

Nonstationary Gabor Frames - Approximately Dual Frames and Reconstruction Errors

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October 31, 2018

Abstract

Nonstationary Gabor frames, recently introduced in adaptive signal analysis, represent a natural generalization of classical Gabor frames by allowing for adaptivity of windows and lattice in either time or frequency. Due to the lack of a complete lattice structure, perfect reconstruction is in general not feasible from coefficients obtained from nonstationary Gabor frames. In this paper it is shown that for nonstationary Gabor frames that are related to some known frames for which dual frames can be computed, good approximate reconstruction can be achieved by resorting to approximately dual frames. In particular, we give constructive examples for so-called almost painless nonstationary frames, that is, frames that are closely related to nonstationary frames with compactly supported windows. The theoretical results are illustrated by concrete computational and numerical examples.

Keywords: adaptive representations, nonorthogonal expansions, irregular Gabor frames, reconstruction, approximately dual frame

1 Introduction

Adapted and adaptive signal representation have received increasing interest over the past few years. As opposed to classical approaches such as the short-time Fourier

^{*}This work was supported by the WWTF project *Audiominer* (MA09-24)

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transform (STFT) or wavelet transform, adaptive representations allow for a variation of parameters such as window width or sampling density over time, frequency or both. Changing parameters in the frequency domain leads, for example, to non-uniform filter banks while adapting window width and sampling density in time is reminiscent of the approach suggested in the construction of nonuniform lapped transforms. Transforms featuring simultaneous adaptivity in time and frequency are notoriously difficult to construct and implement, cp. [15, 7, 12]; however they have shown to be useful in some applications, cf. [14]. On the other hand, fast and efficient implementations exist for representations with adaptivity in only time or frequency. One recent method to obtain this kind of representations is represented by *nonstationary Gabor frames*, first suggested in [11] and further developed in [1, 18, 10]. All the known implementations rely on compactness of the used analysis window in either time or frequency. This assumption allows for usage of tools developed for painless non-orthogonal expansions [6]. While a priori very convenient, the restriction to using compactly supported windows in the domain for which one wishes a flexible representation can be undesirable. As an example, we mention the construction of nonuniform filter banks via nonstationary Gabor frames, in which case this restriction forbids finite impulse response (FIR) filters; the latter are, however, imperative for real-time processing applications.

In the current contribution, we therefore go beyond the results presented in the references above and consider nonstationary Gabor frames with fast decay but unbounded support. The existence of this kind of frames was shown in [8]. Here we are concerned with methods for approximate reconstruction for these adaptive systems.

2 Notation and Preliminaries

Given a non-zero function $g \in L^2(\mathbb{R})$, a modulation, or frequency shift, operator M_{bl} is defined by $M_{bl}g(t) := e^{2\pi iblt}g(t)$, and time shift operator T_{ak} by $T_{ak}g(t) := g(t - ak)$. A composition, $g_{k,l} = M_{bl}T_{ak}g(t) := e^{2\pi iblt}g(t - ak)$ is a time-frequency shift operator.

The set $\mathcal{G}(g, a, b) = \{g_{k,l} : k, l \in \mathbb{Z}\}$ is called a Gabor system for any real, positive a, b . $\mathcal{G}(g, a, b)$ is a Gabor frame for $L^2(\mathbb{R})$, if there exist frame bounds $0 < A \leq B < \infty$ such that for every $f \in L^2(\mathbb{R})$ we have

$$A\|f\|_2^2 \leq \sum_{k,l \in \mathbb{Z}} |\langle f, g_{k,l} \rangle|^2 \leq B\|f\|_2^2. \quad (1)$$

When working with irregular grids, we assume that the sampling points form a separated set: a set of sampling points $\{a_k : k \in \mathbb{Z}\}$ is called δ -separated, if $|a_k - a_m| > \delta$ for a_k, a_m , whenever $k \neq m$. χ_I will denote the characteristic function of the interval I .

A convenient class of window functions for time-frequency analysis on $L^2(\mathbb{R})$ is the Wiener space.

Definition 1. A function $g \in L^\infty(\mathbb{R})$ belongs to the Wiener space $W(L^\infty, \ell^1)$ if

$$\|g\|_{W(L^\infty, \ell^1)} := \sum_{k \in \mathbb{Z}} \operatorname{ess\,sup}_{t \in Q} |g(t+k)| < \infty, \quad Q = [0, 1].$$

For $g \in W(L^\infty, \ell^1)$ and $\delta > 0$ we have [9]

$$\operatorname{ess\,sup}_{t \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |g(t - \delta k)| \leq (1 + \delta^{-1}) \|g\|_{W(L^\infty, \ell^1)}. \quad (2)$$

For $f \in L^2(\mathbb{R})$ we use the following normalization of the Fourier transform, which we denote by \mathcal{F} :

$$\mathcal{F}f(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-2\pi i \omega t} dt.$$

3 Nonstationary Gabor Frames

Nonstationary Gabor systems are a generalization of classical Gabor systems of regular time-frequency shifts of a single window function.

Definition 2. Let $\mathbf{g} = \{g_k \in W(L^\infty, \ell^1) : k \in \mathbb{Z}\}$ be a set of window functions and let $\mathbf{b} = \{b_k : k \in \mathbb{Z}\}$ be a corresponding sequence of frequency-shift parameters. Set $g_{k,l} = M_{b_k l} g_k$. Then, the set

$$\mathcal{G}(\mathbf{g}, \mathbf{b}) = \{g_{k,l} : k, l \in \mathbb{Z}\} \quad (3)$$

is called a *nonstationary Gabor (NSG) system*.

In generalization of regular Gabor frames, for which $g_k = T_{a_k} g$, we will usually assume that the windows g_k are localized around points a_k in a separated set of time-sampling points $\{a_k : k \in \mathbb{Z}\}$. Further, we will always make the assumption that the frequency sampling parameters b_k are positive numbers contained in a closed interval, i.e. $b_k \in [b_L, b_U] \subset \mathbb{R}^+$ for all $k \in \mathbb{Z}$.

To every collection (3) we associate the analysis operator C_g given by $(C_g f)_{k,l} = \langle f, g_{k,l} \rangle$, and synthesis operator U_g , where $U_g c = \sum_{k,l \in \mathbb{Z}} c_{k,l} g_{k,l}$ and $c \in \ell^2$. For two Gabor systems $\mathcal{G}(\mathbf{g}, \mathbf{b})$ and $\mathcal{G}(\boldsymbol{\gamma}, \mathbf{b})$ the composition $S_{g,\gamma} = U_\gamma C_g$,

$$S_{g,\gamma} f = \sum_{k,l \in \mathbb{Z}} \langle f, M_{lb_k} g_k \rangle M_{lb_k} \gamma_k, \quad (4)$$

admits a Walnut representation for all $f \in L^2(\mathbb{R})$, [8]:

$$S_{g,\gamma} f(t) = \sum_{k,l \in \mathbb{Z}} b_k^{-1} \overline{g_k(t - lb_k^{-1})} \gamma_k(t) f(t - lb_k^{-1}). \quad (5)$$

We will frequently use the following correlation functions of a pair of Gabor systems:

$$G_l^{g,\gamma}(t) = \sum_{k \in \mathbb{Z}} b_k^{-1} |g_k(t - lb_k^{-1})| |\gamma_k(t)|, \text{ for } l \in \mathbb{Z}. \quad (6)$$

Note that this definition is asymmetric with respect to \mathbf{g} and $\boldsymbol{\gamma}$. Using this notation, we obtain the following bounds for the frame operator (4):

$$\|S_{g,\gamma}\|^2 \leq \text{ess sup}_{l \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} G_l^{g,\gamma} \cdot \text{ess sup}_{l \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} G_l^{\gamma,g}. \quad (7)$$

By inspection of (5), we note that the summands corresponding to $l \neq 0$ may be seen as the off-diagonal entries of the frame operator. We thus isolate the diagonal part

$$G_0^{g,\gamma}(t) = \sum_{k \in \mathbb{Z}} b_k^{-1} |g_k(t)| |\gamma_k(t)| \quad (8)$$

and denote the off-diagonal entries as follows:

$$R_{g,\gamma} = \text{ess sup}_{l \in \mathbb{Z} \setminus \{0\}} \sum_{k \in \mathbb{Z}} b_k^{-1} |g_k(\cdot - lb_k^{-1})| |\gamma_k(\cdot)|. \quad (9)$$

Note that, if $\mathbf{g} = \boldsymbol{\gamma}$, then the diagonal part of the frame operator $S_{g,g}$ is equal to $G_0^{g,g}$. Using this notation, we obtain the following additional bound:

$$\langle S_{g,\gamma} f, f \rangle \|f\|_2^{-2} \leq \text{ess sup } G_0^{g,\gamma} + \sqrt{R_{g,\gamma} \cdot R_{\gamma,g}}, \quad (10)$$

Bessel sequences are of particular importance in the theory of frames and Riesz bases,[5, 19]. In the regular Gabor case, where $g_k(t) = g(t - ak)$ for some $a > 0$, it is sufficient to assume $g \in W(L^\infty, \ell^1)$ to obtain a Bessel sequence. We next provide a generalization of this property to NSG frames.

Proposition 3.1. *Let $\mathcal{G}(\mathbf{g}, \mathbf{b})$ be a NSG system. If $g_k \in W(L^\infty, \ell^1)$ for all $k \in \mathbb{Z}$ with $\sup_{k \in \mathbb{Z}} \|g_k\|_{W(L^\infty, \ell^1)}$ bounded, and $\sum_{k \in \mathbb{Z}} |g_k(t)| \leq B$ almost everywhere for some $B < \infty$, then the sequence $g_{k,l}$ is a Bessel sequence.*

Proof. Let $f \in L^2(\mathbb{R})$. Then by the assumption on the windows g_k and estimate (7)

$$\begin{aligned}
\sum_{k,l \in \mathbb{Z}} |\langle f, g_{k,l} \rangle|^2 &= \langle S_{g,g} f, f \rangle \leq \|f\|_2^2 \cdot \text{ess sup} \sum_{l \in \mathbb{Z}} G_l^{g,g} \\
&= \|f\|_2^2 \cdot \text{ess sup} \sum_{k \in \mathbb{Z}} b_k^{-1} |g_k(\cdot)| \sum_{l \in \mathbb{Z}} |g_k(\cdot - lb_k^{-1})| \\
&\leq \|f\|_2^2 \cdot \text{ess sup} \sum_{k \in \mathbb{Z}} b_k^{-1} |g_k(\cdot)| (1 + b_k^{-1}) \|g_k\|_{W(L^\infty, \ell^1)} \\
&\leq \|f\|_2^2 \cdot B \cdot \sup_{k \in \mathbb{Z}} [(1 + b_k) \|g_k\|_{W(L^\infty, \ell^1)}].
\end{aligned}$$

□

Given a frame, it is well known that there exists at least one dual frame $\mathcal{G}(\boldsymbol{\gamma}, \mathbf{b})$ such that

$$f = \sum_{k,l \in \mathbb{Z}} \langle f, \gamma_{k,l} \rangle g_{k,l}, \quad \text{for all } f \in L^2(\mathbb{R}). \quad (11)$$

The canonical dual frame is given by $\gamma_{k,l} = S^{-1}g_{k,l}$. In the regular Gabor case, the dual frames are again Gabor frames, i.e., they consist of time-frequency shifted versions of one dual window. This is due to the fact, that the frame operator S commutes with time-frequency shifts, hence $\gamma_{k,l} = S^{-1}g_{k,l} = S^{-1}M_{bl}T_{ak}g = M_{bl}T_{ak}S^{-1}g = M_{bl}T_{ak}\gamma$. In general, we cannot expect, that the dual frame of a NSG frame is again a NSG frame.

However, even in the case of regular Gabor frames, it is often difficult to calculate a dual frame explicitly. For that reason, alternative possibilities to obtain perfect or approximate reconstruction have been proposed, [4, 2]. The following lemma quantifies the reconstruction error using general pairs of Bessel sequences.

Lemma 3.2. *Let $\mathcal{G}(\mathbf{g}, \mathbf{b})$ and $\mathcal{G}(\boldsymbol{\gamma}, \mathbf{b})$ be two Bessel sequences. Then*

$$\|I - S_{g,\gamma}\| \leq \left\| 1 - \sum_{k \in \mathbb{Z}} b_k^{-1} \overline{g_k} \gamma_k \right\|_\infty + \sqrt{R_{g,\gamma} \cdot R_{\gamma,g}}. \quad (12)$$

Proof. Starting from the Walnut representation of $S_{g,\gamma}$, we estimate using Cauchy-Schwartz inequality for sums and integrals and, since all summands have absolute value, Fubini's theorem to justify changing the order of summation and integral:

$$\begin{aligned}
|\langle f - S_{g,\gamma}f, f \rangle| &= \left| \langle f - \sum_{k \in \mathbb{Z}} b_k^{-1} \overline{g_k} \gamma_k f, f \rangle - \left\langle \sum_{l \in \mathbb{Z} \setminus \{0\}} \sum_{k \in \mathbb{Z}} b_k^{-1} \gamma_k(\cdot) \overline{g_k(\cdot - lb_k^{-1})} f(\cdot - lb_k^{-1}), f \right\rangle \right| \\
&\leq \left\| 1 - \sum_{k \in \mathbb{Z}} b_k^{-1} \overline{g_k} \gamma_k \right\|_{\infty} \|f\|_2^2 + \sqrt{R_{g,\gamma} \cdot R_{\gamma,g}} \|f\|_2^2,
\end{aligned} \tag{13}$$

since

$$\begin{aligned}
&\left| \left\langle \sum_{\substack{k \in \mathbb{Z} \\ l \in \mathbb{Z} \setminus \{0\}}} b_k^{-1} \gamma_k(\cdot) \overline{g_k(\cdot - lb_k^{-1})} f(\cdot - lb_k^{-1}), f \right\rangle \right| & (14) \\
&\leq \sum_{\substack{k \in \mathbb{Z} \\ l \in \mathbb{Z} \setminus \{0\}}} b_k^{-1} \int_{\mathbb{R}} |\gamma_k(t)| |g_k(t - lb_k^{-1})| |f(t - lb_k^{-1})| |f(t)| dt \\
&\leq \sum_{\substack{k \in \mathbb{Z} \\ l \in \mathbb{Z} \setminus \{0\}}} b_k^{-1} \left[\int_{\mathbb{R}} |g_k(t - lb_k^{-1})| |\gamma_k(t)| |f(t - lb_k^{-1})|^2 dt \right]^{1/2} \left[\int_{\mathbb{R}} |g_k(t - lb_k^{-1})| |\gamma_k(t)| |f(t)|^2 dt \right]^{1/2} \\
&\leq \left[\int_{\mathbb{R}} |f(t)|^2 \sum_{\substack{k \in \mathbb{Z} \\ l \in \mathbb{Z} \setminus \{0\}}} b_k^{-1} |g_k(t)| |\gamma_k(t - lb_k^{-1})| dt \right]^{1/2} \left[\int_{\mathbb{R}} |f(t)|^2 \sum_{\substack{k \in \mathbb{Z} \\ l \in \mathbb{Z} \setminus \{0\}}} b_k^{-1} |\gamma_k(t)| |g_k(t - lb_k^{-1})| dt \right]^{1/2}.
\end{aligned}$$

□

A special class of NSG systems are collections of compactly supported windows. They were first addressed in [1]. The collection $\mathcal{G}(\mathbf{g}, \mathbf{b})$ with windows g_k being compactly supported with $|\text{supp } g_k| \leq \frac{1}{b_k}$ for all k is a frame for $L^2(\mathbb{R})$ if there exist constants $A > 0$ and $B < \infty$ such that

$$A \leq G_0^{g,g}(t) \leq B \text{ a.e.} \tag{15}$$

In this situation, $\mathcal{G}(\mathbf{g}, \mathbf{b})$ is called *painless NSG frame*. The canonical dual atoms are given by $\gamma_{k,l} = M_{lb_k}(G_0^{g,g})^{-1} g_k$. Note again that, in general, we may have $\gamma_{k,l} = S^{-1}(M_{lb_k} g_k) \neq M_{lb_k}(S^{-1} g_k)$. If $b_k = b$ for all k , then the frame operator commutes with the frequency-shifts and the dual frame is an NSG frame.

The existence of NSG frames with not necessarily compactly supported windows was established in [8]. For these frames, finding canonical dual frames requires the inversion of the frame operator. This computation is expensive since the operator has considerably less structure than the frame operator in the classical, regular Gabor frame case, for which fast algorithms now exist, [17, 13, 16]. To circumvent the problem, we suggest the use of windows other than canonical duals to obtain sufficiently good approximate reconstruction.

4 Approximately dual atoms

The notion of approximately dual pairs was discussed in [4]. For NSG Bessel sequences we adapt their definition as follows.

Definition 3. Two Bessel sequences $\mathcal{G}(\mathbf{g}, \mathbf{b})$ and $\mathcal{G}(\boldsymbol{\gamma}, \mathbf{b})$ are said to be approximately dual frames if $\|I - S_{g,\gamma}\| < 1$ or $\|I - S_{\gamma,g}\| < 1$.

Note that the two conditions given in the definition are equivalent since

$$\|I - S_{g,\gamma}\| = \|I - C_g U_\gamma\| = \|I - U_\gamma^* C_g^*\| = \|I - C_\gamma U_g\| = \|I - S_{\gamma,g}\|.$$

In Definition 3 it is implicitly stated that, if two Bessel sequences are approximately dual frames, then each of them is a frame. This result was proved in [4] for general frames and it will be useful to reformulate the conditions for the existence of NSG frame, given in [8], in the context of approximately dual frames.

Proposition 4.1. *Let $\mathcal{G}(\mathbf{g}, \mathbf{b})$ be a Bessel sequence with Bessel bound B and $0 < A_1 \leq \sum_{k \in \mathbb{Z}} |g_k(t)|^2 \leq A_2 < \infty$ a.e. for some positive constants A_1, A_2 .*

i) The multiplication operator $G_0^{g,g}$ is invertible a.e. and, for $\gamma_k = (G_0^{g,g})^{-1} g_k$,

$$\|I - S_{g,\gamma}\| \leq \frac{R_{g,g}}{\text{ess inf } G_0^{g,g}}. \quad (16)$$

ii) If

$$R_{g,g} < \text{ess inf } G_0^{g,g}, \quad (17)$$

then $\mathcal{G}(\mathbf{g}, \mathbf{b})$ and $\mathcal{G}(\boldsymbol{\gamma}, \mathbf{b})$ are approximately dual frames for $L^2(\mathbb{R})$.

iii) Assume, additionally, for some δ -separated set of time-sampling points $\{a_k : k \in \mathbb{Z}\}$ and constants $0 < p_U, C_L, C_U < \infty$ such that for $p_k \in]2, p_U] \subset \mathbb{R}$, $C_k \in [C_L, C_U]$ we have

$$|g_k(t)| \leq C_k (1 + |t - a_k|)^{-p_k} \text{ for all } k \in \mathbb{Z}. \quad (18)$$

Then there exists a sequence $\{b_k^0\}_{k \in \mathbb{Z}}$, such that for all sequence $b_k \leq b_k^0$, $k \in \mathbb{Z}$, (17) holds.

Remark 1. If (17) holds, $\mathcal{G}(\boldsymbol{\gamma}, \mathbf{b})$ is called a single preconditioning dual system for $\mathcal{G}(\mathbf{g}, \mathbf{b})$.

Proof. Since all frequency modulation parameters b_k are taken from a closed interval in \mathbb{R}^+ , the invertibility of $G_0^{g,g}$ is straightforward and the windows γ_k are well defined. Moreover, since $\mathcal{G}(\mathbf{g}, \mathbf{b})$ is a Bessel sequence, so is $\mathcal{G}(\boldsymbol{\gamma}, \mathbf{b})$, with Bessel bound $(\text{ess inf } G_0^{g,g})^{-2}B$. Substituting $\gamma_k = (G_0^{g,g})^{-1}g_k$ for γ_k in the proof of Lemma 3.2, the first term in (13) vanishes and we obtain (i):

$$\begin{aligned} |\langle f - S_{g,\boldsymbol{\gamma}}f, f \rangle| &\leq \left| \left\langle (G_0^{g,g})^{-1} \sum_{l \in \mathbb{Z} \setminus \{0\}} \sum_{k \in \mathbb{Z}} b_k^{-1} g_k(\cdot) \overline{g_k(\cdot - lb_k^{-1})} f(\cdot - lb_k^{-1}), f \right\rangle \right| \\ &\leq (\text{ess inf } G_0^{g,g})^{-1} \left\langle \sum_{l \in \mathbb{Z} \setminus \{0\}} \sum_{k \in \mathbb{Z}} b_k^{-1} |g_k(\cdot)| |g_k(\cdot - lb_k^{-1})| |f(\cdot - lb_k^{-1})|, |f| \right\rangle \\ &\leq (\text{ess inf } G_0^{g,g})^{-1} R_{g,g} \|f\|_2^2. \end{aligned} \quad (19)$$

(ii) follows directly from Definition 3. Finally, (iii) follows from [8, Theorem 3.4], where it is shown that the assumptions (18) on the windows g_k guarantee the existence of a sequence b_k^0 such that

$$\langle S_{g,g}f, f \rangle \|f\|_2^{-2} \geq \text{ess inf } G_0^{g,g} - R_{g,g} > 0. \quad (20)$$

□

Single preconditioning dual windows are a good choice for reconstruction, whenever the frame operator is close to diagonal. This is the case, if the original windows g_k decay fast and frequency sampling is fast.

If the frame of interest is close to some other frame, which, ideally, is better understood, other approximate dual windows may be derived from this frame. The prototypical situation is a NSG frame which is close to a painless NSG frame in the sense of a small perturbation. Approximately dual frames in the context of perturbation theory were recently studied in [4]. In the following proposition we give error estimates for the reconstruction with approximately dual frames in such a situation. This provides different reconstruction methods apart from single preconditioning which was addressed in Proposition 4.1.

Proposition 4.2. *Assume that $\mathcal{G}(\mathbf{g}, \mathbf{b})$ is a Bessel sequence with bound B and that $\mathcal{G}(\mathbf{h}, \mathbf{b})$ is a NSG frame with lower and upper frame bound A_h and B_h , respectively. We set $\psi_k = h_k - g_k$ and define the following windows:*

$$(a) \gamma_{k,l}^1 = S_{h,h}^{-1} h_{k,l} \quad (\text{canonical dual of } h_{k,l}) \quad (21)$$

$$(b) \gamma_{k,l}^2 = S_{h,h}^{-1} g_{k,l} \quad (22)$$

Then the following hold:

(i)

$$\|I - S_{g,\gamma^1}\| \leq A_h^{-1/2} \|C_\psi\|. \quad (23)$$

If $\text{ess sup} \sum_{l \in \mathbb{Z}} G_l^{\psi,\psi} < A_h$, then $\mathcal{G}(\gamma^1, \mathbf{b})$ and $\mathcal{G}(\mathbf{g}, \mathbf{b})$ are approximately dual frames.

(ii)

$$\|I - S_{g,\gamma^2}\| \leq A_h^{-1} (\sqrt{B_h} + \sqrt{B}) \|C_\psi\|. \quad (24)$$

If $\text{ess sup} \sum_{l \in \mathbb{Z}} G_l^{\psi,\psi} < \frac{A_h^2}{(\sqrt{B_h} + \sqrt{B})^2}$, then $\mathcal{G}(\gamma^2, \mathbf{b})$ and $\mathcal{G}(\mathbf{g}, \mathbf{b})$ are approximately dual frames.

Remark 2. The first statement of Proposition 4.2 is contained in [4].

According to [8], the assumption that $\text{ess sup} \sum_{l \in \mathbb{Z}} G_l^{\psi,\psi} < A_h$ can be satisfied if the functions ψ_k decay polynomially, i.e., $|\psi_k(t)| \leq C_k(1 + |t|)^{-p_k}$ with appropriate constants C_k and decay rates $p_k > 1$.

Proof. From (7) it follows that $\|C_\psi\|^2 \leq \text{ess sup} \sum_{l \in \mathbb{Z}} G_l^{\psi,\psi}$. The same estimate holds for $\|U_\psi\|^2$.

Since $\mathcal{G}(\mathbf{h}, \mathbf{b})$ is a frame with canonical dual frame $\mathcal{G}(\gamma^1, \mathbf{b})$, an upper frame bound of $\mathcal{G}(\gamma^1, \mathbf{b})$ is given by A_h^{-1} . We thus obtain (i) as follows:

$$\|I - S_{g,\gamma^1}\| = \|U_{\gamma^1} C_h - U_{\gamma^1} C_g\| \leq \|U_{\gamma^1}\| \|C_\psi\| \leq A_h^{-1/2} \|C_\psi\|. \quad (25)$$

If $\text{ess sup} \sum_{l \in \mathbb{Z}} G_l^{\psi,\psi} < A_h$, then $\|I - S_{g,\gamma^1}\| < 1$ and $\mathcal{G}(\gamma^1, \mathbf{b})$ and $\mathcal{G}(\mathbf{g}, \mathbf{b})$ are approximately dual frames as claimed.

To show (ii), we note that $U_{\gamma^2} = S_{h,h}^{-1} U_g$ and thus

$$\|I - S_{g,\gamma^2}\| = \|S_{h,h}^{-1} S_{h,h} - S_{h,h}^{-1} S_{g,g}\| = \|S_{h,h}^{-1} (U_h C_h - U_g C_g)\| \quad (26)$$

$$\leq \|S_{h,h}^{-1}\| \|U_h C_h - U_g C_h + U_g C_h - U_g C_g\| = A_h^{-1} \|U_\psi C_h - U_g C_\psi\| \quad (27)$$

$$\leq A_h^{-1} \|C_\psi\| (\sqrt{B_h} + \sqrt{B}) \quad (28)$$

and (24) follows. \square

4.1 Perturbation of painless nonstationary Gabor frames

If a NSG system can be derived as a perturbation of a painless NSG frame, the approximately dual windows given in Proposition 4.2 are particularly simple to compute. In this situation, the frame $\mathcal{G}(\mathbf{h}, \mathbf{b})$ is the painless frame $\mathcal{G}(\mathbf{g}^\circ, \mathbf{b})$ and the frame operator

$S_{h,h} = S_{g^\circ, g^\circ}$ is the multiplication operator $G_0^{g^\circ, g^\circ}$. Moreover, in this particular case, the approximately dual frames, given by $\gamma_{k,l}^1 = (G_0^{g^\circ, g^\circ})^{-1} g_{k,l}^\circ$ and $\gamma_{k,l}^2 = (G_0^{g^\circ, g^\circ})^{-1} g_{k,l}$ are NSG frames. This is a very important asset, since for NSG frames fast algorithms for analysis and reconstruction using FFT exist.

In [8] we constructed a special class of NSG frames, arising from painless NSG frames, which we introduce next.

Definition 4 (Almost painless NSG frames). Let $\mathcal{G}(\mathbf{g}, \mathbf{b})$ be a NSG system, assume that the windows g_k are essentially bounded away from zero on the intervals $I_k = [a_k - (2b_k)^{-1}, a_k + (2b_k)^{-1}]$ and set $g_k^\circ = g_k \chi_{I_k}$. If $\mathcal{G}(\mathbf{g}^\circ, \mathbf{b})$ is a (painless) frame for $L^2(\mathbb{R})$, then we call the system $\mathcal{G}(\mathbf{g}, \mathbf{b})$ an *almost painless NSG system* (or frame).

For almost painless NSG systems, the estimates given in Proposition 4.2 can be written more explicitly.

Corollary 4.3. *Assume that $\mathcal{G}(\mathbf{g}, \mathbf{b})$ is an almost painless NSG system and let $A_0 = \text{ess inf } G_0^{g^\circ, g^\circ}$ to be the lower frame bound of the painless frame $\mathcal{G}(\mathbf{g}^\circ, \mathbf{b})$ and $g_k^r = g_k - g_k \chi_{I_k}$. Then the following hold:*

(i) for $\gamma_k^1 = (G_0^{g^\circ, g^\circ})^{-1} g_k^\circ$,

$$\|I - S_{g, \gamma^1}\| \leq A_0^{-1} \sqrt{R_{g^r, g^\circ} \cdot R_{g^\circ, g^r}} \quad (29)$$

(ii) for $\gamma_k^2 = (G_0^{g^\circ, g^\circ})^{-1} g_k$

$$\|I - S_{g, \gamma^2}\| \leq A_0^{-1} \left(R_{g^r, g^\circ} + R_{g^\circ, g^r} + \text{ess sup} \sum_{l \in \mathbb{Z}} G_l^{g^r, g^r} \right) \quad (30)$$

Proof. The estimates follow from Lemma 3.2. First, substituting $\gamma_k^1 = (G_0^{g^\circ, g^\circ})^{-1} g_k^\circ$ for γ_k in (12), the first term vanishes, since $\sum_{k \in \mathbb{Z}} b_k^{-1} \overline{g_k} g_k^\circ = G_0^{g^\circ, g^\circ}$, and we obtain

$$\|I - S_{g, \gamma^1}\| \leq \sqrt{R_{g, \gamma^1} \cdot R_{\gamma^1, g}} \leq (\text{ess inf } G_0^{g^\circ, g^\circ})^{-1} \sqrt{R_{g^r, g^\circ} \cdot R_{g^\circ, g^r}},$$

since

$$\begin{aligned} R_{g, \gamma^1} &= \text{ess sup} \sum_{l \in \mathbb{Z} \setminus \{0\}} \sum_{k \in \mathbb{Z}} b_k^{-1} |g_k(\cdot - lb_k^{-1})| |(G_0^{g^\circ, g^\circ})^{-1} g_k^\circ(\cdot)| \\ &\leq (\text{ess inf } G_0^{g^\circ, g^\circ})^{-1} \cdot \text{ess sup} \sum_{l \in \mathbb{Z} \setminus \{0\}} \sum_{k \in \mathbb{Z}} b_k^{-1} |g_k^r(\cdot - lb_k^{-1})| |g_k^\circ(\cdot)| \\ &= (\text{ess inf } G_0^{g^\circ, g^\circ})^{-1} R_{g^r, g^\circ}, \end{aligned} \quad (31)$$

similarly for $R_{\gamma^1, g}$.

For (ii) we substitute $\gamma_k^2 = (G_0^{g^o, g^o})^{-1} g_k$ for γ in the proof of Lemma 3.2, $g_k^o + g_k^r$ for g_k and use the fact that g_k^o and g_k^r have disjoint supports. Then, since $\overline{g_k^o(\cdot - lb_k^{-1})} g_k^o(\cdot)$ is zero for $l \neq 0$, using (14) we obtain that

$$\begin{aligned} |\langle f - S_{g, \gamma^2} f, f \rangle| &= \left| \left\langle f - \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \overline{g_k(\cdot - lb_k^{-1})} (G_0^{g^o, g^o})^{-1} g_k(\cdot) f(\cdot - lb_k^{-1}), f \right\rangle \right| \\ &\leq \left| \left\langle f - (G_0^{g^o, g^o})^{-1} \sum_{k \in \mathbb{Z}} b_k^{-1} |g_k^o|^2 f, f \right\rangle \right| \\ &\quad + (\text{ess inf}(G_0^{g^o, g^o})^{-1} (R_{g^r, g^o} + R_{g^o, g^r} + \|S_{g^r, g^r}\|) \|f\|_2^2. \end{aligned}$$

The first term vanishes and by (7), $\|S_{g^r, g^r}\| \leq \text{ess sup} \sum_{l \in \mathbb{Z}} G_l^{g^r, g^r}$. □

5 Examples

We present two examples to illustrate our theory. In the first example we deal with almost painless frames using Gaussian windows. We consider three different approximately dual frames and check their performance in terms of reconstruction.

In both examples, we consider a basic window and dilations by 2 and $\frac{1}{2}$, respectively. Since the dilation parameters take only three different values, there are three kinds of windows, with support size $1/2$, 1 and 2, respectively. Note that, while theoretically possible, sudden changes in the shape and width of adjacent windows turn out to be undesirable for applications, hence we only allow for stepwise change in dilation parameters.

Example 5.1. Let $s_k \in \{-1, 0, 1\}$ with $|s_k - s_{k-1}| \in \{0, 1\}$ for all $k \in \mathbb{Z}$. We consider a sequence of windows g_k that are translated and dilated versions of the Gaussian window $g(t) = e^{-\pi[\sigma t]^2}$: $g_k(t) = T_{a_k} \sqrt{b_k} g(b_k t) = \sqrt{b_k} g(b_k(t - a_k))$, with $\sigma = 2.5$, $b_k = 2^{s_k}$, $a_0 = 0$ and for all $k \in \mathbb{Z}$

$$\begin{aligned} a_{k+1} &= a_k + (2b_k)^{-1} && \text{if } s_k = s_{k+1}, \\ a_{k+1} &= a_k + (3b_{k+1})^{-1} && \text{if } s_k > s_{k+1}, \\ a_{k+1} &= a_k + (3b_k)^{-1} && \text{if } s_k < s_{k+1}. \end{aligned}$$

Here, $b_L = 1/2$, $b_U = 2$ and the $\{a_k : k \in \mathbb{Z}\}$ are separated with minimum distance $\delta = 1/4$. We arrange the windows as follows: after each change of window size, no change is allowed in the next step; in other words, each window has at least one neighbor of the same size.

Let $I_k = [a_k - (2b_k)^{-1}, a_k + (2b_k)^{-1}]$ and define a new set of windows by $g_k^o(t) = g_k(t)\chi_{I_k}$. Then $\{M_{lb_k}g_k^o : k, l \in \mathbb{Z}\}$ is a painless nonstationary Gabor frame with lower frame bound $A_0 = 0.1718$.

The system $\mathcal{G}(\mathbf{g}, \mathbf{b})$ arises from the painless frame $\mathcal{G}(\mathbf{g}^o, \mathbf{b})$, and therefore we are interested in the approximate dual windows proposed in Corollary 4.3. We first consider $\gamma_k^1 = (G_0^{g^o, g^o})^{-1}g_k^o$, the canonical dual frame of $\mathcal{G}(\mathbf{g}^o, \mathbf{b})$. According to (29), we need to calculate R_{g^r, g^o} and R_{g^o, g^r} in order to obtain an estimate of the reconstruction error.

Claim 1: $R_{g^r, g^o} \leq 0.00827 + \mathcal{O}(10^{-6})$.

For fixed $k \in \mathbb{Z}$,

$$b_k^{-1}|g_k^o(t)| \sum_{l \in \mathbb{Z} \setminus \{0\}} |g_k^r(t - lb_k^{-1})| = b_k^{-1/2}|g_k^o(t)| \sum_{l \in \mathbb{Z}} b_k^{-1/2}|g_k^r(t - lb_k^{-1})|$$

since g_k^r and g_k^o have disjoint support. Then, due to b_k^{-1} -periodicity of $\sum_{l \in \mathbb{Z}} b_k^{-1/2}|g_k^r(t - lb_k^{-1})|$, we have

$$\begin{aligned} R_{g^r, g^o} &= \text{ess sup}_{k \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} b_k^{-1/2}|g_k^o(\cdot)| \sum_{l \in \mathbb{Z}} b_k^{-1/2}|g_k^r(\cdot - lb_k^{-1})| \\ &\leq \text{ess sup}_{k \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} b_k^{-1/2}|g_k^o(\cdot)| \cdot \text{ess sup}_{t \in \mathcal{I}_k} \sum_{l \in \mathbb{Z}} b_k^{-1/2}|g_k^r(t - lb_k^{-1})| \end{aligned} \quad (32)$$

In order to obtain more accurate estimates, we split I_k by setting $I_k^+ = [a_k, a_k + (2b_k)^{-1}]$ and $I_k^- = [a_k - (2b_k)^{-1}, a_k]$ and estimate expression (32) by its values at the end points of I_k^+ or I_k^- , respectively. We observe that, for $t \in I_k^+$:

$$\begin{aligned} \sum_{l \in \mathbb{Z}} b_k^{-1/2}|g_k^r(t - lb_k^{-1})| &= \sum_{l=1}^{\infty} \left[e^{-\pi[\sigma b_k(t - a_k - lb_k^{-1})]^2} + e^{-\pi[\sigma b_k(t - a_k + lb_k^{-1})]^2} \right] \\ &\leq \sum_{l=1}^{\infty} e^{-\pi[\sigma l]^2} + \sum_{l=1}^{\infty} e^{-\pi[\sigma(2l-1)/2]^2} \leq 0.00738 + \mathcal{O}(10^{-6}). \end{aligned}$$

By the symmetry of g_k^r with respect to a_k , an analogous estimate holds for $t \in I_k^-$. Furthermore, due to the arrangement of the windows g_k , $\text{ess sup}_{k \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} b_k^{-1/2}|g_k^o(\cdot)| \leq 1.1206$ and *Claim 1* follows.

Claim 2: $R_{g^o, g^r} \leq 0.0157 + \mathcal{O}(10^{-6})$.

Observe that $\sum_{l \in \mathbb{Z} \setminus \{0\}} b_k^{-1/2}|g_k^o(t - lb_k^{-1})| \leq \|b_k^{-1/2}g_k^o\|_{\infty} = 1$. Then, with I_m^+ and

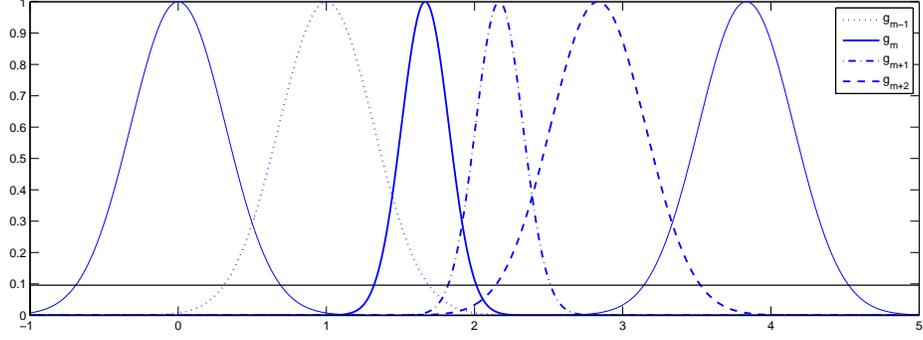


Figure 1: An example of an arrangement of windows g_k .

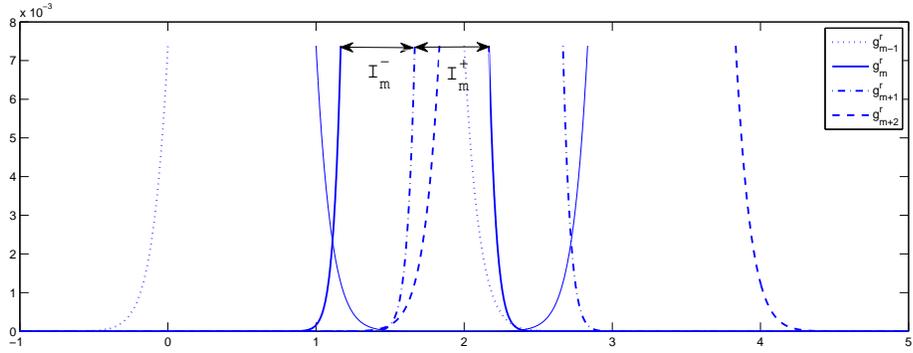


Figure 2: A zoom-in on the tails g_k^r in the arrangement of Fig. 1.

I_m^- defined as before:

$$\begin{aligned}
R_{g^o, g^r} &= \text{ess sup}_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z} \setminus \{0\}} b_k^{-1/2} |g_k^r(\cdot)| \sum_{l \in \mathbb{Z} \setminus \{0\}} b_k^{-1/2} |g_k^o(\cdot - lb_k^{-1})| \leq \text{ess sup}_{k \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} b_k^{-1/2} |g_k^r(\cdot)| \\
&\leq \sup_{m \in \mathbb{Z}} \left\{ \text{ess sup}_{t \in I_m^+} \sum_{k \in \mathbb{Z}} b_k^{-1/2} |g_k^r(t)|, \text{ess sup}_{t \in I_m^-} \sum_{k \in \mathbb{Z}} b_k^{-1/2} |g_k^r(t)| \right\} \\
&= \sup_{m \in \mathbb{Z}} \left\{ \underbrace{\text{ess sup}_{t \in I_m^+} \sum_{k \in \mathbb{Z}} e^{-\pi[\sigma b_k(t-a_k)]^2} \chi_{I_k^c}(t)}_{C^+}, \underbrace{\text{ess sup}_{t \in I_m^-} \sum_{k \in \mathbb{Z}} e^{-\pi[\sigma b_k(t-a_k)]^2} \chi_{I_k^c}(t)}_{C^-} \right\}.
\end{aligned}$$

We bound C^+ and C^- by their maximal values on I_m^+ , respectively I_m^- . The set $\{a_k : k \in \mathbb{Z}\}$ is δ -separated, hence $|a_k - a_m| \geq |k - m|\delta$. By the arrangement of the windows, there are at most two windows g_k^r which assume their maximum $e^{-\pi[\sigma/2]^2}$ in the interval I_m^+ or in I_m^- . Without loss of generality we assume that the two maximal values occur in I_m^+ , corresponding to the windows g_{m-1}^r and g_{m+2}^r . Then, g_{m+1}^r, g_m^r

are zero in I_m^+ , cf. Figure 2 for an example situation. Therefore,

$$\begin{aligned} C^+ &\leq \sum_{k>m+2} e^{-\pi[\sigma b_k(a_m+(2b_m)^{-1}-a_k)]^2} + 2e^{-\pi[\sigma b_k(2b_k)^{-1}]^2} + \sum_{k<m-1} e^{-\pi[\sigma b_k(a_m-a_k)]^2} \\ &\leq 2 \left(e^{-\pi[\sigma/2]^2} + \sum_{k>0; kb_k>2} e^{-\pi[\sigma b_k k \delta]^2} \right). \end{aligned}$$

Since we assumed that two windows g_k^r reached their maximum in I_m^+ , $C^- \leq C^+$. As before, the bound from Claim 2 follows by numerical calculations.

In summary, due to (29), Claim 1 and 2 and the lower frame bound $A_0 = 0.1718$, the reconstruction error using the approximate duals γ_k^1 is bounded by

$$\|I - S_{g,\gamma^1}\| \leq 0.0663 + \mathcal{O}(10^{-6}). \quad (33)$$

Another choice of approximate dual system are the windows $\gamma_k^2 = (G_0^{g^o, g^o})^{-1} g_k$. In this setting we use (30) to derive

$$\|I - S_{g,\gamma^2}\| \leq A_0^{-1} \left(R_{g^r, g^o} + R_{g^o, g^r} + \text{ess sup} \sum_{l \in \mathbb{Z}} G_l^{g^r, g^r} \right) \leq 0.1402 + \mathcal{O}(10^{-6}),$$

since, by previous calculations,

$$\begin{aligned} \text{ess sup} \sum_{l \in \mathbb{Z}} G_l^{g^r, g^r} &= \text{ess sup} \sum_{k \in \mathbb{Z}} b_k^{-1/2} |g_k^r(t)| \sum_{l \in \mathbb{Z}} b_k^{-1/2} |g_k^r(t - lb_k^{-1})| \\ &\leq \text{ess sup} \sum_{k \in \mathbb{Z}} b_k^{-1/2} |g_k^r(t)| \cdot \text{ess sup} \sum_{l \in \mathbb{Z}} b_k^{-1/2} |g_k^r(t - lb_k^{-1})| \\ &\leq 0.0001158 + \mathcal{O}(10^{-6}). \end{aligned} \quad (34)$$

As a third choice of approximate dual windows we consider single preconditioning windows $\gamma_k = (G_0^{g, g})^{-1} g_k$ introduced in Proposition 4.1. It can easily be seen that

$$R_{g, g} \leq R_{g^o, g^r} + R_{g^r, g^o} + R_{g^r, g^r} \leq R_{g^o, g^r} + R_{g^r, g^o} + \text{ess sup} \sum_{l \in \mathbb{Z}} G_l^{g^r, g^r}, \quad (35)$$

and, by previous calculations, it follows

$$R_{g, g} \leq 0.0241 + \mathcal{O}(10^{-6}) < A_0 \leq \text{ess inf} G_0^{g, g}. \quad (36)$$

Therefore, by Proposition 4.1 (ii), $\mathcal{G}(\mathbf{g}, \mathbf{b})$ and $\mathcal{G}(\gamma, \mathbf{b})$ are approximately dual frames. Moreover

$$\|I - S_{g,\gamma}\| \leq \frac{R_{g, g}}{\text{ess inf} G_0^{g, g}} \leq \frac{R_{g, g}}{A_0} \leq 0.1402 + \mathcal{O}(10^{-6}). \quad (37)$$

Remark 3. Observe that from each of the approximately dual frame estimates given in the above example, the frame property of $\mathcal{G}(\mathbf{g}, \mathbf{b})$ follows.

Note that, from (34), the frame property of $\mathcal{G}(\mathbf{g}, \mathbf{b})$ may also be derived by applying results from perturbation theory cf. [3]. Indeed, since $\sum_{k,l \in \mathbb{Z}} |\langle f, g_{k,l} - g_{k,l}^o \rangle|^2 = \sum_{k,l \in \mathbb{Z}} |\langle f, g_{k,l}^r \rangle|^2 \leq \text{ess sup} \sum_{l \in \mathbb{Z}} G_l^{g^r, g^r} \|f\|_2^2 < A_0 \|f\|_2^2$, it follows that $\mathcal{G}(\mathbf{g}, \mathbf{b})$ is a frame with a lower frame bound $A = 0.1630$.

Example 5.2. In our second example, we turn to the situation mentioned in the introduction, namely, the construction of non-uniform filter banks with compactly supported, that is, FIR filters, via NSG frames. In this situation, the frame operator S does not have a Walnut-like structure as given in (5) on the time side. However, S may be considered on the frequency side by applying a Fourier transform. Then, we encounter the same structure as before and may exploit the developed techniques to deduce the frame property and to construct approximate dual frames for reconstruction. The situation is schematically depicted in Figure 3.

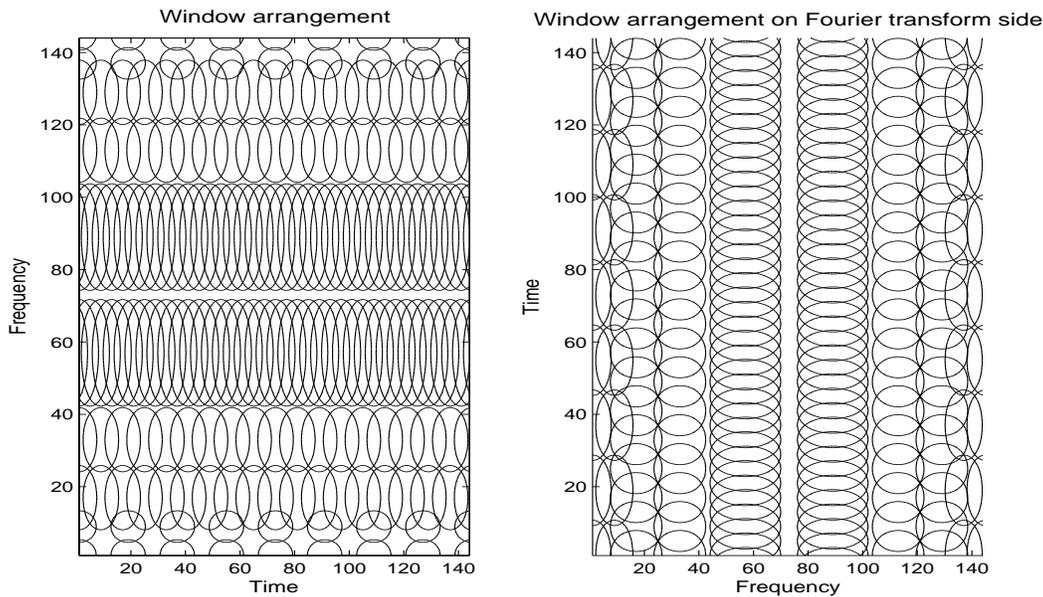


Figure 3: An example for the arrangement of dilated windows in Example 2

It is obvious that, while, in the time domain, various windows g_k with different pass-bands, are shifted to obtain the overall system, applying a Fourier transform yields the known situation: various windows \hat{g}_k are modulated to create an NSG system.

On the other hand, we are interested in using FIR filters, that is, the windows g_k

are all compactly supported, hence not bandlimited. Similar to the construction in Example 5.1, we can cut the windows \hat{g}_k to obtain a painless NSG reference frame. More precisely, we consider the family of windows g_k and a vector of corresponding time-shift parameters a_k . Then, we set $g_{k,l} = T_{a_{kl}}g_k$ and are interested in the frame property of the set of functions $\{g_{k,l} : k, l \in \mathbb{Z}\}$. Considering, due to the lack of structure on the time side as mentioned above, the corresponding frame operator on the frequency side corresponds to investigating the operator $\mathcal{F}S\mathcal{F}^*$, which acts on the Fourier transform of a signal of interest. In other words, we are now dealing with the set of functions $\{\mathcal{F}(g_{k,l}) = M_{a_{kl}}\hat{g}_k : k, l \in \mathbb{Z}\}$.

For the current example, we consider Hanning windows h_k and, as in the previous example, apply dilations by 2^{-1} and 2 , respectively, to obtain various time- and frequency resolutions. The time-shift parameters a_k are chosen in parallel to the choice of the frequency-shift parameters b_k in Example 5.1.

Given the explicit knowledge of the spectral properties of the Hanning windows, explicit error estimates can be derived in a similar manner as in the previous example. Here, we also numerically calculate the errors resulting from reconstruction by means of the three different proposed approximate dual systems. We use the same nomenclature as before, that is, γ_k^1 , γ_k^2 and γ_k^3 denote the canonical duals of the painless frame, the set of windows $(G_0^{g^\circ, g^\circ})^{-1}g_k$ and the single preconditioning windows, respectively. Then

1. $\|I - S_{g, \gamma^1}\| = 0.0210$
2. $\|I - S_{g, \gamma^2}\| = 0.0407$
3. $\|I - S_{g, \gamma^3}\| = 0.0407$

As before, the canonical duals of the painless frame provide the best approximate reconstruction.

The set of windows \hat{g}_k used in this example, together with their true, single preconditioning duals γ_k^3 and the canonical duals of the corresponding painless frame, γ_k^1 , are depicted in Figure 4. On the right plots, zoom-ins are shown to better compare the detailed behavior.

A comparison between a basic window from the true dual frame and the preconditioning duals γ_k^3 and the true dual and the approximate dual γ_k^1 is depicted in Figures 5 and 6, respectively, in both the frequency domain (upper plot) and the time domain (lower plot). It should be noted that, while compactly supported in frequency, γ_k^1

still have better decay in the time domain than both the true dual windows and the other approximate dual windows.

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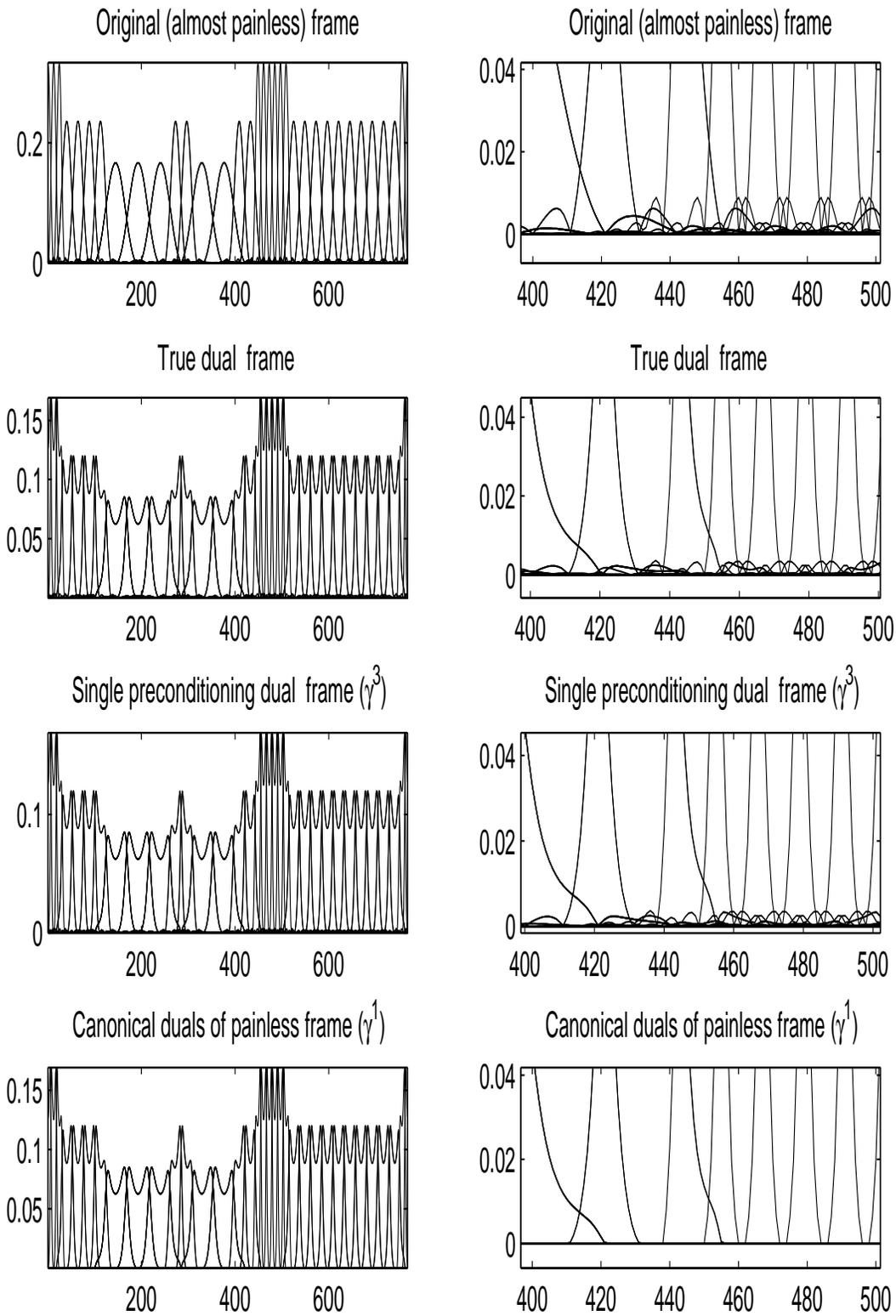


Figure 4: The original windows and the windows from various (approximate) dual frames.

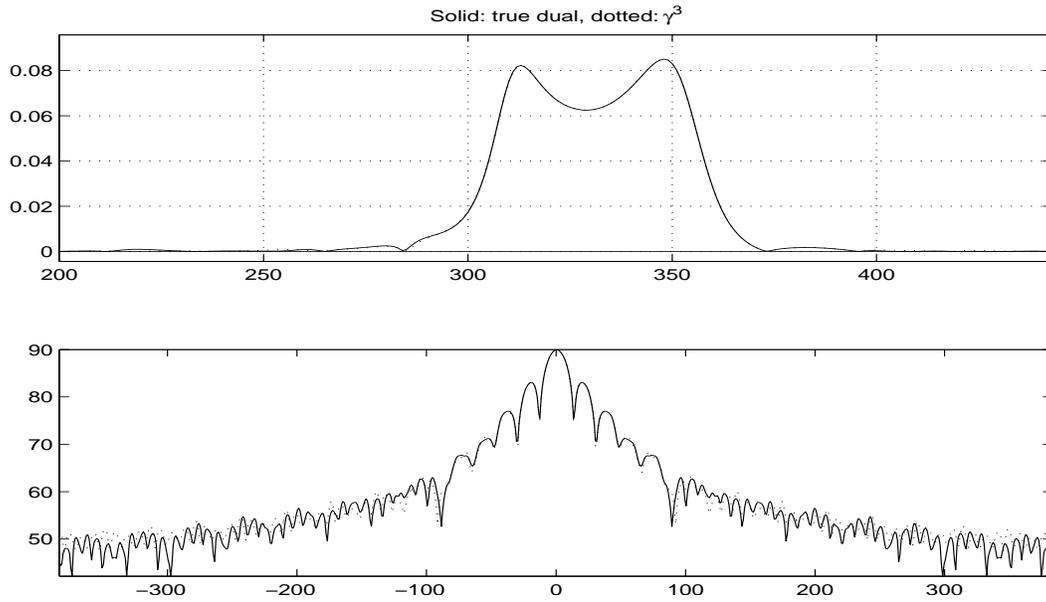


Figure 5: Comparison between true dual and single preconditioning dual γ^3 .

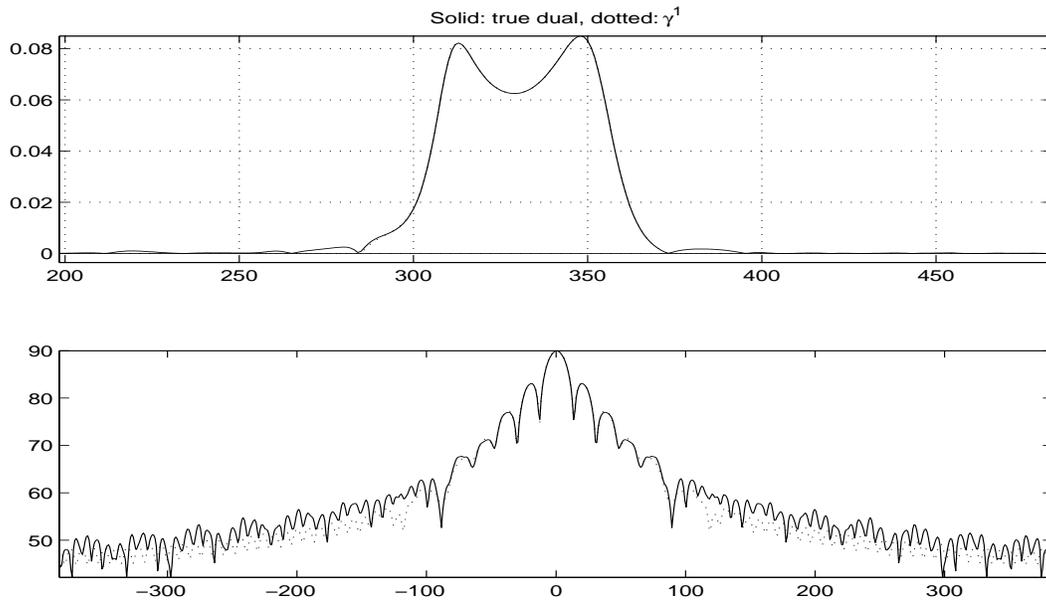


Figure 6: Comparison between true dual and painless approximate dual γ^1 .