-min elements described in Theorem 3.3. In particular, we need no results of Sections 4 and 5.

**Theorem 6.10.** Let B = B'(p) be a base-polyhedron defined by an integer-valued supermodular function p. Then

$$\min\{\sum_{s\in\mathcal{S}} m(s)^2 : m\in\widetilde{B}\} = \max\{\hat{p}(\pi) - \sum_{s\in\mathcal{S}} \left\lfloor \frac{\pi(s)}{2} \right\rfloor \left\lceil \frac{\pi(s)}{2} \right\rceil : \pi\in\mathbb{Z}^S\}.$$
(6.17)

**Proof.** By Proposition 6.9, min  $\geq$  max holds in (6.17) and hence all what we have to prove is that there is an element  $m \in \overline{B}$  and an integral vector  $\pi \in \mathbb{Z}^S$  meeting the two optimality criteria formulated in Proposition 6.9. Let *m* be an arbitrary dec-min element of  $\overline{B}$ . By Property (B) of Theorem 3.3, there is a chain  $(\emptyset \subset) C_1 \subset C_2 \subset \cdots \subset C_{\ell} = S$  of *m*-tight and *m*-top sets for which the restrictions of *m* onto the difference sets  $S_i := C_i - C_{i-1}$   $(i = 1, \dots, \ell)$  are near-uniform in  $S_i$  (where  $C_0 := \emptyset$ ). Note that  $\{S_1, \dots, S_{\ell}\}$  is a partition of S.

For  $i = 1, ..., \ell$ , let  $\beta_i(m) := \max\{m(s) : s \in S_i\}$ . Define  $\pi_m : S \to \mathbb{Z}$  by

$$\pi_m(s) := 2\beta_i(m) - 1$$
 if  $s \in S_i$   $(i = 1, \dots, \ell)$ .

We have

$$\lfloor \pi_m(s)/2 \rfloor = \beta_i(m) - 1 \le m(s) \le \beta_i(m) = \lceil \pi_m(s)/2 \rceil$$

for every  $s \in S_i$ , and hence Optimality criterion (O1) holds for *m* and  $\pi_m$ .

We claim that each strict  $\pi_m$ -top set *Z* is a member of chain *C*. Indeed, as  $\pi_m$  is uniform in each *S*<sub>j</sub>, if *Z* contains an element of *S*<sub>j</sub>, then *Z* includes the whole *S*<sub>j</sub>. Furthermore, since each member of *C* is an *m*-top set, we have  $\beta_1(m) \ge \beta_2(m) \ge \cdots \ge \beta_\ell(m)$ , and hence if *Z* includes *S*<sub>j</sub>, then it includes each *S*<sub>i</sub> with i < j. Therefore every strict  $\pi_m$ -top set is indeed a member of the chain, implying Optimality criterion (O2).

It should be noted that the optimal dual solution  $\pi_m$  obtained in the proof of the theorem is actually an **odd** vector in the sense that each of its component is an odd integer. **Corollary 6.11.** There is an odd dual optimizer  $\pi$  in the min-max formula (6.17), that is, the min-max formula in Theorem 6.10 can be re-written as follows:

$$\min\{\sum_{s\in S} m(s)^2 : m\in \overline{B}\} = \max\{\hat{p}(\pi) - \sum_{s\in S} \frac{\pi(s)^2 - 1}{4} : \pi\in \mathbf{Z}^S, \ \pi \text{ is odd }\}.$$
 (6.18)

We emphasize that for the proof of Theorem 6.10 and Corollary 6.11 we relied only on Theorem 3.3 and did not need the characterization of the set of dec-min elements of  $\vec{B}$  given in Section 5.

In the proof of Theorem 6.10, we chose an arbitrary dec-min element *m* of  $\overline{B}$  and an arbitrary chain of *m*-tight and *m*-top sets such that *m* is near-uniform on each difference set. In Section 5, we proved that there is a single canonical chain  $C^*$  which meets these properties for every dec-min element of  $\overline{B}$ . Therefore the dual optimal  $\pi^*$  assigned to  $C^*$  is also independent of *m*. Namely, consider the canonical *S*-partition  $\{S_1, \ldots, S_q\}$  and the essential value-sequence  $\beta_1 > \cdots > \beta_q$ . Define  $\pi^*$  by

$$\pi^*(s) := 2\beta_i - 1 \quad \text{if} \quad s \in S_i \ (i = 1, \dots, q). \tag{6.19}$$

As we pointed out in the proof of Theorem 6.10, this  $\pi^*$  is also a dual optimum in (6.17). We shall prove in the next section that  $\pi^*$  is actually the unique smallest dual optimum in (6.17).

#### 6.3 The set of optimal duals to integral square-sum minimization

We proved earlier that an element  $m \in B$  is a square-sum minimizer precisely if it is a dec-min element. This and Theorem 5.1 imply that the square-sum minimizers of B are the integral members of a base-polyhedron  $B^{\bullet}$  obtained by intersecting a particular face of B with a special small box. This means that the integral square-sum minimizers form an M-convex set.

Our next goal is to reveal the structure of the set  $\Pi$  of the dual optima in Theorem 6.10 and we provide a description of  $\Pi$  as the integral solution set of feasible potentials in a box. This shows another connection to DCA, which is discussed after the proof of Theorem 6.10.

Recall that the optimality criteria for a dec-min element *m* of B and for an integral vector  $\pi$  were given by (O1) and (O2) in (6.14)–(6.15). These immediately imply the following.

**Proposition 6.12.** For an integral vector  $\pi$ , the following are equivalent.

- (A)  $\pi$  is a dual optimum (that is,  $\pi$  belongs to  $\Pi$ ).
- (B) There is a dec-min element m of  $\overline{B}$  such that m and  $\pi$  meet the optimality criteria.
- (C) For every dec-min m of B, m and  $\pi$  meet the optimality criteria.

Consider the canonical *S*-partition  $\{S_1, \ldots, S_q\}$ , the essential value-sequence  $\beta_1 > \beta_2 > \cdots > \beta_q$ , and the matroids  $M_i$  on  $S_i$   $(i = 1, \ldots, q)$ . We can use the notions and the results of Section 4.5 formulated for  $M_1$  to each  $M_i$   $(i = 1, \ldots, q)$ . To follow the pattern of  $\mathcal{F}_1$  introduced in (4.13), let

$$\mathcal{F}_i := \{ X \subseteq S_i : \beta_i | X | = p_i(X) \}, \tag{6.20}$$

where  $p_i$  was defined by  $p_i(X) = p(C_{i-1} \cup X) - p(C_{i-1})$  for  $X \subseteq S_i$ . Since  $\beta_i|X| \ge p_i(X)$  for every  $X \subseteq S_i$  and  $p_i$  is supermodular,  $\mathcal{F}_i$  is closed under taking intersection and union. Let  $F_i$  denote the unique largest member of  $\mathcal{F}_i$ , that is,  $F_i$  is the union of the members of  $\mathcal{F}_i$ . Both  $F_i = \emptyset$  and  $F_i = S_i$  are possible. As a generalization of Theorems 4.15 and 4.16 we obtain the following.

**Theorem 6.13.** For an element  $s \in S_i$  (i = 1, ..., q), the following properties are pairwise equivalent.

- (A) *s is value-fixed*.
- (B)  $m(s) = \beta_i$  holds for every dec-min element m of B.
- (C)  $s \in F_i$ .
- (D) s is a co-loop of  $M_i$ .

Define a digraph  $D_i = (F_i, A_i)$  on node-set  $F_i$  in which st is an arc if  $s, t \in F_i$  and there is no  $t\overline{s}$ -set in  $\mathcal{F}_i$ . This implies that no arc of  $D_i$  enters any member of  $\mathcal{F}_i$ .

**Theorem 6.14.** An integral vector  $\pi \in \mathbb{Z}^S$  is an optimal dual solution to the integral minimum squaresum problem (that is,  $\pi \in \Pi$ ) if and only if the following three conditions hold for each i = 1, ..., q:

$$\pi(s) = 2\beta_i - 1 \quad \text{for every } s \in S_i - F_i, \tag{6.21}$$

$$2\beta_i - 1 \le \pi(s) \le 2\beta_i + 1 \quad \text{for every } s \in F_i, \tag{6.22}$$

$$\pi(s) - \pi(t) \ge 0$$
 whenever  $s, t \in F_i$  and  $st \in A_i$ . (6.23)

Proof.

Claim 6.15. Optimality criterion (O1) in (6.14) is equivalent to

(O1')  $2m(s) - 1 \le \pi(s) \le 2m(s) + 1$  for  $s \in S$ . (6.24)

**Proof.** When  $\pi(s)$  is even, we have the following equivalences:

$$m(s) \in \left\{ \left\lfloor \frac{\pi(s)}{2} \right\rfloor, \left\lceil \frac{\pi(s)}{2} \right\rceil \right\} \quad \Leftrightarrow \quad \pi(s) = 2m(s)$$
$$\Leftrightarrow \quad 2m(s) - 1 \le \pi(s) \le 2m(s) + 1.$$

When  $\pi(s)$  is odd, we have the following equivalences:

$$m(s) \in \left\{ \left\lfloor \frac{\pi(s)}{2} \right\rfloor, \left\lceil \frac{\pi(s)}{2} \right\rceil \right\} \quad \Leftrightarrow \quad \pi(s) - 1 \le 2m(s) \le \pi(s) + 1$$
$$\Leftrightarrow \quad 2m(s) - 1 \le \pi(s) \le 2m(s) + 1.$$

Suppose first that  $\pi \in \mathbb{Z}^S$  is an optimal dual solution. Then the optimality criteria (O1') and (O2) formulated in (6.24) and (6.15) hold for every dec-min element *m* of  $\overline{B}$ .

Let *s* be an element of  $S_i - F_i$ . Since *s* is not value-fixed, there are dec-min elements *m* and *m'* of  $\overline{B}$  for which  $m(s) = \beta_i - 1$  and  $m'(s) = \beta_i$ . By applying (6.24) to *m* and to *m'*, we obtain that

$$2\beta_i - 1 = 2m'(s) - 1 \le \pi(s) \le 2m(s) + 1 = 2(\beta_i - 1) + 1 = 2\beta_i - 1,$$

from which  $\pi(s) = 2\beta_i - 1$  follows, and hence (6.21) holds indeed.

Let *s* be an element of  $F_i$ . As *s* is value-fixed,  $m(s) = \beta_i$  holds for any dec-min element *m* of B. We obtain from (6.24) that

$$2\beta_i - 1 = 2m(s) - 1 \le \pi(s) \le 2m(s) + 1 = 2\beta_i + 1$$

and hence (6.22) holds.

To derive (6.23), suppose indirectly that *st* is an arc in  $A_i$  for which  $\pi(t) > \pi(s) (\ge 2\beta_i - 1)$ . Let  $Z := \{v \in S : \pi(v) \ge \pi(t)\}$ . Then *Z* is a strict  $\pi$ -top set, and moreover, we have  $C_{i-1} \subseteq Z \subseteq (C_{i-1} \cup F_i) - s$ , where the latter inclusion follows from (6.21) and  $\pi(t) > 2\beta_i - 1$ . By Optimality criterion (O2), *Z* is *m*-tight with respect to *p*. Let  $X := Z \cap S_i$ . Then  $X \subseteq F_i$  and hence

$$p(Z) = \widetilde{m}(Z) = \widetilde{m}(C_{i-1}) + \widetilde{m}(X) = p(C_{i-1}) + \beta_i |X|,$$

from which

$$\beta_i |X| = p(Z) - p(C_{i-1}) = p_i(X),$$

that is, X is in  $\mathcal{F}_i$ , contradicting the definition of  $A_i$  which requires that st enters no member of  $\mathcal{F}_i$ .

Suppose now that  $\pi$  meets the three properties (6.21), (6.22), and (6.23). Let  $m \in B$  be an arbitrary dec-min element. Consider an element *s* of *S*<sub>*i*</sub>. If  $s \in F_i$ , that is, if *s* is value-fixed, then  $m(s) = \beta_i$ . By

(6.22), we have  $2m(s) - 1 \le \pi(s) \le 2m(s) + 1$ , that is, Optimality criterion (O1') holds. If  $s \in S_i - F_i$ , then  $\pi(s) = 2\beta_i - 1$  by (6.21), from which

$$\left\lfloor \frac{\pi(s)}{2} \right\rfloor = \frac{\pi(s) - 1}{2} = \beta_i - 1 \le m(s) \le \beta_i = \frac{\pi(s) + 1}{2} = \left\lceil \frac{\pi(s)}{2} \right\rceil,$$

showing that Optimality criterion (O1') holds.

To prove optimality criterion (O2), let *Z* be a strict  $\pi$ -top set and let  $\mu := \min\{\pi(v) : v \in Z\}$ . Let *i* denote the largest subscript for which  $X := Z \cap S_i \neq \emptyset$ . Then  $\mu \le 2\beta_i + 1 \le 2\beta_{i-1} - 1 \le \pi(u)$  holds for every  $u \in C_{i-1}$ , from which  $C_{i-1} \subseteq Z$  as *Z* is a strict  $\pi$ -top set.

If  $\mu = 2\beta_i - 1$ , then  $S_i \subseteq Z$  as Z is a strict  $\pi$ -top set, from which  $Z = C_i$ , implying that Z is an *m*-tight set in this case. Therefore we suppose  $\mu \ge 2\beta_i$ , from which  $X \subseteq F_i$  follows. Now  $X \in \mathcal{F}_i$ , for otherwise there is an arc  $st \in A_i$   $(s, t \in F_i)$  entering X, and then  $\pi(t) \le \pi(s)$  holds by Property (6.23); this contradicts the assumption that Z is a strict  $\pi$ -top set. By  $X \in \mathcal{F}_i$  we have  $\beta_i |X| = p_i(X)$ , whereas  $m(s) = \beta_i$  for each  $s \in X$  by  $X \subseteq F_i$ . Hence

$$\widetilde{m}(Z) = \widetilde{m}(X) + \widetilde{m}(C_{i-1}) = \beta_i |X| + p(C_{i-1}) = p_i(X) + p(C_{i-1}) = p(X \cup C_{i-1}) - p(C_{i-1}) + p(C_{i-1}) = p(Z),$$

that is, Z is indeed m-tight.

We now relate Theorem 6.10 to a concept from discrete convex analysis, where two kinds of discrete convexity play major roles as mutually 'conjugate' notions of discrete convexity [37, 38]. One of them is M-convexity and the other is called L-convexity. One of the equivalent definitions says that a set *L* of integer vectors is an **L-convex set** if it is the set of integer-valued feasible potentials in the network flow problem. Formally,  $L = \{\pi \in \mathbb{Z}^S : \pi(v) - \pi(u) \le g(uv) \ (u, v \in S)\}$ , where *g* is an integer-valued function on the ordered pairs of elements of *S*. A set of integer vectors is called an  $\mathbb{L}^{\ddagger}$ -convex set (pronounce L-natural convex set) if it is the intersection of an L-convex set with an integral box.

In (6.19), we defined a special dual optimal solution  $\pi^*$  by  $\pi^*(s) = 2\beta_i - 1$  whenever  $s \in S_i$  (i = 1, ..., q). Theorem 6.14 and the definition we use for L<sup>\\[\beta\_-</sup>convex sets immediately implies the following.

**Corollary 6.16.** The set  $\Pi$  of optimal dual integral vectors  $\pi$  in the min-max formula (6.17) of Theorem 6.10 is an L<sup>\\(\beta\)</sup>-convex set. The unique smallest element of  $\Pi$  (that is, the unique smallest dual optimum) is  $\pi^*$ .

It will be worth mentioning that  $L^{\natural}$ -convexity of the set of optimal dual integral vectors is a general phenomenon that is true in separable convex function minimization on an M-convex set; see Section 5 of [12]. Indeed, this is a consequence of conjugacy between M-convexity and L-convexity. It is also known that every  $L^{\natural}$ -convex set has a unique smallest (and a unique largest) element.

# 7 Continuous versus discrete

In this paper, we have concentrated on discrete decreasing minimization whose continuous counterpart had been investigated earlier. In this section, we briefly look at the relationship between the continuous (fractional) and discrete (integral) settings.

First of all, there is a fundamental difference between the problems of finding a dec-min element of a base-polyhedron and that of an M-convex set (the set of integral elements of an integral base-polyhedron). In the former case (investigated by Fujishige [17, 18]), there alway exists a single, unique dec-min element, while in the latter, the dec-min elements of an M-convex set have an elegant matroidal structure. Namely, Theorem 5.7 shows that the set of dec-min elements of an M-convex set arises from the bases of a matroid by translating their incidence vectors with an integral vector.

In spite of such fundamental difference, a dec-min element can be characterized, in either case, as a square-sum minimizer. In the continuous case, it is well-known [17, 18] that the unique decmin element of a base-polyhedron coincides with the minimum norm point of the base-polyhedron, whereas, in the discrete case, Corollary 6.4 shows that the square-sum minimizers are exactly the dec-min elements of an M-convex set. Furthermore, a dec-min element can be characterized by a symmetric strictly convex function, which is stated in Theorem 6.2 for the discrete case as a discrete counterpart of a result of Maruyama [34] for the continuous case. See also Nagano [39, Corollary 13]. Symmetric convex function minimization is studied, mainly for the continuous case, in the literature of majorization [2, 33].

In the following, we show links between the continuous and discrete versions of decreasing minimization by considering an integral base-polyhedron B and the asociated M-convex set  $\overline{B}$ . The following theorems, given in [12], formalize the intuitive feeling that the minimum norm points (dec-min elements) of an M-convex set  $\overline{B}$  and the unique minimum norm point (dec-min element) of the corresponding base-polyhedron B are 'close' to each other.

**Theorem 7.1** ([12, Theorem 6.6]). Let  $m_{\mathbf{R}}$  be the minimum norm point of an integral base-polyhedron *B*. Then every dec-min element *m* of  $\overline{B}$  satisfies  $\lfloor m_{\mathbf{R}} \rfloor \le m \le \lceil m_{\mathbf{R}} \rceil$ .

**Theorem 7.2** ([12, Theorem 6.7]). *The minimum norm point of an integral base-polyhedron B can be represented as a convex combination of the dec-min elements of*  $\overline{B}$ .

Since the (unique) minimum norm point of base-polyhedron B is the (unique) dec-min element of B, Theorem 7.2 can be reformulated in the following equivalent form.

**Theorem 7.3.** The (unique) dec-min element of an integral base-polyhedron B can be represented as a convex combination of the dec-min elements of  $\overline{B}$ .

This result looks quite natural and even straightforward. However, one has to be cautious with such a naive intuition. In fact, the analogous statement fails to hold for an M<sub>2</sub>-convex set (the intersection of two M-convex sets), as is demonstrated in the following example.

Example 7.1. Consider the following two M-convex sets:

In their intersection  $B_1 \cap B_2 = \{(1,0,0,0), (0,-1,1,1)\}$ , the element z = (1,0,0,0) is the unique dec-min element. In the continuous version,  $B_1 \cap B_2$  is the line segment connecting (1,0,0,0) and (0,-1,1,1). The middle point x = (1/2, -1/2, 1/2, 1/2) is the unique dec-min element of  $B_1 \cap B_2$ . (Note that x is decreasingly smaller than z.) We cannot represent x as a convex combination of a single element z.

Not only the dec-min elements for B and B are related as above, but the 'dual objects' (namely, the chains and partitions) for B and B are related as follows. Actually, Theorems 7.1 and 7.2 are proved in [12] on the basis of this relationship between the 'dual objects.' Recall that the principal partition is the continuous counterpart of the canonical partition, while the critical values are the continuous counterpart of essential values (see [18, Section 7.2], [19] for notions related to the principal partition).

**Theorem 7.4** ([12, Theorem 6.5]). Let B be an integral base-polyhedron.

(1) An integer  $\beta$  is an essential value of B if and only if there exists a critical value  $\lambda$  of B satisfying  $\beta \ge \lambda > \beta - 1$ .

(2) The essential values  $\beta_1 > \beta_2 > \cdots > \beta_q$  of B are obtained from the critical values  $\lambda_1 > \lambda_2 > \cdots > \lambda_r$  of *B* as the distinct members of the rounded-up integers  $\lceil \lambda_1 \rceil \ge \lceil \lambda_2 \rceil \ge \cdots \ge \lceil \lambda_r \rceil$ .

(3) The canonical partition  $\{S_1, S_2, \dots, S_q\}$  of  $\overline{B}$  is obtained from the principal partition  $\{\hat{S}_1, \hat{S}_2, \dots, \hat{S}_r\}$  of B as an aggregation as

$$S_j = \bigcup_{i \in I(j)} \hat{S}_i \qquad (j = 1, 2, \dots, q),$$

where  $I(j) = \{i : [\lambda_i] = \beta_j\}$  for j = 1, 2, ..., q.

(4) The canonical chain  $\{C_j\}$  of  $\overline{B}$  is a subchain of the principal chain  $\{\hat{C}_i\}$  of B, which is given by  $C_j = \hat{C}_i$  with the maximum index i in I(j).

# 8 Conclusion

The present work is the first member of a series of papers concerning discrete decreasing minimization. In the companion paper [13] we give a strongly polynomial algorithm for finding a dec-min element of an M-convex set and discuss applications of discrete decreasing minimization to the 'background problems' mentioned in Section 1.1.

While the present framework of decreasing minimization on an M-convex set is effective for a fairly wide class of graph orientation problems [13], there are other important graph orientation problems that do not fit in this framework. For example, for strong orientations of mixed graphs, dec-min orientations and inc-max orientations do not coincide. The reason behind this phenomenon is that the set of in-degree vectors of strong orientations of a mixed graph is not an M-convex set anymore. It is, in fact, the intersection of two M-convex sets. By investigating the decreasing minimization problem over the intersection of two M-convex sets we can solve a broader class of graph orientation problems, see [16].

Decreasing minimization on an M-convex set contains the integer version of Megiddo's problem [35] of finding a maximum flow that is 'lexicographically optimal' on the set of edges leaving the source node. In [14] this problem is generalized to the problem of finding an integral feasible flow that is decreasing minimal on an arbitrarily specified subset of edges. The structure of decreasingly minimal integral feasible flows is clarified and a strongly polynomial algorithm for finding such a dec-min flow is developed. A further generalization to integral submodular flows is reported in [16].

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