Spline Wavelets on the Interval with Homogeneous Boundary Conditions

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Abstract

In this paper we investigate spline wavelets on the interval with homogeneous boundary conditions. Starting with a pair of families of B-splines on the unit interval, we give a general method to explicitly construct wavelets satisfying the desired homogeneous boundary conditions. On the basis of a new development of multiresolution analysis, we show that these wavelets form Riesz bases of certain Sobolev spaces. The wavelet bases investigated in this paper are suitable for numerical solutions of ordinary and partial differential equations.

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§1. Introduction

In this paper we investigate spline wavelets on the interval $[0, 1]$ with homogeneous boundary conditions. In [3] Chui and Wang initiated the study of semi-orthogonal wavelets generated from cardinal splines. Following their work, Chui and Quak [4] constructed semi-orthogonal spline wavelets on the interval [0,1]. In [10] Jia modified the construction of boundary wavelets in [4] and established the stability of wavelet bases in Sobolev spaces. Concerning applications of wavelets to numerical solutions of ordinary and partial differential equations, we are interested in wavelets on the interval $[0, 1]$ with homogeneous boundary conditions. Using Hermite cubic splines, Jia and Liu in [11] constructed wavelet bases on the interval $[0, 1]$ and applied those wavelets to numerical solutions of the Sturm-Liouville equations with the Dirichlet boundary condition. In this paper, starting with cardinal B-splines, we will construct a family of wavelets on the interval $[0, 1]$ which satisfy homogeneous boundary conditions of arbitrary order.

Let us introduce some notation. We use $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ to denote the set of positive integers, integers, real numbers, and complex numbers, respectively. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For a complex number c, we use $\overline{c}$ to denote its complex conjugate.

For a complex-valued (Lebesgue) measurable function f on $\mathbb{R}$, let

$$\|f\|_p := \left(\int_{\mathbb{R}} |f(x)|^p \, dx \right)^{1/p} \qquad \text{for } \ 1 \le p < \infty,$$

and let $\|f\|_\infty$ denote the essential supremum of $|f|$ on $\mathbb{R}$. For $1 \le p \le \infty$, by $L_p(\mathbb{R})$ we denote the Banach space of all measurable functions f on $\mathbb{R}$ such that $\|f\|_p < \infty$. In particular, $L_2(\mathbb{R})$ is a Hilbert space with the inner product given by

$$\langle f, g \rangle := \int_{\mathbb{R}} f(x)\overline{g(x)} \, dx, \quad f, g \in L_2(\mathbb{R}).$$

The Fourier transform of a function $f \in L_1(\mathbb{R})$ is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-ix\xi} \, dx, \quad \xi \in \mathbb{R}.$$

The Fourier transform can be naturally extended to functions in $L_2(\mathbb{R})$. For $\mu > 0$, we denote by $H^\mu(\mathbb{R})$ the Sobolev space of all functions $f \in L_2(\mathbb{R})$ such that the seminorm

$$|f|_{H^\mu(\mathbb{R})} := \left(\frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 |\xi|^{2\mu} \, d\xi \right)^{1/2}$$

is finite. The space $H^\mu(\mathbb{R})$ is a Hilbert space with the inner product given by

$$\langle f, g \rangle_{H^\mu(\mathbb{R})} := \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) \, \overline{\hat{g}(\xi)} \big(1 + |\xi|^{2\mu}\big) \, d\xi, \quad f, g \in H^\mu(\mathbb{R}).$$

The corresponding norm in $H^\mu(\mathbb{R})$ is given by $\|f\|_{H^\mu(\mathbb{R})} := \sqrt{\|f\|^2_{L_2(\mathbb{R})} + |f|^2_{H^\mu(\mathbb{R})}}$.

Let (a, b) be a nonempty open interval of the real line $\mathbb{R}$. By $C_c^\infty(a, b)$ we denote the space of all infinitely differentiable functions on $\mathbb{R}$ whose support is compact and contained in (a, b). For $\mu > 0$, we use $H_0^\mu(a, b)$ to denote the closure of $C_c^\infty(a, b)$ in $H^\mu(\mathbb{R})$. For $\mu = 0$, $H_0^\mu(a, b)$ is interpreted as the closure of $C_c^\infty(a, b)$ in $L_2(\mathbb{R})$. We identify this space with $L_2(a, b)$.

Let J be a (finite or infinite) countable set. By $\ell(J)$ we denote the linear space of all complex-valued sequences $(u_j)_{j\in J}$. Let $\ell_0(J)$ denote the linear space of all sequences $(u_j)_{j\in J}$ with only finitely many nonzero terms. We use $\ell_2(J)$ to denote the linear space of all sequences $u = (u_j)_{j\in J}$ such that $\|u\|_2 := \big(\sum_{j\in J} |u_j|^2\big)^{1/2} < \infty$. For $u = (u_j)_{j\in J}$ and $v = (v_j)_{j\in J}$, the inner product of u and v is defined as

$$\langle u, v\rangle := \sum_{j\in J} u_j \overline{v_j}.$$

Equipped with this inner product, $\ell_2(J)$ becomes a Hilbert space.

Let H be a Hilbert space. The inner product of two elements f and g in H is denoted by $\langle f, g\rangle$. The norm of an element f in H is given by $\|f\| := \sqrt{\langle f, f\rangle}$. If $\langle f, g\rangle = 0$, we say that f is orthogonal to g and write $f \perp g$. For a subset G of H, we define $G^\perp := \{f \in H : \langle f, g\rangle = 0 \ \forall\, g \in G\}$. It is easily seen that $G^\perp$ is a closed subspace of H.

A countable set F in H is said to be a **Riesz sequence** if there exist two positive constants A and B such that the inequalities

$$A\Big(\sum_{f\in F} |c_f|^2\Big)^{1/2} \le \Big\|\sum_{f\in F} c_f f\Big\| \le B\Big(\sum_{f\in F} |c_f|^2\Big)^{1/2} \tag{1.1}$$

hold true for every sequence $(c_f)_{f\in F}$ in $\ell_0(F)$. If this is the case, then the series $\sum_{f\in F} c_f f$ converges unconditionally for every $(c_f)_{f\in F}$ in $\ell_2(F)$, and the inequalities in (1.1) are valid for all $(c_f)_{f\in F}$ in $\ell_2(F)$. We call A a **Riesz lower bound** and B a **Riesz upper bound**. If F is a Riesz sequence in H, and if the linear span of F is dense in H, then F is a **Riesz basis** of H.

In Section 2, we will establish a general theory of multiresolution analysis induced by a pair of nested families of closed subspaces of a Hilbert space. This theory provides a general method to construct Riesz bases of a Hilbert space.

For a positive integer m, let M_m denote the B-spline of order m, which is the convolution of m copies of $\chi_{[0,1]}$, the characteristic function of the interval $[0, 1]$. More precisely, $M_1 := \chi_{[0,1]}$ and, for $m \ge 2$,

$$M_m(x) = \int_0^1 M_{m-1}(x - t)\, dt, \quad x \in \mathbb{R}.$$

It follows from the definition immediately that M_m is supported on $[0, m]$, $M_m(x) > 0$ and $M_m(m - x) = M_m(x)$ for $0 < x < m$. The Fourier transform of M_m is given by

$$\hat{M}_m(\xi) = \Big(\frac{1 - e^{-i\xi}}{i\xi}\Big)^m, \quad \xi \in \mathbb{R}.$$

For $m \geq 2$, M_m has continuous derivatives of order up to $m-2$. Moreover, $M_m \in H_0^\mu(0,m)$ for $0 < \mu < m - 1/2$.

Suppose that $r, s \in \mathbb{N}$, $r \geq s$, and $r+s$ is an even integer. Let n_0 be the least integer such that $2^{n_0} \geq r+s$. For $j \in \mathbb{Z}$, let

$$\phi_{n,j}(x) := 2^{n/2} M_r(2^n x - j) \quad \text{and} \quad \tilde{\phi}_{n,j}(x) := 2^{n/2} M_s(2^n x - j - (r-s)/2), \quad x \in \mathbb{R}. \tag{1.2}$$

If $n \geq n_0$ and $j \in I_n := \{0, 1, \ldots, 2^n - r\}$, then $\phi_{n,j}(x) = 0$ and $\tilde{\phi}_{n,j}(x) = 0$ for $x \in \mathbb{R} \setminus [0,1]$. Let $V_n := \operatorname{span}\{\phi_{n,j} : j \in I_n\}$ and $\tilde{V}_n := \operatorname{span}\{\tilde{\phi}_{n,j} : j \in I_n\}$, where $\operatorname{span} E$ denotes the linear span of the set E in a linear space. Then $\dim(V_n) = \dim(\tilde{V}_n) = 2^n - r + 1$. Evidently, $V_n \subset V_{n+1}$ and $\tilde{V}_n \subset \tilde{V}_{n+1}$ for $n \geq n_0$. Moreover, V_n is a subspace of $H_0^\mu(0,1)$ for $0 \leq \mu < r - 1/2$. For $r \geq 2$, each function f in V_n satisfies the homogeneous boundary conditions

$$f^{(k)}(0) = f^{(k)}(1) = 0, \quad k = 0, 1, \ldots, r-2.$$

Some properties of the pair of families $(\phi_{n,j})_{n \geq n_0, j \in I_n}$ and $(\tilde{\phi}_{n,j})_{n \geq n_0, j \in I_n}$ will be discussed in Section 3.

For $n \geq n_0$, let $W_n := V_{n+1} \cap \tilde{V}_n^\perp$ and $\tilde{W}_n := \tilde{V}_{n+1} \cap V_n^\perp$. It is easily seen that V_{n+1} is the direct sum of V_n and W_n, and $\tilde{V}_{n+1}$ is the direct sum of $\tilde{V}_n$ and $\tilde{W}_n$. Moreover, $\dim(W_n) = \dim(\tilde{W}_n) = 2^n$. Let $J_n := \{1, \ldots, 2^n\}$. A desire to construct bases for W_n and $\tilde{W}_n$ will lead us to study slant matrices in Section 4.

In Section 5, we will give a general method to construct a basis $\{\psi_{n,j} : j \in J_n\}$ of W_n for each $n = n_0, n_0+1, \ldots$. Finally, in Sections 6 and 7, we will complete the proof of the main result that the set

$$\{2^{-n_0\mu}\phi_{n_0,j} : j \in I_{n_0}\} \cup \cup_{n=n_0}^\infty \{2^{-n\mu}\psi_{n,j} : j \in J_n\}$$

forms a Riesz basis of $H_0^\mu(0,1)$ for $0 \leq \mu < r - 1/2$.

For the two important cases $s = 1$ and $s = 2$, we are able to give explicit formulation of wavelet bases as follows. The corresponding wavelets on $\mathbb{R}$ were first constructed by Jia, Wang , and Zhou in [13].

Suppose r is an odd positive integer and $s = 1$. Let

$$\psi(x) := \sum_{k=0}^{r} \frac{(-1)^k}{2} \big[M_{r+1}(k) + M_{r+1}(k+1)\big] M_r(2x - k), \quad x \in \mathbb{R}.$$

For $j = 1, \ldots, (r-1)/2$, let

$$\psi_j(x) := \sum_{k=0}^{2j-1} \frac{(-1)^k}{2} \big[M_{r+1}(2j-1-k) + M_{r+1}(2j-k)\big] M_r(2x-k), \quad x \in \mathbb{R}.$$

For $n \geq n_0$ and $x \in \mathbb{R}$, define

$$\psi_{n,j}(x) = \begin{cases} 2^{n/2}\psi_j(2^n x) & j = 1, \ldots, (r-1)/2, \\ 2^{n/2}\psi(2^n x - j + (r+1)/2) & j = (r+1)/2, \ldots, 2^n - (r-1)/2, \\ 2^{n/2}\psi_{2^n - j + 1}\big(2^n(1-x)\big) & j = 2^n - (r-3)/2, \ldots, 2^n. \end{cases}$$

Theorem 1.1. *For $n \geq n_0$ and $j \in J_n$, let $\psi_{n,j}$ be the functions as constructed above. Then the set*

$$\{2^{-n_0\mu}\phi_{n_0,j} : j \in I_{n_0}\} \cup \cup_{n=n_0}^{\infty} \{2^{-n\mu}\psi_{n,j} : j \in J_n\}$$

forms a Riesz basis of $H_0^\mu(0,1)$ for $0 \leq \mu < r - 1/2$.

For example, in the case when $r = 3$ and $s = 1$, we have

$$\psi(x) = \frac{1}{12}M_3(2x) - \frac{5}{12}M_3(2x-1) + \frac{5}{12}M_3(2x-2) - \frac{1}{12}M_3(2x-3), \quad x \in \mathbb{R},$$

and

$$\psi_1(x) = \frac{5}{12}M_3(2x) - \frac{1}{12}M_3(2x-1), \quad x \in \mathbb{R}.$$

Now suppose that r is an even positive integer and $s = 2$. Let

$$a(k) := \frac{1}{4}\big[M_{r+2}(k-1) + 2M_{r+2}(k) + M_{r+2}(k+1)\big], \quad k \in \mathbb{Z},$$

and

$$\psi(x) := \sum_{k=0}^{r+2}(-1)^k a(k) M_r(2x-k), \quad x \in \mathbb{R}.$$

For $j = 1, \ldots, r/2$ and $x \in \mathbb{R}$, let

$$\psi_j(x) := \sum_{k=0}^{2j}(-1)^k a(2j-k) M_r(2x-k) - \frac{a(2j+1)}{a(1)}\, a(0) M_r(2x).$$

For $n \geq n_0$ and $x \in \mathbb{R}$, define

$$\psi_{n,j}(x) = \begin{cases} 2^{n/2}\psi_j(2^n x) & j = 1, \ldots, r/2, \\ 2^{n/2}\psi(2^n x - j + r/2 + 1) & j = r/2+1, \ldots, 2^n - r/2, \\ 2^{n/2}\psi_{2^n-j+1}\big(2^n(1-x)\big) & j = 2^n - r/2 + 1, \ldots, 2^n. \end{cases}$$

Theorem 1.2. *For $n \geq n_0$ and $j \in J_n$, let $\psi_{n,j}$ be the functions as constructed above. Then the set*

$$\{2^{-n_0\mu}\phi_{n_0,j} : j \in I_{n_0}\} \cup \cup_{n=n_0}^{\infty} \{2^{-n\mu}\psi_{n,j} : j \in J_n\}$$

forms a Riesz basis of $H_0^\mu(0,1)$ for $0 \leq \mu < r - 1/2$.

For example, in the case when $r = 2$ and $s = 2$, we have

$$\psi(x) = \frac{1}{24}M_2(2x) - \frac{1}{4}M_2(2x-1) + \frac{5}{12}M_2(2x-2) - \frac{1}{4}M_2(2x-3) + \frac{1}{24}M_2(2x-4), \quad x \in \mathbb{R},$$

and

$$\psi_1(x) = \frac{3}{8}M_2(2x) - \frac{1}{4}M_2(2x-1) + \frac{1}{24}M_2(2x-2), \quad x \in \mathbb{R}.$$

Let us consider the case when $r = 4$ and $s = 2$. In this case, we have

$$\psi(x) = \frac{1}{480}\big[M_4(2x) - 28M_4(2x-1) + 119M_4(2x-2) - 184M_4(2x-3) \\ + 119M_4(2x-4) - 28M_4(2x-5) + M_4(2x-6)\big], \quad x \in \mathbb{R}.$$

Moreover, for $x \in \mathbb{R}$, we have

$$\psi_1(x) = \frac{1}{480}\left[\frac{787}{7}M_4(2x) - 28M_4(2x-1) + M_4(2x-2)\right]$$

and

$$\psi_2(x) = \frac{1}{480}\big[118M_4(2x) - 184M_4(2x-1) + 119M_4(2x-2) - 28M_4(2x-3) + M_4(2x-4)\big].$$

§2. Multiresolution Analysis

In this section we establish a general theory of multiresolution analysis induced by a pair of nested families of closed subspaces of a Hilbert space. This theory is a further development of the results in §2 of [13].

Let $A = (a_{jk})_{j\in I, k\in J}$ be a matrix with its entries being complex numbers, where I and J are countable sets. The transpose of A is denoted by A^T. For an element $u = (u_k)_{k\in J}$ in $\ell(J)$, let $v = (v_j)_{j\in I}$ be the element in $\ell(I)$ given by

$$v_j := \sum_{k\in J} a_{jk}u_k, \quad j \in I,$$

provided the above series converges absolutely for every $j \in I$. We use the same letter A to denote the linear mapping $u \mapsto v$ from $\ell(J)$ to $\ell(I)$. In particular, if J is a finite set, then the linear mapping A is well defined. In this case, we use $\ker A$ to denote the linear space of all elements $u \in \ell(J)$ such that $Au = 0$.

Now suppose that Au is well defined and lies in $\ell_2(I)$ for every u in $\ell_2(J)$. Then A is a linear mapping from $\ell_2(J)$ to $\ell_2(I)$ and its norm is defined by

$$\|A\| := \sup_{\|u\|_2\le 1} \|Au\|_2.$$

A sequence $(f_j)_{j\in J}$ in a Hilbert space H is said to be a **Bessel sequence** if there exists a constant K such that

$$\sum_{j\in J} \big|\langle f, f_j\rangle\big|^2 \le K\|f\|^2 \quad \forall f \in H,$$

or equivalently, the inequality

$$\bigg\|\sum_{j\in J} c_j f_j\bigg\|^2 \le K \sum_{j\in J} |c_j|^2$$

holds for every sequence $(c_j)_{j\in J}$ in $\ell_2(J)$. This happens if and only if the norm of the matrix $(\langle f_j, f_k\rangle)_{j,k\in J}$ is no bigger than K. Similarly, the norm of the *inverse* of the matrix $(\langle f_j, f_k\rangle)_{j,k\in J}$ is no bigger than K if and only if the inequality

$$\sum_{j\in J} |c_j|^2 \le K \left\| \sum_{j\in J} c_j f_j \right\|^2$$

holds for every sequence $(c_j)_{j\in J}$ in $\ell_2(J)$. See the book [18] for discussions on Bessel sequences and Riesz sequences.

Let H be a Hilbert space. Suppose that $(V_n)_{n=1,2,\ldots}$ and $(\tilde{V}_n)_{n=1,2,\ldots}$ are two nested families of closed subspaces of H:

$$V_1 \subset V_2 \subset \cdots \quad \text{and} \quad \tilde{V}_1 \subset \tilde{V}_2 \subset \cdots .$$

For $n = 1, 2, \ldots$, let $W_n := V_{n+1} \cap \tilde{V}_n^{\perp}$ and $\tilde{W}_n := \tilde{V}_{n+1} \cap V_n^{\perp}$. Let $W_0 := V_1$ and $\tilde{W}_0 := \tilde{V}_1$.

For each $n \in \mathbb{N}$, let I_n be a countable index set. We assume that $I_1 \subset I_2 \subset \cdots$. Let $J_0 := I_1$ and $J_n := I_{n+1} \setminus I_n$, $n = 1, 2, \ldots$. For each $n \in \mathbb{N}$, suppose that $\{\phi_{n,j} : j \in I_n\}$ and $\{\tilde{\phi}_{n,j} : j \in I_n\}$ are Riesz bases of V_n and $\tilde{V}_n$, respectively. For each $n \in \mathbb{N}_0$, suppose that $\{\psi_{n,j} : j \in J_n\}$ and $\{\tilde{\psi}_{n,j} : j \in J_n\}$ are Riesz bases of W_n and $\tilde{W}_n$, respectively. We assume that $\psi_{0,j} = \phi_{1,j}$ and $\tilde{\psi}_{0,j} = \tilde{\phi}_{1,j}$ for $j \in J_0 = I_1$.

Lemma 2.1. *If there exists a constant K independent of n such that the norms of the matrices $(\langle \phi_{n,j}, \phi_{n,k}\rangle)_{j,k\in I_n}$, $(\langle \tilde{\phi}_{n,j}, \tilde{\phi}_{n,k}\rangle)_{j,k\in I_n}$, $(\langle \psi_{n,j}, \psi_{n,k}\rangle)_{j,k\in J_n}$, $(\langle \tilde{\psi}_{n,j}, \tilde{\psi}_{n,k}\rangle)_{j,k\in J_n}$, $(\langle \phi_{n,j}, \tilde{\phi}_{n,k}\rangle)_{j,k\in I_n}$, and their inverses are bounded by K for all $n \in \mathbb{N}$, then the norms of the matrix $(\langle \psi_{n,j}, \tilde{\psi}_{n,k}\rangle)_{j,k\in J_n}$ and its inverse are bounded by a constant depending only on K.*

Proof. First, we assert that, for each $n \in \mathbb{N}$, $\{\phi_{n,j} : j \in I_n\} \cup \{\psi_{n,j} : j \in J_n\}$ is a Riesz basis of V_{n+1} with Riesz bounds depending only on K.

Given $f \in H$, we consider the system of linear equations

$$\sum_{j\in I_n} a_{n,j} \langle \phi_{n,j}, \tilde{\phi}_{n,k}\rangle = \langle f, \tilde{\phi}_{n,k}\rangle, \quad k \in I_n. \tag{2.1}$$

Since the matrix $(\langle \phi_{n,j}, \tilde{\phi}_{n,k}\rangle)_{j,k\in I_n}$ is invertible and the norm of its inverse is bounded by K, the above system of linear equations has a unique solution for $(a_{n,j})_{j\in I_n}$, and

$$\sum_{j\in I_n} |a_{n,j}|^2 \le K^2 \sum_{k\in I_n} \left|\langle f, \tilde{\phi}_{n,k}\rangle\right|^2. \tag{2.2}$$

Let $g := \sum_{j\in I_n} a_{n,j}\phi_{n,j}$ and $h := f - g$. Then (2.1) implies $h \perp \tilde{V}_n$. If f lies in V_{n+1}, then $h \in V_{n+1} \cap \tilde{V}_n^{\perp} = W_n$. This shows that V_{n+1} is the direct sum of V_n and W_n. Similarly, $\tilde{V}_{n+1}$ is the direct sum of $\tilde{V}_n$ and $\tilde{W}_n$. We may write $h = \sum_{j\in J_n} b_{n,j}\psi_{n,j}$.

By our assumption, the norms of the matrices $(\langle \phi_{n,j}, \phi_{n,k} \rangle)_{j,k\in I_n}$, $(\langle \psi_{n,j}, \psi_{n,k} \rangle)_{j,k\in J_n}$ and their inverses are bounded by K. Hence, we have

$$K^{-1} \sum_{j\in I_n} |a_{n,j}|^2 \le \|g\|^2 \le K \sum_{j\in I_n} |a_{n,j}|^2 \tag{2.3}$$

and

$$K^{-1} \sum_{j\in J_n} |b_{n,j}|^2 \le \|h\|^2 \le K \sum_{j\in J_n} |b_{n,j}|^2. \tag{2.4}$$

Consequently,

$$\|f\|^2 = \|g+h\|^2 \le 2\|g\|^2 + 2\|h\|^2 \le 2K \left[\sum_{j\in I_n} |a_{n,j}|^2 + \sum_{j\in J_n} |b_{n,j}|^2 \right]. \tag{2.5}$$

Moreover, since the norm of the matrix $(\langle \tilde{\phi}_{n,j}, \tilde{\phi}_{n,k} \rangle)_{j,k\in I_n}$ is bounded by K, we have $\sum_{k\in I_n} |\langle f, \tilde{\phi}_{n,k} \rangle|^2 \le K\|f\|^2$. This together with (2.2) yields

$$\sum_{j\in I_n} |a_{n,j}|^2 \le K^3 \|f\|^2. \tag{2.6}$$

Taking (2.3) into account, we obtain $\|g\|^2 \le K^4 \|f\|^2$. It follows that $\|g\| \le K^2 \|f\|$ and $\|h\| \le \|f\| + \|g\| \le (1+K^2)\|f\|$. This in connection with (2.4) gives

$$\sum_{j\in J_n} |b_{n,j}|^2 \le K\|h\|^2 \le K(1+K^2)^2 \|f\|^2. \tag{2.7}$$

Thus, our assertion is verified by (2.5), (2.6), and (2.7). In the same fashion it can be proved that $\{\tilde{\phi}_{n,j} : j \in I_n\} \cup \{\tilde{\psi}_{n,j} : j \in J_n\}$ is a Riesz basis of $\tilde{V}_{n+1}$ with Riesz bounds depending only on K.

In order to complete the proof, we set

$$\Phi_n := \big(\langle \phi_{n,j}, \tilde{\phi}_{n,k} \rangle\big)_{j,k\in I_n} \quad \text{and} \quad \Psi_n := \big(\langle \psi_{n,j}, \tilde{\psi}_{n,k} \rangle\big)_{j,k\in J_n}.$$

There exists a complex-valued matrix $D_{n+1} = (d_{jk})_{j,k\in I_{n+1}}$ such that

$$\phi_{n,j} = \sum_{k\in I_{n+1}} d_{jk} \phi_{n+1,k},\ j \in I_n \quad \text{and} \quad \psi_{n,j} = \sum_{k\in I_{n+1}} d_{jk} \phi_{n+1,k},\ j \in J_n.$$

We observe that the set $\{\phi_{n+1,j} : j \in I_{n+1}\}$ is a Riesz basis of V_{n+1} with Riesz bounds depending only on K, and so is the set $\{\phi_{n,j} : j \in I_n\} \cup \{\psi_{n,j} : j \in J_n\}$. Therefore, the norms of D_{n+1} and its inverse are bounded by a constant depending only on K. Similarly, there exists a complex-valued matrix $\tilde{D}_{n+1} = (\tilde{d}_{jk})_{j,k\in I_{n+1}}$ such that

$$\tilde{\phi}_{n,j} = \sum_{k\in I_{n+1}} \tilde{d}_{jk} \tilde{\phi}_{n+1,k},\ j \in I_n \quad \text{and} \quad \tilde{\psi}_{n,j} = \sum_{k\in I_{n+1}} \tilde{d}_{jk} \tilde{\phi}_{n+1,k},\ j \in J_n.$$

The norms of $\tilde{D}_{n+1}$ and its inverse are bounded by a constant depending only on K. Taking account of the fact that $V_n \perp \tilde{W}_n$ and $\tilde{V}_n \perp W_n$, we obtain

$$\begin{bmatrix} \Phi_n & 0 \\ 0 & \Psi_n \end{bmatrix} = D_{n+1}\Phi_{n+1}\overline{\tilde{D}_{n+1}}^T.$$

This shows that the norms of Ψ_n and Ψ_n^{-1} are bounded by a constant depending only on K. $\square$

The following lemma extends Theorem 3.1 of [13] to the general case.

Lemma 2.2. *Suppose that $(\psi_{n,j})_{n\in\mathbb{N}_0,j\in J_n}$ and $(\tilde{\psi}_{n,j})_{n\in\mathbb{N}_0,j\in J_n}$ are Bessel sequences in a Hilbert space H with the property that $\psi_{m,j} \perp \tilde{\psi}_{n,k}$ whenever $m \neq n$. If the norm of the inverse of the matrix $(\langle\psi_{n,j}, \tilde{\psi}_{n,k}\rangle)_{j,k\in J_n}$ is bounded by a constant independent of n, then $\{\psi_{n,j} : n \in \mathbb{N}_0, j \in J_n\}$ is a Riesz sequence in H.*

Proof. By our assumption, there exists a positive constant C_1 such that the inequalities

$$\sum_{n=0}^{\infty}\sum_{j\in J_n}\left|\langle f, \psi_{n,j}\rangle\right|^2 \le C_1\|f\|^2 \quad \text{and} \quad \sum_{n=0}^{\infty}\sum_{j\in J_n}\left|\langle f, \tilde{\psi}_{n,j}\rangle\right|^2 \le C_1\|f\|^2$$

are valid for all $f \in H$. Let $f = \sum_{n=0}^{\infty}\sum_{j\in J_n} b_{n,j}\psi_{n,j}$. Since $\psi_{m,j} \perp \tilde{\psi}_{n,k}$ for $m \neq n$, we have

$$\langle f, \tilde{\psi}_{n,k}\rangle = \sum_{j\in J_n} b_{n,j}\langle\psi_{n,j}, \tilde{\psi}_{n,k}\rangle.$$

But the norm of the inverse of the matrix $(\langle\psi_{n,j}, \tilde{\psi}_{n,k}\rangle)_{j,k\in J_n}$ is bounded by a constant independent of n. Hence, there exists a positive constant C_2 such that

$$\sum_{j\in J_n}|b_{n,j}|^2 \le C_2\sum_{k\in J_n}\left|\langle f, \tilde{\psi}_{n,k}\rangle\right|^2 \quad \forall\, n \in \mathbb{N}_0.$$

It follows that

$$\sum_{n=0}^{\infty}\sum_{j\in J_n}|b_{n,j}|^2 \le C_2\sum_{n=0}^{\infty}\sum_{k\in J_n}\left|\langle f, \tilde{\psi}_{n,k}\rangle\right|^2 \le C_1C_2\|f\|^2.$$

This shows that $\{\psi_{n,j} : n \in \mathbb{N}_0, j \in J_n\}$ is a Riesz sequence in H. $\square$

§3. Splines on the Interval

Suppose that r and s are positive integers and $r \geq s$. Recall that n_0 is the least integer such that $2^{n_0} \geq r+s$, and $I_n = \{0, 1, \ldots, 2^n - r\}$. For $n \geq n_0$ and $j \in \mathbb{Z}$, let $\phi_{n,j}$ and $\tilde{\phi}_{n,j}$ be the functions defined in (1.2). Under the condition that $r+s$ is an even integer, we will show that the norms of the matrices $(\langle \phi_{n,j}, \phi_{n,k} \rangle)_{j,k \in I_n}$, $(\langle \phi_{n,j}, \tilde{\phi}_{n,k} \rangle)_{j,k \in I_n}$, $(\langle \tilde{\phi}_{n,j}, \tilde{\phi}_{n,k} \rangle)_{j,k \in I_n}$ and their inverses are bounded by a constant independent of n.

Let us recall the concept of bracket products from [12] and [1]. The **bracket product** of two compactly supported functions f and g in $L_2(\mathbb{R})$ is given by

$$[f, g](\xi) := \sum_{j \in \mathbb{Z}} \langle f, g(\cdot - j) \rangle e^{-ij\xi} = \sum_{k \in \mathbb{Z}} \hat{f}(\xi + 2k\pi) \overline{\hat{g}(\xi + 2k\pi)}, \quad \xi \in \mathbb{R}.$$

Clearly, $[f, g]$ is a 2π-periodic function on $\mathbb{R}$.

For a compactly supported function ϕ in $L_2(\mathbb{R})$, define

$$\gamma_\phi := \min_{\xi \in [0, 2\pi]} \left\{ \sqrt{[\phi, \phi](\xi)} \right\} \quad \text{and} \quad \Gamma_\phi := \max_{\xi \in [0, 2\pi]} \left\{ \sqrt{[\phi, \phi](\xi)} \right\}.$$

We have

$$\gamma_\phi \|u\|_2 \leq \left\| \sum_{j \in \mathbb{Z}} u(j) \phi(\cdot - j) \right\|_2 \leq \Gamma_\phi \|u\|_2 \quad \forall\, u \in \ell_2(\mathbb{Z}).$$

In particular, if ϕ is the B-spline M_r, then $\Gamma_\phi = 1$ and

$$\gamma_\phi^2 = \sum_{k \in \mathbb{Z}} \left| \hat{\phi}(\pi + 2k\pi) \right|^2 = 2 \left(\frac{2}{\pi} \right)^{2r} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2r}} =: \gamma_{2r}.$$

This shows that $\{M_r(\cdot - j) : j \in \mathbb{Z}\}$ is a Riesz sequence in $L_2(\mathbb{R})$.

Consider the matrix

$$\Phi_n := \big(\langle \phi_{n,j}, \tilde{\phi}_{n,k} \rangle \big)_{j,k \in I_n}.$$

We have

$$\begin{aligned} \langle \phi_{n,j}, \tilde{\phi}_{n,k} \rangle &= \int_{\mathbb{R}} M_r(x - j) M_s(x - k - (r-s)/2) \, dx \\ &= M_{r+s}(r + j - k - (r-s)/2) = M_{r+s}((r+s)/2 + j - k). \end{aligned}$$

Consequently, Φ_n is a real symmetric matrix.

Lemma 3.1. *The matrix Φ_n and its inverse are bounded. More precisely, for all n, $\|\Phi_n\| \leq 1$ and $\|\Phi_n^{-1}\| \leq 1/\gamma_{r+s}$.*

Proof. For $u = (u_j)_{j \in I_n} \in \ell_2(I_n)$ we have

$$\left\| \sum_{j \in I_n} u_j M_{(r+s)/2}(\cdot - j) \right\|_2^2 = \sum_{j \in I_n} \sum_{k \in I_n} u_j M_{r+s}((r+s)/2 + j - k) \overline{u_k}.$$

It follows that

$$\gamma_{r+s}\|u\|_2^2 \le \sum_{j\in I_n}\sum_{k\in I_n} u_j M_{r+s}((r+s)/2+j-k)\overline{u_k} \le \|u\|_2^2 \quad \forall\, u \in \ell_2(I_n).$$

Therefore, $\|\Phi_n\| \le 1$ and $\|\Phi_n^{-1}\| \le 1/\gamma_{r+s}$. $\square$

Similarly, we see that the norms of the matrices $(\langle \phi_{n,j}, \phi_{n,k}\rangle)_{j,k\in I_n}$, $(\langle \tilde\phi_{n,j}, \tilde\phi_{n,k}\rangle)_{j,k\in I_n}$, and their inverses are bounded by a constant independent of n.

Let V_n be the linear span of $\{\phi_{n,j} : j \in I_n\}$ and let $\tilde V_n$ be the linear span of $\{\tilde\phi_{n,j} : j \in I_n\}$. Clearly, $\{\phi_{n,j} : j \in I_n\}$ and $\{\tilde\phi_{n,j} : j \in I_n\}$ are Riesz bases of V_n and $\tilde V_n$, respectively. For $n \ge n_0$, let $W_n := V_{n+1} \cap \tilde V_n^\perp$ and $\tilde W_n := \tilde V_{n+1} \cap V_n^\perp$. A function g in V_{n+1} is a linear combination of $\{\phi_{n+1,k} : k \in I_{n+1}\}$. It lies in W_n if and only if g is orthogonal to $\tilde\phi_{n,j}$ for all $j \in I_n$. This motivates us to consider the inner product $\langle \phi_{n+1,k}, \tilde\phi_{n,j}\rangle$. We have

$$\begin{aligned}
\langle \phi_{n+1,k}, \tilde\phi_{n,j}\rangle &= 2^{n+1/2}\int_{\mathbb{R}} M_r(2^{n+1}x-k)M_s(2^n x - j - (r-s)/2)\,dx \\
&= 2^{1/2}\int_{\mathbb{R}} M_r(2x-k)M_s(x-j-(r-s)/2)\,dx \\
&= 2^{1/2}\int_{\mathbb{R}} M_r(2x+2j+r-s-k)M_s(x)\,dx \\
&= 2^{1/2}a(s-1+k-2j),
\end{aligned}$$

where a is the sequence on $\mathbb{Z}$ given by

$$a(k) := \int_{\mathbb{R}} M_r(2x+r-1-k)M_s(x)\,dx, \quad k \in \mathbb{Z}. \tag{3.1}$$

It is easily seen that $a(k) > 0$ for $k = 0, 1, \ldots, m$, where $m := r + 2s - 2$. Moreover, $a(k) = 0$ for $k < 0$ or $k > m$, and $a(k) = a(m-k)$ for all $k \in \mathbb{Z}$. Consequently,

$$\langle \phi_{n+1,k}, \tilde\phi_{n,j}\rangle = a(r+s-1+2j-k).$$

Let

$$S_n := \big(a(r+s-1+2j-k)\big)_{j\in I_n, k\in I_{n+1}}. \tag{3.2}$$

Lemma 3.2. *The matrix S_n is of full rank. Consequently, the dimension of its kernel space is 2^n.*

The proof of this lemma is based on properties of Euler-Frobenius polynomials. For $r = 1, 2, \ldots$, let

$$E_r(z) := r!\sum_{j=1}^{r} M_{r+1}(j)z^{j-1}, \quad z \in \mathbb{C}.$$

Then $E_r(z)$ is called the Euler-Frobenius polynomial of degree $r-1$. The leading coefficient of $E_r(z)$ is 1. The zeros $\lambda_1, \ldots, \lambda_{r-1}$ of $E_r(z)$ are simple and negative. We label them so that

$$\lambda_{r-1} < \lambda_{r-2} < \cdots < \lambda_1 < 0.$$

Moreover, $\lambda_j \lambda_{r-j} = 1$ for $j = 1, \ldots, r-1$. If r is an odd integer, all the zeros of $E_r(z)$ are different from -1. For these results and other properties of Euler-Frobenius polynomials, the reader is referred to the book [16] of Schoenberg.

The B-spline M_s satisfies the following refinement equation:

$$M_s(x) = \sum_{k \in \mathbb{Z}} 2^{1-s} \binom{s}{k} M_s(2x-k), \quad x \in \mathbb{R}. \tag{3.3}$$

By (3.1) and (3.3) we have

$$\begin{aligned}
a(j) &= \int_{\mathbb{R}} M_s(x) M_r(2x + r - 1 - j)\, dx \\
&= \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} 2^{1-s} \binom{s}{k} M_s(2x-k) M_r(2x + r - 1 - j)\, dx \\
&= \frac{1}{2} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} 2^{1-s} \binom{s}{k} M_s(x-k) M_r(x + r - 1 - j)\, dx \\
&= \frac{1}{2} \sum_{k \in \mathbb{Z}} 2^{1-s} \binom{s}{k} M_{r+s}(1 + j - k).
\end{aligned}$$

Consider the polynomial $P(z) := \sum_{j \in \mathbb{Z}} a(j) z^j$. We have

$$\begin{aligned}
P(z) &= \frac{1}{2} \sum_{k \in \mathbb{Z}} 2^{1-s} \binom{s}{k} \sum_{j \in \mathbb{Z}} M_{r+s}(1 + j - k) z^j \\
&= \frac{1}{2} \sum_{k \in \mathbb{Z}} 2^{1-s} \binom{s}{k} z^k \sum_{j \in \mathbb{Z}} M_{r+s}(1 + j - k) z^{j-k} \\
&= \left(\frac{1+z}{2}\right)^s \frac{E_{r+s-1}(z)}{(r+s-1)!}.
\end{aligned}$$

Since all of the zeros of the Euler-Frobenius polynomial are negative real numbers, $P(z)$ and $P(-z)$ do not have common zeros. Hence, the polynomials

$$\sum_{j \in \mathbb{Z}} a(2j) z^j \quad \text{and} \quad \sum_{j \in \mathbb{Z}} a(2j-1) z^j$$

do not have common zeros. Moreover, it follows from $P(-1) = 0$ that

$$\sum_{j \in \mathbb{Z}} (-1)^j a(j) = 0. \tag{3.4}$$

§4. Slant Matrices

The matrix S_n is a slant matrix, according to the definition given by Goodman, Jia, and Micchelli in [5], where the spectral properties of slant matrices were investigated. In this section, on the basis of the work of Micchelli [15] on banded matrices with banded inverses, we give a self-contained treatment of invertibility of slant matrices. Also, see the work of Goodman and Micchelli [6] on refinement equations related to slant matrices.

Let $\mathbb{C}[z]$ denote the ring of polynomials over $\mathbb{C}$. If $p(z) = c_0 + c_1 z + \cdots + c_k z^k$ with $c_k \neq 0$, then we say that k is the degree of p and write $k = \deg p$. If p is the zero polynomial, then we shall use the convention that $\deg p = -\infty$. For $k \in \mathbb{N}_0$, we use Π_k to denote the linear space of all polynomials of degree at most k.

Lemma 4.1. *Let p_0, p_1, and f be polynomials in $\mathbb{C}[z]$ such that $\deg p_0 = m_0 \geq 0$, $\deg p_1 = m_1 \geq 0$, and $\deg f < m_0 + m_1$. If p_0 and p_1 have no common zeros, then there exist $q_0, q_1 \in \mathbb{C}[z]$ with $\deg q_0 < m_1$ and $\deg q_1 < m_0$ such that*

$$f = p_0 q_0 + p_1 q_1.$$

Proof. The proof proceeds with induction on $m := m_0 + m_1$. Suppose that $f \in \mathbb{C}[z]$ and $\deg f < m_0 + m_1$. If $m = 0$, then $f = 0$. In this case, one may choose $q_0 = 0$ and $q_1 = 0$.

Let $m > 0$ and suppose the lemma has been established for $m' < m$. Without loss of any generality we may assume $m_1 \leq m_0$. If $m_1 = 0$, then p_1 is a nonzero constant; hence we may write $f = p_1 q_1$ with $q_1 := p_1^{-1} f \in \mathbb{C}[z]$ and $\deg q_1 = \deg f < m_0$. Thus, in what follows, we assume $1 \leq m_1 \leq m_0$. By using the Euclidean algorithm, we can find g and h in $\mathbb{C}[z]$ such that

$$f = p_1 g + h,$$

with $\deg g \leq \deg f - \deg p_1 < m_0$ and $\deg h < \deg p_1 = m_1$. Furthermore, we can find η and θ in $\mathbb{C}[z]$ such that

$$p_0 = p_1 \eta + \theta$$

with $\deg \eta \leq \deg p_0 - \deg p_1 = m_0 - m_1$ and $\deg \theta < \deg p_1 = m_1$. Since p_0 and p_1 have no common zeros, we deduce that $\theta \neq 0$ and the polynomials p_1 and θ have no common zeros. We have $\deg h < m_1 = \deg p_1$. By the induction hypothesis, there exist τ_0 and τ_1 in $\mathbb{C}[z]$ with $\deg \tau_0 < m_1$ and $\deg \tau_1 < m_0$ such that

$$h = \theta \tau_0 + p_1 \tau_1.$$

It follows that

$$\begin{aligned} f &= p_1 g + \theta \tau_0 + p_1 \tau_1 = p_1 g + (p_0 - p_1 \eta)\tau_0 + p_1 \tau_1 \\ &= p_0 \tau_0 + p_1 (g + \tau_1 - \eta \tau_0). \end{aligned}$$

Choose $q_0 := \tau_0$ and $q_1 := g + \tau_1 - \eta \tau_0$. Then $\deg q_0 < m_1$ and $\deg q_1 < m_0$. This completes the induction procedure. $\square$

Let a be a sequence of complex numbers on $\mathbb{Z}$. Suppose that $a(0) \neq 0$, $a(m) \neq 0$ for some $m \in \mathbb{N}$, and $a(j) = 0$ for $j < 0$ or $j > m$. Let

$$p_0(z) := \sum_{j \in \mathbb{Z}} a(2j) z^j \quad \text{and} \quad p_1(z) := \sum_{j \in \mathbb{Z}} a(2j-1) z^j, \quad z \in \mathbb{C}.$$

Lemma 4.2. *Let n be an integer such that $n \geq m$. If the polynomials p_0 and p_1 have no common zeros, then the matrices*

$$\big(a(2j-k)\big)_{0\leq j\leq n, 0\leq k\leq 2n-m} \quad \text{and} \quad \big(a(1+2j-k)\big)_{0\leq j\leq n-1, 0\leq k\leq 2n-m}$$

are of full rank.

Proof. Let us first consider the matrix $(a(2j-k))_{0\leq j\leq n, 0\leq k\leq 2n-m}$. Note that $2n-m \geq n$. In order to prove that this matrix is of full rank, it suffices to show that its column vectors span $\mathbb{C}^{\{0,1,\ldots,n\}}$. We associate each column vector $[c_0, c_1, \ldots, c_n]^T$ with the polynomial $\sum_{j=0}^{n} c_j z^j$. Thus, it suffices to show that the polynomials corresponding to the columns of the matrix span Π_n. Let $f \in \Pi_n$.

Suppose $m = 2l$ is an even integer. Then the polynomials corresponding to the columns of the matrix are

$$p_0(z),\, p_1(z),\, zp_0(z),\, zp_1(z),\, \ldots,\, z^{n-l-1}p_1(z),\, z^{n-l}p_0(z). \tag{4.1}$$

We have $\deg p_0 = l$ and $\deg p_1 \leq l$. By using the Euclidean algorithm, we may write $f = p_0 g + h$, where $\deg g = \deg f - \deg p_0 \leq n-l$ and $\deg h < l$. By Lemma 4.1, there exist $q_0, q_1 \in \mathbb{C}[z]$ such that $h = p_0 q_0 + p_1 q_1$, where $\deg q_0 < l$ and $\deg q_1 < l$. Hence, the polynomials in (4.1) span f.

Suppose $m = 2l+1$ is an odd integer. Then the polynomials corresponding to the columns of the matrix are

$$p_0(z),\, p_1(z),\, zp_0(z),\, zp_1(z),\, \ldots,\, z^{n-l-1}p_0(z),\, z^{n-l-1}p_1(z). \tag{4.2}$$

We have $\deg p_0 \leq l$ and $\deg p_1 = l+1$. By using the Euclidean algorithm, we may write $f = p_1 g + h$, where $\deg g = \deg f - \deg p_1 \leq n-(l+1) = n-l-1$ and $\deg h \leq l$. By Lemma 4.1, there exist $q_0, q_1 \in \mathbb{C}[z]$ such that $h = p_0 q_0 + p_1 q_1$, where $\deg q_0 \leq l$ and $\deg q_1 < l$. Hence, the polynomials in (4.2) span f.

An analogous argument shows that the matrix $(a(1+2j-k))_{0\leq j\leq n-1, 0\leq k\leq 2n-m}$ is of full rank. □

Lemma 3.2 is a consequence of Lemma 4.2. Indeed, let us consider the following augmented matrix of S_n:

$$T_n := \big(a(r+s-1+2j-k)\big)_{-t\leq j\leq 2^n-r+t, 0\leq k\leq 2^{n+1}-r},$$

where $t := (r+s-2)/2$. We have

$$T_n = \big(a(1+2j-k)\big)_{0\leq j\leq 2^n+s-2, 0\leq k\leq 2^{n+1}-r}.$$

Note that $2^n+s-1 \geq (r+s)+s-1 > r+2s-2 = m$ and $2(2^n+s-1)-m = 2^{n+1}-r$. By Lemma 4.2, the matrix T_n is of full rank, that is, its row vectors are linearly independent. Consequently, the row vectors of S_n are linearly independent.

Choosing $n = m$ in Lemma 4.2, we see that the matrix $(a(2j-k))_{0\leq j,k\leq m}$ is invertible. But $a(2m-k) = 0$ for $0 \leq k < m$. In other words, the last row of this matrix has exactly one nonzero entry at the position (m, m). Therefore, the matrix $(a(2j-k))_{0\leq j,k\leq m-1}$ is also invertible. This result was already established in Lemma 1 of [17].

§5. Spline Wavelets on the Interval

Various methods of construction of spline wavelets on the real line were discussed in [3], [14], [13], and [7]. In this section, we give a general method to construct spline wavelets on the interval $[0,1]$ with homogeneous boundary conditions.

Let r and s be two positive integers such that $r \geq s$ and $r+s$ is even. Recall that n_0 is the least integer satisfying $2^{n_0} \geq r+s$. For $n \geq n_0$, V_n is the linear span of $\{\phi_{n,j} : j \in I_n\}$, where $I_n = \{0, \ldots, 2^n - r\}$, and $\phi_{n,j}(x) = 2^{n/2} M_r(2^n x - j)$, $x \in \mathbb{R}$. Moreover, $\tilde{V}_n$ is the linear span of $\{\tilde{\phi}_{n,j} : j \in I_n\}$, where $\tilde{\phi}_{n,j}(x) = 2^{n/2} M_s(2^n x - j - (r-s)/2)$, $x \in \mathbb{R}$. For $n \geq n_0$, $W_n = V_{n+1} \cap \tilde{V}_n^{\perp}$ and $\tilde{W}_n = \tilde{V}_{n+1} \cap V_n^{\perp}$. By Lammas 2.1 and 3.1, V_{n+1} is the direct sum of V_n and W_n, and $\tilde{V}_{n+1}$ is the direct sum of $\tilde{V}_n$ and $\tilde{W}_n$. Consequently, $\dim(W_n) = \dim(\tilde{W}_n) = 2^n$. In this section we will give a method to construct a basis for W_n. Let $g \in V_{n+1}$. Then g can be represented as $\sum_{k \in I_{n+1}} w(k)\phi_{n+1,k}$. According to the analysis given in §3, g lies in W_n if and only if $w \in \ker S_n$, where S_n is the matrix given in (3.2). Thus, for our purpose, it suffices to find a suitable basis for $\ker S_n$.

Let $a \in \ell_0(\mathbb{Z})$. Then we have

$$\sum_{k \in \mathbb{Z}} (-1)^k a(2j+1-k)a(k) = 0 \quad \forall\, j \in \mathbb{Z}. \tag{5.1}$$

Indeed, making the change of indices $k \to 2j+1-k$ in the above sum, we obtain

$$\sum_{k \in \mathbb{Z}} (-1)^k a(2j+1-k)a(k) = \sum_{k \in \mathbb{Z}} (-1)^{2j+1-k} a(k)a(2j+1-k) = -\sum_{k \in \mathbb{Z}} (-1)^k a(k)a(2j+1-k),$$

from which (5.1) follows.

Now let a be the sequence given in (3.1). With $m = r + 2s - 2$ we have $a(k) > 0$ for $0 \leq k \leq m$ and $a(k) = 0$ for $k < 0$ or $k > m$. Let b be the sequence on $\mathbb{Z}$ given by

$$b(k) := (-1)^k a(k), \quad k \in \mathbb{Z}.$$

It follows from (3.4) that $\sum_{k \in \mathbb{Z}} b(k) = 0$. Moreover, by (5.1) we have

$$\sum_{k \in \mathbb{Z}} a(2j+1-k)b(k) = 0 \quad \forall\, j \in \mathbb{Z}. \tag{5.2}$$

It was demonstrated in §3 that $\sum_{k=0}^{\infty} a(2k)z^k$ and $\sum_{k=0}^{\infty} a(2k+1)z^k$ are two polynomials having no common zeros. Consequently, $\sum_{k=0}^{\infty} b(2k)z^k$ and $\sum_{k=0}^{\infty} b(2k+1)z^k$ are two polynomials having no common zeros.

For a real number x, we use $\lfloor x \rfloor$ to denote the integer such that $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$. For $n \geq n_0$ and $j \in \mathbb{Z}$, let $u_{n,j}$ and $v_{n,j}$ be the elements in $\ell(I_{n+1})$ given by

$$u_{n,j}(k) := a(r+s-1+2j-k) \quad \text{and} \quad v_{n,j}(k) := b(r+s-2j+k), \quad k \in I_{n+1}. \tag{5.3}$$

We claim that the vectors $v_{n,j}$ ($j = \lfloor (3-s)/2 \rfloor, \ldots, 2^n + \lfloor s/2 \rfloor$) are linearly independent. In order to justify our claim, it suffices to show that the matrix

$$B_n := \big(b(r+s-2j+k)\big)_{\lfloor (3-s)/2 \rfloor \leq j \leq 2^n + \lfloor s/2 \rfloor, 0 \leq k \leq 2^{n+1}-r}$$

is of full rank. In light of the definition of the sequence b, this is equivalent to saying that the matrix

$$A_n := \big(a(r+s-2j+k)\big)_{\lfloor (3-s)/2\rfloor \le j \le 2^n+\lfloor s/2\rfloor, 0\le k\le 2^{n+1}-r}$$

is of full rank. Note that $a(r+s-2j+k) = a(m-(r+s-2j+k)) = a(s-2+2j-k)$. If both r and s are even integers, then

$$A_n = \big(a(s-2+2j-k)\big)_{(2-s)/2\le j\le 2^n+s/2, 0\le k\le 2^{n+1}-r} = \big(a(2j-k)\big)_{0\le j\le 2^n+s-1, 0\le k\le 2^{n+1}-r}.$$

If both r and s are odd integers, then

$$A_n = \big(a(1+2j-k)\big)_{0\le j\le 2^n+s-2, 0\le k\le 2^{n+1}-r}.$$

By Lemma 4.2, the matrix A_n is of full rank in both cases. This justifies our claim.

We observe that $u_{n,j}$ ($j \in I_n$) are the row vectors of the matrix S_n. Note that $b(r+s-2j+k) = 0$ for $k \notin I_{n+1}$ and $(r+s)/2 \le j \le 2^n - (r+s)/2 + 1$. Hence, by (5.2) we have

$$\langle u_{n,j'}, v_{n,j}\rangle = \sum_{k\in\mathbb{Z}} a(r+s-1+2j'-k)b(r+s-2j+k) = 0 \quad \forall\, j' \in I_n.$$

This shows $v_{n,j} \in \ker S_n$ for $(r+s)/2 \le j \le 2^n - (r+s)/2 + 1$.

Let $J := \{j \in \mathbb{Z} : \lfloor (3-s)/2\rfloor \le j \le (r+s)/2 - 1\}$. For $j \in J$ and $k > 2^{n+1} - r$ we have

$$r+s-2j+k \ge k+2 \ge 2^{n+1} - r + 3 \ge 2(r+s) - r + 3 = m+1,$$

and hence $b(r+s-2j+k) = 0$. Moreover, for $j' > \lfloor (s-2)/2\rfloor$ and $k < 0$ we have $a(r+s-1+2j'-k) = 0$. Consequently, for $j' > \lfloor (s-2)/2\rfloor$ and $j \in J$,

$$\langle u_{n,j'}, v_{n,j}\rangle = \sum_{k\in\mathbb{Z}} a(r+s-1+2j'-k)b(r+s-2j+k) = 0,$$

where (5.2) has been used to derive the last equality. This shows

$$\operatorname{span}\{v_{n,j} : j \in J\} \cap \ker S_n = \operatorname{span}\{v_{n,j} : j \in J\} \cap \big(\operatorname{span}\{u_{n,j} : j = 0, \ldots, \lfloor (s-2)/2\rfloor\}\big)^{\perp}.$$

For $j \in J$, let v_j be the sequence on $\mathbb{Z}$ given by $v_j(k) := b(r+s-2j+k)$ for $k \ge 0$ and $v_j(k) = 0$ for $k < 0$. Then $v_{n,j} = v_j|_{I_{n+1}}$. For $j = 0, \ldots, \lfloor (s-2)/2\rfloor$, let u_j be the sequence on $\mathbb{Z}$ given by $u_j(k) := a(r+s-1+2j-k)$ for $k \ge 0$ and $u_j(k) = 0$ for $k < 0$. Then $u_{n,j} = u_j|_{I_{n+1}}$. Consider the linear space

$$\operatorname{span}\{v_j : j \in J\} \cap \big(\operatorname{span}\{u_j : j = 0, \ldots, \lfloor (s-2)/2\rfloor\}\big)^{\perp}.$$

Choose a basis $\{w_1, \ldots, w_t\}$ for this space. Then we have

$$t \ge (r+s)/2 - \lfloor (3-s)/2\rfloor - \lfloor (s-2)/2\rfloor - 1 = (r+s-2)/2.$$

For $n \geq n_0$ and $j = 1, \ldots, t$, let $w_{n,j} := w_j|_{I_{n+1}}$. In light of the above discussion, $\{w_{n,1}, \ldots, w_{n,t}\}$ is a basis of the linear space

$$\text{span}\,\{v_{n,j} : j \in J\} \cap \ker S_n.$$

For $j = 2^n - t + 1, \ldots, 2^n$, let

$$w_{n,j}(k) := w_{2^n+1-j}(2^{n+1} - r - k), \quad k \in I_{n+1}.$$

By symmetry, we see that $\{w_{n,2^n-t+1}, \ldots, w_{n,2^n}\}$ is a basis of the linear space

$$\text{span}\,\{v_{n,2^n+1-j} : j \in J\} \cap \ker S_n.$$

Consequently,

$$\{w_{n,1}, \ldots, w_{n,t}\} \cup \{v_{n,(r+s)/2}, \ldots, v_{n,2^n-(r+s)/2+1}\} \cup \{w_{n,2^n-t+1}, \ldots, w_{n,2^n}\}$$

is a linearly independent set of vectors in the kernel space $\ker S_n$. Since $\dim(\ker S_n) = 2^n$, we have $2t + 2^n - (r+s) + 2 \leq 2^n$. It follows that $t \leq (r+s-2)/2$. On the other hand, $t \geq (r+s-2)/2$. Therefore, $t = (r+s-2)/2$. Let $w_{n,j} := v_{n,j}$ for $j = t+1, \ldots, 2^n - t$. Thus, $\{w_{n,1}, \ldots, w_{n,2^n}\}$ is a basis of $\ker S_n$. For $j \in J_n = \{1, \ldots, 2^n\}$, let

$$\psi_{n,j} := \sum_{k \in I_{n+1}} w_{n,j}(k)\phi_{n+1,k}.$$

We conclude that $\{\psi_{n,j} : j \in J_n\}$ is a basis of W_n.

The above discussion can be summarized as follows. Let $\{w_1, \ldots, w_t\}$ be a basis of the linear space

$$\text{span}\,\{v_j : j = \lfloor (3-s)/2 \rfloor, \ldots, (r+s)/2 - 1\} \cap \big(\text{span}\,\{u_j : j = 0, \ldots, \lfloor (s-2)/2 \rfloor\}\big)^{\perp}.$$

Then $t = (r+s)/2 - 1$. For $j = 1, \ldots, t$, let

$$\psi_j(x) := \sum_{k \in \mathbb{Z}} w_j(k) 2^{1/2} M_r(2x - k), \quad x \in \mathbb{R}.$$

Let

$$\psi(x) := \sum_{k \in \mathbb{Z}} b(k) 2^{1/2} M_r(2x - k), \quad x \in \mathbb{R}.$$

For $n \geq n_0$ and $x \in \mathbb{R}$, let

$$\psi_{n,j}(x) = \begin{cases} 2^{n/2}\psi_j(2^n x) & \text{for } j = 1, \ldots, t, \\ 2^{n/2}\psi(2^n x - j + t + 1) & \text{for } j = t+1, \ldots, 2^n - t, \\ 2^{n/2}\psi_{2^n+1-j}\big(2^n(1-x)\big) & \text{for } j = 2^n - t + 1, \ldots, 2^n. \end{cases}$$

Theorem 5.1. *The set* $\{\phi_{n_0,j} : j \in I_{n_0}\} \cup \cup_{n=n_0}^{\infty} \{\psi_{n,j} : j \in J_n\}$ *forms a Riesz basis of* $L_2(0,1)$.

First, we claim that $(\phi_{n_0,j})_{j \in I_{n_0}}$ and $(\psi_{n,j})_{n \geq n_0, j \in J_n}$ are Bessel sequences in $L_2(0,1)$. Indeed, [9, Lemma 3.2] tells us that $(\phi_{n_0,j})_{j \in I_{n_0}}$ is a Bessel sequence in $L_2(0,1)$. Since $\sum_{k \in \mathbb{Z}} b(k) = 0$, we have $\int_0^1 \psi_{n,j}(x)\,dx = 0$ for $n \geq n_0$ and $j = t+1, \ldots, 2^n - t$. Hence, by [9, Theorem 1.1], $(\psi_{n,j})_{n \geq n_0, j = t+1, \ldots, 2^n - t}$ is a Bessel sequence in $L_2(0,1)$. The following lemma shows that $(\psi_{n,j})_{n \geq n_0, j \in \{1, \ldots, t\}}$ is a Bessel sequences in $L_2(0,1)$. By symmetry, $(\psi_{n,j})_{n \geq n_0, j \in \{2^n - t + 1, \ldots, 2^n\}}$ is also a Bessel sequence in $L_2(0,1)$. This justifies our claim.

Lemma 5.2. *Let ϕ be a compactly supported function on $\mathbb{R}$ and $K := \|\phi\|_\infty < \infty$. For $n \in \mathbb{N}$, let $\phi_n(x) := 2^{n/2}\phi(2^n x)$, $x \in \mathbb{R}$. Then $(\phi_n)_{n\in\mathbb{N}}$ is a Bessel sequence in $L_2(\mathbb{R})$.*

Proof. Suppose that ϕ is supported on an interval $[a, b]$ and $L := b - a$. For $n, n' \in \mathbb{N}$ we have

$$\langle \phi_n, \phi_{n'} \rangle = 2^{(n+n')/2} \int_{\mathbb{R}} \phi(2^n x)\overline{\phi(2^{n'} x)}\, dx = 2^{(n-n')/2} \int_{\mathbb{R}} \phi(2^{n-n'} x)\overline{\phi(x)}\, dx.$$

It follows that

$$\big|\langle \phi_n, \phi_{n'} \rangle\big| \le 2^{(n-n')/2} K^2 L.$$

Similarly,

$$\big|\langle \phi_n, \phi_{n'} \rangle\big| \le 2^{(n'-n)/2} K^2 L.$$

Hence, there exists a positive constant C such that

$$\sum_{n'\in\mathbb{N}} \big|\langle \phi_n, \phi_{n'} \rangle\big| \le C \quad \forall\, n \in \mathbb{N} \quad \text{and} \quad \sum_{n\in\mathbb{N}} \big|\langle \phi_n, \phi_{n'} \rangle\big| \le C \quad \forall\, n' \in \mathbb{N}.$$

By [9, Lemma 4.1], the norm of the matrix $(\langle \phi_n, \phi_{n'} \rangle)_{n,n'\in\mathbb{N}}$ is no bigger than C. Therefore, $(\phi_n)_{n\in\mathbb{N}}$ is a Bessel sequence in $L_2(\mathbb{R})$. □

In §6 we will prove that there exist two positive constants C_1 and C_2 independent of n such that, for all $n \ge n_0$, C_1 is a lower Riesz bound for the basis $\{\psi_{n,j} : j \in J_n\}$ of the space W_n and C_2 is an upper Riesz bound for this basis.

By using an analogous argument, for $n \ge n_0$, we can find a Riesz basis $\{\tilde\psi_{n,j} : j \in J_n\}$ for $\tilde W_n$ with Riesz bounds being independent of n. Moreover, the bases can be so chosen that $(\tilde\phi_{n_0,j})_{j\in I_{n_0}}$ and $(\tilde\psi_{n,j})_{n\ge n_0, j\in J_n}$ are Bessel sequences in $L_2(0,1)$. Thus, in light of Lemmas 2.1 and 2.2, $\{\phi_{n_0,j} : j \in I_{n_0}\} \cup \cup_{n=n_0}^\infty \{\psi_{n,j} : j \in J_n\}$ is a Riesz sequence in $L_2(0,1)$. The proof of Theorem 5.1 will be complete after we establish in §7 the result that $\cup_{n=n_0}^\infty V_n$ is dense in $L_2(0,1)$.

Before concluding this section we investigate two important cases: $s = 1$ and $s = 2$. For the case $s = 1$, we have $t = (r-1)/2$. In this case, we may choose w_j as v_j for $j = 1, \ldots, t$. More precisely, $w_j(k) = 0$ for $k < 0$ and, for $k \ge 0$,

$$w_j(k) = b(k + r + 1 - 2j) = (-1)^k a(k + r + 1 - 2j) = (-1)^k a(2j - 1 - k).$$

For the case $s = 2$, we have $m = r + 2$. In this case, for $k \ge 0$,

$$v_j(k) = b(r + 2 - 2j + k) = (-1)^k a(r + 2 - 2j + k) = (-1)^k a(2j - k)$$

and $u_0(k) = a(r + 1 - k) = a(k + 1)$. A basis $\{w_1, \ldots, w_t\}$ of the linear space

$$\operatorname{span}\{v_j : j = 0, \ldots, r/2\} \cap (\operatorname{span}\{u_0\})^\perp$$

can be constructed as follows:

$$w_j := v_j - \frac{b(2j+1)}{b(1)} v_0 = v_j - \frac{a(2j+1)}{a(1)} v_0, \quad j = 1, \ldots, t,$$

where $t = r/2$. Indeed, since $a(k+1) = 0$ for $k < -1$, we have

$$\langle u_0, v_j \rangle = \sum_{k=0}^{\infty} a(k+1)b(2j-k) = \sum_{k=-\infty}^{\infty} a(k+1)b(2j-k) - a(0)b(2j+1).$$

By (5.2), $\sum_{k=-\infty}^{\infty} a(k+1)b(2j-k) = 0$. It follows that

$$\langle u_0, v_j \rangle = -a(0)b(2j+1).$$

Consequently, $w_1, \ldots, w_t$ are linearly independent vectors orthogonal to u_0.

§6. Riesz Sequences

In this section we will prove that $\{\psi_{n,j} : j \in J_n\}$ is a Riesz basis of W_n for $n \geq n_0$ with Riesz bounds being independent of n. For this purpose, it suffices to show that $(w_{n,j})_{j \in J_n}$ is a Riesz sequence in $\ell_2(I_{n+1})$ with Riesz bounds being independent of n.

Let $v_{n,j}$ ($j \in \mathbb{Z}$) be the elements in $\ell_2(I_{n+1})$ given in (5.3). Recall that $w_{n,j} = v_{n,j}$ for $j = t+1, \ldots, 2^n - t$, where $t = (r+s-2)/2$. In what follows, δ_{jk} stands for the Kronecker sign: $\delta_{jk} = 1$ for $j = k$ and $\delta_{jk} = 0$ for $j \neq k$.

Lemma 6.1. *There exists a sequence d on $\mathbb{Z}$ with the following properties: $d(j) = 0$ for $j < 0$ or $j > m-1$, and the sequences $y_{n,j}$ ($j \in \mathbb{Z}$) given by*

$$y_{n,j}(k) := d(r+s-2j+k), \quad k \in I_{n+1},$$

satisfy $\langle v_{n,j'}, y_{n,j} \rangle = \delta_{j'j}$ for all $j' \in \mathbb{Z}$ and $j = t+1, \ldots, 2^n - t$.

Proof. We divide our attention into two cases: both r and s are odd, and both r and s are even.

First, suppose that both r and s are odd integers. In this case, $m = r + 2s - 2$ is also an odd integer. Let $l := (m-1)/2$. We claim that the matrix $(b(m-1-2j+k))_{0 \leq j,k \leq m-1}$ is invertible. Indeed, by making change of indices $j \to m-1-j'$ and $k \to m-1-k'$, we see that $m-1-2j+k = 2j'-k'$. By Lemma 4.2, the matrix $(a(2j'-k'))_{0 \leq j',k' \leq m-1}$ is invertible. It follows that the matrix $(b(2j'-k'))_{0 \leq j',k' \leq m-1}$ is invertible. Hence, the matrix $(b(m-1-2j+k))_{0 \leq j,k \leq m-1}$ is invertible. Thus, there exists a sequence d on $\mathbb{Z}$ such that $d(k) = 0$ for $k < 0$ or $k > m-1$ and, for $j = 0, \ldots, m-1$,

$$\sum_{k=0}^{m-1} b(m-1-2j+k)d(k) = \delta_{jl}. \tag{6.1}$$

For $j < 0$ and $k \geq 0$, we have $m-1-2j+k \geq m+1$, and hence $b(m-1-2j+k) = 0$. For $j > m-1$ and $k \leq m-1$, we have $m-1-2j+k \leq m-1-2m+m-1 \leq -2$, and hence $b(m-1-2j+k) = 0$. This shows that (6.1) is valid for all $j \in \mathbb{Z}$.

For $j, j' \in \mathbb{Z}$ we have

$$\langle v_{n,j'}, y_{n,j} \rangle = \sum_{k \in I_{n+1}} b(r+s-2j'+k)d(r+s-2j+k).$$

But, for $j = t+1, \ldots, 2^n - t$, $d(k+r+s-2j) = 0$ for $k < 0$ or $k > 2^{n+1} - r$. Therefore,

$$\begin{aligned}
\langle v_{n,j'}, y_{n,j} \rangle &= \sum_{k \in \mathbb{Z}} b(r+s-2j'+k)d(r+s-2j+k) \\
&= \sum_{k \in \mathbb{Z}} b(2j-2j'+k)d(k) \\
&= \sum_{k \in \mathbb{Z}} b\big(m-1-2(j'+l-j)+k\big)d(k) \\
&= \delta_{j'+l-j,l} = \delta_{j'j}.
\end{aligned}$$

Second, suppose that both r and s are even integers. In this case, $m = r + 2s - 2$ is also an even integer. Let $l := m/2$. By Lemma 4.2, the matrix $(b(m-2j+k))_{0 \le j,k \le m-1}$ is invertible. There exists a sequence d on $\mathbb{Z}$ such that $d(k) = 0$ for $k < 0$ or $k > m-1$ and, for $j = 0, \ldots, m-1$,

$$\sum_{k=0}^{m-1} b(m-2j+k)d(k) = \delta_{jl}. \tag{6.2}$$

It is easily seen that (6.2) is valid for all $j \in \mathbb{Z}$. In the same way as above, we can show that $\langle v_{n,j'}, y_{n,j} \rangle = \delta_{j'j}$ for all $j' \in \mathbb{Z}$ and $j = t+1, \ldots, 2^n - t$. $\square$

Recall that

$$w_{n,j} \in \operatorname{span}\{v_{n,k} : k \in J\} \quad \text{for } j = 1, \ldots, t,$$

where $J = \{k \in \mathbb{Z} : \lfloor (3-s)/2 \rfloor \le k \le (r+s)/2 - 1\}$. Moreover,

$$w_{n,j} \in \operatorname{span}\{v_{n,2^n+1-k} : k \in J\} \quad \text{for } j = 2^n - t + 1, \ldots, 2^n.$$

Therefore, by Lemma 6.1 we obtain

$$\langle w_{n,j'}, y_{n,j} \rangle = \delta_{j'j} \quad \text{for } j' \in J_n \text{ and } j = t+1, \ldots, 2^n - t. \tag{6.3}$$

Lemma 6.2. *For $n \ge n_0$, $(w_{n,j})_{j \in J_n}$ is a Riesz sequence in $\ell_2(I_{n+1})$ with Riesz bounds being independent of n.*

Proof. By using the method in the proof of Lemma 3.2 of [9], we assert that there exists a constant B independent of n such that the inequality

$$\Big\| \sum_{j \in J_n} c_{n,j} w_{n,j} \Big\|_2 \le B \Big(\sum_{j \in J_n} |c_{n,j}|^2 \Big)^{1/2} \tag{6.4}$$

holds true for every sequence $(c_{n,j})_{j \in J_n}$ in $\ell_2(J_n)$.

In order to establish the lower bound, we set $g := \sum_{j \in J_n} c_{n,j} w_{n,j}$. Then it follows from (6.3) that $c_{n,j} = \langle g, y_{n,j} \rangle$ for $j = t+1, \ldots, 2^n - t$. By using the method in the proof of Lemma 3.1 of [9], we see that there exists a constant C_1 independent of n such that

$$\Big(\sum_{j=t+1}^{2^n - t} |c_{n,j}|^2 \Big)^{1/2} = \Big(\sum_{j=t+1}^{2^n - t} \big| \langle g, y_{n,j} \rangle \big|^2 \Big)^{1/2} \le C_1 \|g\|_2. \tag{6.5}$$

Let

$$g_0 := \sum_{j=t+1}^{2^n-t} c_{n,j} w_{n,j} \quad \text{and} \quad g_1 := \sum_{j=1}^{t} c_{n,j} w_{n,j} + \sum_{j=2^n-t+1}^{2^n} c_{n,j} w_{n,j}.$$

Then $g = g_0 + g_1$. The total number of terms in the last two sums is equal to $2t = r+s-2$, which is independent of n. Since $\{w_{n,j} : j \in \{1, \dots, t\} \cup \{2^n - t + 1, \dots, 2^n\}\}$ is linearly independent, there exists a constant C_2 independent of n such that

$$\left(\sum_{j=1}^{t} |c_{n,j}|^2 + \sum_{j=2^n-t+1}^{2^n} |c_{n,j}|^2 \right)^{1/2} \leq C_2 \|g_1\|_2 \leq C_2 \big(\|g\|_2 + \|g_0\|_2 \big). \tag{6.6}$$

But it follows from (6.4) that

$$\|g_0\|_2 \leq B \left(\sum_{j=t+1}^{2^n-t} |c_{n,j}|^2 \right)^{1/2}. \tag{6.7}$$

Combining the estimates (6.5), (6.6), and (6.7) together, we conclude that there exists a constant C independent of n such that

$$\left(\sum_{j \in J_n} |c_{n,j}|^2 \right)^{1/2} \leq C \|g\|_2.$$

This together with (6.4) shows that $(w_{n,j})_{j \in J_n}$ is a Riesz sequence with Riesz bounds being independent of n. □

§7. Wavelet Bases in Sobolev Spaces

In this section we first establish some approximation properties of a scaled family of B-splines on the interval $[0,1]$. This study is based on the work of de Boor and Fix [2] on quasiinterpolants, and the work of Jia [8] on quasi-projection operators. With the help of the results on spline approximation, we complete the proof of the main result that $\{2^{-n_0\mu} \phi_{n_0,j} : j \in I_{n_0}\} \cup \cup_{n=n_0}^{\infty} \{2^{-n\mu} \psi_{n,j} : j \in J_n\}$ forms a Riesz basis of the space $H_0^\mu(0,1)$ for $0 \leq \mu < r - 1/2$.

For a function $f \in L_p(\mathbb{R})$ $(1 \leq p \leq \infty)$, the modulus of continuity of f is defined by

$$\omega(f,t)_p := \sup_{|h| \leq t} \|\nabla_h f\|_p, \quad 0 \leq t < \infty,$$

where ∇_h is the difference operator given by $\nabla_h f := f - f(\cdot - h)$. For $m \in \mathbb{N}$, the m-th order modulus of smoothness of f is defined by

$$\omega_m(f,t)_p := \sup_{|h| \leq t} \|\nabla_h^m f\|_p, \quad 0 \leq t < \infty.$$

We observe that the B-spline M_r is supported on $[0,r]$. Consequently, $M_r(\cdot - k)$ vanishes on $(0,1)$ for $k \le -r$ or $k \ge 1$. But the set $\{M_r(\cdot - k)|_{(0,1)} : 1-r \le k \le 0\}$ is linearly independent (see [2]). Hence, we can find a function $u \in C(\mathbb{R})$ supported on $[0,1]$ such that $\langle u, M_r(\cdot - k)\rangle = \delta_{0k}$ for all $k \in \mathbb{Z}$. Similarly, we can find a function $v \in C(\mathbb{R})$ supported on $[r-1,r]$ such that $\langle v, M_r(\cdot - k)\rangle = \delta_{0k}$ for all $k \in \mathbb{Z}$.

For $n \in \mathbb{N}$ and $j \in \mathbb{Z}$, let $\phi_{n,j}$ and $\tilde{\phi}_{n,j}$ be the functions defined in (1.2). For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, let

$$\tilde{\varphi}_{n,k}(x) := \begin{cases} 2^{n/2}u(2^n x - k) & \text{for } k \le 2^n - r, \\ 2^{n/2}v(2^n x - k) & \text{for } k > 2^n - r. \end{cases}$$

It is easily seen that $\langle \phi_{n,j}, \tilde{\varphi}_{n,k}\rangle = \delta_{jk}$.

Let us consider the quasi-projection operator Q_n given by

$$Q_n f := \sum_{j\in\mathbb{Z}} \langle f, \tilde{\varphi}_{n,j}\rangle \phi_{n,j},$$

where f is a locally integrable function on $\mathbb{R}$. We have $Q_n(\phi_{n,j}) = \phi_{n,j}$. Moreover, the splines $\phi_{n,j}$ $(j \in \mathbb{Z})$ reproduce all polynomials of degree at most $r-1$. Hence, $Q_n p = p$ for all $p \in \Pi_{r-1}$. By using the method in the proof of [8, Theorem 3.2] we see that there exists a constant C_1 independent of n such that

$$\|f - Q_n f\|_2 \le C_1 \omega_r(f, 2^{-n})_2 \quad \forall\, f \in L_2(\mathbb{R}). \tag{7.1}$$

We claim that $\cup_{n=n_0}^{\infty} V_n$ is dense in $L_2(0,1)$, where $V_n = \operatorname{span}\{\phi_{n,j} : j \in I_n\}$. Indeed, for $k < 0$ and $x \ge 0$, we have $2^n x - k \ge 1$; hence $\tilde{\varphi}_{n,k}(x) = 2^{n/2}u(2^n x - k) = 0$. For $k > 2^n - r$ and $x \le 1$, we have $2^n x - k \le 2^n - (2^n - r + 1) = r - 1$; hence $\tilde{\varphi}_{n,k}(x) = 2^{n/2}v(2^n x - k) = 0$. This shows that $\tilde{\varphi}_{n,k}(x) = 0$ for $x \in [0,1]$, provided $k < 0$ or $k > 2^n - r$. Thus, if f is a function in $L_2(\mathbb{R})$ supported on $[0,1]$, then $\langle f, \tilde{\varphi}_{n,k}\rangle = 0$ unless $0 \le k \le 2^n - r$. Therefore, $Q_n f \in V_n$. It follows from (7.1) that

$$\lim_{n\to\infty} \|Q_n f - f\|_2 = 0.$$

This justifies our claim.

For $f \in L_2(0,1)$ and $n \ge n_0$, let $P_n f$ be the unique element in V_n such that

$$\langle P_n f, \tilde{\phi}_{n,k}\rangle = \langle f, \tilde{\phi}_{n,k}\rangle \quad \forall\, k \in I_n.$$

It is easily seen that P_n is a projector from $L_2(0,1)$ onto V_n. Since the norms of the matrices $(\langle \phi_{n,j}, \phi_{n,k}\rangle)_{j,k\in I_n}$, $(\langle \phi_{n,j}, \tilde{\phi}_{n,k}\rangle)_{j,k\in I_n}$, $(\langle \tilde{\phi}_{n,j}, \tilde{\phi}_{n,k}\rangle)_{j,k\in I_n}$ and their inverses are bounded by a constant independent of n, we have $K := \sup_{n\ge n_0} \|P_n\| < \infty$. Moreover,

$$\|f - P_n f\|_2 = \|(f - Q_n f) - P_n(f - Q_n f)\|_2 \le (1 + \|P_n\|)\|f - Q_n f\|_2.$$

This together with (7.1) gives

$$\|f - P_n f\|_2 \le (1 + K) C_1 \omega_r(f, 2^{-n})_2 \quad \forall\, n \ge n_0. \tag{7.2}$$

For $n \ge n_0$, $P_{n+1}f - P_nf$ lies in $V_{n+1} \cap \tilde{V}_n^{\perp} = W_n$. Hence, there exist complex numbers $c_{n,j}$ $(j \in J_n)$ such that

$$P_{n+1}f - P_nf = \sum_{j \in J_n} c_{n,j}\psi_{n,j}.$$

Moreover, there exist complex numbers $b_{n_0,j}$ $(j \in I_{n_0})$ such that

$$P_{n_0}f = \sum_{j \in I_{n_0}} b_{n_0,j}\phi_{n_0,j}.$$

By (7.2) we have

$$f = P_{n_0}f + \sum_{n=n_0}^{\infty}(P_{n+1}f - P_nf) = \sum_{j \in I_{n_0}} b_{n_0,j}\phi_{n_0,j} + \sum_{n=n_0}^{\infty}\sum_{j \in J_n} c_{n,j}\psi_{n,j}, \tag{7.3}$$

with the convergence being in the L_2-norm.

If $\mu > 0$ and m is an integer greater than μ, the Besov space $B_{2,2}^{\mu}(\mathbb{R})$ is the collection of those functions $f \in L_2(\mathbb{R})$ for which the following seminorm is finite:

$$|f|_{B_{2,2}^{\mu}(\mathbb{R})} := \left(\sum_{k \in \mathbb{Z}} \left[2^{k\mu}\omega_m(f, 2^{-k})_2\right]^2\right)^{1/2}.$$

It is well known that $H^{\mu}(\mathbb{R}) = B_{2,2}^{\mu}(\mathbb{R})$. Moreover, the seminorms $|f|_{H^{\mu}(\mathbb{R})}$ and $|f|_{B_{2,2}^{\mu}(\mathbb{R})}$ are equivalent.

Recall that $H_0^{\mu}(0,1)$ is the closure of $C_c^{\infty}(0,1)$ in $H^{\mu}(\mathbb{R})$. We have $V_n \subset H_0^{\mu}(0,1)$ for $0 < \mu < r - 1/2$.

Theorem 7.1. *The set*

$$\{2^{-n_0\mu}\phi_{n_0,j} : j \in I_{n_0}\} \cup \cup_{n=n_0}^{\infty}\{2^{-n\mu}\psi_{n,j} : j \in J_n\} \tag{7.4}$$

is a Riesz basis of the Sobolev space $H_0^{\mu}(0,1)$ for $0 < \mu < r - 1/2$.

Proof. Let $f \in H_0^{\mu}(0,1)$. Suppose f has a representation as in (7.3). By Theorem 1.2 of [9], there exists a constant B such that

$$|f|_{H^{\mu}} \le B\left(\sum_{j \in I_{n_0}} \left|2^{n_0\mu}b_{n_0,j}\right|^2 + \sum_{n=n_0}^{\infty}\sum_{j \in J_n}\left|2^{n\mu}c_{n,j}\right|^2\right)^{1/2}, \tag{7.5}$$

provided the right-hand side of the above inequality is finite.

By (7.2) we have

$$\|P_{n+1}f - P_nf\|_2 \le \|P_{n+1}f - f\|_2 + \|f - P_nf\|_2 \le 2(1+K)C_1\omega_r(f, 2^{-n})_2.$$

Hence, there exists a constant C_2 such that

$$\left(\sum_{n=n_0}^{\infty}\left[2^{n\mu}\|P_{n+1}f-P_nf\|_2\right]^2\right)^{1/2} \le 2(1+K)C_1\left(\sum_{n=n_0}^{\infty}\left[2^{n\mu}\omega_r(f,2^{-n})_2\right]^2\right)^{1/2} \le C_2|f|_{H^\mu}.$$

Since $\{\psi_{n,j} : j \in J_n\}$ is a Riesz basis of W_n with Riesz bounds being independent of n, there exists a constant C_3 independent of n such that

$$\left(\sum_{j\in J_n}|c_{n,j}|^2\right)^{1/2} \le C_3\|P_{n+1}f-P_nf\|_2.$$

It follows that

$$\left(\sum_{n=n_0}^{\infty}\sum_{j\in J_n}\left|2^{n\mu}c_{n,j}\right|^2\right)^{1/2} \le C_3\left(\sum_{n=n_0}^{\infty}\left[2^{n\mu}\|P_{n+1}f-P_nf\|_2\right]^2\right)^{1/2} \le C_2C_3|f|_{H^\mu}. \quad (7.6)$$

Furthermore, there exists a constant C_4 such that

$$\left(\sum_{j\in I_{n_0}}|b_{n_0,j}|^2\right)^{1/2} \le C_4\|P_{n_0}f\|_2 \le C_4K\|f\|_2. \quad (7.7)$$

Combining (7.5), (7.6) and (7.7) together, we conclude that the set in (7.4) is a Riesz sequence in $H_0^\mu(0,1)$. The following theorem shows that $\cup_{n=n_0}^{\infty}V_n$ is sense in $H_0^\mu(0,1)$. This completes the proof of the theorem. □

Theorem 7.2. *For $f \in H_0^\mu(0,1)$, $0<\mu<r-1/2$,*

$$\lim_{n\to\infty}\|P_nf-f\|_{H^\mu}=0. \quad (7.8)$$

Consequently, $\cup_{n=n_0}^{\infty}V_n$ is dense in $H_0^\mu(0,1)$.

Proof. Let $f \in H_0^\mu(0,1)$, $0<\mu<r-1/2$. Suppose that N_1 and N_2 are two integers with $N_2 > N_1 \ge n_0$. We have

$$P_{N_2}f-P_{N_1}f=\sum_{n=N_1}^{N_2-1}(P_{n+1}f-P_nf)=\sum_{n=N_1}^{N_2-1}\sum_{j\in J_n}c_{n,j}\psi_{n,j}.$$

By (7.5) we have

$$|P_{N_2}f-P_{N_1}f|_{H^\mu} \le B\left(\sum_{n=N_1}^{N_2-1}\sum_{j\in J_n}\left|2^{n\mu}c_{n,j}\right|^2\right)^{1/2}.$$

But (7.6) tells us that the series $\sum_{n=n_0}^{\infty}\sum_{j\in J_n}|2^{n\mu}c_{n,j}|^2$ converges. Hence,

$$\lim_{N_1,N_2\to\infty}|P_{N_2}f-P_{N_1}f|_{H^\mu}=0.$$

In other words, $(P_nf)_{n\ge n_0}$ is a Cauchy sequence in $H_0^\mu(0,1)$. Consequently, there exists a function $g \in H_0^\mu(0,1)$ such that

$$\lim_{n\to\infty}\|f_n-g\|_{H^\mu}=0,$$

where $f_n := P_nf \in V_n$. On the other hand, $\lim_{n\to\infty}\|f_n-f\|_2=0$. Therefore, $g=f$ and $\lim_{n\to\infty}\|f_n-f\|_{H^\mu}=0$. This shows that $\cup_{n=n_0}^{\infty}V_n$ is dense in $H_0^\mu(0,1)$. □

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