An approximation of Daubechies wavelet matrices by perfect reconstruction filter banks with rational coefficients

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#### Abstract

It is described how the coefficients of Daubechies wavelet matrices can be approximated by rational numbers in such a way that the perfect reconstruction property of the filter bank be preserved exactly.


Keywords: Daubechies wavelets, paraunitary matrix polynomials
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## 1. Introduction

Daubechies wavelet matrices $D_{N}$ are perfect reconstruction orthogonal filter banks to which there correspond the orthonormal bases of compactly supported wavelet functions $\left\{2^{-\frac{i}{2}} \psi\left(2^{i} x-j\right)\right\}_{i, j \in \mathbb{Z}}$. However, in most of practical applications of these wavelets, what matters is the coefficients of $D_{N}$ and not the form of the corresponding function $\psi$. In mere approximation of the irrational coefficients of $D_{N}$ by rational numbers, which is desirable in order to simplify the related calculations on a digital computer, the perfect reconstruction property of the filter bank $D_{N}$ (which is its most important property) is not preserved in general. Obtained in this way $\hat{D}_{N} \approx D_{N}$ is a perfect reconstruction orthogonal filter bank only approximately. In the present paper, we describe a procedure of approximation of Daubechies wavelet matrices $D_{N}$ by filter banks $\hat{D}_{N}$ with rational coefficients which have the perfect reconstruction property exactly. This approach depends on a recent parametrization of compact wavelet matrices [6] which was developed in parallel with a new matrix spectral factorization method [7], (8). 1

The paper is organized as follows. The necessary notation and definitions are introduced in the next section. In Section III, an exact formulation of the problem solved is given. In Section IV, the mathematical background of the proposed method is provided and the method itself is described in Section V. Some results of numerical simulations are presented in Section VI.

## 2. Notation and Basic Definitions

The sets of integer, rational, real and complex numbers are denoted by $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$, respectively. $\mathbb{D}:=\{z \in \mathbb{C}:|z|<1\}$ and $\mathbb{T}:=\partial \mathbb{D}$. $\delta_{i j}$ stands for the Kronecker delta, $\delta_{i j}=1$ if $i=j$ and 0 otherwise, and $\operatorname{Id}_{X}$ is the identity map on a set $X$.

Let $\mathcal{M}_{F}(m \times N)$ be the set of $m \times N$ matrices with entries from a field $F$ (if $F$ is omitted it is always assumed that $F=\mathbb{C}$ ). A row of coefficients $\left(c_{0}, c_{1}, \ldots, c_{N-1}\right)$ will be sometimes called a filter and an $m \times N$ matrix will be called an $m$-channel filter bank (with $N$ taps).

For $M \in \mathcal{M}(m \times N)$, let $\bar{M}$ be the matrix with conjugate entries and $M^{*}=\bar{M}^{T}$.

[^0]$U \in \mathcal{M}(m \times m)$ is called unitary if $U U^{*}=U^{*} U=I_{m}$ where $I_{m}$ stands for the $m \times m$ identity matrix, and the set of unitary matrices is denoted by $\mathcal{U}(m)$.
$\mathcal{P}[F]$ denotes the set of Laurent polynomials with coefficients from a field $F$, and $\mathcal{P}_{N}[F]:=\left\{\sum_{k=-N}^{N} c_{k} z^{k}: c_{k} \in F, k=-N, \ldots, N\right\}$. If we write just $\mathcal{P}$, the field of coefficients will be clear from the context, in most cases $\mathcal{P}=\mathcal{P}[\mathbb{C}] . \quad \mathcal{P}^{+} \subset \mathcal{P}$ is the set of polynomials (with non-negative powers of $z, \sum_{k=0}^{N} c_{k} z^{k} \in \mathcal{P}^{+}$) and $\mathcal{P}^{-} \subset \mathcal{P}$ is the set of Laurent polynomials with negative powers of $z, \sum_{k=1}^{N} c_{k} z^{-k} \in \mathcal{P}^{-}$. We emphasize that constant functions belong only to $\mathcal{P}^{+}$so that $\mathcal{P}^{+} \cap \mathcal{P}^{-}=\emptyset$. Let also $\mathcal{P}_{N}^{ \pm}=\mathcal{P}^{ \pm} \cap \mathcal{P}_{N}$.

For power series $f(z)=\sum_{k=-\infty}^{\infty} c_{k} z^{k}$ and $N \geq 1$, let $[f(z)]^{-},[f(z)]^{+},[f(z)]_{N}^{-}$, and $[f(z)]_{N}^{+}$, denote, respectively, $\sum_{k=-\infty}^{-1} c_{k} z^{k}, \sum_{k=0}^{\infty} c_{k} z^{k}, \sum_{k=-N}^{-1} c_{k} z^{k}$, and $\sum_{k=0}^{N} c_{k} z^{k}$ and we assume the corresponding functions under these expressions if the convergence domains of these power series are known.
$\mathcal{P}(m \times n)$ denotes the set of $m \times n$ (polynomial) matrices with entries from $\mathcal{P}$, and the sets $\mathcal{P}^{+}(m \times n), \mathcal{P}_{N}^{-}[F](m \times n)$, etc. are defined similarly. The elements of these sets $\mathbf{P}(z)=\left[p_{i j}(z)\right]$ are called polynomial matrix functions. When we speak about continuous maps between these sets, we mean that they are equipped with a usual topology.

For $p(z)=\sum_{k=-N}^{N} c_{k} z^{k} \in \mathcal{P}$, let $\widetilde{p}(z)=\sum_{k=-N}^{N} \bar{c}_{k} z^{-k}$ and for $P(z)=\left[p_{i j}(z)\right] \in$ $\mathcal{P}(m \times n)$ let $\widetilde{P}(z)=\left[\widetilde{p}_{i j}(z)\right]^{T} \in \mathcal{P}(n \times m)$. Note that $\widetilde{P}(z)=(P(z))^{*}$ when $z \in \mathbb{T}$. Thus usual relations for adjoint matrices like $\widetilde{P_{1}+P_{2}}(z)=\widetilde{P}_{1}(z)+\widetilde{P}_{2}(z), \widetilde{P_{1} P_{2}}(z)=$ $\widetilde{P}_{2}(z) \widetilde{P}_{1}(z)$, etc. hold.

A polynomial matrix function $\mathbf{U}(z) \in \mathcal{P}(m \times m)$ is called paraunitary if $\mathbf{U}(z) \widetilde{\mathbf{U}}(z)=$ $I_{m}$ for each $z \in \mathbb{C} \backslash\{0\}$, and the set of all paraunitary polynomial matrices is denoted by $\mathcal{P U}(m)$. Note that if $\mathbf{U} \in \mathcal{P} \mathcal{U}(m)$, then $\mathbf{U}(z) \in \mathcal{U}(m)$ for each $z \in \mathbb{T}$.

An $S \in \mathcal{P}_{N}(m \times m)$ is called positive definite if $S(z)$ is such $\left(X S(z) X^{*}>0\right.$ for each $0 \neq X \in \mathcal{M}(1 \times m)$ ) for almost every $z \in \mathbb{T}$. The polynomial matrix spectral factorization theorem (see e.g [3], 5]) asserts that every positive definite $S \in \mathcal{P}_{N}(m \times$ $m$ ) can be factorized as

$$
\begin{equation*}
S(z)=S_{+}(z) \widetilde{S_{+}}(z), \quad z \in \mathbb{C} \backslash\{0\} \tag{1}
\end{equation*}
$$

where $S_{+} \in \mathcal{P}_{N}^{+}(m \times m)$ and $\operatorname{det} S_{+}(z) \neq 0$ for each $z \in \mathbb{D}$. The representation (1) is unique in a sense that if $S(z)=S_{+}^{\prime}(z) \widetilde{S_{+}^{\prime}}(z)$, then there exists $U \in \mathcal{U}(m)$ such that $S_{+}(z)=S_{+}^{\prime}(z) U$.

A matrix $A \in \mathcal{M}_{F}(m \times m N)$,

$$
A=\left(\begin{array}{c}
\mathbf{a}^{0}  \tag{2}\\
\mathbf{a}^{1} \\
\vdots \\
\mathbf{a}^{m-1}
\end{array}\right)=\left(\begin{array}{cccc}
a_{0}^{0} & a_{1}^{0} & \cdots & a_{m N-1}^{0} \\
a_{0}^{1} & a_{1}^{1} & \cdots & a_{m N-1}^{1} \\
\vdots & \vdots & \vdots & \vdots \\
a_{0}^{m-1} & a_{1}^{m-1} & \cdots & a_{m N-1}^{m-1}
\end{array}\right),
$$

is said to be a wavelet matrix of rank $m$ and genus $N, A \in W M(m, N, F)$ (see [10, p. 41]) if the shifted versions of the rows of $A$ by arbitrary multiples of $m$ form an
orthogonal set, that is

$$
\begin{equation*}
\sum_{k=0}^{m N-1} a_{k+m r}^{i} \overline{a_{k+m s}^{j}}=m \delta_{i j} \delta_{r s}, \quad r, s \in \mathbb{Z}, \quad i, j \in\{0,1, \ldots, m-1\} \tag{3}
\end{equation*}
$$

(it is assumed that $a_{k}^{i}=0$ whenever $k<0$ or $k \geq m N$ ) and

$$
\begin{equation*}
\sum_{k=0}^{m N-1} a_{k}^{i}=m \delta_{i 0}, \quad 0 \leq i<m \tag{4}
\end{equation*}
$$

The conditions (3) and (4) are referred to as the quadratic and linear conditions, respectively, defining a wavelet matrix.

Wavelet matrices with genus 1 (and rank $m$ ) are called Haar wavelet matrices, $H(m, F):=W M(m, 1, F)$. It can be shown that the first row of any $H \in H(m, F)$ consists of just 1s (see [10, Lemma 4.4.2]) and if $H_{1}, H_{2} \in H(m, \mathbb{C})$, then $H_{1}=$ $\left(\begin{array}{cc}1 & 0 \\ 0 & U\end{array}\right) H_{2}$ where $U \in \mathcal{U}(m-1)$ (see [10, Corollary 4.4.3]). Consequently, the only real Haar wavelet matrix of rank 2 with determinant 1 is

$$
H_{2}=\left(\begin{array}{cc}
1 & 1  \tag{5}\\
-1 & 1
\end{array}\right)
$$

For the matrix (2), we consider the Fourier series (the $z$-transform) of its rows

$$
\begin{equation*}
h_{s}(z)=\sum_{k=0}^{m N-1} a_{k}^{s} z^{k}, \quad s=0,1, \ldots, m-1, \tag{6}
\end{equation*}
$$

and the corresponding polyphase matrix polynomial $A(z) \in \mathcal{P}_{F}^{+}(m \times m)$ defined by

$$
\begin{equation*}
A(z)=\sum_{k=0}^{N-1} A_{k} z^{k} \tag{7}
\end{equation*}
$$

where $A_{k}=\left[a_{k m+j}^{i}\right] \in \mathcal{M}_{F}(m \times m), k=0,1, \ldots, N-1$, e.g. $A_{0}=\left(\begin{array}{ccc}a_{0}^{0} & \cdots & a_{m-1}^{0} \\ \vdots & \vdots & \vdots \\ a_{0}^{m-1} & \cdots & a_{m-1}^{m-1}\end{array}\right)$.
We will heavily use the fact that the quadratic condition (3) is equivalent to the condition on (7),

$$
\begin{equation*}
A(z) \widetilde{A}(z)=m I_{m} \tag{8}
\end{equation*}
$$

i.e. $A(z)$ is a constant multiplier of paraunitary matrix function. This equivalence can be checked by direct computations (see [10, p. 43]). Consequently, $A(1)$ is always a Haar wavelet matrix (see [10, p. 49]).

It is well known as well that the quadratic condition (3) is also equivalent to the following condition on (61) (see [10, p. 96])

$$
\begin{equation*}
\sum_{k=0}^{m-1} h_{r}\left(z z_{0}^{k}\right) \tilde{h}_{s}\left(z z_{0}^{k}\right)=m^{2} \delta_{r s} \tag{9}
\end{equation*}
$$

In signal processing applications, if we split a function (signal) $f: \mathbb{Z} \rightarrow \mathbb{C}$ into $m$ parts

$$
\begin{equation*}
f_{r}=\sum_{s=-\infty}^{\infty} \frac{1}{m}\left\langle f, \mathbf{a}_{s m}^{r}\right\rangle \mathbf{a}_{s m}^{r}, \quad r=0,1, \ldots, m-1 \tag{10}
\end{equation*}
$$

where $\mathbf{a}_{s}^{r}(\cdot)=\mathbf{a}^{r}(\cdot-s), s \in \mathbb{Z}$, (see (2)) and $\left\langle f, \mathbf{a}_{s}^{r}\right\rangle=\sum_{k=-\infty}^{\infty} f(k) \overline{\mathbf{a}}_{s}^{r}(k)$, (it is assumed that $\mathbf{a}^{r}(k)=0$ whenever $k$ is outside the range $\{0,1, \ldots, m N-1\}$, so that only finitely many products in the above sums differ from 0 ) which corresponds to the filtering by each of the rows $\mathbf{a}^{r}, r=0,1, \ldots, m-1$, followed by downsampling with rate $m$, then each of the equivalent conditions (3), (8), and (9) guarantees that $f$ can be reconstructed exactly as follows (see [10, Theorem 4.4.23])

$$
\begin{equation*}
f_{r}=\sum_{r=0}^{m-1} f_{r} \tag{11}
\end{equation*}
$$

For this reason, a wavelet matrix is a perfect reconstruction filter bank.
The linear condition (4) implies that a constant signal $f: \mathbb{Z} \rightarrow \mathbb{C}$ emerges from the first filter in the representation (11).

It is said that a wavelet matrix (2) has a polynomial-regularity degree $d$ if

$$
\begin{equation*}
\sum_{k=0}^{m N-1} k^{p} a_{k}^{i}=0, \quad p=0,1, \ldots, d, \quad i=1,2, \ldots, m-1 \tag{12}
\end{equation*}
$$

The higher this degree, the more zero coefficients appear in the representation (10) of a smooth signal $f$. Note that every wavelet matrix has a polynomial regularity degree equal at least to 0 .

## 3. Formulation of the problem

The Daubechies wavelet matrix $D_{N}$ (with $2 N$ taps) is the two-channel filter bank with real coefficients

$$
D_{N}=\binom{h_{0}}{h_{1}}=\left(\begin{array}{ccccccc}
a_{0} & b_{0} & a_{1} & b_{1} & \ldots & a_{N-1} & b_{N-1}  \tag{13}\\
-b_{N-1} & a_{N-1} & -b_{N-2} & a_{N-2} & \ldots & -b_{0} & a_{0}
\end{array}\right)
$$

which together with quadratic and linear conditions (cf. (9) and (4))

$$
\begin{equation*}
\left|h_{0}(z)\right|^{2}+\left|h_{0}(-z)\right|^{2}=4 \quad \text { when }|z|=1 \tag{14}
\end{equation*}
$$

and

$$
\begin{equation*}
h_{0}(1)=2 ; \quad h_{1}(1)=0 \tag{15}
\end{equation*}
$$

where $h_{j}(z)=\sum_{k=0}^{2 N-1} h_{j}[k] z^{k}, j=0,1$, has the polynomial-regularity degree $N-1$ (12)

$$
\begin{equation*}
\sum_{k=0}^{2 N-1} h_{1}[k] \cdot k^{p}=0 \text { for } p=0,1, \ldots, N-1 \tag{16}
\end{equation*}
$$

The way of construction of such matrices $D_{N}$ (computation of coefficients in (13)) was first established by Daubechies [1] and is described in most books on wavelets [2], [9], [10]. To each matrix $D_{N}$ there corresponds the Daubechies wavelet $\psi=\psi_{N}$ which is a supported in $[-N+1, N]$ continuous function of certain smoothness (depending on $N$ ) such that the system $\left\{2^{-\frac{i}{2}} \psi\left(2^{i} x-j\right)\right\}, i, j \in \mathbb{Z}$, forms an orthonormal basis of $L_{2}(\mathbb{R})$. However the forms of wavelet functions $\psi_{N}$ are mostly of theoretical interest, while the numerical values of the coefficients of $D_{N}$ are very important for applications. Since they are irrational numbers in general, during the actual calculations on digital computers, these coefficients are quantized and thus $D_{N}$ is approximated by $\hat{D}_{N}$. It may then happen that $\hat{D}_{N}$ satisfies the quadratic condition (14) only approximately. As it has been explained in the preceding section, the quadratic condition on a wavelet matrix determines the perfect reconstruction property of a filter bank.

In the present paper, we propose a method of approximation of $D_{N}$ by $\hat{D}_{N}$ which has rational coefficients and satisfies the quadratic and linear conditions (14) and (15) exactly. It is obvious that $\hat{D}_{N}$ will have the maximal polynomial regularity property (16) only approximately.

## 4. Mathematical Background of the Method

The following theorem, which plays a crucial role in the established method, was actually proved in [7]. We present here the simplified proof of this theorem.

Theorem 1. Let $N \geq 1$. For any $0 \not \equiv \varphi \in \mathcal{P}_{N}^{-}$, there exists a unique pair of functions $\alpha, \beta \in \mathcal{P}_{N}^{+}$such that

$$
\begin{gather*}
\alpha(z) \widetilde{\alpha}(z)+\beta(z) \widetilde{\beta}(z)=1, \quad z \in \mathbb{C} \backslash\{0\}  \tag{17}\\
\alpha(1)=1 ; \quad \beta(1)=0 \tag{18}
\end{gather*}
$$

and

$$
\left(\begin{array}{ll}
1 & 0  \tag{19}\\
\varphi & 1
\end{array}\right)\left(\begin{array}{cc}
\alpha & \beta \\
-\widetilde{\beta} & \widetilde{\alpha}
\end{array}\right) \in \mathcal{P}^{+}(2 \times 2)
$$

Moreover, $\alpha$ and $\beta$ satisfy the condition

$$
\begin{equation*}
|\alpha(0)|+|\beta(0)|>0 \tag{20}
\end{equation*}
$$

Lemma 1. Let (19) be satisfied for $\varphi \in \mathcal{P}_{N}^{-}$and $\alpha, \beta \in \mathcal{P}_{N}^{+}$. Then

$$
\begin{equation*}
\alpha(z) \widetilde{\alpha}(z)+\beta(z) \widetilde{\beta}(z)=\text { Const, } \quad z \in \mathbb{C} \backslash\{0\} \tag{21}
\end{equation*}
$$

Note that this constant should be positive (for $\alpha, \beta \not \equiv 0$ ) since $\alpha(z) \widetilde{\alpha}(z)+\beta(z) \widetilde{\beta}(z)=$ $|\alpha(z)|^{2}+|\beta(z)|^{2}$ for each $z \in \mathbb{T}$.
Proof. It follows from (19) that

$$
\left\{\begin{array}{l}
\varphi \alpha-\widetilde{\beta}=: \Phi_{1} \in \mathcal{P}^{+}  \tag{22}\\
\varphi \beta+\widetilde{\alpha}=: \Phi_{2} \in \mathcal{P}^{+}
\end{array}\right.
$$

Hence

$$
\alpha \widetilde{\alpha}+\beta \widetilde{\beta}=\Phi_{2} \alpha-\Phi_{1} \beta=: \Phi \in \mathcal{P}^{+}
$$

and since $\Phi=\widetilde{\Phi}$ it follows that $\Phi$ is constant.
Proof of Theorem 1. Let $\varphi(z)=\sum_{k=1}^{N} \gamma^{k} z^{-k}$ be given. We provide a constructive proof how to find $\alpha$ and $\beta$. First we seek for nontrivial polynomials

$$
\begin{equation*}
\alpha(z)=\sum_{k=0}^{N} x_{k} z^{k} ; \quad \beta(z)=\sum_{k=0}^{N} y_{k} z^{k} \tag{23}
\end{equation*}
$$

which satisfy (22) and hence (19). If we equate all coefficients of negative powers of $z$ of functions $\Phi_{1}$ and $\Phi_{2}$ in (22) to 0 and their 0 th coefficients to 0 and 1 respectively, then we get the following system of equations in the block matrix form

$$
\left\{\begin{array}{c}
\Theta X-\bar{Y}=\mathbf{0}  \tag{24}\\
\Theta Y+\bar{X}=\mathbf{1}
\end{array}\right.
$$

where

$$
\Theta=\left(\begin{array}{cccccc}
0 & \gamma_{1} & \gamma_{2} & \cdots & \gamma_{N-1} & \gamma_{N} \\
\gamma_{1} & \gamma_{2} & \gamma_{3} & \cdots & \gamma_{N} & 0 \\
\gamma_{2} & \gamma_{3} & \gamma_{4} & \cdots & 0 & 0 \\
\cdot & \cdot & \cdot & \cdots & . & . \\
\gamma_{N} & 0 & 0 & \cdots & 0 & 0
\end{array}\right), X=\left(\begin{array}{c}
x_{0} \\
x_{1} \\
x_{2} \\
\vdots \\
x_{N}
\end{array}\right), Y=\left(\begin{array}{c}
y_{0} \\
y_{1} \\
y_{2} \\
\vdots \\
y_{N}
\end{array}\right), \mathbf{0}=\left(\begin{array}{c}
0 \\
0 \\
0 \\
\vdots \\
0
\end{array}\right), \mathbf{1}=\left(\begin{array}{c}
1 \\
0 \\
0 \\
\vdots \\
0
\end{array}\right)
$$

If we substitute the first equation of (24)

$$
\begin{equation*}
Y=\overline{\Theta X} \tag{25}
\end{equation*}
$$

into the second equation, we get $\Theta \bar{\Theta} \bar{X}+\bar{X}=\mathbf{1}$, which is equivalent to

$$
\begin{equation*}
\left(\bar{\Theta} \Theta+I_{N+1}\right) X=\mathbf{1} \tag{26}
\end{equation*}
$$

The system (26) is nonsingular as $\Theta$ is symmetric and $\bar{\Theta} \Theta=\Theta^{*} \Theta$ is positive definite. (Furthermore, all eigenvalues of $\Delta:=\bar{\Theta} \Theta+I_{N+1}$ are grater than or equal to 1 , and hence $\left\|\Delta^{-1}\right\| \leq 1$ as well.) Hence, the coefficients $x_{k}$ and $y_{k}, k=0,1, \ldots, N$, in (23) can be determined from (26) and (25), and the constructed $\alpha$ and $\beta$ will satisfy (19). The equation (21) will be accomplished by Lemma 1 and we can achieve (17) by normalization.

If now the unitary matrix

$$
U=\left(\begin{array}{cc}
\alpha(1) & \beta(1)  \tag{27}\\
-\widetilde{\beta}(1) & \widetilde{\alpha}(1)
\end{array}\right)
$$

is not the identity matrix (note that $\operatorname{det} U=1$ by virtue of (17)), we can redefine $\alpha$ and $\beta$ by the equation

$$
\left(\begin{array}{cc}
\alpha & \beta \\
-\widetilde{\beta} & \widetilde{\alpha}
\end{array}\right):=\left(\begin{array}{cc}
\alpha & \beta \\
-\widetilde{\beta} & \widetilde{\alpha}
\end{array}\right) \cdot U^{-1}
$$

and thus the determined $\alpha$ and $\beta$ will satisfy the conditions (17)-(19).
Since the determinant of the product in (19) is 1 , we have $\alpha(z) \Phi_{2}(z)-\beta(z) \Phi_{1}(z)=1$ for each $z \in \mathbb{C}($ see (221)). Hence (20) holds as well.

The uniqueness of a pair of polynomials $\alpha$ and $\beta$ follows from the uniqueness of spectral factorization (see the Introduction) since $\left(\begin{array}{ll}1 & 0 \\ \varphi & 1\end{array}\right)\left(\begin{array}{cc}\alpha & \beta \\ -\widetilde{\beta} & \widetilde{\alpha}\end{array}\right)$ is the spectral factor of $\left(\begin{array}{ll}1 & 0 \\ \varphi & 1\end{array}\right)\left(\begin{array}{cc}1 & \varphi^{*} \\ 0 & 1\end{array}\right)$.

Every process described during the construction of $\alpha$ and $\beta$ in the proof of Theorem 1 is stable under small perturbations of the data, which implies the validity of the following

Corollary 1. Let $N \geq 1$, and let $\amalg: \mathcal{P}_{N}^{-} \rightarrow \mathcal{P}_{N}^{+} \times \mathcal{P}_{N}^{+}$be the map defined according to Theorem 1 which assigns a pair of polynomials $\alpha$ and $\beta$ to each $\varphi \in \mathcal{P}_{N}^{-}$. Then $\amalg$ is a continuous map.

If we take the coefficients of $\varphi$ rational, then the proof goes through without any change and the obtained coefficients of $\alpha$ and $\beta$ are rational as well. Thus we have the following
Corollary 2. If $\varphi \in \mathcal{P}_{N}^{-}[\mathbb{Q}]$, then the corresponding polynomials $\alpha$ and $\beta$ are from $\mathcal{P}_{N}^{+}[\mathbb{Q}]$.
Theorem 2. Let $N \geq 1$. For any pair of polynomials $\alpha, \beta \in \mathcal{P}_{N}^{+}$which satisfy (17) and (20) there exists a unique $\varphi \in \mathcal{P}_{N}^{-}$such that (19) holds.

The proof of this theorem is also constructive.
Proof. Define the function $f$ in a deleted neighborhood of 0 as (see (20))

$$
f(z)=\left\{\begin{array}{l}
\frac{1}{\alpha(z)} \widetilde{\beta}(z) \quad \text { if } \quad \alpha(0) \neq 0 \\
-\frac{1}{\beta(z)} \widetilde{\alpha}(z) \quad \text { if } \quad \beta(0) \neq 0
\end{array}\right.
$$

and let us show that

$$
\varphi(z)=[f(z)]^{-}
$$

satisfies (19). Indeed, $\varphi \in \mathcal{P}_{N}^{-}$(as $\left.[f(z)]^{-}=[f(z)]_{N}^{-}\right)$and

$$
\varphi(z)=f(z)-[f(z)]^{+}
$$

in a deleted neighborhood of 0 . Consider the case $\alpha(0) \neq 0$ (the case $\beta(0) \neq 0$ can be treated analogously). Then

$$
\varphi \alpha-\widetilde{\beta}=\left(f-[f]^{+}\right) \alpha-\widetilde{\beta}=\widetilde{\beta}-[f]^{+} \alpha-\widetilde{\beta}=-[f]^{+} \alpha
$$

and

$$
\varphi \beta+\widetilde{\alpha}=\left(f-[f]^{+}\right) \beta+\widetilde{\alpha}=\frac{\widetilde{\beta} \beta+\alpha \widetilde{\alpha}}{\alpha}-[f]^{+} \beta=\frac{1}{\alpha}-[f]^{+} \beta
$$

which shows that the functions $\varphi \alpha-\widetilde{\beta}$ and $\varphi \beta+\widetilde{\alpha}$ have removable singularities at 0 . Since we know that these functions are from $\mathcal{P}$, we conclude that actually they belong to $\mathcal{P}^{+}$. Thus (22) and consequently (19) hold. Observe that

$$
\begin{equation*}
\varphi(z)=\left[\left[\frac{1}{\alpha}\right]_{N}^{+} \widetilde{\beta}(z)\right]^{-} \quad \text { or } \quad \varphi(z)=-\left[\left[\frac{1}{\beta}\right]_{N}^{+} \widetilde{\alpha}(z)\right]^{-}, \tag{28}
\end{equation*}
$$

so that we need to compute the first $N$ coefficients of $1 / \alpha(z)$ or $1 / \beta(z)$ in order to construct $\varphi(z)$.

To show the uniqueness of $\varphi(z)$ observe that, by virtue of the Bezout theorem (see e.g. [9, Theorem 7.6]), there are $\alpha_{0}, \beta_{0} \in \mathcal{P}^{+}$such that $\alpha(z) \alpha_{0}(z)+\beta(z) \beta_{0}(z)=1$. Hence, if $\varphi \in \mathcal{P}^{-}$satisfies (22), then

$$
\alpha_{0}(z)(\varphi(z) \alpha(z)-\widetilde{\beta}(z))+\beta_{0}(\varphi(z) \beta(z)+\widetilde{\alpha}(z)) \in \mathcal{P}^{+}
$$

and

$$
\begin{equation*}
\varphi(z)=\left[\alpha_{0}(z) \widetilde{\beta}(z)-\beta_{0}(z) \widetilde{\alpha}(z)\right]^{-} \tag{29}
\end{equation*}
$$

As in Theorem 1, the process of construction of $\varphi$ (see (28)) is stable under small perturbations of $\alpha$ and $\beta$. Thus we come to

Corollary 3. Let $N \geq 1$ and $\Omega_{N} \subset \mathcal{P}_{N}^{+} \times \mathcal{P}_{N}^{+}$be the set of pairs $(\alpha, \beta)$ which satisfy (17) and (20). Then the map $\Pi: \Omega_{N} \rightarrow \mathcal{P}_{N}^{-}$defined according to Theorem 2, which assigns $\varphi \in \mathcal{P}_{N}^{-}$to each $(\alpha, \beta) \in \Omega_{N}$, is continuous.

We can combine Corollaries 1 and 3 as follows
Corollary 4. If $\Omega_{N}^{0} \subset \Omega_{n}$ is the set of $(\alpha, \beta) \in \Omega_{N}$ which in addition satisfy (18), then $\amalg$ is a continuous one-to-one map from $\mathcal{P}_{n}^{-}$onto $\Omega_{N}^{0}$ such that $\Pi \circ \amalg=\operatorname{Id}_{\mathcal{P}_{N}^{-}}$

Proof. If $\varphi \in \mathcal{P}_{N}^{-}$and $(\alpha, \beta) \in \Omega_{N}^{0}$ are such that (19) holds, then $\coprod(\varphi)=(\alpha, \beta)$ and $\prod(\alpha, \beta)=\varphi$.

Corollary 5. For any $(\alpha, \beta) \in \Omega_{N}$, let $\prod(\alpha, \beta)=\varphi$ and $\coprod(\varphi)=\left(\alpha^{\prime}, \beta^{\prime}\right)$. Then

$$
\begin{equation*}
(\alpha, \beta)=\left(\alpha^{\prime}, \beta^{\prime}\right) U \tag{30}
\end{equation*}
$$

where the matrix $U$ is defined by the equation (27).
Proof. Because of (17), $U \in \mathcal{U}(2)$ and $U^{-1}=U^{*}$. Thus

$$
\left(\begin{array}{cc}
\alpha & \beta \\
-\widetilde{\beta} & \widetilde{\alpha}
\end{array}\right) U^{-1}=\left(\begin{array}{cc}
\widetilde{\alpha}(1) \alpha+\widetilde{\beta}(1) \beta & -\beta(1) \alpha+\alpha(1) \beta \\
-(-\beta(1) \widetilde{\alpha+\alpha}(1) \beta) & \widetilde{\alpha}(1) \alpha+\widetilde{\beta}(1) \beta
\end{array}\right)
$$

and $(\widetilde{\alpha}(1) \alpha+\widetilde{\beta}(1) \beta, \beta(1) \alpha+\alpha(1) \beta) \in \Omega_{N}^{0}$. Since (see (19))

$$
\left(\begin{array}{cc}
1 & 0 \\
\varphi & 1
\end{array}\right)\left(\begin{array}{cc}
\alpha & \beta \\
-\widetilde{\beta} & \widetilde{\alpha}
\end{array}\right) U^{-1} \in \mathcal{P}^{+}(2 \times 2)
$$

we have $(\alpha, \beta) U^{-1}=\coprod(\varphi)$. Thus $(\alpha, \beta) U^{-1}=\left(\alpha^{\prime}, \beta^{\prime}\right)$ and (30) follows.

## 5. Description of the Method

Using the coefficients of (13), we can define

$$
\alpha(z)=\frac{1}{\sqrt{2}} \sum_{k=0}^{N-1} a_{k} z^{k} ; \quad \beta(z)=\frac{1}{\sqrt{2}} \sum_{k=0}^{N-1} b_{k} z^{k}
$$

As has been mentioned in Section 2, the quadratic condition (14) is equivalent to the condition for the matrix

$$
\frac{1}{\sqrt{2}} A(z)=\left(\begin{array}{cc}
\alpha(z) & \beta(z) \\
-z^{N-1} \widetilde{\beta}(z) & z^{N-1} \widetilde{\alpha}(z)
\end{array}\right)
$$

to be paraunitary (see (7), (8)), and hence (17) holds, while (15) impies that $\frac{1}{\sqrt{2}} A(1)=$ $\frac{1}{\sqrt{2}} H_{2}$, where $H_{2}$ is the Haar wavelet matrix of rank 2 defined by (5).

Since $(\alpha(z), \beta(z)) \in \Omega_{N-1}$, we can construct $\varphi=\prod(\alpha, \beta)$ according to Theorem 2, and

$$
\begin{equation*}
(\alpha, \beta)=\coprod(\varphi) \cdot \frac{1}{\sqrt{2}} H_{2} \Longrightarrow\left(\sum_{k=0}^{N-1} a_{k} z^{k}, \sum_{k=0}^{N-1} b_{k} z^{k}\right)=\coprod(\varphi) \cdot H_{2} \tag{31}
\end{equation*}
$$

because of Corollary 5. If we approximate $\varphi$ by $\varphi_{\mathbb{Q}} \in \mathcal{P}_{N-1}^{-}[\mathbb{Q}], \varphi \approx \varphi_{\mathbb{Q}}$, and construct $\coprod\left(\varphi_{\mathbb{Q}}\right)$, then $\coprod\left(\varphi_{\mathbb{Q}}\right) \in \Omega_{N-1}^{0}(\mathbb{Q})$ by Corollary 2 , and $\coprod(\varphi) \approx \coprod\left(\varphi_{\mathbb{Q}}\right)$ by Corollary 1 . Thus the coefficients of $\coprod\left(\varphi_{\mathbb{Q}}\right) \cdot H_{2}$ will be rational and they will approximate $a_{k}$ and $b_{k}, k=0,1, \ldots, N-1$ (see 31). In this way, we can approximate (13) by a matrix with rational coefficients which satisfy (14) and (15) exactly.

The proposed method can be generalized for wavelet matrices (2) of any rank $m$ since the generalization of the main result Theorem 1 used in the method is valid for $m$-dimensional matrices as well (see [8, Theorem 1]) and at least the formula (29) for obtaining $\varphi$ can be generalized as well (see [4]). However, not for any $m$, there exists a Haar wavelet matrix of rank $m$ with rational coefficients which would provide the generalization of formula (31). Consequently, we can approximate any wavelet matrix $A$ by $\hat{A}$ with rational coefficients for which equivalent quadratic conditions (3), (8) and (9) hold exactly, while (4) only approximately. As has been explained in Section 2, the quadratic condition on a filter bank is decisive for it to have the perfect reconstruction property.

## 6. Computer Simulations and Results

To construct explicitly the fractions which are close to coefficients of Daubechies wavelet matrices $D_{N}$ (see (13) ), a program was written in Mathematica 8. A complete screening of all possible options has been performed in order to select the fractions with minimal denominator in the given range. On a 2 GHz Intel Core 2 Duo system with 2GB RAM running Ubuntu 11 the calculations took less than a second. As it was explained in preceding sections, constructed approximate filter banks $\hat{D}_{N}$ have the perfect reconstruction property. The results of different approximate computations of the coefficients of Daubechies scaling vectors (the first rows of $D_{N}$ ) for genus $N=2$ and $N=3$ are presented in the tables below. The $p$ th moments of these coefficients,
$M_{p}=\sum_{k=0}^{2 N-1} h_{1}[k] k^{p}$ (see (16)), which are not exactly 0 anymore because of approximation, are also computed and located in the table. These tables are presented only for illustrative purposes and the interested readers can produce the different rational approximations which might be more suitable for their specific reasons.

Table 1. $N=2$

| $k=0$ | 0.683012701892219 | $\frac{12}{17} \approx 0.70588$ | $\frac{3008}{4385} \approx 0.68597$ | $\frac{192000}{280913} \approx 0.68348$ |
| :--- | :---: | :---: | :---: | :---: |
| $k=1$ | 1.18301270189222 | $\frac{20}{17} \approx 1.17647$ | $\frac{5184}{4385} \approx 1.18221$ | $\frac{332288}{280913} \approx 1.18288$ |
| $k=2$ | 0.316987298107781 | $\frac{5}{17} \approx 0.29411$ | $\frac{1377}{4385} \approx 0.31402$ | $\frac{88913}{280913} \approx 0.31651$ |
| $k=3$ | -0.183012701892219 | $-\frac{3}{17} \approx-0.17647$ | $-\frac{799}{4385} \approx-0.18221$ | $-\frac{51375}{280913} \approx-0.18288$ |
| $M_{1} \approx$ | 0.0 | 0.59 | 0.008 | 0.001 |

Table 2. $N=3$

| $k=0$ | 0.470467207784164 | $\frac{2888}{5249} \approx 0.5502$ | $\frac{2132672}{4439725} \approx 0.48036$ | $\frac{2677170944}{5703228401} \approx 0.46941$ |
| :--- | :---: | :---: | :---: | :---: |
| $k=1$ | 1.14111691583144 | $\frac{5944}{5249} \approx 1.1324$ | $\frac{5059904}{4439725} \approx 1.13968$ | $\frac{6509075712}{5703228401} \approx 1.14129$ |
| $k=2$ | 0.650365000526232 | $\frac{3104}{5249} \approx 0.5913$ | $\frac{572096}{887945} \approx 0.64429$ | $\frac{3712561536}{5703228401} \approx 0.65095$ |
| $k=3$ | -0.190934415568327 | $-\frac{1056}{5249} \approx-0.2011$ | $-\frac{170688}{887945} \approx-0.19222$ | $-\frac{1088205184}{5703228401} \approx-0.19080$ |
| $k=4$ | -0.120832208310396 | $-\frac{743}{5249} \approx-0.1415$ | $-\frac{553427}{4439725} \approx-0.12465$ | $-\frac{686504079}{5703228401} \approx-0.12037$ |
| $k=5$ | 0.0498174997368838 | $\frac{361}{5249} \approx 0.0687$ | $\frac{233261}{4439725} \approx 0.05253$ | $\frac{282357873}{5703228401} \approx 0.04950$ |
| $k$ | 0.0 | 0.256 | 0.0357 | -0.0040 |
| $M_{1} \approx$ | 0.0 | 1.622 | 0.2169 | -0.0239 |
| $M_{2} \approx$ | 0.0 |  |  |  |

## References

[1] I. Daubechies, "Orthonormal bases of compactly supported wavelets," Commun. Pure Appl. Math., vol. 41, pp. 909-996, 1988.
[2] I. Daubechies, Ten Lectures on Wavelets, SIAM, Philadelhia, PA,1992.
[3] P. Delsarte, Y. Gelin, and Y. Kamp, "A simple approach to spectral factorization", IEEE Trans. Circuits Syst., vol. 25, pp. 943-946, 1978.
[4] L. Ephremidze, G. Janashia and E. Lagvilava, On the factorization of unitary matrix-functions, Proc. A. Razmadze Math. Inst. 116, 1998, 101-106.
[5] ——, "A simple proof of matrix-valued Fejér-Riesz theorem", J. Fourier Anal. Appl. vol. 14, pp. 124-127, 2009 (DOI: 10.1007/s00041-008-9051-z).
[6] L. Ephremidze and E. Lagvilava, "On parameterization of compact wavelet matrices," Bull. Georgian Nat. Acad. Sci., vol. 2, no. 4, pp. 23-27, 2008.
[7] G. Janashia and E. Lagvilava, "A method of approximate factorization of positive definite matrix functions," Studia Math., vol. 137, no. 1, pp. 93-100, 1999.
[8] G. Janashia, E. Lagvilava, and L. Ephremidze "A new method of matrix spectral factorization", IEEE Trans. Inform. Theory, vol. 57, no. 4, 2011 (DOI: 10.1109/TIT.2011.2112233)
[9] S. Mallat, A wavelet Tour of Signal Processing, Academic Press, Ney York, 1998.
[10] H. L. Resnikoff and R. O. Wells, Wavelet Analysis, Springer-Verlag, 1998.
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[^0]:    ${ }^{1}$ This method is currently patent pending.

