# Area preserving maps and volume preserving maps between a class of polyhedrons and a sphere 

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#### Abstract

For a class of polyhedrons denoted $\mathbb{K}_{n}(r, \varepsilon)$, we construct a bijective continuous area preserving map from $\mathbb{K}_{n}(r, \varepsilon)$ to the sphere $\mathbb{S}^{2}(r)$, together with its inverse. Then we investigate for which polyhedrons $\mathbb{K}_{n}\left(r^{\prime}, \varepsilon\right)$ the area preserving map can be used for constructing a bijective continuous volume preserving map from $\overline{\mathbb{K}}_{n}\left(r^{\prime}, \varepsilon\right)$ to the ball $\overline{\mathbb{S}^{2}}(r)$. These maps can be further used in constructing uniform and refinable grids on the sphere and on the ball, starting from uniform and refinable grids of the polyhedrons $\mathbb{K}_{n}(r, \varepsilon)$ and $\overline{\mathbb{K}}_{n}\left(r^{\prime}, \varepsilon\right)$, respectively. In particular, we show that HEALPix grids can be obtained by mappings polyhedrons $\mathbb{K}_{n}(r, \varepsilon)$ onto the sphere.


## 1. Introduction

A uniform grid on a two(three)-dimensional domain $D$ is a grid all of whose cells have the same area(volume). This is required in statistical applications and in construction of wavelet bases of the space $L^{2}(D)$, when one wishes to use the standard inner product and 2-norm, instead of a weighted norm dependent on the grid [8]. A refinement process is needed for multiresolution analysis or for multigrid methods, when a grid is not fine enough to

[^0]solve a problem accurately. A refinement of a grid is called uniform when each cell is divided into a given number of smaller cells having the same measure. To be efficient in practice, a refinement procedure should also be a simple one. In many applications, especially in geosciences, one requires simple, uniform and refinable (hierarchical) grids on the 2D-sphere or on the 3D-ball (i.e. the solid sphere). One simple method to construct grids on the 2D-sphere is to transfer existing planar grids, while for the 3D-ball a simple way would be to transfer polyhedral grids.

Partitions of the 2D-sphere $\mathbb{S}^{2}$ into regions of small diameter and equal area has already been constructed by Alexander [1]. In [6] Leopardi derives a recursive zonal equal area partitioning algorithm for the unit sphere $\mathbb{S}^{d}$ embedded in $\mathbb{R}^{d+1}$. The partition for the particular case $d=2$ consists of polar cups and rectilinear regions that are arranged in zonal collars. Besides the fact that that the regions have different shapes, his partition is not suitable for applications where one must avoid that vertices of spherical rectangles lie on edges of neighbor rectangles.

In astronomy, the most used construction of equal area partitions of $\mathbb{S}^{2}$, is the HEALPix grid [2], providing a hierarchical equal area iso-latitude pixelization. Other constructions are the truncated icosahedron-method of Snyder [13], the small circle subdivision method introduced in [14], the icosahedron-based method by Tegmark [16], see also [17]. Section 1 in [10] contains a larger list of uniform spherical grids, together with their properties. A complete description of all known spherical projections from a sphere or parts of a sphere to the plane, used in cartography, is realized in [3, 15]. We should mention that most of the existing constructions of spherical hierarchical (i.e. refinable) grids do not provide an equal area partition. However, in [9, 10, 11] we have already constructed some area preserving maps onto the sphere, using some Lambert azimuthal equal area projections. By transporting onto the sphere uniform and refinable planar grids, we could obtain uniform and refinable spherical grids. Also, in [4] we have constructed an octahedral equal area partition of the sphere without make use of Lambert projection, and we have studied some properties of the corresponding configurations of points. Regarding the equal volume partition of the 3D-ball, to our knowledge there was no uniform and refinable grid. This was the motivation for the work in [12], where we have constructed a volume preserving map from the 3D-cube to the 3D-ball. This allowed us not only the construction of uniform and refinable grids on the 3D-ball, but also a uniform sampling in the space of 3 D rotations with, with applications in texture analysis.

In this paper we consider a class of convex polyhedrons $\mathbb{K}_{n}(r, \varepsilon)$ and first we construct an area preserving map from $\mathbb{K}_{n}(r, \varepsilon)$ to the 2 D -sphere of radius $r$. Then, using this map, we construct a volume preserving projection from the interior of the polyhedrons to the 3D-ball of radius $r$. The spherical grids in [2] and [4] can be obtained by mapping grids on $\mathbb{K}_{n}(r, \varepsilon)$ to the 2D-sphere.

The paper is structured as follows. In Section 2 we describe the class of polyhedrons $\mathbb{K}_{n}(r, \varepsilon)$ which will be projected onto the sphere and we give some formulas which will be useful in the next sections. In Section 3 we construct a continuous area preserving map from $\mathbb{K}_{n}(r, \varepsilon)$ to the sphere, together with its inverse. In Section 3.5 we show how our maps can be useful for the construction of uniform and refinable grids on the sphere. In particular, we show that the HEALPix grids [2] can be obtained as continuous images of polyhedrons $\mathbb{K}_{n}(r, \varepsilon)$. Finally, in Section 4 , we construct a volume preserving map from the interior of the polyhedrons $\mathbb{K}_{n}(r, \varepsilon)$ to the 3D-ball of radius $r$, using the new constructed area preserving map.

## 2. Preliminaries

Let $r>0$ and consider the sphere $\mathbb{S}^{2}(r)=\mathbb{S}^{2}$ of radius $r$ centered at the origin $O$, of parametric equations

$$
\begin{align*}
& x=r \cos \theta \sin \varphi, \\
& y=r \sin \theta \sin \varphi,  \tag{1}\\
& z=r \cos \varphi,
\end{align*}
$$

where $\varphi \in[0, \pi]$ is the colatitude and $\theta \in[0,2 \pi)$ is the longitude. A simple calculation shows that the area element of the sphere is

$$
\begin{equation*}
d S=r^{2} \sin \varphi d \theta d \varphi \tag{2}
\end{equation*}
$$

Let $\varepsilon \in[0,1)$ be fixed. We intersect the sphere $\mathbb{S}^{2}$ with the planes $z=$ $\pm \varepsilon r$, and we denote with $\mathbb{S}_{r \varepsilon}^{+}=\mathbb{S}^{+}$the spherical cap situated above the plane $z=\varepsilon r$, with $\mathbb{S}_{\varepsilon r}^{-}=\mathbb{S}^{-}$the spherical cap situated below the plane $z=-\varepsilon r$, and with $\mathbb{E}_{\varepsilon r}=\mathbb{E}$ the equatorial belt between the planes $z= \pm \varepsilon r$. Their areas are

$$
\mathcal{A}\left(\mathbb{S}^{+}\right)=\mathcal{A}\left(\mathbb{S}^{-}\right)=2 \pi(1-\varepsilon) r^{2}, \quad \mathcal{A}(\mathbb{E})=4 \pi \varepsilon r^{2}
$$

and since $\mathbb{S}^{+} \cup \mathbb{S}^{-} \cup \mathbb{E}=\mathbb{S}^{2}$ and these portions are pairwise disjoint, one has

$$
\mathcal{A}\left(\mathbb{S}^{+}\right)+\mathcal{A}\left(\mathbb{S}^{-}\right)+\mathcal{A}(\mathbb{E})=\mathcal{A}\left(\mathbb{S}^{2}\right)=4 \pi r^{2}
$$



Figure 1: The polyhedron $\mathbb{K}_{n}(r, \varepsilon)$ which will be projected on the sphere. Here $n=6$.

For a fixed integer $n, n \geq 3$, let $\mathbb{K}_{n}=\mathbb{K}_{n}(r, \varepsilon)$ be a polyhedron formed by a regular prism $\mathbb{B}_{n}=\mathbb{B}_{n}(r, \varepsilon)$ of height $2 \varepsilon r$ and two congruent pyramids $\mathbb{P}_{n}^{+}(r, \varepsilon)=\mathbb{P}_{n}^{+}$and $\mathbb{P}_{n}^{-}(r, \varepsilon)=\mathbb{P}_{n}^{-}$of lateral area $2 \pi(1-\varepsilon) r^{2}$, such that the two bases of the prism $\mathbb{B}_{n}$ coincide with the bases of the two pyramids (see Figure (1). Let $\mathcal{P}_{n}$ denotes the regular polygon which is the base of $\mathbb{P}_{n}^{+}$and the upper base of $\mathbb{B}_{n}$. We place $\mathbb{K}_{n}$ such that $\mathbb{B}_{n}$ is centered at $O$ and symmetric with respect to the plane $O X Y$ and such that one of its vertical edge is situated in the plane $O X Z$. Further, we divide the space $\mathbb{R}^{3}$ onto $2 n$ zones $I_{i}^{ \pm}, i=0, \ldots, n-1$, as follows:

$$
\begin{aligned}
I_{0}^{+} & =\left\{(x, y, z) \in \mathbb{R}^{3}, z \geq 0,0 \leq y \leq x \tan \frac{2 \pi}{n}\right\} \\
I_{0}^{-} & =\left\{(x, y, z) \in \mathbb{R}^{3}, z \leq 0,0 \leq y \leq x \tan \frac{2 \pi}{n}\right\}
\end{aligned}
$$

and further, the other zones $I_{i}^{ \pm}$are obtained by rotating $I_{0}^{ \pm}$with the angle $\alpha_{i}=\frac{2 i \pi}{n}$ as

$$
I_{i}^{ \pm}=\left\{\mathcal{R}_{i} \cdot(x, y, z)^{T},(x, y, z) \in I_{0}^{ \pm}\right\}
$$

where $\mathcal{R}_{i}$ is the $3 D$ rotation matrix around $O Z$,

$$
\mathcal{R}_{i}=\left(\begin{array}{ccc}
\cos \alpha_{i} & -\sin \alpha_{i} & 0 \\
\sin \alpha_{i} & \cos \alpha_{i} & 0 \\
0 & 0 & 1
\end{array}\right)
$$

Thus, each face of the pyramids $\mathbb{P}_{n}^{ \pm}$will be situated in one of the domains $I_{i}^{ \pm}$(see Figure 2).


Figure 2: The portions $I_{0}^{+}$and $I_{0}^{-}$.
In Section 3 we will construct a map $\mathcal{T}_{n, r}=\mathcal{T}_{n}: \mathbb{S}^{2} \rightarrow \mathbb{K}_{n}$ which preserves areas, in the sense that

$$
\begin{equation*}
\mathcal{A}(D)=\mathcal{A}\left(\mathcal{T}_{n}(D)\right), \text { for all } D \subseteq \mathbb{S}^{2} \tag{3}
\end{equation*}
$$

where $\mathcal{A}(D)$ denotes the area of a domain $D$. For an arbitrary point $(x, y, z) \in$ $\mathbb{S}^{2}$ we denote

$$
\begin{equation*}
(X, Y, Z)=\mathcal{T}_{n}(x, y, z) \in \mathbb{K}_{n} \tag{4}
\end{equation*}
$$

The restrictions of $\mathcal{T}_{n}$ to $\mathbb{S}^{+}, \mathbb{S}^{-}, \mathbb{E}$ will be denoted by $\mathcal{T}_{n}^{+}, \mathcal{T}_{n}^{-}$and $\mathcal{T}_{n}^{e}$, respectively. More precisely, we will deduce the formulas for the area preserving maps $\mathcal{T}_{n}^{+}: \mathbb{S}^{+} \rightarrow \mathbb{P}_{n}^{+}, \mathcal{T}_{n}^{-}: \mathbb{S}^{-} \rightarrow \mathbb{P}_{n}^{-}$and $\mathcal{T}_{n}^{e}: \mathbb{E}^{+} \rightarrow \mathbb{B}_{n}$.

We introduce the following notations:

$$
\begin{aligned}
R_{n} & =\text { the radius of the circle circumscribed to } \mathcal{P}_{n}, \\
r_{n} & =\text { the radius of the circle inscribed in } \mathcal{P}_{n}, \\
b_{n} & =\text { the altitude of the pyramid } \mathbb{P}_{n}^{+}, \\
\ell_{n} & =\text { the edge of the polygon } \mathcal{P}_{n}, \\
a_{n} & =\text { the slant height of the pyramid } \mathbb{P}_{n}^{+} .
\end{aligned}
$$

We impose that the lateral area of $\mathbb{B}_{n}$ equals the area of the equatorial belt $\mathbb{E}$, therefore

$$
\begin{equation*}
\ell_{n}=\frac{2 \pi r}{n} \tag{5}
\end{equation*}
$$

On the other hand, since $\ell_{n}=2 R_{n} \sin \frac{\pi}{n}$, we deduce that

$$
\begin{equation*}
R_{n}=\frac{\pi r}{n \sin \frac{\pi}{n}} \tag{6}
\end{equation*}
$$

We also have

$$
\begin{equation*}
r_{n}=R_{n} \cos \frac{\pi}{n} \tag{7}
\end{equation*}
$$

The area of a face of the pyramid $\mathbb{P}_{n}^{+}$is

$$
\begin{equation*}
\mathcal{A}_{n}=\frac{a_{n} \cdot \ell_{n}}{2} \tag{8}
\end{equation*}
$$

and by imposing that $\mathcal{A}\left(\mathbb{P}_{n}^{+}\right)=\mathcal{A}\left(\mathbb{S}^{+}\right)$, we find that $a_{n}=2(1-\varepsilon) r$ and

$$
\begin{equation*}
\mathcal{A}_{n}=\frac{2 \pi(1-\varepsilon) r^{2}}{n}=\ell_{n}(1-\varepsilon) r . \tag{9}
\end{equation*}
$$

Using the equality $a_{n}^{2}=r_{n}^{2}+b_{n}^{2}$, we obtain

$$
\begin{equation*}
b_{n}=r\left(4(1-\varepsilon)^{2}-\frac{\pi^{2}}{n^{2}} \cot ^{2} \frac{\pi}{n}\right)^{1 / 2} \tag{10}
\end{equation*}
$$

## 3. Construction of the area preserving $\operatorname{map} \mathcal{T}_{n}: \mathbb{S}^{2} \rightarrow \mathbb{K}_{n}$ and its inverse

### 3.1. Construction of the map $\mathcal{T}_{n}^{+}$

For $h \in(1-\varepsilon, 1)$, we denote by $\mathbb{S}_{r h}$ the spherical cap situated above the plane of equation $z=r h$. A simple calculation shows that

$$
\begin{equation*}
\mathcal{A}\left(\mathbb{S}_{r h}\right)=2 \pi(1-h) r^{2} . \tag{11}
\end{equation*}
$$

Now, we calculate $H>0$ such that the portion $\mathbb{P}_{n}^{+}(H)$ of the pyramid $\mathbb{P}_{n}^{+}$ situated above the plane $z=H$ has the same area $\mathcal{A}\left(\mathbb{S}_{r h}\right)$. For the small pyramid $\mathbb{P}_{n}^{+}(H)$, let $\widetilde{R}_{n}$ be the radius of the circumscribed circle of its base. Then, one has

$$
\frac{\widetilde{R}_{n}}{R_{n}}=\frac{b_{n}-H+\varepsilon r}{b_{n}}
$$

and the area $\mathcal{A}(H)$ of a face of $\mathbb{P}_{n}^{+}(H)$ is therefore

$$
\mathcal{A}_{n}(H)=\mathcal{A}_{n} \cdot\left(1-\frac{H-\varepsilon r}{b_{n}}\right)^{2}=\frac{2 \pi(1-\varepsilon) r^{2}}{n}\left(1-\frac{H-\varepsilon r}{b_{n}}\right)^{2}
$$

Imposing now $\mathcal{A}\left(\mathbb{S}_{r h}\right)=n \mathcal{A}(H)$, we obtain that

$$
2 \pi(1-h) r^{2}=2 \pi(1-\varepsilon) r^{2}\left(1-\frac{H-\varepsilon r}{b_{n}}\right)^{2}
$$

whence after some calculations we obtain

$$
H=\varepsilon r+b_{n}\left(1-\sqrt{\frac{1-h}{1-\varepsilon}}\right) .
$$

In conclusion, for $z \geq \varepsilon r$ we can define $Z$ from (4) as

$$
\begin{equation*}
Z=\varepsilon r+b_{n}\left(1-\sqrt{\frac{1-z / r}{1-\varepsilon}}\right), \tag{12}
\end{equation*}
$$

meaning that the parallel circles of the sphere map onto polygons obtained as intersections of $\mathbb{P}_{n}^{+}$with planes parallel to $X O Y$.

For the formulas for $X$ and $Y$ we proceed as follows. We focus on the portion $\mathcal{F}_{0}^{+}$of $\mathbb{S}^{2}$, situated in $I_{0}^{+}$, and the face of the pyramid $\mathbb{P}_{n}^{+}$situated in $I_{0}^{+}$will be denoted with $F_{0}^{+}$. We denote $A\left(\sqrt{1-\varepsilon^{2}} r, 0, \varepsilon r\right), C(0,0, r)$ and we consider the vertical plane of equation $y=x \tan \alpha$, with $\alpha \in(0,2 \pi / n)$ (see Figure 3, left). We denote by $\widetilde{M C}$ its intersection with $\mathcal{F}_{0}^{+}$. More precisely, $M\left(\sqrt{1-\varepsilon^{2}} r \cos \alpha, \sqrt{1-\varepsilon^{2}} r \sin \alpha, \varepsilon r\right)$. The area of the spherical domain $A M C$ equals $\alpha(1-\varepsilon) r^{2}$. Now we intersect the face $F_{0}^{+}$of the pyramid with the vertical plane of equation $y=x \tan \beta$ and we denote by $M^{\prime}\left(X^{\prime}, Y^{\prime}\right)$ its intersection with the edge $A^{\prime} B^{\prime}$, where $A^{\prime}\left(R_{n}, 0, \varepsilon r\right)$ and $B^{\prime}\left(R_{n} \cos 2 \pi / n, R_{n} \sin 2 \pi / n, \varepsilon r\right)$ (see Figure 3, right).

For finding the coordinates $X^{\prime}, Y^{\prime}$ we use the fact that $M^{\prime}$ is the intersection of the line $A^{\prime} B^{\prime}$ with the plane of equation $Y^{\prime}=X^{\prime} \tan \beta$, so solving the system

$$
\frac{X^{\prime}-R_{n}}{R_{n}\left(\cos \frac{2 \pi}{n}-1\right)}=\frac{Y^{\prime}-0}{R_{n} \sin \frac{2 \pi}{n}}, \quad Y^{\prime}=X^{\prime} \tan \beta
$$

we obtain

$$
\begin{equation*}
X^{\prime}=\frac{R_{n}}{1+\tan \beta \tan \frac{\pi}{n}}, \quad Y^{\prime}=\frac{R_{n} \tan \beta}{1+\tan \beta \tan \frac{\pi}{n}} . \tag{13}
\end{equation*}
$$

Further,

$$
A^{\prime} M^{\prime}=\frac{Y^{\prime}}{\cos \frac{\pi}{n}},
$$



Figure 3: The portion $A M C$ of the spherical cap $\mathbb{S}^{+}$and its image triangle $A^{\prime} M^{\prime} C^{\prime}=$ $\mathcal{T}_{n}(A M C)$ on the pyramid $\mathbb{P}_{n}^{+}$.
and therefore, for the area of the planar triangle $A^{\prime} C^{\prime} M^{\prime}$ we find

$$
\mathcal{A}\left(A^{\prime} C^{\prime} M^{\prime}\right)=\frac{A^{\prime} M^{\prime} \cdot a_{n}}{2}=\frac{2 \pi(1-\varepsilon) r^{2} \tan \beta}{n \sin \frac{2 \pi}{n}\left(1+\tan \beta \tan \frac{\pi}{n}\right)}
$$

If we impose that this area equals the area of the spherical domain $A M C$, which is $\alpha(1-\varepsilon) r^{2}$, we finally find that

$$
\begin{equation*}
\tan \beta=\frac{\alpha n \sin \frac{2 \pi}{n}}{2\left(\pi-\alpha n \sin ^{2} \frac{\pi}{n}\right)} \tag{14}
\end{equation*}
$$

This is in fact the relation between $\alpha$ and $\beta$, such that the slice $\widetilde{A C M}$ of $\mathcal{F}_{0}^{+}$ has the same area with the triangular slice $A^{\prime} C^{\prime} M^{\prime}$ of the face $F_{0}^{+}$.

Consider now $V=(m, q, p) \in \mathcal{F}_{0}^{+}$and denote $N=(u, s, t):=\mathcal{T}_{n}(V)$. The point $V$ is the intersection of $\mathcal{F}_{0}^{+}$with the planes

$$
\left(P_{1}\right): z=p \text { and }\left(P_{2}\right): y=\frac{q}{m} x, \quad \text { where } \frac{q}{m}=\tan \alpha .
$$

The point $N$ will be the intersection of $\mathcal{T}_{n}\left(P_{1}\right), \mathcal{T}_{n}\left(P_{2}\right)$ and the face $F_{0}^{+}$, therefore

$$
\begin{aligned}
\mathcal{T}_{n}\left(P_{1}\right) & : \quad t=\varepsilon r+b_{n}\left(1-\sqrt{\frac{1-p / r}{1-\varepsilon}}\right) \\
\mathcal{T}_{n}\left(P_{2}\right) & : s=\frac{n \sin \frac{2 \pi}{n} \cdot \arctan \frac{q}{m}}{2\left(\pi-n \arctan \frac{q}{m} \sin ^{2} \frac{\pi}{n}\right)} \cdot u \\
F_{0}^{+} & : u \cdot b_{n}+s \cdot b_{n} \tan \frac{\pi}{n}+t \cdot R_{n}=\left(b_{n}+\varepsilon r\right) R_{n} .
\end{aligned}
$$

Solving this system we find

$$
\begin{aligned}
u & =R_{n} \sqrt{\frac{1-p / r}{1-\varepsilon}}\left(1-\frac{n}{\pi} \sin ^{2} \frac{\pi}{n} \arctan \frac{q}{m}\right) \\
s & =R_{n} \sqrt{\frac{1-p / r}{1-\varepsilon}} \frac{n \sin \frac{2 \pi}{n} \arctan \frac{q}{m}}{2 \pi} \\
t & =\varepsilon r+b_{n}\left(1-\sqrt{\frac{1-p / r}{1-\varepsilon}}\right)
\end{aligned}
$$

In conclusion, using (6) and with the notations in (4), for $(x, y, z) \in \mathcal{F}_{0}^{+}$, we define the restriction of $\mathcal{T}_{n}^{+}$to $\mathcal{F}_{0}^{+}$as

$$
\begin{align*}
X & =\sqrt{\frac{r(r-z)}{1-\varepsilon}}\left(\frac{\pi}{n \sin \frac{\pi}{n}}-\sin \frac{\pi}{n} \arctan \frac{y}{x}\right)  \tag{15}\\
Y & =\sqrt{\frac{r(r-z)}{1-\varepsilon}} \cos \frac{\pi}{n} \arctan \frac{y}{x}  \tag{16}\\
Z & =\varepsilon r+b_{n}\left(1-\sqrt{\frac{1-z / r}{1-\varepsilon}}\right) \tag{17}
\end{align*}
$$

with $b_{n}$ given in (10).
In general, for $i=0, \ldots, n-1$, let $\mathcal{F}_{i}^{ \pm}$be the portion of $\mathbb{S}^{2}$ situated in $I_{i}^{ \pm}$. The the restriction of $\mathcal{T}_{n}^{+}$to $\mathcal{F}_{i}^{+}$will be calculated as

$$
\begin{align*}
& (X, Y, Z)^{T}=\mathcal{R}_{i} \cdot \mathcal{T}_{n}^{+}\left(\mathcal{R}_{i}^{T} \cdot(x, y, z)^{T}\right), \quad \text { for }(x, y, z) \in \mathcal{F}_{i}^{+}, \text {i.e. }  \tag{18}\\
X & \left.=\sqrt{\frac{r(r-z)}{1-\varepsilon}}\left(\frac{\pi \cos \alpha_{i}}{n \sin \frac{\pi}{n}}-\sin \left(\alpha_{i}+\frac{\pi}{n}\right) \cdot \arctan \frac{-x \sin \alpha_{i}+y \cos \alpha_{i}}{x \cos \alpha_{i}+y \sin \alpha_{i}}\right) 1,9\right) \\
Y & \left.=\sqrt{\frac{r(r-z)}{1-\varepsilon}}\left(\frac{\pi \sin \alpha_{i}}{n \sin \frac{\pi}{n}}+\cos \left(\alpha_{i}+\frac{\pi}{n}\right) \cdot \arctan \frac{-x \sin \alpha_{i}+y \cos \alpha_{i}}{x \cos \alpha_{i}+y \sin \alpha_{i}}\right) 2,0\right) \\
Z & =\varepsilon r+b_{n}\left(1-\sqrt{\frac{1-z / r}{1-\varepsilon}}\right) . \tag{21}
\end{align*}
$$

In spherical coordinates $(\varphi, \theta)$, the map $\mathcal{T}_{n}^{+}$restricted to $\mathcal{F}_{0}^{+}$writes as

$$
\begin{aligned}
X & =r \sqrt{\frac{2}{1-\varepsilon}}\left(\frac{\pi}{n \sin \frac{\pi}{n}}-\theta \sin \frac{\pi}{n}\right) \sin \frac{\varphi}{2} \\
Y & =r \sqrt{\frac{2}{1-\varepsilon}} \theta \cos \frac{\pi}{n} \sin \frac{\varphi}{2} \\
Z & =\varepsilon r+b_{n}\left(1-\sqrt{\frac{2}{1-\varepsilon}} \sin \frac{\varphi}{2}\right) .
\end{aligned}
$$

For the portion $\mathcal{F}_{i}^{+}$the map is defined by

$$
\begin{aligned}
X & =r \sqrt{\frac{2}{1-\varepsilon}}\left(\frac{\pi \cos \alpha_{i}}{n \sin \frac{\pi}{n}}-\sin \frac{\pi(2 i+1)}{n} \cdot\left(\theta-\alpha_{i}\right)\right) \sin \frac{\varphi}{2} \\
Y & =r \sqrt{\frac{2}{1-\varepsilon}}\left(\frac{\pi \sin \alpha_{i}}{n \sin \frac{\pi}{n}}+\cos \frac{\pi(2 i+1)}{n} \cdot\left(\theta-\alpha_{i}\right)\right) \sin \frac{\varphi}{2}, \\
Z & =\varepsilon r+b_{n}\left(1-\sqrt{\frac{2}{1-\varepsilon}} \sin \frac{\varphi}{2}\right) .
\end{aligned}
$$

If we evaluate the coefficients of the first fundamental order of this surface

$$
\begin{aligned}
& E^{\prime}=\left(X_{\varphi}^{\prime}\right)^{2}+\left(Y_{\varphi}^{\prime}\right)^{2}+\left(Z_{\varphi}^{\prime}\right)^{2}=\frac{r^{2}}{2(1-\varepsilon)}\left[4(1-\varepsilon)^{2}+\left(\theta-\frac{(2 i+1) \pi}{n}\right)^{2}\right] \cos ^{2} \frac{\varphi}{2} \\
& F^{\prime}=X_{\varphi}^{\prime} X_{\theta}^{\prime}+Y_{\varphi}^{\prime} Y_{\theta}^{\prime}+Z_{\varphi}^{\prime} Z_{\theta}^{\prime}=\frac{r^{2}}{2(1-\varepsilon)}\left(\theta-\frac{(2 i+1) \pi}{n}\right) \sin \varphi \\
& G^{\prime}=\left(X_{\theta}^{\prime}\right)^{2}+\left(Y_{\theta}^{\prime}\right)^{2}+\left(Z_{\theta}^{\prime}\right)^{2}=\frac{2 r^{2}}{1-\varepsilon} \sin ^{2} \frac{\varphi}{2}
\end{aligned}
$$

then we find that $E^{\prime} G^{\prime}-\left(F^{\prime}\right)^{2}=r^{4} \sin ^{2} \varphi$, which is equal to $E G-F^{2}$ for the sphere (see (2)). This property holds for all the portions $\mathcal{F}_{i}^{ \pm}$, in conclusion, $\sqrt{E G-F^{2}}$ is invariant under $\mathcal{T}_{n}^{+}$, whence $\mathcal{T}_{n}^{+}$is an area preserving map.

Remark 1. The map $\mathcal{T}_{n}^{-}: \mathbb{S}^{-} \rightarrow \mathbb{P}_{n}^{-}$can be obtained by symmetry. More
precisely, the formulas are

$$
\begin{align*}
X & =\sqrt{\frac{r(r+z)}{1-\varepsilon}}\left(\frac{\pi}{n \sin \frac{\pi}{n}}-\sin \frac{\pi}{n} \arctan \frac{y}{x}\right)  \tag{22}\\
Y & =\sqrt{\frac{r(r+z)}{1-\varepsilon}} \cos \frac{\pi}{n} \arctan \frac{y}{x}  \tag{23}\\
Z & =\varepsilon r+b_{n}\left(1-\sqrt{\frac{1+z / r}{1-\varepsilon}}\right) \tag{24}
\end{align*}
$$

### 3.2. The inverse of the map $\mathcal{T}_{n}^{+}$

Let $(X, Y, Z) \in \mathbb{P}_{n}^{+}$. First, from (21) we immediately deduce that

$$
\begin{equation*}
z=r-r(1-\varepsilon)\left(1-\frac{Z-\varepsilon r}{b_{n}}\right)^{2} \tag{25}
\end{equation*}
$$

Then, from (19) and (20) we find that

$$
\frac{Y}{X}=\frac{\frac{\pi \sin \alpha_{i}}{n \sin \frac{\pi}{n}}+\cos \left(\alpha_{i}+\frac{\pi}{n}\right) \cdot \arctan \frac{-x \sin \alpha_{i}+y \cos \alpha_{i}}{x \cos \alpha_{i}+y \sin \alpha_{i}}}{\frac{\pi \cos \alpha_{i}}{n \sin \frac{\pi}{n}}-\sin \left(\alpha_{i}+\frac{\pi}{n}\right) \cdot \arctan \frac{-x \sin \alpha_{i}+y \cos \alpha_{i}}{x \cos \alpha_{i}+y \sin \alpha_{i}}},
$$

whence

$$
\arctan \frac{-x \sin \alpha_{i}+y \cos \alpha_{i}}{x \cos \alpha_{i}+y \sin \alpha_{i}}=\frac{\pi}{n \sin \frac{\pi}{n}} \cdot \frac{-X \sin \alpha_{i}+Y \cos \alpha_{i}}{X \cos \left(\alpha_{i}+\frac{\pi}{n}\right)+Y \sin \left(\alpha_{i}+\frac{\pi}{n}\right)}=: \lambda
$$

Therefore,

$$
y=x \frac{\tan \lambda \cos \alpha_{i}+\sin \alpha_{i}}{\cos \alpha_{i}-\tan \lambda \sin \alpha_{i}}
$$

and then, taking into account that $x^{2}+y^{2}+z^{2}=r^{2}$, after simple calculations we find

$$
\begin{align*}
x & =\sqrt{r^{2}-z^{2}} \cdot \cos \left(\frac{\pi}{n \sin \frac{\pi}{n}} \cdot \frac{-X \sin \alpha_{i}+Y \cos \alpha_{i}}{X \cos \left(\alpha_{i}+\frac{\pi}{n}\right)+Y \sin \left(\alpha_{i}+\frac{\pi}{n}\right)}+\alpha_{i}\right)  \tag{26}\\
y & =\sqrt{r^{2}-z^{2}} \cdot \sin \left(\frac{\pi}{n \sin \frac{\pi}{n}} \cdot \frac{-X \sin \alpha_{i}+Y \cos \alpha_{i}}{X \cos \left(\alpha_{i}+\frac{\pi}{n}\right)+Y \sin \left(\alpha_{i}+\frac{\pi}{n}\right)}+\alpha_{i}\right) . \tag{27}
\end{align*}
$$

where $z$ is given in (25).

Remark 2. For the inverse of $\mathcal{T}_{n}^{-}$, the formulas for $x$ and $y$ are as in (26) and (27), respectively, while for $z$ formula (25) changes into

$$
\begin{equation*}
z=r(1-\varepsilon)\left(1-\frac{Z-\varepsilon}{b_{n}}\right)^{2}-r \tag{28}
\end{equation*}
$$

Remark 3. Formulas (26) and (27) can also be obtained from the formulas for $i=0$, similarly to (18).

### 3.3. Construction of the maps $\left(\mathcal{T}_{n}^{e}\right)^{-1}$ and $\mathcal{T}_{n}^{e}$

The prism $\mathbb{B}_{n}$ is regular with height $2 \varepsilon r$ and edge $\ell_{n}$ given in (5). The face $B_{0}$ of $\mathbb{B}_{n}$ situated in the zone $I_{0}=I_{0}^{+} \cup I_{0}^{-}$has the equation

$$
\begin{equation*}
Y \cdot \sin \frac{\pi}{n}+X \cdot \cos \frac{\pi}{n}=R_{n} \cos \frac{\pi}{n} \tag{29}
\end{equation*}
$$

or, equivalently, $Y=\left(R_{n}-X\right) \cot \frac{\pi}{n}$. Then, in order to project the face $B_{0}$ onto the plane $O Y Z$, we make a translation with $R_{n}$ along the axis $O X$, followed by a rotation of angle $-\frac{\pi}{n}$ around the axis $O Z$. Thus, the point

$$
\left(X,\left(R_{n}-X\right) \cot \frac{\pi}{n}, Z\right) \in \mathbb{B}_{n}
$$

will be first translated to

$$
\left(X-R_{n},\left(R_{n}-X\right) \cot \frac{\pi}{n}, Z\right) \in \mathbb{B}_{n}
$$

and then rotated around $O Z$ it will be mapped onto the point

$$
P\left(0, \frac{R_{n}-X}{\sin \frac{\pi}{n}}, Z\right)=P\left(0, \frac{Y}{\cos \frac{\pi}{n}}, Z\right)
$$

As a translation followed by rotation, this map will be area preserving. Next we use the inverse Lambert cylindrical equal area projection (see i.e. [3]), the point $P$ being further mapped onto

$$
\begin{equation*}
\left(\sqrt{r^{2}-Z^{2}} \cos \frac{Y}{r \cos \frac{\pi}{n}}, \sqrt{r^{2}-Z^{2}} \sin \frac{Y}{r \cos \frac{\pi}{n}}, Z\right) \in \mathbb{E} \subset \mathbb{S}^{2} \tag{30}
\end{equation*}
$$

In conclusion, the map $\left(\mathcal{T}_{n}^{e}\right)^{-1}: \mathbb{B}_{n} \rightarrow \mathbb{E}$ is area preserving and maps the point $(X, Y, Z) \in \mathbb{B}_{n}$ onto the point given in (30).

In general, similarly to (18), the face of the prism $\mathbb{B}_{n}$ situated in $I_{i}=$ $I_{i}^{+} \cup I_{i}^{-}$is mapped on the sphere by

$$
\begin{align*}
x & =\sqrt{r^{2}-Z^{2}} \cos \left(\frac{Y \cos \alpha_{i}-X \sin \alpha_{i}}{r \cos \frac{\pi}{n}}+\alpha_{i}\right)  \tag{31}\\
y & =\sqrt{r^{2}-Z^{2}} \sin \left(\frac{Y \cos \alpha_{i}-X \sin \alpha_{i}}{r \cos \frac{\pi}{n}}+\alpha_{i}\right)  \tag{32}\\
z & =Z \tag{33}
\end{align*}
$$

For the direct application $\mathcal{T}_{n}^{e}: \mathbb{E} \rightarrow \mathbb{B}_{n}$, the calculations give

$$
\begin{align*}
X & =R_{n} \cos \alpha_{i}-r \sin \left(\alpha_{i}+\frac{\pi}{n}\right) \arctan \frac{-x \sin \alpha_{i}+y \cos \alpha_{i}}{x \cos \alpha_{i}+y \sin \alpha_{i}}  \tag{34}\\
Y & =R_{n} \sin \alpha_{i}+r \cos \left(\alpha_{i}+\frac{\pi}{n}\right) \arctan \frac{-x \sin \alpha_{i}+y \cos \alpha_{i}}{x \cos \alpha_{i}+y \sin \alpha_{i}}  \tag{35}\\
Z & =z \tag{36}
\end{align*}
$$

### 3.4. The continuity of $\mathcal{T}_{n}$

Proposition 4. The map $\mathcal{T}_{n}: \mathbb{S}^{2} \rightarrow \mathbb{K}_{n}$ is continuous.
Proof. It is enough to restrict ourselves to the first octant $I_{0}^{+}$and to prove the continuity of $\mathcal{T}_{n}^{-1}$ on $\mathbb{B}_{n} \cap \mathbb{P}_{n}^{+} \cap I_{0}^{+}$.

For $Z=r \varepsilon$, the point projected onto the border of $\mathbb{E}$ is

$$
\begin{equation*}
\left(r \sqrt{1-\varepsilon^{2}} \cos \frac{Y}{r \cos \frac{\pi}{n}}, r \sqrt{1-\varepsilon^{2}} \sin \frac{Y}{r \cos \frac{\pi}{n}}, \varepsilon r\right) . \tag{37}
\end{equation*}
$$

On the other hand, let $(X, Y, \varepsilon r) \in \mathbb{P}_{n}^{+}$. From (26), (27) and (25), its image on the sphere, $\mathcal{T}_{n}(X, Y, \varepsilon r)$, will be

$$
\begin{aligned}
x & =r \sqrt{1-\varepsilon^{2}} \cos \frac{Y \pi}{n \sin \frac{\pi}{n} \cdot\left(X \cos \frac{\pi}{n}+Y \sin \frac{\pi}{n}\right)} \\
y & =r \sqrt{1-\varepsilon^{2}} \sin \frac{Y \pi}{n \sin \frac{\pi}{n} \cdot\left(X \cos \frac{\pi}{n}+Y \sin \frac{\pi}{n}\right)} \\
z & =\varepsilon r .
\end{aligned}
$$

But $(X, Y, \varepsilon r)$ is also a point of $\mathbb{B}_{n}$, so it satisfies (29), and finally, replacing $R_{n}$ from (6), the above formulas transform into

$$
\begin{aligned}
x & =r \sqrt{1-\varepsilon^{2}} \cos \frac{Y}{r \cos \frac{\pi}{n}}, \\
y & =r \sqrt{1-\varepsilon^{2}} \sin \frac{Y}{r \cos \frac{\pi}{n}} \\
z & =\varepsilon r,
\end{aligned}
$$

i.e. the same as in formula (37).

Similarly one can prove the same equality for a point in $\mathbb{P}_{n}^{+} \cap \mathbb{B}_{n}$ and then in $\mathbb{P}_{n}^{-} \cap \mathbb{B}_{n}$. In conclusion, the map $\mathcal{T}_{n}^{-1}$ is continuous, therefore $\mathcal{T}_{n}$ is also continuous.

### 3.5. Uniform grids on the sphere

The most important application of our area preserving projection maps is the construction of spherical grids by mapping grids from the polyhedrons $\mathbb{K}_{n}(r, \varepsilon)$ to the sphere $\mathbb{S}^{2}(r)$. The major advantage is the possibility of construction of uniform and refinable spherical grids, since uniform and refinable grids are easier obtainable on polyhedrons.

In the particular case $n=4$ and $\varepsilon=0$, the polyhedron $\mathbb{K}_{4}(r, 0)$ is an octahedron, but not a regular one, as in [4]. However, one can easily see that, if we make uniform triangular refinements by dividing each planar triangle into four small triangles by lines parallel to the edges, one obtains exactly the grids in [4].

Next we will show that the HEALPix grids (see [2]) can also be obtained by mapping onto the sphere grids on $\mathbb{K}_{n}(\varepsilon, r)$. However, one advantage of our map is that it allows us to transport onto the sphere some already constructed continuous wavelet bases on polyhedrons, by a similar technique as the one described in [7]. Compared with the construction in [7], one can use now the usual scalar product in $L^{2}\left(\mathbb{S}^{2}\right)$ instead of a weighted scalar product, since the map is area preserving. Moreover, the transportation of continuous functions on $\mathbb{K}_{n}(\varepsilon, r)$ yields continuous non-distorted functions on the sphere, and this does not usually happen when functions defined on rectangles are mapped onto the sphere with the area preserving maps used in [2]. When transporting a function $f:[0, a] \times[-b, b] \rightarrow \mathbb{R}$ onto the sphere using the maps used in [2], the transported function defined on $\mathbb{S}^{2}$ is usually distorted, and it remains continuous on $\mathbb{S}^{2}$ only when it satisfies some periodicity conditions (see [7]).


Figure 4: The grids on the polyhedron $\mathbb{K}_{6}(r, p /(p+1))$, for $p=2$ (left) and $p=4$ (right). The dashed lines represent the edges of the prism.

### 3.5.1. Obtaining the value of $\varepsilon$

In order to obtain the HEALPix grids on the sphere, we consider $r=1$ for simplicity, and we construct the grids on the polyhedron $\mathbb{K}_{n}(1, \varepsilon)$ as follows.

Let $p \in \mathbb{N}$. For $p$ even, on each face of the prism $\mathbb{B}_{n}$ we first draw horizontal lines to divide each face into $p / 2$ rectangles having the same area. Then, for each of the rectangle we draw the diagonals. Together with the triangular faces of the pyramids of $\mathbb{K}_{n}$, they will form a grid consisting in $n \cdot p$ cells of rhombic shape (see Figure (4), and we will impose that all this cells have the same area $A_{\text {cell }}$, and this area will be the same after projecting the grids onto the sphere. Therefore we have

$$
\begin{equation*}
\mathcal{A}\left(\mathbb{E}_{\varepsilon}\right)=4 \pi \varepsilon=n p \mathcal{A}_{\text {cell }}, \quad \mathcal{A}\left(\mathbb{S}_{\varepsilon}^{+}\right)=2 \pi(1-\varepsilon)=\frac{n}{2} \mathcal{A}_{\text {cell }} \tag{38}
\end{equation*}
$$

Elimining $A_{\text {cell }}$ from these equalities we obtain $\varepsilon=\frac{p}{p+1}$. For $p$ odd, we first rotate the prism $\mathbb{P}_{n}^{-}$with $\pi / n$, around the axis $O z$ and on each face of the prism we first draw $(p-1) / 2$ horizontal lines, such that the resulting rectangles have the same area, except the "lowest" ones, which have only half of the area. Then we draw the rhombic cells as for the case $p$ even, except the "lowest" part of the prism (see Figure 6). The equalities in (38) also hold, so in this case we must have again $\varepsilon=p /(p+1)$.

Further, for $k \in \mathbb{N}$, we perform a subdivision of each rhombic cell of the grid on $\mathbb{K}_{n}(1, p /(p+1))$ into $k^{2}$ cells with the same area, by drawing $k-1$ equidistant parallel lines to each of the edges (see Figure 3).


Figure 5: The grids on the polyhedron $\mathbb{K}_{6}(r, p /(p+1))$, for $p=3$ (left) and $p=5$ (right). The dashed lines represent the edges of the prism.


Figure 6: The refined grids on the polyhedron $\mathbb{K}_{6}(r, p /(p+1))$, obtained after a subdivision with $k=2$, for $p=2$ (left) and $p=3$ (right).

Next we write the equations of the lines of our grids and we show that they coincide with the ones deduced in [2] for the HEALPix grids.

### 3.5.2. The grid on the pyramid

We parametrize the triangle with vertices

$$
(0,0),\left(R_{n} \cos \alpha_{i}, R_{n} \sin \alpha_{i}\right),\left(R_{n} \cos \alpha_{i+1}, R_{n} \sin \alpha_{i+1}\right)
$$

as

$$
\begin{aligned}
X & =u \cdot R_{n} \cos \alpha_{i}+v \cdot R_{n} \cos \alpha_{i+1}, \\
Y & =u \cdot R_{n} \sin \alpha_{i}+v \cdot R_{n} \sin \alpha_{i+1},
\end{aligned}
$$

with $u \in[0,1], v \in[0,1-u]$. For the parametrization of the face of the pyramid contained in $I_{i}^{+}$, we need to express $Z$ from the equation of the plane containing the points $\left(0,0, b_{n}+\varepsilon\right),\left(R_{n} \cos \alpha_{i}, R_{n} \sin \alpha_{i}, \varepsilon\right),\left(R_{n} \cos \alpha_{i+1}, R_{n} \sin \alpha_{i+1}, \varepsilon\right)$. Thus, we obtain

$$
Z=\varepsilon+b_{n}(1-u-v) .
$$

The spherical images have the equations

$$
\begin{aligned}
x & =\sqrt{1-z^{2}} \cos \left(\frac{2 \pi}{n} \cdot \frac{v}{u+v}+\alpha_{i}\right), \\
y & =\sqrt{1-z^{2}} \sin \left(\frac{2 \pi}{n} \cdot \frac{v}{u+v}+\alpha_{i}\right), \\
z & =1-(1-\varepsilon)(u+v)^{2} .
\end{aligned}
$$

If we map the curves $u=\mathcal{C}_{1}, v \in[0,1-u]$ and $v=\mathcal{C}_{2}, u \in[0,1-v]$ (with $\mathcal{C}_{1,2}$ constants chosen properly), we obtain the spherical curves in the HEALPix grid. Indeed, if we take for example $n=4$ and $p=2$ (i.e. $\varepsilon=2 / 3$ ), the above equations give

$$
\begin{align*}
& z=1-\frac{1}{3}(u+v)^{2}  \tag{39}\\
& \theta=\frac{\pi}{2} \cdot \frac{v}{u+v}+\frac{\pi i}{2} \tag{40}
\end{align*}
$$

where $\theta=\arctan (y / x)$. Let us consider $\theta_{t}=\theta \bmod \frac{\pi}{2}=\frac{\pi}{2} \cdot \frac{v}{u+v}$. From (39) and (40) we obtain

$$
z=1-\frac{1}{3} u^{2} \cdot\left(\frac{\pi}{2 \theta_{t}-\pi}\right)^{2}, \text { and also } z=1-\frac{1}{3} v^{2} \cdot\left(\frac{\pi}{2 \theta_{t}}\right)^{2} .
$$

Now, if we consider $u=\frac{\ell}{k}=: u_{\ell}$, for each $\ell \in\{0,1, \ldots, k\}$ we obtain the family of curves

$$
z=1-\frac{1}{3} u_{\ell}^{2} \cdot\left(\frac{\pi}{2 \theta_{t}-\pi},\right)^{2}
$$

and this is exactly the equation (20) from [2]. Note that $N_{\text {side }}$ and $k$ in [2] are in our notations $k$ and $\ell$, respectively.

The other class of curves of our grid are

$$
\begin{aligned}
& v=\frac{\ell}{k}=: v_{\ell} \\
& z=1-\frac{1}{3} v_{\ell}^{2} \cdot\left(\frac{\pi}{2 \theta_{t}}\right)^{2}
\end{aligned}
$$

which is exacly the equation (19) from [2], with $v_{\ell}=k / N_{\text {side }}$.

### 3.5.3. The grid on the prism

The point $U \in \mathbb{P}_{n}^{+} \cap \mathbb{B}_{n} \cap I_{i}^{+}$of coordinates

$$
\begin{aligned}
x_{U} & =\left(1-\frac{\ell}{k}\right) R_{n} \cos \frac{2 \pi i}{n}+\frac{\ell}{k} R_{n} \cos \frac{2(i+1) \pi}{n} \\
y_{U} & =\left(1-\frac{\ell}{k}\right) R_{n} \sin \frac{2 \pi i}{n}+\frac{\ell}{k} R_{n} \sin \frac{2(i+1) \pi}{n} \\
z_{U} & =\varepsilon
\end{aligned}
$$

and $Q \in \mathbb{P}_{n}^{-} \cap \mathbb{B}_{n} \cap I_{i}^{-}$of coordinates

$$
\begin{aligned}
& x_{Q}=\left(1-\frac{\ell}{k} \mp \frac{p}{2}\right) R_{n} \cos \frac{2 \pi i}{n}+\left(\frac{\ell}{k} \pm \frac{p}{2}\right) R_{n} \cos \frac{2(i+1) \pi}{n} \\
& y_{Q}=\left(1-\frac{\ell}{k} \mp \frac{p}{2}\right) R_{n} \sin \frac{2 \pi i}{n}+\left(\frac{\ell}{k} \pm \frac{p}{2}\right) R_{n} \sin \frac{2(i+1) \pi}{n} \\
& z_{Q}=-\varepsilon
\end{aligned}
$$

belong to the same line of the grid of the prism. A point $(X, Y, Z)$ of the grid of the prism satisfies the equation of the line passing through $U$ and $Q$

$$
\frac{X-x_{U}}{x_{Q}-x_{U}}=\frac{Y-y_{U}}{y_{Q}-y_{U}}=\frac{Z-z_{U}}{z_{Q}-z_{U}}=t
$$

When we calculate its image on the sphere using formulas (31), (32) and (33), we obtain

$$
\begin{aligned}
& z=\varepsilon-2 \varepsilon t \\
& x=\sqrt{1-z^{2}} \cos \left(\frac{2 \pi}{n} \cdot\left(i+\frac{\ell}{k} \pm \frac{p t}{2}\right)\right) \\
& y=\sqrt{1-z^{2}} \sin \left(\frac{2 \pi}{n} \cdot\left(i+\frac{\ell}{k} \pm \frac{p t}{2}\right)\right)
\end{aligned}
$$

In the particular case $p=2, n=4, \varepsilon=2 / 3$ we obtain

$$
\begin{aligned}
z & =\frac{2}{3}-\frac{4 t}{3} \\
\theta & =\frac{\pi}{2}\left(i+\frac{\ell}{k} \pm t\right)
\end{aligned}
$$

If we take again $\theta_{t}=\theta \bmod \frac{\pi}{2}=\frac{\pi}{2} \cdot\left(\frac{\ell}{k} \pm t\right)$, we further obtain

$$
z=\frac{2}{3} \pm \frac{4}{3}\left(\frac{2 \theta_{t}}{\pi}-\frac{\ell}{k}\right)
$$

which agree with formulas (22), (A2), (A3) in [2].

## 4. Volume preserving map from the interior of a polyhedron $\mathbb{K}_{n}\left(r^{\prime}, \varepsilon\right)$ to the ball of radius $r$.

We will use the area preserving map constructed in the previous section for the construction of a volume preserving map from the interior of a polyhedron $\mathbb{K}_{n}$ onto the ball of radius $r$.

We fix $\varepsilon$, and for $\rho>0$ we define

$$
\begin{gathered}
\overline{\mathbb{K}}_{n}(\rho, \varepsilon)=\operatorname{int}\left(\mathbb{K}_{n}(\rho, \varepsilon)\right) \cup \mathbb{K}_{n}(\rho, \varepsilon) \quad \text { and } \\
\overline{\mathbb{S}^{2}}(\rho)=\left\{(x, y, z) \in \mathbb{R}^{3}, x^{2}+y^{2}+z^{2} \leq \rho^{2}\right\} .
\end{gathered}
$$

Their volumes are

$$
\operatorname{vol}\left(\overline{\mathbb{K}}_{n}(\rho, \varepsilon)\right)=\rho^{3} \gamma, \quad \operatorname{vol}\left(\overline{\mathbb{S}^{2}}(\rho)\right)=\rho^{3} \beta
$$

with

$$
\gamma=2\left(\varepsilon+\frac{c(\varepsilon)}{3}\right) \frac{\pi^{2}}{n} \cot \frac{\pi}{n}, \quad c(\varepsilon)=\sqrt{4(1-\varepsilon)^{2}-\frac{\pi^{2}}{n^{2}} \cot ^{2} \frac{\pi}{n}}, \quad \beta=\frac{4 \pi}{3} .
$$

We also need to define the domains

$$
\begin{aligned}
\overline{\mathbb{S}^{+}}(\rho) & =\overline{\mathbb{S}^{2}}(\rho) \cap\{(x, y, z), z \geq \varepsilon \rho\} \\
\overline{\mathbb{E}}(\rho) & =\overline{\mathbb{S}^{2}}(\rho) \cap\{(x, y, z),|z| \leq \varepsilon \rho\}
\end{aligned}
$$

4.1. Construction of the volume preserving map $\mathcal{V}_{n}: \overline{\mathbb{S}}^{2}(r) \rightarrow \overline{\mathbb{K}}_{n}(r \sqrt[3]{\beta / \gamma}, \varepsilon)$

Next we fix $r>0, n \in \mathbb{N}, n \geq 3$, and we will investigate the possibility of construction of a map $\mathcal{V}_{n}: \overline{\mathbb{S}^{2}}(r) \rightarrow \overline{\mathbb{K}}_{n}(r \sqrt[3]{\beta / \gamma}, \varepsilon)$ which is volume preserving, i.e.

$$
\operatorname{vol}(D)=\operatorname{vol}\left(\mathcal{V}_{n}(D)\right), \quad \text { for all } D \subseteq \overline{\mathbb{S}^{2}}(r)
$$

We propose the following construction, which has two steps.
Step 1. We denote $\xi=\sqrt[3]{\beta / \gamma}$ and define the map $\mathcal{L}: \overline{\mathbb{S}^{2}}(r) \rightarrow \overline{\mathbb{S}^{2}}(\xi r)$,

$$
\mathcal{L}(x, y, z)=\xi \cdot(x, y, z), \text { for all }(x, y, z) \in \overline{\mathbb{S}^{2}}(r)
$$

Step 2. Let $(\widetilde{x}, \widetilde{y}, \widetilde{z}) \in \overline{\mathbb{S}^{2}}(r \xi)$. Then $(\widetilde{x}, \widetilde{y}, \widetilde{z}) \in \mathbb{S}^{2}(\widetilde{\rho})$, with $\widetilde{\rho}=\sqrt{\widetilde{x}^{2}+\widetilde{y}^{2}+\widetilde{z}^{2}}=$ $\xi \rho$. Now we consider the area preserving map $\mathcal{T}_{n, \tilde{\rho}}: \mathbb{S}^{2}(\widetilde{\rho}) \rightarrow \mathbb{K}_{n}(\widetilde{\rho}, \varepsilon)$ defined in Section 3 (in formulas (19)-(21), (22)-(24) and (34)-(36) we have to take $\tilde{\rho}$ instead of $r$ ).

Finally, we define the map $\mathcal{V}_{n}: \overline{\mathbb{S}^{2}}(r) \rightarrow \overline{\mathbb{K}}_{n}(r \sqrt[3]{\beta / \gamma}, \varepsilon)$ by

$$
\mathcal{V}_{n}(x, y, z)=\left(\mathcal{T}_{n, \tilde{\rho}} \circ \mathcal{L}\right)(x, y, z)=\left(\mathcal{T}_{n, \xi\left(x^{2}+y^{2}+z^{2}\right)^{1 / 2}} \circ \mathcal{L}\right)(x, y, z)
$$

Next we investigate wether the Jacobian of $\mathcal{V}_{n}$ ca be $\pm 1$.
Case 1. For $(x, y, z) \in \overline{\mathbb{S}^{+}}(r) \cap I_{0}^{+}$the formulas for $(X, Y, Z)=\mathcal{V}_{n}(x, y, z)$ are

$$
\begin{align*}
X & =\xi \sqrt{\frac{\rho(\rho-z)}{1-\varepsilon}}\left(\frac{\pi}{n \sin \frac{\pi}{n}}-\sin \frac{\pi}{n} \arctan \frac{y}{x}\right)  \tag{41}\\
Y & =\xi \sqrt{\frac{\rho(\rho-z)}{1-\varepsilon}} \cos \frac{\pi}{n} \arctan \frac{y}{x}  \tag{42}\\
Z & =\xi \rho\left(\varepsilon+c(\varepsilon)\left(1-\sqrt{\frac{1-z / \rho}{1-\varepsilon}}\right)\right) \tag{43}
\end{align*}
$$

where $\rho=\sqrt{x^{2}+y^{2}+z^{2}}$. If we evaluate the Jacobian of $\mathcal{V}_{n}$ restricted to $I_{0}^{+}$ we find

$$
J\left(\mathcal{V}_{n}\right)=\left|\begin{array}{lll}
\frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} & \frac{\partial X}{\partial z} \\
\frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} & \frac{\partial Y}{\partial z} \\
\frac{\partial Z}{\partial x} & \frac{\partial Z}{\partial y} & \frac{\partial Z}{\partial z}
\end{array}\right|=\frac{c(\varepsilon)+\varepsilon}{(1-\varepsilon)(c(\varepsilon)+3 \varepsilon)}>0 .
$$

If we impose the volume preserving condition $J\left(\mathcal{V}_{n}\right)=1$ we obtain the solution $\varepsilon=0$ for all $n \geq 3$. Another case when $J\left(\mathcal{V}_{n}\right)=1$ is when $c(\varepsilon)=2-3 \varepsilon$, which reduces to the equation

$$
\begin{equation*}
2-3 \varepsilon=\sqrt{4(1-\varepsilon)^{2}-\frac{\pi^{2}}{n^{2}} \cot ^{2} \frac{\pi}{n}} \tag{44}
\end{equation*}
$$

This equation gives solutions only in the case when $n=3,4,5$, and the solutions are
for $n=3$,

$$
\varepsilon=\frac{1}{45}\left(18-\sqrt{3\left(108-5 \pi^{2}\right)}\right), \text { i.e. } \varepsilon=0.105226 \ldots
$$

for $n=4$,

$$
\varepsilon_{1,2}=\frac{1}{20}\left(8 \pm \sqrt{\left.64-5 \pi^{2}\right)}\right), \text { i.e. } \varepsilon_{1}=0.20861 \ldots, \varepsilon_{2}=0.59139 \ldots
$$

for $n=5$,

$$
\varepsilon_{1,2}=\frac{1}{25}\left(10 \pm \sqrt{\left.100-5 \pi^{2}-2 \sqrt{5} \pi^{2}\right)}\right), \text { i.e. } \varepsilon_{1}=0.297912 \ldots, \varepsilon_{2}=0.502088 \ldots
$$

Case 2. For $(x, y, z) \in \overline{\mathbb{E}}(r) \cap I_{0}^{+}$the formulas for $(X, Y, Z)=\mathcal{V}_{n}(x, y, z)$ are

$$
\begin{aligned}
X & =\xi \rho\left(\frac{\pi}{n \sin \frac{\pi}{n}}-\sin \frac{\pi}{n} \arctan \frac{y}{x}\right) \\
Y & =\xi \rho \cos \frac{\pi}{n} \arctan \frac{y}{x} \\
Z & =\xi z
\end{aligned}
$$

In this case we calculate that

$$
J\left(\mathcal{V}_{n}\right)=\frac{2}{3 \varepsilon+c(\varepsilon)},
$$

and its value is 1 when $c(\varepsilon)=2-3 \varepsilon$.
Of course, it is immediate that the Jacobian is 1 for the whole ball $\overline{\mathbb{S}^{2}}(r)$.

Remark 5. The method that we have used above, which makes use of the area preserving map, can be applied only in the cases when all the polyhedrons $\mathbb{K}_{n}(\widetilde{\rho}, \varepsilon)$, with $\widetilde{\rho} \in(0, r \xi]$ admit inside a sphere which is tangent to all the faces. Indeed, if we split $\mathbb{K}_{n}(\widetilde{\rho}, \varepsilon)$ with $\varepsilon \neq 0$ into small pyramids with apex at the origin $O$ and bases the faces of $\mathbb{K}_{n}(\widetilde{\rho}, \varepsilon)$, there exists such a sphere if the distance from $O$ to a face of $\mathbb{P}_{n}^{+}$equals to the distance from $O$ to a face of $\mathbb{B}_{n}$, i.e.

$$
\begin{equation*}
\frac{\left(b_{n}+\varepsilon \widetilde{\rho}\right) \pi}{2(1-\varepsilon) n} \cot \frac{\pi}{n}=\frac{\pi \widetilde{\rho}}{n} \cot \frac{\pi}{n}=: d_{n} \tag{45}
\end{equation*}
$$

In this case, the sum of the volumes of the small pyramids is

$$
\operatorname{vol}\left(\mathbb{K}_{n}(\widetilde{\rho}, \varepsilon)\right)=\frac{d_{n} \mathcal{A}\left(\mathbb{K}_{n}(\widetilde{\rho}, \varepsilon)\right)}{3}=\frac{d_{n} \mathcal{A}\left(\mathbb{S}^{2}(\widetilde{\rho})\right)}{3}=\frac{d_{n} 4 \pi \widetilde{\rho}^{2}}{3}
$$

This equals the volume of the sphere $\mathbb{S}^{2}(\widetilde{\rho} / \xi)$ if $d_{n}=\widetilde{\rho} / \xi^{3}$, which reduces to $c(\varepsilon)=2-3 \varepsilon$.

On the other hand, condition (45) further gives $b_{n}=\widetilde{\rho}(2-3 \varepsilon)$, i.e. again $c(\varepsilon)=2-3 \varepsilon$. This condition is exactly the condition obtained by imposing $J\left(\mathcal{V}_{n}\right)=1$.

We remind also the construction in [12], where a similar method worked for the cube, which is also a polyhedron which can be circumscribed to a sphere.
4.2. The inverse $\mathcal{V}_{n}^{-1}: \overline{\mathbb{K}}_{n}(r \sqrt[3]{\beta / \gamma}, \varepsilon) \rightarrow \overline{\mathbb{S}^{2}}(r)$
4.2.1. The case $\varepsilon=0$

Let $(X, Y, Z) \in \overline{\mathbb{K}}_{n}(r \sqrt[3]{\beta / \gamma}, 0)$. First we have to find the domain $I_{i}^{s}$ which contains $(X, Y, Z)$. Suppose, for simplicity, that $(X, Y, Z) \in I_{0}^{+}$. We have to find $\bar{\rho} \in(0, r \xi]$ such that $(X, Y, Z) \in \mathbb{K}_{n}(\bar{\rho}, 0)$. The equation of the plane containing the face of $\mathbb{P}_{n}^{+} \cap I_{0}^{+}$is

$$
\begin{equation*}
X+Y \tan \frac{\pi}{n}+Z \frac{\pi}{n \sin \frac{\pi}{n} \sqrt{4-\frac{\pi^{2}}{n^{2}} \cot ^{2} \frac{\pi}{n}}}=\frac{\pi \bar{\rho}}{n \sin \frac{\pi}{n}} \tag{46}
\end{equation*}
$$

Therefore, when $(X, Y, Z)$ is given, then $\bar{\rho}$ can be calculated from formula (46). Further, $\mathcal{V}_{n}^{-1}(X, Y, Z)$ is defined as

$$
\begin{equation*}
\mathcal{V}_{n}^{-1}(X, Y, Z)=\left(\mathcal{L}^{-1} \circ \mathcal{T}_{n, \bar{\rho}}^{-1}\right)(X, Y, Z) \tag{47}
\end{equation*}
$$

which can be immediately be calculated using the formulas (25), (26), (27) for $\alpha_{i}=0, \varepsilon=0$, and $r=\bar{\rho}$. The formulas for the general case $(X, Y, Z) \in I_{i}^{ \pm}$ cand be deduced similarly using the same ideas as in Section 3.

### 4.2.2. The case $\varepsilon \neq 0$

For simplicity, suppose again that $(X, Y, Z) \in I_{0}^{+} \cup I_{0}^{-}$. In this case we have to find not only the radius $\bar{\rho}$ of the polyhedron $\mathbb{K}_{n}(\bar{\rho}, \varepsilon)$ which contains the point $(X, Y, Z)$, but we must also find whether $(X, Y, Z)$ belongs to the pyramid $\mathbb{P}_{n}^{+}(\bar{\rho}, \varepsilon)$ or to the prism $\mathbb{B}_{n}(\bar{\rho}, \varepsilon)$. First we observe that the pyramids $\mathbb{P}_{n}^{+}(\rho, \varepsilon), \rho>0$ are situated above the conical surfaces generated by lines that make a constant angle $\beta=\arccos \varepsilon$ with the axis $O Z$. This means that $(X, Y, Z) \in \mathbb{P}_{n}^{+}(\bar{\rho}, \varepsilon)$ if it satisfies the equations

$$
X^{2}+Y^{2} \leq\left(\frac{1}{\varepsilon^{2}}-1\right) Z^{2}, \quad Z \geq 0
$$

and $(X, Y, Z) \in \mathbb{P}_{n}^{-}(\bar{\rho}, \varepsilon)$ if it satisfies the first inequality and $Z \leq 0$. If

$$
X^{2}+Y^{2} \geq\left(\frac{1}{\varepsilon^{2}}-1\right) Z^{2}
$$

then $(X, Y, Z) \in \mathbb{B}_{n}(\bar{\rho}, \varepsilon)$.
Further we have to calculate $\bar{\rho}$. In the case when $(X, Y, Z) \in \mathbb{P}_{n}^{ \pm}(\bar{\rho}, \varepsilon)$, the determination of $\bar{\rho}$ was already described for the case $\varepsilon=0$, but in this case $\bar{\rho}$ is calculated from the formula

$$
X+Y \tan \frac{\pi}{n}+Z \frac{\pi}{c(\varepsilon) n \sin \frac{\pi}{n}}=\bar{\rho}\left(\frac{\varepsilon}{c(\varepsilon)}+1\right) \frac{\pi}{n \sin \frac{\pi}{n}}
$$

So it remains to consider the case $(X, Y, Z) \in \mathbb{B}_{n}(\bar{\rho}, \varepsilon)$. The equation of the vertical plane which contains $(X, Y, Z)$ can be written as

$$
\begin{equation*}
X+Y \tan \frac{\pi}{n}=\frac{\pi \bar{\rho}}{n \sin \frac{\pi}{n}} \tag{48}
\end{equation*}
$$

and this formula allows us to calculate the value of $\bar{\rho}$.
Finally, after the determination of $\bar{\rho}$, the inverse can be calculated with formula (47).

### 4.3. Transporting uniform and refinable grids from the polyhedrons $\overline{\mathbb{K}}_{n}\left(r^{\prime}, \varepsilon\right)$ onto the ball

The most important application of the volume preserving map $\mathcal{V}_{n}$ is that it allows the construction of uniform and refinable grids on the ball starting
from similar grids of the polyhedron. In a first step we construct $2 n$ tetrahedrons (triangular prisms), each having the vertex $O$ and base a face of the pyramids $\mathbb{P}_{n}^{ \pm}$. For the prism $\mathbb{B}_{n}$, we triangularize each face of $\mathbb{B}_{n}$ and then construct again $2 n$ tetrahedrons with vertex $O$ and base a triangle. In the second step, we split every of the $4 n$ tetrahedrons into four smaller tetrahedrons having the same volume, this simple procedure being described in [5]. The refinement can be repeated, therefore we end up with a uniform and refinable grid of $\overline{\mathbb{K}}_{n}\left(r^{\prime}, \varepsilon\right)$, whose image on the ball $\overline{S^{2}}(r)$ will be a uniform and refinable grid.

## References

[1] R. Alexander, On the sum of distances between $N$ points on the sphere, Acta. Math. Hungar., 23 (1972), 443-448.
[2] K. M. Górski, B. D. Wandelt, E. Hivon, A. J. Banday, B. D. Wandelt, F. K. Hansen, Reinecke, M. M. Bartelmann, HEALPix: A framework for high-resolution discretization and fast analysis of data distributed on the sphere, Astrophys. J., 622 (2005), 759-771.
[3] E. W. Grafared and F. W. Krumm, Map Projections, Cartographic Information Systems, Springer-Verlag, Berlin, 2006.
[4] A. Holhoş, D. Roşca, An octahedral equal area partition of the sphere and near optimal configurations of points, Comput. Math. Appl., vol. 67, 5 (2014), 1092-1107.
[5] A. Holhoş, D. Roşca, Equal-volume subdivisions of regular convex polyhedrons and balls, submitted.
[6] P. Leopardi, A partition of the unit sphere into regions of equal area and small diameter, Electron. Trans. Numer. Anal. 25 (2006), 309-327.
[7] D. Roşca, Locally supported rational spline wavelets on the sphere, Math. Comp. 74, 252 (2005), 1803-1829.
[8] D. Roşca, On a norm equivalence in $L^{2}\left(\mathbb{S}^{2}\right)$, Result. Math. 53, 3-4 (2009), 399-405.
[9] D. Roşca, New uniform grids on the sphere, Astron. Astrophys., 520 (2010), A63.
[10] D. Roşca, G. Plonka, Uniform spherical grids via equal area projection from the cube to the sphere, J. Comput. Appl. Math., 236, 6 (2011), 1033-1041.
[11] D. Roşca, G. Plonka, An area preserving projection from the regular octahedron to the sphere, Result. Math., 63, 2 (2012), 429-444.
[12] D. Roşca, A. Morawiec and M. De Graef, A new method of constructing a grid in the space of $3 D$ rotations and its applications to texture analysis, Modelling Simul. Mater. Sci. Eng. 22 (2014) 075013 (17pp).
[13] J. P. Snyder, An equal-area map projection for polyhedral globes, Cartographica: The International Journal for Geographic Information and Geovisualization, 29, 1 (1992), 10-21.
[14] L. Song, A. J. Kimerling, and K. Sahr, Developing an equal area global grid by small circle subdivision, in Discrete Global Grids, M. Goodchild and A. J. Kimerling, eds., National Center for Geographic Information \& Analysis, Santa Barbara, CA, USA, 2002.
[15] J. P. Snyder, Flattening the Earth, University of Chicago Press, 1990.
[16] M. Tegmark, An icosahedron-based method for pixelizing the celestial sphere, ApJ. Letters, 470 (1996), L81.
[17] N.A. Teanby, An icosahedron-based method for even binning of globally distributed remote sensing data, Comput. \& Geosci. 32, 9 (2006), 14421450.


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