Analysis of the grad-div stabilization for the time-dependent Navier–Stokes equations with inf-sup stable finite elements

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Abstract

This paper studies inf-sup stable finite element discretizations of the evolutionary Navier–Stokes equations with a grad-div type stabilization. The analysis covers both the case in which the solution is assumed to be smooth and consequently has to satisfy nonlocal compatibility conditions as well as the practically relevant situation in which the nonlocal compatibility conditions are not satisfied. The constants in the error bounds obtained do not depend on negative powers of the viscosity. Taking into account the loss of regularity suffered by the solution of the Navier–Stokes equations at the initial time in the absence of nonlocal compatibility conditions of the data, error bounds of order $\mathcal{O}(h^2)$ in space are proved. The analysis is optimal for quadratic/linear inf-sup stable pairs of finite elements. Both the continuous-in-time case and the fully discrete scheme with the backward Euler method as time integrator are analyzed.

Keywords Incompressible Navier–Stokes equations; inf-sup stable finite element methods; grad-div stabilization; error bounds independent of the viscosity; nonlocal compatibility condition; backward Euler method

1 Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded domain with polyhedral and Lipschitz boundary $\partial\Omega$. The incompressible Navier–Stokes equations model the conservation of linear momentum and the conservation of mass (continuity equation) by

$$\begin{aligned}
\partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } (0,T)\times\Omega, \\
\nabla\cdot\boldsymbol{u} &= 0 && \text{in } (0,T)\times\Omega, \\
\boldsymbol{u}(0,\cdot) &= \boldsymbol{u}_0(\cdot) && \text{in } \Omega,
\end{aligned} \tag{1}$$

where $\boldsymbol{u}$ is the velocity field, p the pressure, $\nu > 0$ the viscosity coefficient, $\boldsymbol{u}_0$ a given initial velocity, and $\boldsymbol{f}$ represents external forces acting on the fluid. The Navier–Stokes equations (1) are equipped with homogeneous Dirichlet boundary conditions $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$.

The interest of this paper is the case of small viscosity or, equivalently, high Reynolds number. To this end, a Galerkin finite element method augmented with a grad-div stabilization term for (1) is considered. Grad-div stabilization adds a penalty term with respect to the continuity equation to the momentum equation. It was originally proposed in [16] to improve the conservation of mass in finite element methods. There are a number of papers analyzing the grad-div stabilization for steady-state problems, e.g., [21, 27, 28]. On the one hand, it is known that while grad-div stabilization improves mass conservation, the computed finite element velocities are by far not divergence-free [24]. On the other hand, it was observed in the simulation of turbulent flows that using exclusively grad-div stabilization resulted in stable simulations, compare [23, Fig. 3] and [29, Fig. 7]. This observation is one of the motivations for the present paper: to derive error bounds for the Galerkin finite element method with grad-div stabilization whose constants do not depend on inverse powers of ν. The analysis will be performed for pairs of finite element spaces that satisfy a discrete inf-sup condition. Error bounds with constants independent of ν were previously obtained in [15] for the evolutionary Oseen equations. Contrary to the present paper, the wind velocity in the convective term of the Oseen equations is divergence-free and this property considerably simplifies the analysis. Besides extending the analysis from [15], more realistic conditions on the initial data are assumed in the present paper, conditions which affect the regularity near the initial time.

An analysis of inf-sup stable elements with divergence-free approximations of the Navier–Stokes equations is presented in [30]. There, error bounds independent of negative powers of ν were proved for the Galerkin method without any stabilization, utilizing ideas, e.g., from [15]. Adding a grad-div stabilization term as in the present paper allows the use of more general, not necessarily divergence-free, finite elements.

Some related works analyzing stabilized finite element approximations to the Navier–Stokes equations include [8], where the continuous interior penalty method is studied and [4, 12], where the local projection stabilization (LPS) method is studied. It is discussed in [26] that the case of the Navier–Stokes equations with grad-div stabilization but without LPS method can be considered as a special case of the analysis presented in [4]. Notice however that the error bounds in [4] depend explicitly on inverse powers of the viscosity parameter ν, unless grids are taken sufficiently fine ($h \lesssim \sqrt{\nu}$, where h is the mesh width), whereas this is not the case in

the present paper. In [7], error bounds for stabilized finite element approximations to the Navier–Stokes equations are obtained depending on an exponential factor proportional to the $L^\infty(\Omega)$ norm of the gradient of the large eddies instead of the gradient of the full velocity $\boldsymbol{u}$ in the case that Ω is the unit square and the boundary conditions are periodic. An analysis of a fully discrete method based on LPS in space and the Euler method in time is carried out in [3]. The error bounds in [3] are not independent of negative powers of ν. In all these papers, some stabilization terms are added to the Galerkin formulation. In particular, all these methods, save the method studied in [26], include a stabilization for the convective term. The aim of the present paper consists in deriving error bounds that are independent of inverse powers of the viscosity parameter for finite element approximations that do not include a stabilization of the convective term.

In the present paper, optimal error bounds with constants that do not depend explicitly on inverse powers of the viscosity parameter will be obtained for the $L^2(\Omega)$ norm of the divergence of the velocity, which measures the closeness of the velocity approximation of being divergence-free, and the $L^2(\Omega)$ norm of the pressure, assuming that the solution is sufficiently smooth. In addition, an error bound for $\nu^{1/2}$ times the gradient of the velocity is proved. This error bound is optimal in the viscosity-dominated regime, although it is a weak term in the convection-dominated regime. Note that all error bounds might depend implicitly on the viscosity through the dependency on higher order Sobolev norms of the solution of the continuous problem.

In Section 3, it will be assumed that the solution satisfies nonlocal compatibility conditions. The analysis is valid for pairs of inf-sup stable mixed finite elements of any degree. In the case of first order mixed finite elements, the error bound for the pressure can be proved only in two spatial dimensions.

Due to the increasing use of higher order methods in computational fluid dynamics, the question of optimal approximation of the Navier–Stokes equations under realistic assumptions of the data has become important. The regularity customarily hypothesized in the error analysis for parabolic problems generally cannot be expected for the Navier–Stokes equations. No matter how regular the initial data are, solutions of the Navier–Stokes equations cannot be assumed to have more than second order spatial derivatives bounded in $L^2(\Omega)$ up to the initial time $t = 0$. Higher regularity requires the solution to satisfy some nonlocal compatibility conditions that are unlikely to be fulfilled in practical situations [18, 19]. Taking into account this loss of regularity at $t = 0$ locally in time, the optimal rate of convergence of the grad-div mixed finite element method is studied in Section 4. The analysis of [3, 4, 8, 7, 30] assumes that the solution satisfies nonlocal compatibility conditions. To the best of our knowledge, the present paper is the first one where error bounds independent of the viscosity parameter are proved without those assumptions, and the best bounds that we obtain are not better than $\mathcal{O}(h^2)$. In the literature, [13, 14, 18, 19], error bounds up to $\mathcal{O}(h^5|\log(h)|)$ have been obtained for both standard and two-grid mixed finite element methods without assuming nonlocal compatibility conditions. But contrary to the $\mathcal{O}(h^2)$ bounds in the present paper, the error constants in those $\mathcal{O}(h^5|\log(h)|)$ bounds depend on ν^{-1}.

In Section 5, the analysis of the fully discrete case is presented. For the time integration, the implicit Euler method is considered. Again, both the regular case

and the case in which nonlocal compatibility conditions are not assumed are analyzed. In this last case, the errors are shown to be $\mathcal{O}(h^2|\log(\Delta t)|^{1/2} + (\Delta t)^{1/2})$, where Δt is the size of the time step.

Section 6 provides numerical studies supporting the analytical results and a summary finishes the paper.

2   Preliminaries and notation

Throughout the paper, $W^{s,p}(D)$ will denote the Sobolev space of real-valued functions defined on the domain $D \subset \mathbb{R}^d$ with distributional derivatives of order up to s in $L^p(D)$. These spaces are endowed with the usual norm denoted by $\|\cdot\|_{W^{s,p}(D)}$. If s is not a positive integer, $W^{s,p}(D)$ is defined by interpolation [1]. In the case $s = 0$, it is $W^{0,p}(D) = L^p(D)$. As it is standard, $W^{s,p}(D)^d$ will be endowed with the product norm and, since no confusion can arise, it will be denoted again by $\|\cdot\|_{W^{s,p}(D)}$. The case $p = 2$ will be distinguished by using $H^s(D)$ to denote the space $W^{s,2}(D)$. The space $H_0^1(D)$ is the closure in $H^1(D)$ of the set of infinitely differentiable functions with compact support in D. For simplicity, $\|\cdot\|_s$ (resp. $|\cdot|_s$) is used to denote the norm (resp. seminorm) both in $H^s(\Omega)$ or $H^s(\Omega)^d$. The exact meaning will be clear by the context. The inner product of $L^2(\Omega)$ or $L^2(\Omega)^d$ will be denoted by $(\cdot,\cdot)$ and the corresponding norm by $\|\cdot\|_0$. The norm of the space of essentially bounded functions $L^\infty(\Omega)$ will be denoted by $\|\cdot\|_\infty$. For vector-valued functions, the same conventions will be used as before. The norm of the dual space $H^{-1}(\Omega)$ of $H_0^1(\Omega)$ is denoted by $\|\cdot\|_{-1}$. As usual, $L^2(\Omega)$ is always identified with its dual, so one has $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$ with compact injection.

Using the function spaces $V = H_0^1(\Omega)^d$, and

$$Q = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) : (q, 1) = 0 \right\},$$

the weak formulation of problem (1) is as follows: Find $(\boldsymbol{u}, p) \in V \times Q$ such that for all $(\boldsymbol{v}, q) \in V \times Q$,

$$(\partial_t \boldsymbol{u}, \boldsymbol{v}) + \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + ((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}, p) + (\nabla \cdot \boldsymbol{u}, q) = (\boldsymbol{f}, \boldsymbol{v}). \tag{2}$$

The Hilbert space

$$H^{\mathrm{div}} = \{ \boldsymbol{u} \in L^2(\Omega)^d \mid \nabla \cdot \boldsymbol{u} = 0, \ \boldsymbol{u} \cdot \mathbf{n}|_{\partial\Omega} = 0 \}$$

will be endowed with the inner product of $L^2(\Omega)^d$ and the space

$$V^{\mathrm{div}} = \{ \boldsymbol{u} \in V \mid \nabla \cdot \boldsymbol{u} = 0 \}$$

with the inner product of V.

Let $\Pi : L^2(\Omega)^d \to H^{\mathrm{div}}$ be the Leray projector that maps each function in $L^2(\Omega)^d$ onto its divergence-free part (see e.g. [11, Chapter IV]. The Stokes operator in Ω is given by

$$A : \mathcal{D}(A) \subset H^{\mathrm{div}} \to H^{\mathrm{div}}, \quad A = -\Pi\Delta, \quad \mathcal{D}(A) = H^2(\Omega)^d \cap V^{\mathrm{div}}.$$

The following Sobolev's embedding [1] will be used in the analysis: For $1 \le p < d/s$ let q be such that $\frac{1}{q} = \frac{1}{p} - \frac{s}{d}$. There exists a positive constant C, independent of s, such that

$$\|v\|_{L^{q'}(\Omega)} \le C\|v\|_{W^{s,p}(\Omega)}, \quad \frac{1}{q'} \ge \frac{1}{q}, \quad v \in W^{s,p}(\Omega). \tag{3}$$

If $p > d/s$ the above relation is valid for $q' = \infty$. A similar embedding inequality holds for vector-valued functions.

Let $V_h \subset V$ and $Q_h \subset Q$ be two families of finite element spaces composed of piecewise polynomials of degrees at most k and l, respectively, that correspond to a family of partitions $\mathcal{T}_h$ of Ω into mesh cells with maximal diameter h. In this paper, we will only consider pairs of finite element spaces satisfying the discrete inf-sup condition,

$$\inf_{q_h \in Q_h} \sup_{\boldsymbol{v}_h \in V_h} \frac{(\nabla \cdot \boldsymbol{v}_h, q_h)}{\|\nabla \boldsymbol{v}_h\|_0 \|q_h\|_0} \geq \beta_0, \tag{4}$$

with $\beta_0 > 0$, a constant independent of the mesh size h. For example, for the MINI element it is $k = l = 1$ and for the Hood–Taylor element one has $l = k-1$. Since the error bounds for the pressure depend both on the mixed finite element used and on the regularity of the solution, and in general it will be assumed that $p \in Q \cap H^k(\Omega)$ with $l \geq k-1$, in the sequel the error bounds will be written depending only on k.

It will be assumed that the family of meshes is quasi-uniform and that the following inverse inequality holds for each $v_h \in V_h$, see e.g., [10, Theorem 3.2.6],

$$\|\boldsymbol{v}_h\|_{W^{m,p}(K)} \leq C_{\mathrm{inv}} h_K^{n-m-d\left(\frac{1}{q}-\frac{1}{p}\right)} \|\boldsymbol{v}_h\|_{W^{n,q}(K)}, \tag{5}$$

where $0 \leq n \leq m \leq 1$, $1 \leq q \leq p \leq \infty$, and h_K is the size (diameter) of the mesh cell $K \in \mathcal{T}_h$.

The space of discrete divergence-free functions is denoted by

$$V_h^{\mathrm{div}} = \{\boldsymbol{v}_h \in V_h \mid (\nabla \cdot \boldsymbol{v}_h, q_h) = 0 \quad \forall q_h \in Q_h\},$$

and by $A_h : V_h^{\mathrm{div}} \to V_h^{\mathrm{div}}$ is denoted the following linear operator

$$(A_h \boldsymbol{v}_h, \boldsymbol{w}_h) = (\nabla \boldsymbol{v}_h, \nabla \boldsymbol{w}_h) \quad \forall \boldsymbol{v}_h, \boldsymbol{w}_h \in V_h^{\mathrm{div}}. \tag{6}$$

Note that from this definition, it follows for $\boldsymbol{v}_h \in V_h^{\mathrm{div}}$ that

$$\|A_h^{1/2} \boldsymbol{v}_h\|_0 = \|\nabla \boldsymbol{v}_h\|_0, \quad \|\nabla A_h^{-1/2} \boldsymbol{v}_h\|_0 = \|\boldsymbol{v}_h\|_0.$$

Additionally, two linear operators $C_h : V_h^{\mathrm{div}} \to V_h^{\mathrm{div}}$ and $D_h : L^2(\Omega) \to V_h^{\mathrm{div}}$ are defined by

$$(C_h \boldsymbol{v}_h, \boldsymbol{w}_h) = (\nabla \cdot \boldsymbol{v}_h, \nabla \cdot \boldsymbol{w}_h) \quad \forall \boldsymbol{v}_h, \boldsymbol{w}_h \in V_h^{\mathrm{div}}, \tag{7}$$

$$(D_h p, \boldsymbol{v}_h) = -(p, \nabla \cdot \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in V_h^{\mathrm{div}}. \tag{8}$$

In what follows, $\Pi_h^{\mathrm{div}} : L^2(\Omega)^d \to V_h^{\mathrm{div}}$ will denote the so-called discrete Leray projection, which is the orthogonal projection of $L^2(\Omega)^d$ onto V_h^{div}

$$\left(\Pi_h^{\mathrm{div}} \boldsymbol{v}, \boldsymbol{w}_h\right) = (\boldsymbol{v}, \boldsymbol{w}_h) \quad \forall \boldsymbol{w}_h \in V_h^{\mathrm{div}}. \tag{9}$$

By definition, it is clear that the projection is stable in the $L^2(\Omega)^d$ norm: $\|\Pi_h^{\mathrm{div}} \boldsymbol{v}\|_0 \leq \|\boldsymbol{v}\|_0$ for all $\boldsymbol{v} \in L^2(\Omega)^d$. The following well-known bound will be used

$$\|(I - \Pi_h^{\mathrm{div}})\boldsymbol{v}\|_0 + h\|(I - \Pi_h^{\mathrm{div}})\boldsymbol{v}\|_1 \leq Ch^{j+1}\|\boldsymbol{v}\|_{j+1} \quad \forall \boldsymbol{v} \in V^{\mathrm{div}} \cap H^{j+1}(\Omega)^d, \tag{10}$$

for $j = 0, \ldots, k$.

Denoting by π_h the $L^2(\Omega)$ projection onto Q_h, one has that for $0 \le m \le 1$

$$\|q - \pi_h q\|_m \le Ch^{j+1-m}\|q\|_{j+1} \quad \forall q \in H^{j+1}(\Omega), \quad j = 0, \ldots, l. \tag{11}$$

For simplicity of presentation, the notation π_h will be used instead of $\pi_h p$ for the pressure p in (1).

In the error analysis, the Poincaré–Friedrichs inequality

$$\|\boldsymbol{v}\|_0 \le C\|\nabla \boldsymbol{v}\|_0 \quad \forall \boldsymbol{v} \in H_0^1(\Omega)^d, \tag{12}$$

will be used.

In the sequel, $I_h \boldsymbol{u} \in V_h$ will denote the Lagrange interpolant of a continuous function $\boldsymbol{u}$. The following bound can be found in [6, Theorem 4.4.4]

$$|\boldsymbol{u} - I_h\boldsymbol{u}|_{W^{m,p}(K)} \le c_{\text{int}} h^{n-m}|\boldsymbol{u}|_{W^{n,p}(K)}, \quad 0 \le m \le n \le k+1, \tag{13}$$

where $n > d/p$ when $1 < p \le \infty$ and $n \ge d$ when $p = 1$.

In the analysis, the Stokes problem

$$\begin{aligned} -\nu \Delta \boldsymbol{u} + \nabla p &= \boldsymbol{g} \quad \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \partial\Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 \quad \text{in } \Omega, \end{aligned} \tag{14}$$

will be considered. Let us denote by $(\boldsymbol{u}_h, p_h) \in V_h \times Q_h$ the mixed finite element approximation to (14), given by

$$\begin{aligned} \nu(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) - (\nabla \cdot \boldsymbol{v}_h, p_h) &= (\boldsymbol{g}, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in V_h, \\ (\nabla \cdot \boldsymbol{u}_h, q_h) &= 0 \qquad \forall q_h \in Q_h. \end{aligned}$$

Following [17, 22], one gets the estimates

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 \le C \left(\inf_{\boldsymbol{v}_h \in V_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_1 + \nu^{-1} \inf_{q_h \in Q_h} \|p - q_h\|_0 \right), \tag{15}$$

$$\|p - p_h\|_0 \le C \left(\nu \inf_{\boldsymbol{v}_h \in V_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_1 + \inf_{q_h \in Q_h} \|p - q_h\|_0 \right), \tag{16}$$

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_0 \le Ch \left(\inf_{\boldsymbol{v}_h \in V_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_1 + \nu^{-1} \inf_{q_h \in Q_h} \|p - q_h\|_0 \right). \tag{17}$$

It can be observed that the error bounds for the velocity depend on negative powers of ν.

For the analysis, it will be advantageous to use a projection of $(\boldsymbol{u}, p)$ into $V_h \times Q_h$ with uniform in ν, optimal, bounds for the velocity. In [15] a projection with this property was introduced. Let $(\boldsymbol{u}, p)$ be the solution of the Navier–Stokes equations (1) with $\boldsymbol{u} \in V \cap H^{k+1}(\Omega)^d$, $p \in Q \cap H^k(\Omega)$, $k \ge 1$, and observe that $(\boldsymbol{u}, 0)$ is the solution of the Stokes problem (14) with right-hand side

$$\boldsymbol{g} = \boldsymbol{f} - \partial_t \boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nabla p. \tag{18}$$

6

Denoting the corresponding Galerkin approximation in $V_h \times Q_h$ by $(\boldsymbol{s}_h, l_h)$, one obtains from (15)–(17)

$$\|\boldsymbol{u} - \boldsymbol{s}_h\|_0 + h\|\boldsymbol{u} - \boldsymbol{s}_h\|_1 \le Ch^{j+1}\|\boldsymbol{u}\|_{j+1}, \quad 0 \le j \le k, \tag{19}$$

$$\|l_h\|_0 \le C\nu h^j \|\boldsymbol{u}\|_{j+1}, \quad 0 \le j \le k, \tag{20}$$

where the constant C does not depend on ν.

Remark 1 Assuming the necessary smoothness in time and considering (14) with

$$\boldsymbol{g} = \partial_t \left(\boldsymbol{f} - \partial_t \boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nabla p \right),$$

one can derive an error bound of the form (19) also for $\partial_t(\boldsymbol{u} - \boldsymbol{s}_h)$. One can proceed similarly for higher order derivatives in time. In Section 4, where boundedness of derivatives up to $t = 0$ is not assumed, the bound (19) is also valid, but then the quantities assumed to be bounded up to $t = 0$ are $t^{(j-1)/2}\|\boldsymbol{u}(t)\|_{j+1}$, $t^{(j+1)/2}\|\partial_t\boldsymbol{u}(t)\|_{j+1}$, $t^{(j+3)/2}\|\partial_{tt}\boldsymbol{u}(t)\|_{j+1}$, etc. Note that for a given $t_0 > 0$, the assumptions in the present section hold for $t \ge t_0$, and those of Section 4 for $0 \le t \le t_0$.

Following [9], one can also obtain the following bound for $\boldsymbol{s}_h$

$$\|\nabla(\boldsymbol{u} - \boldsymbol{s}_h)\|_\infty \le C\|\nabla\boldsymbol{u}\|_\infty, \tag{21}$$

where C does not depend on ν.

The method that will be studied for the approximation of the solution of the Navier–Stokes equations (1) is obtained by adding to the Galerkin equations a control of the divergence constraint (grad-div stabilization). More precisely, the following grad-div method will be considered: Find $(\boldsymbol{u}_h, p_h) : (0, T] \to V_h \times Q_h$ such that

$$\begin{aligned}
(\partial_t \boldsymbol{u}_h, \boldsymbol{v}_h) + \nu(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) + b(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) - (p_h, \nabla \cdot \boldsymbol{v}_h,) \\
+ (\nabla \cdot \boldsymbol{u}_h, q_h) + \mu(\nabla \cdot \boldsymbol{u}_h, \nabla \cdot \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h),
\end{aligned} \tag{22}$$

for all $(\boldsymbol{v}_h, q_h) \in V_h \times Q_h$, with $\boldsymbol{u}_h(0) = I_h\boldsymbol{u}_0$. Here, and in the rest of the paper,

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = (B(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w}) \quad \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in H_0^1(\Omega)^d,$$

where,

$$B(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} + \frac{1}{2}(\nabla \cdot \boldsymbol{u})\boldsymbol{v} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in H_0^1(\Omega)^d$$

Notice the well-known property

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) \quad \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V, \tag{23}$$

such that, in particular, $b(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{w}) = 0$ for all $\boldsymbol{u}, \boldsymbol{w} \in V$.

3 The regular continuous-in-time case

In this section, error bounds for the continuous-in-time discretization are derived for the case in which regularity up to time $t = 0$ is assumed. Some of the lemmas are written in such a way that can also be applied in Section 4 for the analysis of the situation without compatibility assumptions.

3.1 Error bound for the velocity

Using test functions in V_h^{div} and applying definitions (6)–(9), one finds that (22) implies that $\boldsymbol{u}_h$ satisfies

$$\partial_t \boldsymbol{u}_h + \nu A_h \boldsymbol{u}_h + B_h(\boldsymbol{u}_h, \boldsymbol{u}_h) + \mu C_h \boldsymbol{u}_h = \Pi_h^{\text{div}} \boldsymbol{f}, \tag{24}$$

where

$$B_h(\boldsymbol{u}, \boldsymbol{v}) = \Pi_h^{\text{div}} B(\boldsymbol{u}, \boldsymbol{v}), \quad \boldsymbol{u}, \boldsymbol{v} \in H_0^1(\Omega)^d,$$

and C_h is defined in (7). Notice that Π_h^{div} can be extended from $L^2(\Omega)^d$ to $H^{-1}(\Omega)$ in such a way that $B_h(\boldsymbol{u}, \boldsymbol{v})$ is well defined for $\boldsymbol{u}, \boldsymbol{v} \in H_0^1(\Omega)^d$.

The following two lemmas will be used in Sections 3 and 4.

Lemma 1 *Let* $\boldsymbol{w}_h : [0, T] \to V_h^{\text{div}}$ *be an arbitrary function piecewise differentiable with respect time. Let* $\boldsymbol{u}_h$ *be the mixed finite element approximation to the velocity defined in (24). Define the truncation errors* $\boldsymbol{t}_{1,h} : [0, T] \to V_h^{\text{div}}$ *and* $t_2 : [0, T] \to L^2(\Omega)$ *such that the following equation is satisfied*

$$\partial_t \boldsymbol{w}_h + \nu A_h \boldsymbol{w}_h + B_h(\boldsymbol{w}_h, \boldsymbol{w}_h) + \mu C_h \boldsymbol{w}_h = \Pi_h^{\text{div}} \boldsymbol{f} + \boldsymbol{t}_{1,h} - D_h t_2, \tag{25}$$

where D_h *has been defined in (8). Then, if the function*

$$g(t) = 1 + 2\|\nabla \boldsymbol{w}_h(t)\|_\infty + \frac{\|\boldsymbol{w}_h(t)\|_\infty^2}{2\mu} \tag{26}$$

is integrable in $(0, T)$, *i.e.,* $\nabla \boldsymbol{w}_h \in L^1(0, T; L^\infty)$ *and* $\boldsymbol{w}_h \in L^2(0, T; L^\infty)$, *the error* $\boldsymbol{e}_h = \boldsymbol{u}_h - \boldsymbol{w}_h$ *can be bounded as follows*

$$\|\boldsymbol{e}_h(t)\|_0^2 + \int_0^t e^{K(t,s)} \left(2\nu \|\nabla \boldsymbol{e}_h(s)\|_0^2 + \mu \|\nabla \cdot \boldsymbol{e}_h(s)\|_0^2 \right) \, ds$$

$$\leq e^{K(t,0)} \|\boldsymbol{e}_h(0)\|_0^2 + \int_0^t e^{K(t,s)} \left(\|\boldsymbol{t}_{1,h}\|_0^2 + \frac{2}{\mu} \|t_2\|_0^2 \right) \, ds,$$

where

$$K(t, s) = \int_s^t \left(1 + 2\|\nabla \boldsymbol{w}_h\|_\infty + \frac{\|\boldsymbol{w}_h\|_\infty^2}{2\mu} \right) \, dr.$$

Proof Subtracting (24) from (25), taking the inner product with $\boldsymbol{e}_h \in V_h^{\text{div}}$, and performing some standard computations yields

$$\frac{1}{2} \frac{d}{dt} \|\boldsymbol{e}_h\|_0^2 + \nu \|\nabla \boldsymbol{e}_h\|_0^2 + \mu \|\nabla \cdot \boldsymbol{e}_h\|_0^2 + b(\boldsymbol{w}_h, \boldsymbol{w}_h, \boldsymbol{e}_h) - b(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{e}_h)$$
$$= (\boldsymbol{t}_{1,h}, \boldsymbol{e}_h) + (t_2, \nabla \cdot \boldsymbol{e}_h). \tag{27}$$

Observe that

$$b(\boldsymbol{w}_h, \boldsymbol{w}_h, \boldsymbol{e}_h) - b(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{e}_h) = -b(\boldsymbol{e}_h, \boldsymbol{w}_h, \boldsymbol{e}_h) - b(\boldsymbol{u}_h, \boldsymbol{e}_h, \boldsymbol{e}_h) = -b(\boldsymbol{e}_h, \boldsymbol{w}_h, \boldsymbol{e}_h), \tag{28}$$

where in the last step it was used that, due to (23), $b(\boldsymbol{u}_h, \boldsymbol{e}_h, \boldsymbol{e}_h) = 0$. Applying Hölder's inequality one finds

$$
|b(\boldsymbol{e}_h, \boldsymbol{w}_h, \boldsymbol{e}_h)| \leq \|\nabla \boldsymbol{w}_h\|_\infty \|\boldsymbol{e}_h\|_0^2 + \frac{1}{2}\|\nabla \cdot \boldsymbol{e}_h\|_0 \|\boldsymbol{w}_h\|_\infty \|\|\boldsymbol{e}_h\|_0
$$

$$
\leq \|\nabla \boldsymbol{w}_h\|_\infty \|\boldsymbol{e}_h\|_0^2 + \frac{\mu}{4}\|\nabla \cdot \boldsymbol{e}_h\|_0^2 + \frac{\|\boldsymbol{w}_h\|_\infty^2}{4\mu}\|\boldsymbol{e}_h\|_0^2. \tag{29}
$$

Thus, from (27), using the Cauchy–Schwarz and Young's inequalities, taking into account the definition of function g in (26), and rearranging terms, it follows that

$$
\frac{d}{dt}\|\boldsymbol{e}_h\|_0^2 + 2\nu\|\nabla \boldsymbol{e}_h\|_0^2 + \mu\|\nabla \cdot \boldsymbol{e}_h\|_0^2 \leq g\|\boldsymbol{e}_h\|_0^2 + \|\boldsymbol{t}_{1,h}\|_0^2 + \frac{2}{\mu}\|t_2\|_0^2.
$$

Multiplying by the integrating factor $\exp(-K(t,0))$ and integrating in time, the result follows in a standard way. $\qquad\square$

The following lemma will be used in the proof of the main results of the paper.

Lemma 2 *There exists a positive constant C such that for any $\boldsymbol{w}_h \in V_h$ and $\boldsymbol{v} \in V \cap H^2(\Omega)^d$ the following bound holds*

$$
\|B(\boldsymbol{w}_h, \boldsymbol{w}_h) - B(\boldsymbol{v}, \boldsymbol{v})\|_0 \leq C\left(\|\boldsymbol{w}_h\|_\infty + \|\nabla \cdot \boldsymbol{w}_h\|_{L^{2d/(d-1)}(\Omega)} + \|\boldsymbol{v}\|_2\right)\|\boldsymbol{w}_h - \boldsymbol{v}\|_1.
$$

Proof Since $V_h \subset L^\infty(\Omega)$ and by the well known Sobolev embedding $H^2(\Omega) \subset L^\infty(\Omega)$ (see e.g., [1]), it follows that $B(\boldsymbol{w}_h, \boldsymbol{w}_h), B(\boldsymbol{v}, \boldsymbol{v}) \in L^2(\Omega)$. Then, the application of the Hölder inequality yields

$$
\|B(\boldsymbol{w}_h, \boldsymbol{w}_h) - B(\boldsymbol{v}, \boldsymbol{v})\|_0 = \|B(\boldsymbol{w}_h, \boldsymbol{w}_h - \boldsymbol{v}) + B(\boldsymbol{w}_h - \boldsymbol{v}, \boldsymbol{v})\|_0
$$

$$
\leq \|\boldsymbol{w}_h\|_\infty \|\boldsymbol{w}_h - \boldsymbol{v}\|_1 + \frac{1}{2}\|\nabla \cdot \boldsymbol{w}_h\|_{L^{2d/(d-1)}(\Omega)}\|\boldsymbol{w}_h - \boldsymbol{v}\|_{L^{2d}(\Omega)}
$$

$$
+ \|\boldsymbol{w}_h - \boldsymbol{v}\|_{L^{2d}(\Omega)}\|\nabla \boldsymbol{v}\|_{L^{2d/(d-1)}(\Omega)} + \frac{1}{2}\|\nabla \cdot (\boldsymbol{w}_h - \boldsymbol{v})\|_0\|\boldsymbol{v}\|_\infty.
$$

The statement of the lemma follows from (3). $\qquad\square$

The proof of the error estimate is based on the comparison of the Galerkin approximation to the velocity $\boldsymbol{u}_h$ in (22) with the approximation $\boldsymbol{s}_h$ defined at the end of Section 2. The pair $(\boldsymbol{s}_h, l_h) \in V_h \times Q_h$ solves

$$
\nu(\nabla \boldsymbol{s}_h, \nabla \boldsymbol{v}_h) - (l_h, \nabla \cdot \boldsymbol{v}_h) - (\nabla \cdot \boldsymbol{s}_h, q_h)
$$
$$
= (\boldsymbol{f}, \boldsymbol{v}_h) - (\partial_t \boldsymbol{u}, \boldsymbol{v}_h) - b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_h) + (p, \nabla \cdot \boldsymbol{v}_h). \tag{30}
$$

Adding and subtracting terms gives

$$
\partial_t \boldsymbol{s}_h + \nu A_h \boldsymbol{s}_h + B_h(\boldsymbol{s}_h, \boldsymbol{s}_h) + \mu C_h \boldsymbol{s}_h = \Pi_h^{\mathrm{div}}\boldsymbol{f} - \Pi_h^{\mathrm{div}}(\partial_t \boldsymbol{u} - \partial_t \boldsymbol{s}_h)
$$
$$
- (B_h(\boldsymbol{u}, \boldsymbol{u}) - B_h(\boldsymbol{s}_h, \boldsymbol{s}_h)) - D_h(p - \pi_h) + \mu C_h(\boldsymbol{s}_h).
$$

Taking into account (7) and $\nabla \cdot \boldsymbol{u} = 0$, one can see that Lemma 1 can be applied with $\boldsymbol{w}_h = \boldsymbol{s}_h$, $\boldsymbol{t}_{1,h} = \Pi_h^{\mathrm{div}}(\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2)$, and $t_2 = \tau_3 + \tau_4$, where

$$
\boldsymbol{\tau}_1 = \partial_t \boldsymbol{u} - \partial_t \boldsymbol{s}_h, \quad \boldsymbol{\tau}_2 = B(\boldsymbol{u}, \boldsymbol{u}) - B(\boldsymbol{s}_h, \boldsymbol{s}_h),
$$
$$
\tau_3 = p - \pi_h, \qquad \tau_4 = \mu(\nabla \cdot (\boldsymbol{s}_h - \boldsymbol{u})). \tag{31}
$$

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Let $\boldsymbol{u}$ satisfy the hypothesis in Theorem 1 below. In order to apply Lemma 1, the integrability in $(0, T)$ of the function g defined in (26), with $\boldsymbol{w}_h = \boldsymbol{s}_h$, has to be proved. To this end, it will be shown that the two terms $\|\boldsymbol{s}_h\|_\infty^2$ and $\|\nabla \boldsymbol{s}_h\|_\infty$ are bounded by an integrable function in $(0, T)$. For the latter, one can simply apply (21). For the former term, one first observes that from the assumed regularity of $\boldsymbol{u}$ it follows that $\boldsymbol{u}$ is continuous and, hence, $\|I_h(\boldsymbol{u})\|_\infty \leq C\|\boldsymbol{u}\|_\infty$ for some $C > 0$. Then, one can write

$$\|\boldsymbol{s}_h\|_\infty \leq \|\boldsymbol{s}_h - I_h(\boldsymbol{u})\|_\infty + \|I_h(\boldsymbol{u})\|_\infty \leq C_{\text{inv}} h^{-d/2} \|\boldsymbol{s}_h - I_h(\boldsymbol{u})\|_0 + \|\boldsymbol{u}\|_\infty,$$

where in the last inequality inverse inequality (5) has been applied. Applying (13), (19), and (3), one gets

$$\|\boldsymbol{s}_h\|_\infty \leq C\|\boldsymbol{u}\|_2, \quad \|\nabla \boldsymbol{s}_h\|_\infty \leq C\|\nabla \boldsymbol{u}\|_\infty, \tag{32}$$

where the constants are independent of ν.

Thus, by applying Lemma 1 with $\boldsymbol{e}_h = \boldsymbol{u}_h - \boldsymbol{s}_h$, one obtains

$$\|\boldsymbol{e}_h(t)\|_0^2 + \int_0^t e^{K(t,s)} \left(2\nu \|\nabla \boldsymbol{e}_h(s)\|_0^2 + \mu \|\nabla \cdot \boldsymbol{e}_h(s)\|_0^2 \right) \, ds$$

$$\leq e^{K(t,0)} \|\boldsymbol{e}_h(0)\|_0^2 + \int_0^t e^{K(t,s)} \left(\|\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2\|_0^2 + \frac{2}{\mu} \|\tau_3 + \tau_4\|_0^2 \right) \, ds. \tag{33}$$

From (11) and (19) (see also Remark 1) one gets

$$\|\tau_3 + \tau_4\|_0^2 \leq C h^{2k} (\|p\|_k^2 + \mu^2 \|\boldsymbol{u}\|_{k+1}^2), \tag{34}$$

and

$$\|\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2\|_0^2 \leq C h^{2k} \|\partial_t \boldsymbol{u}\|_k^2 + 2 \|\boldsymbol{\tau}_2\|_0^2. \tag{35}$$

For $\boldsymbol{\tau}_2$, the application of Lemma 2 gives

$$\|\boldsymbol{\tau}_2\|_0 \leq C \left(\|\boldsymbol{s}_h\|_\infty + \|\nabla \cdot \boldsymbol{s}_h\|_{L^{2d/(d-1)}(\Omega)} + \|\boldsymbol{u}\|_2 \right) \|\boldsymbol{u} - \boldsymbol{s}_h\|_1. \tag{36}$$

To bound $\|\nabla \cdot \boldsymbol{s}_h\|_{L^{2d/(d-1)}(\Omega)}$, one finds with the inverse inequality (5) that

$$\|\nabla \cdot \boldsymbol{s}_h\|_{L^{2d/(d-1)}(\Omega)} \leq C h^{-1/2} \|\nabla \cdot \boldsymbol{s}_h\|_0,$$

and with (19) it follows that

$$\|\nabla \cdot \boldsymbol{s}_h\|_{L^{2d/(d-1)}(\Omega)} \leq C h^{-1/2} \|\nabla \cdot (\boldsymbol{u} - \boldsymbol{s}_h)\|_0 \leq C h^{1/2} \|\boldsymbol{u}\|_2. \tag{37}$$

Altogether, from (36), using also (32) and that h is bounded (at least from the diameter of Ω), one obtains

$$\|\boldsymbol{\tau}_2\|_0 \leq C \|\boldsymbol{u}\|_2 \|\boldsymbol{u} - \boldsymbol{s}_h\|_1 \leq C h^k \|\boldsymbol{u}\|_2 \|\boldsymbol{u}\|_{k+1}. \tag{38}$$

In view of (32), one has $1 \leq \exp(K(t,s)) \leq C \exp(L(T))$ with

$$L(T) = \int_0^T \left(1 + 2 \|\nabla \boldsymbol{u}(s)\|_\infty + \frac{\|\boldsymbol{u}(s)\|_2^2}{2\mu} \right) \, ds, \tag{39}$$

where C is independent of ν. From (33), (34)–(38) and taking into account that $\|\boldsymbol{e}_h(0)\| \leq C h^k \|\boldsymbol{u}\|_k$, one derives the following error estimate for the velocity.

Theorem 1 *For $T > 0$ let us assume for the solution $(\boldsymbol{u}, p)$ of (2) that*

$$\boldsymbol{u} \in L^2(0, T, H^{k+1}(\Omega)) \cap L^2(0, T, W^{1,\infty}(\Omega)) \cap L^\infty(0, T, H^2(\Omega)),$$

$\boldsymbol{u}(0) \in H^{\max\{2,k\}}(\Omega)$, $\partial_t \boldsymbol{u} \in L^2(0, T, H^k(\Omega))$, and $p \in L^2(0, T, H^k(\Omega)/\mathbb{R})$ with $k \geq 1$. Then there exists a positive constant C depending on

$$\|\boldsymbol{u}(0)\|_k^2 + \int_0^T \left(\frac{\|p(t)\|_{H^k/\mathbb{R}}^2}{\mu} + \|\partial_t \boldsymbol{u}(t)\|_k^2 + (\mu + \|\boldsymbol{u}(t)\|_2^2)\|\boldsymbol{u}(t)\|_{k+1}^2 \right) dt, \quad (40)$$

but not directly on inverse powers of ν, such that the following bound holds for $\boldsymbol{e}_h = \boldsymbol{u}_h - \boldsymbol{s}_h$ and $t \in [0, T]$

$$\|\boldsymbol{e}_h(t)\|_0^2 + \int_0^t (\nu\|\nabla\boldsymbol{e}_h(s)\|_0^2 + \mu\|\nabla \cdot \boldsymbol{e}_h(s)\|_0^2) \, ds \leq C \exp(L(T))h^{2k}, \quad (41)$$

where $L(T)$ is defined in (39).

Remark 2 Note that Theorem 1 is formulated for the most common choice of inf-sup stable finite element spaces where the polynomial degree of the velocity space is larger by one than the degree of the pressure space. In this situation, the constant C in (41) depends on μ^{-1} and on μ, see (40). Thus, the asymptotic optimal choice of the stabilization parameter is $\mu \sim 1$, which is a well-known result for this situation.

For pairs of inf-sup stable spaces with the same polynomial degree, like the MINI element, the same regularity with respect to the polynomial degree for velocity and pressure is usually assumed and the estimates for proving the error bound can be adapted accordingly. In particular, one gets instead of (34)

$$\|\tau_3 + \tau_4\|_0^2 \leq Ch^{2k}(h^2\|p\|_{k+1}^2 + \mu^2\|\boldsymbol{u}\|_{k+1}^2),$$

such that equilibrating the two terms containing μ gives the choice $\mu \sim h$, which is known from the literature [21]. However, also choosing $\mu \sim 1$ or $\mu \sim h^2$ leads for the MINI element to optimal error bounds with constants independent of ν. Altogether, there is some freedom for the choice of μ and choosing this parameter to be a constant is a valid option also for the MINI element.

Remark 3 By writing

$$(\boldsymbol{u}_h - \boldsymbol{u}) = \boldsymbol{e}_h + (\boldsymbol{s}_h - \boldsymbol{u}),$$

applying the triangle inequality, Theorem 1, and (19), it follows that the bound (41) holds true changing $\boldsymbol{e}_h$ by $\boldsymbol{u}_h - \boldsymbol{u}$.

3.2   Error bound for the pressure

The error bound for the pressure will be obtained now using the same arguments as used in [15].

Applying the inf-sup condition (4), substituting in the numerator (22) and (30), adding and subtracting terms, and using the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned}
\beta_0\|p_h - \pi_h\|_0 \leq\ & \nu\|\nabla\boldsymbol{e}_h\|_0 + \|B(\boldsymbol{u}_h, \boldsymbol{u}_h) - B(\boldsymbol{s}_h, \boldsymbol{s}_h)\|_{-1} + \|\partial_t\boldsymbol{e}_h\|_{-1} \\
& + \mu\|\nabla \cdot \boldsymbol{e}_h\|_0 + \|\boldsymbol{\tau}_1\|_{-1} \\
& + \|\boldsymbol{\tau}_2\|_{-1} + \|\tau_3\|_0 + \|\tau_4\|_0 + \|l_h\|_0.
\end{aligned} \quad (42)$$

Note that, due to (41), the presence of the terms $\nu\|\nabla e_h\|_0$ and $\mu\|\nabla \cdot e_h\|$ on the right-hand side of (42) limits the maximum convergence rate to $\mathcal{O}(h^k)$. The same convergence rate is obtained for the term $\|\tau_4\|_0$, which is estimated with (19).

The fifth term is bounded with (35)

$$\|\boldsymbol{\tau}_1\|_{-1} \le C\|\boldsymbol{\tau}_1\|_0 \le Ch^k \|\partial_t \boldsymbol{u}\|_k,$$

and the sixth term, using (38), by

$$\|\boldsymbol{\tau}_2\|_{-1} \le C\|\boldsymbol{\tau}_2\|_0 \le Ch^k\|\boldsymbol{u}\|_2 \|\boldsymbol{u}\|_{k+1}. \tag{43}$$

For the second term on the right-hand side of (42), the skew-symmetry of $b(\cdot,\cdot,\cdot)$ gives

$$
\begin{aligned}
\|B(\boldsymbol{u}_h,\boldsymbol{u}_h) - B(\boldsymbol{s}_h,\boldsymbol{s}_h)\|_{-1} &= \sup_{\|\boldsymbol{\phi}\|_1=1} |b(\boldsymbol{u}_h,\boldsymbol{e}_h,\boldsymbol{\phi}) + b(\boldsymbol{e}_h,\boldsymbol{s}_h,\boldsymbol{\phi})| \\
&= \sup_{\|\boldsymbol{\phi}\|_1=1} |b(\boldsymbol{u}_h,\boldsymbol{\phi},\boldsymbol{e}_h) + b(\boldsymbol{e}_h,\boldsymbol{\phi},\boldsymbol{s}_h)|.
\end{aligned} \tag{44}
$$

Using now Hölder's inequality and the Sobolev embedding (3), one finds the bound

$$
\begin{aligned}
|b(\boldsymbol{u}_h,\boldsymbol{\phi},\boldsymbol{e}_h)| &\le \|\boldsymbol{e}_h\|_0\|\boldsymbol{u}_h\|_\infty\|\boldsymbol{\phi}\|_1 + \|\boldsymbol{e}_h\|_0\|\nabla \cdot \boldsymbol{u}_h\|_{L^{2d/(d-1)}(\Omega)}\|\boldsymbol{\phi}\|_{L^{2d}(\Omega)} \\
&\le C\left(\|\boldsymbol{u}_h\|_\infty + \|\nabla \cdot \boldsymbol{u}_h\|_{L^{2d/(d-1)}(\Omega)}\right)\|\boldsymbol{e}_h\|_0\|\boldsymbol{\phi}\|_1.
\end{aligned}
$$

For the second term on the right-hand of (44), arguing similarly, one gets

$$|b(\boldsymbol{e}_h,\boldsymbol{\phi},\boldsymbol{s}_h)| \le \|\boldsymbol{e}_h\|_0\|\boldsymbol{s}_h\|_\infty\|\boldsymbol{\phi}\|_1 + C\|\nabla \cdot \boldsymbol{e}_h\|_0\|\boldsymbol{s}_h\|_{L^{2d/(d-1)}(\Omega)}\|\boldsymbol{\phi}\|_1,$$

such that

$$
\begin{aligned}
\|B(\boldsymbol{u}_h,\boldsymbol{u}_h) &- B(\boldsymbol{s}_h,\boldsymbol{s}_h)\|_{-1} \\
&\le C\left(\|\boldsymbol{u}_h\|_\infty + \|\nabla \cdot \boldsymbol{u}_h\|_{L^{2d/(d-1)}(\Omega)} + \|\boldsymbol{s}_h\|_\infty\right)\|\boldsymbol{e}_h\|_0 \\
&\quad + C\|\boldsymbol{s}_h\|_{L^{2d/(d-1)}(\Omega)}\|\nabla \cdot \boldsymbol{e}_h\|_0.
\end{aligned} \tag{45}
$$

Next, the terms between parentheses will be bounded. Applying (5) and (32), one finds

$$
\begin{aligned}
\|\boldsymbol{u}_h\|_\infty \le \|\boldsymbol{e}_h\|_\infty + \|\boldsymbol{s}_h\|_\infty &\le Ch^{-d/2}\|\boldsymbol{e}_h\|_0 + \|\boldsymbol{s}_h\|_\infty \\
&\le C\left(h^{-d/2}\|\boldsymbol{e}_h\|_0 + \|\boldsymbol{u}\|_2\right).
\end{aligned} \tag{46}
$$

Recalling (5) and (37) yields

$$
\begin{aligned}
\|\nabla \cdot \boldsymbol{u}_h\|_{L^{2d/(d-1)}(\Omega)} &\le \|\nabla \cdot \boldsymbol{e}_h\|_{L^{2d/(d-1)}(\Omega)} + \|\nabla \cdot \boldsymbol{s}_h\|_{L^{2d/(d-1)}(\Omega)} \\
&\le Ch^{-1/2}\|\nabla \cdot \boldsymbol{e}_h\|_0 + Ch^{1/2}\|\boldsymbol{u}\|_2.
\end{aligned} \tag{47}
$$

Remark 4 The right-hand side of (46) is bounded for $h \to 0$ always for $d = 2$. It follows from (41) that for $d = 3$ the term is bounded for $k \ge 2$. Note that most inf-sup stable pairs of finite element spaces have velocity spaces which are at least of second order so that this is not a big restriction. On the other hand, one can deduce from (47) and (41) that the term $\int_0^t \|\nabla \cdot \boldsymbol{u}_h(s)\|^2_{L^{2d/(d-1)}(\Omega)}\, ds$ is bounded.

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With (45) and using in addition (5), (32), and (37), one obtains

$$\|B(\boldsymbol{u}_h, \boldsymbol{u}_h) - B(\boldsymbol{s}_h, \boldsymbol{s}_h)\|_{-1} \leq C \|\boldsymbol{u}\|_2 (\|\boldsymbol{e}_h\|_0 + \|\nabla \cdot \boldsymbol{e}_h\|_0)$$
$$+ C \left(h^{-d/2} \|\boldsymbol{e}_h\|_0 + h^{-1/2} \|\nabla \cdot \boldsymbol{e}_h\|_0 \right) \|\boldsymbol{e}_h\|_0. \quad (48)$$

Next, the third term on the right-hand side of (42) will be bounded. Arguing as in [15], it will be shown first that $\|\partial_t \boldsymbol{e}_h\|_{-1}$ can be estimated by bounding $\|A_h^{-1/2} \partial_t \boldsymbol{e}_h\|_0$. From [5, Lemma 3.11] it is known that

$$\|\partial_t \boldsymbol{e}_h\|_{-1} \leq Ch\|\partial_t \boldsymbol{e}_h\|_0 + C\|A^{-1/2}\Pi\partial_t \boldsymbol{e}_h\|_0, \quad (49)$$

where Π is the Leray projector defined in Section 2. Applying [5, (2.15)] one gets

$$\|A^{-1/2}\Pi\partial_t \boldsymbol{e}_h\|_0 \leq Ch\|\partial_t \boldsymbol{e}_h\|_0 + \|A_h^{-1/2}\partial_t \boldsymbol{e}_h\|_0, \quad (50)$$

with A_h defined in (6). With (49), (50), the symmetry of A_h, and the inverse inequality (5), one obtains

$$\|\partial_t \boldsymbol{e}_h\|_{-1} \leq Ch\|\partial_t \boldsymbol{e}_h\|_0 + C\|A_h^{-1/2}\partial_t \boldsymbol{e}_h\|_0$$
$$= Ch\|A_h^{1/2} A_h^{-1/2}\partial_t \boldsymbol{e}_h\|_0 + C\|A_h^{-1/2}\partial_t \boldsymbol{e}_h\|_0$$
$$= Ch\|\nabla(A_h^{-1/2}\partial_t \boldsymbol{e}_h)\|_0 + C\|A_h^{-1/2}\partial_t \boldsymbol{e}_h\|_0$$
$$\leq C\|A_h^{-1/2}\partial_t \boldsymbol{e}_h\|_0.$$

Taking into account that $\|A_h^{-1/2}\Pi_h^{\mathrm{div}}\mathbf{g}\|_0 \leq \|\mathbf{g}\|_{-1}$, for all $\mathbf{g} \in L^2(\Omega)^d$, see [5, (2.16)], and arguing as in [15], the following estimate for $\|A_h^{-1/2}\partial_t \boldsymbol{e}_h\|_0$ can be derived

$$\|A_h^{-1/2}\partial_t \boldsymbol{e}_h\|_0 \leq \nu\|A_h^{1/2}\boldsymbol{e}_h\|_0 + \|B(\boldsymbol{u}_h, \boldsymbol{u}_h) - B(\boldsymbol{s}_h, \boldsymbol{s}_h)\|_{-1} + C\mu\|\nabla \cdot \boldsymbol{e}_h\|_0$$
$$+ \|\partial_t(\boldsymbol{u} - \boldsymbol{s}_h)\|_{-1} + \|B(\boldsymbol{u}, \boldsymbol{u}) - B(\boldsymbol{s}_h, \boldsymbol{s}_h)\|_{-1} \quad (51)$$
$$+ C\mu\|\nabla \cdot (\boldsymbol{u} - \boldsymbol{s}_h)\|_0 + C\|p - \pi_h\|_0.$$

All velocity-related terms on the right-hand side were already estimated in this section.

The pressure terms in (42) and (51) are estimated with (11) and (20). Then, arguing exactly as in [15], one concludes the following estimate.

Theorem 2 *Under the assumptions of Theorem 1 there exists a positive constant C such that the following bound holds*

$$\beta_0^2 \int_0^T \|(p_h - \pi_h)(t)\|_0^2 \, dt \leq Ch^{2k}, \quad (52)$$

where in the case $d = 3$ the bound is valid for $k \geq 2$.

Remark 5 By splitting

$$p_h - p = (p_h - \pi_h) + (\pi_h - p),$$

applying the triangle inequality, Theorem 2, and (11), it follows that the bound (52) holds true replacing $p_h - \pi_h$ by $p_h - p$.

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4 The continuous-in-time case: analysis without nonlocal compatibility conditions

It is well known that, no matter how regular the data are, solutions of the Navier–Stokes equations cannot be assumed to have more than second order spatial derivatives bounded in $L^2(\Omega)$ up to initial time $t = 0$, since higher regularity requires the data to satisfy nonlocal compatibility conditions which are not likely to happen in practical situations [18, 19]. The analysis of this section takes into account the lack of regularity at $t = 0$.

Along the section it is assumed that inf-sup stable mixed finite elements of second order are used, for example the Hood–Taylor element consisting of continuous piecewise quadratic polynomials for the velocity and continuous piecewise linear polynomials for the pressure.

It shall be assumed that for some $T > 0$

$$M_1 = \max_{0 \le t \le T} \|\boldsymbol{u}(t)\|_1 < +\infty, \quad M_2 = \max_{0 \le t \le T} \|\boldsymbol{u}(t)\|_2 < +\infty. \tag{53}$$

Also, according to [18, Theorems 2.4 and 2.5], and assuming the right-hand side $\boldsymbol{f}$ in (1) is smooth enough, it shall be assumed that, for $k \ge 2$,

$$M_k = \max_{0 \le t \le T} t^{(k-2)/2} \left(\|\boldsymbol{u}(t)\|_k + \|\partial_t \boldsymbol{u}(t)\|_{k-2} + \|p(t)\|_{H^{k-1}/\mathbb{R}} \right) < +\infty, \tag{54}$$

and, for $k \ge 3$

$$K_k^2 = \int_0^T t^{k-3} \left(\|\boldsymbol{u}(t)\|_k^2 + \|\partial_s \boldsymbol{u}(t)\|_{k-2}^2 + \|p(t)\|_{H^{k-1}/\mathbb{R}}^2 + \|\partial_s p(t)\|_{H^{k-3}/\mathbb{R}}^2 \right) ds < +\infty. \tag{55}$$

Remark 6 Observe that in view of Remark 2, for the case $k = 1$ in Theorem 1 (which covers the case of the so-called MINI element) the constant C in (40) and the function $L(T)$ from (39) depend on $M_2^2(1 + T(\mu^{-1} + \mu + M_2^2)) + K_3^2$ and $T + 2T^{1/2}K_3 + TM_2^2\mu^{-1}/2$, respectively, where no negative powers of t appear. Thus, in the absence of nonlocal compatibility conditions at $t = 0$, the analysis of the previous section applies to the case $k = 1$, but it does not apply to the case $k \ge 2$ since $\|\partial_t \boldsymbol{u}(t)\|_k^2$ is not integrable near $t = 0$.

4.1 An auxiliary function

For the analysis, the auxiliary function $\hat{\boldsymbol{u}}_h : [0, T] \to V_h$ satisfying

$$\partial_t \hat{\boldsymbol{u}}_h + \nu A_h \hat{\boldsymbol{u}}_h = \Pi_h^{\mathrm{div}}(\boldsymbol{f} - \nabla p) - B_h(\boldsymbol{u}, \boldsymbol{u}), \quad \hat{\boldsymbol{u}}_h(0) = \Pi_h^{\mathrm{div}} \boldsymbol{u}_0, \tag{56}$$

will be considered and the following notations will be used

$$\boldsymbol{z}_h = \Pi_h^{\mathrm{div}} \boldsymbol{u} - \hat{\boldsymbol{u}}_h, \quad \boldsymbol{\theta}_h = \Pi_h^{\mathrm{div}} \boldsymbol{u} - \boldsymbol{s}_h.$$

Notice that in view of the triangle inequality, (19), the approximation property of the $L^2(\Omega)$ projection, and (54)–(55) it follows that for $0 < t \le T$,

$$\|\boldsymbol{\theta}_h(t)\|_j \le M_3 \frac{h^{3-j}}{t^{1/2}}, \quad j = 0, 1, \tag{57}$$

$$\int_0^t \left(\|\boldsymbol{\theta}_h\|_j^2 + s^2 \|\partial_s \boldsymbol{\theta}_h\|_j^2 \right) ds \le C \left(K_3^2 + K_5^2 \right) h^{6-2j}, \quad j = 0, 1, \tag{58}$$

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for some positive constant C independent of ν. Observe also that projecting (1) onto V_h^{div}, using the definition of s_h with (14) and the right-hand side given in (18), yields

$$\Pi_h^{\text{div}} \partial_t \boldsymbol{u} + \nu A_h \boldsymbol{s}_h + B_h(\boldsymbol{u}, \boldsymbol{u}) + \Pi_h^{\text{div}} \nabla p = \Pi_h^{\text{div}} \boldsymbol{f},$$

so that

$$\Pi_h^{\text{div}} \partial_t \boldsymbol{u} + \nu A_h \Pi_h^{\text{div}} \boldsymbol{u} = \Pi_h^{\text{div}} \boldsymbol{f} - B_h(\boldsymbol{u}, \boldsymbol{u}) - \Pi_h^{\text{div}} \nabla p + \nu A_h \boldsymbol{\theta}_h.$$

Subtracting now (56) and applying the commutation of the Leray projection and the temporal derivative, one finds that the error $\boldsymbol{z}_h$ satisfies

$$\partial_t \boldsymbol{z}_h + \nu A_h \boldsymbol{z}_h = \nu A_h \boldsymbol{\theta}_h, \quad \boldsymbol{z}_h(0) = \boldsymbol{0}. \tag{59}$$

Lemma 3 *There exists a positive constant C independent of ν such that the error $\boldsymbol{z}_h = \Pi_h^{\text{div}} \boldsymbol{u} - \hat{\boldsymbol{u}}_h$ of the discrete velocity $\hat{\boldsymbol{u}}_h$ defined by (56) satisfies the following bounds for $0 < t \le T$:*

$$\|\boldsymbol{z}_h(t)\|_0^2 + \int_0^t \|\boldsymbol{z}_h(s)\|_1^2 \, ds \le C K_3^2 h^4, \tag{60}$$

$$\|A_h^{-1/2} \boldsymbol{z}_h(t)\|_0^2 + \int_0^t \|\boldsymbol{z}_h(s)\|_0^2 \, ds \le C K_3^2 h^6, \tag{61}$$

$$\|\boldsymbol{z}_h(t)\|_j \le \frac{C}{t^{1/2}} (K_3 + K_5 + M_3) h^{3-j}, \quad j = 0, 1. \tag{62}$$

Proof Multiplying (59) by $\boldsymbol{z}_h$, integrating on Ω, applying the Cauchy–Schwarz inequality, and Young's inequality gives

$$\frac{1}{2} \frac{d}{dt} \|\boldsymbol{z}_h\|_0^2 + \nu \|\nabla \boldsymbol{z}_h\|_0^2 = \nu (A_h^{1/2} \boldsymbol{z}_h, A_h^{1/2} \boldsymbol{\theta}_h) \le \frac{\nu}{2} \|\nabla \boldsymbol{z}_h\|_0^2 + \frac{\nu}{2} \|\nabla \boldsymbol{\theta}_h\|_0^2.$$

Using integration in time and taking into account that $\boldsymbol{z}_h(0) = \boldsymbol{0}$, it follows that

$$\|\boldsymbol{z}_h(t)\|_0^2 + \nu \int_0^t \|\nabla \boldsymbol{z}_h(s)\|_0^2 \, ds \le \nu \int_0^t \|\nabla \boldsymbol{\theta}_h(s)\|_0^2 \, ds.$$

Now, applying (58) and the Poincaré–Friedrichs inequality (12), the bound (60) follows directly. Repeating these arguments but multiplying by $A_h^{-1} \boldsymbol{z}_h$ instead of $\boldsymbol{z}_h$ gives (61).

To prove (62), multiply (59) by $t A_h^{-1} \partial_t \boldsymbol{z}_h$ and integrate in Ω to get

$$t \left\| A_h^{-1/2} \partial_t \boldsymbol{z}_h \right\|_0^2 + \frac{\nu}{2} \frac{d}{dt} \left(t \|\boldsymbol{z}_h\|_0^2 \right) = \nu (t \partial_t \boldsymbol{z}_h, \boldsymbol{\theta}_h) + \frac{\nu}{2} \|\boldsymbol{z}_h\|_0^2.$$

Integrating between 0 and t and integrating the term arising from $\nu (t \partial_t \boldsymbol{z}_h, \boldsymbol{\theta}_h)$ by parts, one gets

$$\int_0^t s \left\| A_h^{-1/2} \partial_s \boldsymbol{z}_h \right\|_0^2 \, ds + \frac{\nu}{2} t \|\boldsymbol{z}_h(t)\|_0^2$$
$$= \nu t (\boldsymbol{z}_h(t), \boldsymbol{\theta}_h(t)) - \nu \int_0^t (\boldsymbol{z}_h, \boldsymbol{\theta}_h + s \partial_s \boldsymbol{\theta}_h) \, ds + \frac{\nu}{2} \int_0^t \|\boldsymbol{z}_h\|_0^2 \, ds.$$

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Applying the Cauchy–Schwarz inequality and Young's inequality to the first two terms on the right-hand side and rearranging terms it follows that

$$
\int_0^t s \left\| A_h^{-1/2} \partial_s \boldsymbol{z}_h \right\|_0^2 ds + \frac{\nu}{4} t \left\| \boldsymbol{z}_h(t) \right\|_0^2
$$
$$
\leq \ \nu t \left\| \boldsymbol{\theta}_h(t) \right\|_0^2 + \nu \int_0^t \left(\left\| \boldsymbol{\theta}_h \right\|_0^2 + s^2 \left\| \partial_s \boldsymbol{\theta}_h \right\|_0^2 \right) ds + \nu \int_0^t \left\| \boldsymbol{z}_h \right\|_0^2 ds.
$$

The bound (62) for $j = 0$ now follows by applying (57)–(58) and (61). With the same arguments, but multiplying by $t\partial_t \boldsymbol{z}_h$ instead of $tA_h^{-1}\partial_t \boldsymbol{z}_h$ the bound (62) for $j = 1$ is obtained. $\qquad \square$

Remark 7 For piecewise polynomials of degree higher than two, it is possible to obtain higher order bounds, but with negative powers of ν. For example, for piecewise cubics, by repeating the arguments in the proof of Lemma 3, it is possible to show that

$$
\left\| \boldsymbol{z}_h(t) \right\|_j \leq \frac{C}{\nu^{1/2} t} \left(K_4 + K_6 + M_4 \right) h^{4-j}, \quad j = 0, 1, \quad 0 \leq t \leq T,
$$

using as test function $t^2 A_h^{-1} \partial_t \boldsymbol{z}_h$. Similar negative powers of ν are obtained also with some other techniques like those in [13]. At the moment, it is an open question whether different techniques could be applied to get higher order bounds with constants independent of inverse powers of ν. For this reason, only piecewise quadratics for the velocity are considered in this section.

4.2 Error bound for the velocity

Observe that the first equation in (56) can be rewritten as

$$
\partial_t \hat{\boldsymbol{u}}_h + \nu A_h \hat{\boldsymbol{u}}_h + B_h(\hat{\boldsymbol{u}}_h, \hat{\boldsymbol{u}}_h) + \mu C_h \hat{\boldsymbol{u}}_h = \Pi_h^{\text{div}} \boldsymbol{f} + (B_h(\hat{\boldsymbol{u}}_h, \hat{\boldsymbol{u}}_h) - B_h(\boldsymbol{u}, \boldsymbol{u}))
$$
$$
- D_h p + \mu C_h \hat{\boldsymbol{u}}_h.
$$

Lemma 1 will be applied with $\boldsymbol{w}_h = \hat{\boldsymbol{u}}_h$, $\boldsymbol{t}_{1,h} = \widehat{\boldsymbol{\tau}}_2$, and $t_2 = \tau_3 + \hat{\tau}_4$, where

$$
\widehat{\boldsymbol{\tau}}_2 = B_h(\hat{\boldsymbol{u}}_h, \hat{\boldsymbol{u}}_h) - B_h(\boldsymbol{u}, \boldsymbol{u}), \quad \hat{\tau}_4 = \mu \nabla \cdot (\hat{\boldsymbol{u}}_h - \boldsymbol{u}),
$$

and where τ_3 is defined in (31). The application of this lemma requires to show that both $\left\| \hat{\boldsymbol{u}}_h \right\|_\infty^2$ and $\left\| \nabla \hat{\boldsymbol{u}}_h \right\|_\infty$ are integrable in $(0, T)$.

To bound $\left\| \hat{\boldsymbol{u}}_h \right\|_\infty$, apply the triangle inequality and the inverse inequality (5) to get

$$
\left\| \hat{\boldsymbol{u}}_h \right\|_\infty \leq C_{\text{inv}} h^{-d/2} \left\| \hat{\boldsymbol{u}}_h - I_h \boldsymbol{u} \right\|_0 + \left\| I_h \boldsymbol{u} \right\|_\infty
$$
$$
\leq C_{\text{inv}} h^{-d/2} \left(\left\| \hat{\boldsymbol{u}}_h - \Pi_h^{\text{div}} \boldsymbol{u} \right\|_0 + \left\| \Pi_h^{\text{div}} \boldsymbol{u} - \boldsymbol{u} \right\|_0 + \left\| \boldsymbol{u} - I_h \boldsymbol{u} \right\|_0 \right) + \left\| I_h \boldsymbol{u} \right\|_\infty.
$$

Since $\left\| I_h \boldsymbol{u} \right\|_\infty \leq C \left\| \boldsymbol{u} \right\|_\infty$, utilizing (3) gives

$$
\left\| I_h \boldsymbol{u} \right\|_\infty \leq C \left\| \boldsymbol{u} \right\|_2 \leq C M_2.
$$

Applying (13), (10), and (60) yields

$$
\left\| \hat{\boldsymbol{u}}_h \right\|_\infty \leq C \left((K_3 + M_2) h^{(4-d)/2} + M_2 \right). \tag{63}
$$

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The bound of $\|\nabla \hat{\boldsymbol{u}}\|_\infty$ will be shown for the more difficult and practically more relevant case $d = 3$. With the triangle inequality and the inverse inequality (5), one obtains

$$
\begin{aligned}
\|\nabla \hat{\boldsymbol{u}}_h\|_\infty &\leq C_{\mathrm{inv}} h^{-5/2} \|\hat{\boldsymbol{u}}_h - \boldsymbol{s}_h\|_0 + \|\nabla \boldsymbol{s}_h\|_\infty \\
&\leq C_{\mathrm{inv}} h^{-5/2} \left(\|\hat{\boldsymbol{u}}_h - \Pi_h^{\mathrm{div}} \boldsymbol{u}\|_0 + \|\Pi_h^{\mathrm{div}} \boldsymbol{u} - \boldsymbol{u}\|_0 + \|\boldsymbol{u} - \boldsymbol{s}_h\|_0 \right) + \|\nabla \boldsymbol{s}_h\|_\infty.
\end{aligned}
\tag{64}
$$

Applying (21), (3), and (54) yields

$$
\|\nabla \boldsymbol{s}_h\|_\infty \leq C M_3 t^{-1/2}.
$$

Arguing as before and applying (62), (10), and (19) gives

$$
\|\nabla \hat{\boldsymbol{u}}_h\|_\infty \leq C t^{-1/2} \left((K_3 + K_5 + M_3) h^{1/2} + M_3 \right).
\tag{65}
$$

Thus, from (63) and (65) it follows that

$$
\begin{aligned}
g(t) = 1 + 2\|\nabla \hat{\boldsymbol{u}}_h(t)\|_\infty &+ \frac{\|\hat{\boldsymbol{u}}_h(t)\|_\infty^2}{2\mu} \\
&\leq C \left(\frac{K_3 + K_5 + M_3}{\tau^{1/2}} + \frac{(K_3 + M_2)^2}{2\mu} \right) \leq \frac{L_1}{t^{1/2}},
\end{aligned}
$$

where

$$
L_1 = C \left(K_3 + K_5 + M_3 \right) + \frac{(K_3 + M_2)^2}{2\mu}.
$$

Then

$$
K(t, s) \;=\; \int_s^t g(r)\,dr \leq L_1 \int_s^t \frac{1}{r^{1/2}}\,dr \leq 2L_1 (t^{1/2} - s^{1/2}).
\tag{66}
$$

Lemma 1 with $\boldsymbol{w}_h = \hat{\boldsymbol{u}}_h$ gives for $\hat{\boldsymbol{e}}_h = \boldsymbol{u}_h - \hat{\boldsymbol{u}}_h$

$$
\begin{aligned}
\|\hat{\boldsymbol{e}}_h(t)\|_0^2 + \int_0^t & e^{K(t,s)} \left(2\nu \|\nabla \hat{\boldsymbol{e}}_h(s)\|_0^2 + \mu \|\nabla \cdot \hat{\boldsymbol{e}}_h(s)\|_0^2 \right) ds \\
&\leq e^{K(t,0)} \|\boldsymbol{e}_h(0)\|_0^2 + \int_0^t e^{K(t,s)} \left(\|\widehat{\boldsymbol{\tau}}_2\|_0^2 + \frac{2}{\mu} \|\tau_3 + \hat{\tau}_4\|_0^2 \right) ds.
\end{aligned}
$$

To estimate the truncation errors first apply Lemma 2 to get

$$
\|\widehat{\boldsymbol{\tau}}_2\|_0 \leq C \left(\|\hat{\boldsymbol{u}}_h\|_\infty + \|\nabla \cdot \hat{\boldsymbol{u}}_h\|_{L^{2d/(d-1)}} + \|\boldsymbol{u}\|_2 \right) \|\boldsymbol{u} - \hat{\boldsymbol{u}}_h\|_1.
\tag{67}
$$

Using the triangle inequality, the inverse inequality (5), (37), (60), (10), (19), and (53) one gets

$$
\begin{aligned}
\|\nabla \cdot \hat{\boldsymbol{u}}_h\|_{L^{2d/(d-1)}} & \\
&\leq C_{\mathrm{inv}} h^{-3/2} \|\hat{\boldsymbol{u}}_h - \boldsymbol{s}_h\|_0 + \|\nabla \cdot \boldsymbol{s}_h\|_{L^{2d/(d-1)}} \\
&\leq C h^{-3/2} \left(\|\hat{\boldsymbol{u}}_h - \Pi_h^{\mathrm{div}} \boldsymbol{u}\|_0 + \|\Pi_h^{\mathrm{div}} \boldsymbol{u} - \boldsymbol{u}\|_0 + \|\boldsymbol{u} - \boldsymbol{s}_h\|_0 \right) + C h^{1/2} \|\boldsymbol{u}\|_2 \\
&\leq C h^{1/2} (K_3 + \|\boldsymbol{u}\|_2) \leq C(K_3 + M_2).
\end{aligned}
\tag{68}
$$

By inserting (68) and (63) in (67) it follows that

$$\|\widehat{\boldsymbol{\tau}}_2\|_0 \leq C(K_3 + M_2)\|\boldsymbol{u} - \hat{\boldsymbol{u}}_h\|_1. \tag{69}$$

As in (34) one also gets

$$\|\tau_3 + \hat{\tau}_4\|_0^2 \leq C\left(\|p\|_{H^2/\mathbb{R}}^2 h^4 + \mu^2\|\boldsymbol{u} - \hat{\boldsymbol{u}}_h\|_1^2\right). \tag{70}$$

To bound $\|\boldsymbol{u} - \hat{\boldsymbol{u}}_h\|_1$ in (69) and (70) one adds and subtracts $\Pi_h^{\text{div}}\boldsymbol{u}$. Applying then (55), (10), and (60) leads to the estimate

$$\int_0^t \left(\|\widehat{\boldsymbol{\tau}}_2\|_0^2 + \frac{2}{\mu}\|\tau_3 + \hat{\tau}_4\|_0^2\right) ds \leq CK_3^2\left(\mu + \mu^{-1} + (K_3 + M_2)^2\right)h^4$$
$$= C_0 h^4. \tag{71}$$

Collecting all estimates and applying at the initial time the triangle inequality, the interpolation estimate (13) and (10), the following theorem is proved.

Theorem 3 *For $T > 0$, assuming the solution $(\boldsymbol{u}, p)$ of (1) satisfies (53), (54) and (55) the following bound holds for the error $\hat{\boldsymbol{e}}_h = \boldsymbol{u}_h - \hat{\boldsymbol{u}}_h$, $t \in [0, T]$:*

$$\|\hat{\boldsymbol{e}}_h(t)\|_0^2 + \int_0^t \left(2\nu\|\nabla\hat{\boldsymbol{e}}_h(s)\|_0^2 + \mu\|\nabla \cdot \hat{\boldsymbol{e}}_h(s)\|_0^2\right) ds \leq e^{K(t,0)}\left(M_2^2 + C_0\right)h^4, \tag{72}$$

where $K(t, s)$ is defined in (66) and C_0 is the constant in (71).

Remark 8 By decomposing

$$\boldsymbol{u}_h - \boldsymbol{u} = \hat{\boldsymbol{e}}_h + (\hat{\boldsymbol{u}}_h - \boldsymbol{u}) = \hat{\boldsymbol{e}}_h - \boldsymbol{z}_h + \left(\Pi_h^{\text{div}}\boldsymbol{u} - \boldsymbol{u}\right),$$

and applying the triangle inequality, Theorem 3, (60), and (10), it follows that the bound (72) holds true changing $\hat{\boldsymbol{e}}_h$ by $\boldsymbol{u}_h - \boldsymbol{u}$.

Remark 9 Observe that it is the factor $h^{-5/2}$ in (64) that prevents the analysis in the present section to apply to the case $k = 1$. On the other hand, the analysis in Section 3 applies to $k = 1$ since one compares $\boldsymbol{u}_h$ with $\boldsymbol{s}_h$ for which the bounds (32) are available. The comparison with $\boldsymbol{s}_h$ in Section 3, however, induces the truncation error $\boldsymbol{\tau}_1$, which, as commented in Remark 6, prevents the extension of the analysis in that section to the case $k > 1$.

4.3   Error bound for the pressure

The analysis follows closely that of Section 3.2.

Again, using the inf-sup condition (4), a straightforward calculation leads to

$$\beta_0\|p_h - \pi_h\|_0 \leq \nu\|\nabla\hat{\boldsymbol{e}}_h\|_0 + \|B(\boldsymbol{u}_h, \boldsymbol{u}_h) - B(\hat{\boldsymbol{u}}_h, \hat{u}_h)\|_{-1} + \|\partial_t\hat{\boldsymbol{e}}_h\|_{-1}$$
$$+ \mu\|\nabla \cdot \hat{\boldsymbol{e}}_h\|_0 + \|B(\boldsymbol{u}, \boldsymbol{u}) - B(\hat{\boldsymbol{u}}_h, \hat{u}_h)\|_{-1}$$
$$+ \mu\|\nabla \cdot (\boldsymbol{u} - \hat{\boldsymbol{u}}_h)\|_0 + \|p - \pi_h\|_0 + \|l_h\|_0,$$

where l_h denotes here the discrete pressure corresponding to the formulation of (56) in V_h. Repeating the arguments used when obtaining (43) and (69), one gets

$$\|B(\boldsymbol{u}, \boldsymbol{u}) - B(\hat{\boldsymbol{u}}_h, \hat{\boldsymbol{u}}_h)\|_{-1} \le C(K_3 + M_2)\|\boldsymbol{u} - \hat{\boldsymbol{u}}_h\|_1.$$

In the same way as for (45), one obtains

$$\|B(\boldsymbol{u}_h, \boldsymbol{u}_h) - B(\hat{\boldsymbol{u}}_h, \hat{\boldsymbol{u}}_h)\|_{-1} \le (\|\boldsymbol{u}_h\|_\infty + \|\nabla \cdot \boldsymbol{u}_h\|_{L^{2d/(d-1)}} + \|\hat{\boldsymbol{u}}_h\|_\infty)\|\hat{\boldsymbol{e}}_h\|_0$$
$$+ \|\hat{\boldsymbol{u}}_h\|_{L^{2d/(d-1)}}\|\nabla \cdot \hat{\boldsymbol{e}}_h\|_0.$$

Using the inverse inequality (5), (68), (46), and (63) leads to

$$\|B(\boldsymbol{u}_h, \boldsymbol{u}_h) - B(\hat{\boldsymbol{u}}_h, \hat{\boldsymbol{u}}_h)\|_{-1} \le C(K_3 + M_2)\left(\|\hat{\boldsymbol{e}}_h\|_0 + \|\nabla \cdot \hat{\boldsymbol{e}}_h\|_0\right)$$
$$+ C\left(h^{-d/2}\|\hat{\boldsymbol{e}}_h\|_0 + h^{-1/2}\|\nabla \cdot \hat{\boldsymbol{e}}_h\|_0\right)\|\hat{\boldsymbol{e}}_h\|_0.$$

Arguing as in Section 3.2, one gets for $\|\partial_t \hat{\boldsymbol{e}}_h\|_{-1}$

$$\|\partial_t \hat{\boldsymbol{e}}_h\|_{-1} \le \nu\|A_h^{1/2}\hat{\boldsymbol{e}}_h\|_0 + \|B(\boldsymbol{u}_h, \boldsymbol{u}_h) - B(\hat{\boldsymbol{u}}_h, \hat{\boldsymbol{u}}_h)\|_{-1} + C\mu\|\nabla \cdot \hat{\boldsymbol{e}}_h\|_0$$
$$+ \|B(\boldsymbol{u}, \boldsymbol{u}) - B(\hat{\boldsymbol{u}}_h, \hat{\boldsymbol{u}}_h)\|_{-1}$$
$$+ C\mu\|\nabla \cdot (\boldsymbol{u} - \hat{\boldsymbol{u}}_h)\|_0 + C\|p - \pi_h\|_0.$$

All terms on the right-hand side of this estimate have already been bounded. Arguing like at the end of Section 3.2, one derives the following estimate.

Theorem 4 *Under the assumptions of Theorem 3 there exists a positive constant C such that the following bound holds*

$$\beta_0^2 \int_0^t \|(p_h - \pi_h)(s)\|_0^2 \le Ch^4. \tag{73}$$

Remark 10 By writing

$$p_h - p = (p_h - \pi_h) + (\pi_h - p),$$

applying the triangle inequality, Theorem 4, and (11), it follows that the bound (73) holds true replacing $p_h - \pi_h$ by $p_h - p$.

5 A fully discrete method

We now analyze the discretization of (24) by the implicit Euler method. For this purpose, we consider a partition $0 = t_0 < t_1 < \ldots < t_N = T$ of the interval $[0, T]$, and for each time level we look for approximations $\boldsymbol{U}_h^n \approx \boldsymbol{u}(t_n)$ in V_h^{div} and $p_h^n \approx p(t_n)$ in Q_h, satisfying

$$(D_t\boldsymbol{U}_h^n, \boldsymbol{v}_h) + \nu(\nabla \boldsymbol{U}_h^n, \nabla \boldsymbol{v}_h) + b(\boldsymbol{U}_h, \boldsymbol{U}_h^n, \boldsymbol{v}_h) - (p_h^n, \nabla \cdot \boldsymbol{v}_h^n)$$
$$+ (\nabla \cdot \boldsymbol{u}_h^n, q_h) + \mu(\nabla \cdot \boldsymbol{U}_h^n, \nabla \cdot \boldsymbol{v}_h) = (\boldsymbol{f}(t_n), \boldsymbol{v}_h), \tag{74}$$

for all $(\boldsymbol{v}_h, q_h) \in (V_h \times Q_h)$, for $n = 1, \ldots, N$, where $\boldsymbol{U}_h^0 \in V_h$ is given, and

$$D_t\boldsymbol{U}_h^n = \frac{\boldsymbol{U}_h^n - \boldsymbol{U}_h^{n-1}}{t_n - t_{n-1}}.$$

In what follows we will take $\boldsymbol{U}_h^0 = \boldsymbol{s}_h(0)$ and consider for simplicity constant step sizes, that is

$$t_n - t_{n-1} = \Delta t, \quad n = 1, \dots, N.$$

The changes for variable step sizes as well as for other consistent initial approximations are straightforward. Also, other time integrators can be considered and the analysis can be carried out arguing essentially as in the next lines.

The existence of the approximation can be proved with the help of Brouwer's fixed point theorem as in [31]. The approximations $\boldsymbol{U}_h^n$ satisfy

$$D_t \boldsymbol{U}_h^n + \nu A_h \boldsymbol{U}_h^n + B_h(\boldsymbol{U}_h, \boldsymbol{U}_h^n) + \mu C_h \boldsymbol{U}_h^n = \Pi_h^{\mathrm{div}} \boldsymbol{f}(t_n), \tag{75}$$

where we keep the notation of previous sections.

To obtain error bounds, we will use the following discrete Gronwall lemma that can be found in [20].

Lemma 4 *Let k, B, and a_n, b_n, c_n, γ_n be nonnegative numbers such that*

$$a_n + k \sum_{j=0}^{n} b_n \leq k \sum_{j=0}^{n} \gamma_n a_n + k \sum_{j=0}^{n} c_n + B, \quad n \geq 1.$$

Suppose that $k\gamma_n < 1$, for all n, and set $\sigma_n = (1 - k\gamma_n)^{-1}$. Then, the following bound holds

$$a_n + k \sum_{j=0}^{n} b_n \leq \exp\left(k \sum_{j=0}^{n} \sigma_j \gamma_j\right)\left(k \sum_{j=0}^{n} c_n + B\right), \quad n \geq 1.$$

Lemma 5 *Fix $\gamma \in (0,1)$, and let $(\boldsymbol{w}_h^n)_{n=0}^{\infty}$, $(\boldsymbol{t}_{h,0}^n)_{n=1}^{\infty}$ and $(\boldsymbol{t}_{h,1}^n)_{n=1}^{\infty}$ be series in V_h^{div} and let $(\boldsymbol{t}_{h,2}^n)_{n=1}^{\infty}$ a series in $L^2(\Omega)$ satisfying*

$$D_t \boldsymbol{w}_h^n + \nu A_h \boldsymbol{w}_h^n + B_h(\boldsymbol{w}_h^n, \boldsymbol{w}_h^n) + \mu C_h \boldsymbol{w}_h^n = \Pi_h^{\mathrm{div}} \boldsymbol{f}(t_n) + \nu A_h \boldsymbol{t}_{h,0}^n + \boldsymbol{t}_{h,1}^n + D_h \boldsymbol{t}_{h,2}^n. \tag{76}$$

Assume $\Delta t g_h^n < \gamma$ where

$$g_h^n = 1 + 2\left\|\nabla \boldsymbol{w}_h^n\right\|_{\infty} + \frac{\left\|\boldsymbol{w}_h^n\right\|_{\infty}^2}{2\mu}, \quad n = 1, 2, \dots. \tag{77}$$

There exists a positive constant C depending only on γ such that the following bound holds for the differences $\boldsymbol{e}_h^n = \boldsymbol{w}_h^n - \boldsymbol{U}_h^n$:

$$\|\boldsymbol{e}_h^n\|_0^2 + \sum_{j=1}^{n} \|\boldsymbol{e}_h^j - \boldsymbol{e}_h^{j-1}\|_0^2 + \Delta t \sum_{j=1}^{n} \left(\nu\|\nabla \boldsymbol{e}_h^j\|_0^2 + \mu\|\nabla \cdot \boldsymbol{e}_h^j\|_0^2\right)$$

$$\leq \quad C \exp\left(\Delta t \sum_{j=1}^{n-1} g_j\right)\left(\|\boldsymbol{e}_h^0\|_0^2 + \Delta t \sum_{j=1}^{n}\left(\nu\|\nabla \boldsymbol{t}_{h,0}^j\|_0^2 + \|\boldsymbol{t}_{h,1}^j\|_0^2 + \frac{1}{\mu}\|\boldsymbol{t}_{h,2}^j\|_0^2\right)\right).$$

Proof A direct calculation shows that

$$(\boldsymbol{e}_h^n - \boldsymbol{e}_n^{n-1}, \boldsymbol{e}_h^n) = \frac{1}{2}\|\boldsymbol{e}_h^n\|_0^2 - \frac{1}{2}\|\boldsymbol{e}_h^{n-1}\|_0^2 + \frac{1}{2}\|\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1}\|_0^2.$$

so that subtracting (75) from (76), taking the inner product in $L^2(\Omega)$ with $2e_h^n$, adding $0 = 2\mu(\nabla \cdot \boldsymbol{w}_h^n, \nabla \cdot \boldsymbol{e}_h^n) - 2\mu(\nabla \cdot (\boldsymbol{w}_h^n - \boldsymbol{u}(t_n)), \nabla \cdot \boldsymbol{e}_h^n)$ and after some rearrangements we have

$$\frac{1}{\Delta t}\left(\|\boldsymbol{e}_h^n\|_0^2 - \|\boldsymbol{e}_h^{n-1}\|_0^2 + \|\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1}\|_0^2\right) + \nu\|\nabla \boldsymbol{e}_h^n\|_0^2 + \frac{3}{2}\mu\|\nabla \cdot \boldsymbol{e}_h^n\|_0^2$$

$$\leq \quad \nu\|\nabla \boldsymbol{t}_{h,0}^n\|_0^2 - 2b(\boldsymbol{e}_h^n, \boldsymbol{w}_h^n, \boldsymbol{e}_h^n) + \|\boldsymbol{t}_{h,1}^n\|_0^2 + \frac{2}{\mu}\|t_{h,2}^n\|_0^2 + \|\boldsymbol{e}_h^n\|_0^2, \qquad (78)$$

where the product $(\boldsymbol{e}_h^n, B_h(\boldsymbol{w}_h^n, \boldsymbol{w}_h^n) - B_h(\boldsymbol{U}_h^n, \boldsymbol{U}_h^n)) = b(\boldsymbol{w}_h^n, \boldsymbol{w}_h^n, \boldsymbol{e}_h^n) - b(\boldsymbol{U}_h^n, \boldsymbol{U}_h^n, \boldsymbol{e}_h^n)$ has been treated as in (28). Arguing as in (29) we may write

$$b(\boldsymbol{e}_h^n, \boldsymbol{w}_h^n, \boldsymbol{e}_h^n) \leq \left(\|\nabla \boldsymbol{w}_h^n\|_\infty + \frac{\|\boldsymbol{w}_h^n\|_\infty^2}{4\mu}\right)\|\boldsymbol{e}_h^n\|_0^2 + \frac{\mu}{4}\|\nabla \cdot \boldsymbol{e}_h^n\|_0^2.$$

Thus, multiplying by Δt in (78) it follows that

$$\|\boldsymbol{e}_h^n\|_0^2 - \|\boldsymbol{e}_h^{n-1}\|_0^2 + \|\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1}\|_0^2 + \nu\Delta t\|\nabla \boldsymbol{e}_h^n\|_0^2 + \mu\Delta t\|\nabla \cdot \boldsymbol{e}_h^n\|_0^2 \leq c_n + \Delta t g_h^n\|\boldsymbol{e}_h^n\|_0^2, \quad (79)$$

where

$$c_n = \Delta t\left(\nu\|\nabla \boldsymbol{t}_{h,0}^n\|_0^2 + \|\boldsymbol{t}_{h,1}^n\|_0^2 + \frac{2}{\mu}\|t_{h,2}^n\|_0^2\right).$$

Adding the expression in (79) to those corresponding to $n-1$, $n-2$, etc, down to 1, we have

$$(1 - \Delta t g_h^n)\|\boldsymbol{e}_h^n\|_0^2 + \sum_{j=1}^n \|\boldsymbol{e}_h^j - \boldsymbol{e}_h^{j-1}\|_0^2 + \Delta t \sum_{j=1}^n \left(\nu\|\nabla \boldsymbol{e}_h^j\|_0^2 + \mu\|\nabla \cdot \boldsymbol{e}_h^j\|_0^2\right)$$

$$\leq \quad \|\boldsymbol{e}_h^0\|_0^2 + \sum_{j=1}^n c_j + \sum_{j=1}^{n-1} \Delta t g_h^j\|\boldsymbol{e}_h^j\|_0^2.$$

Since we are assuming that $\Delta t g_h^n \leq \gamma$, we have $(1 - \Delta t g_h^n) > 1 - \gamma > 0$ and the proof is finished by applying Lemma 4. $\qquad \square$

5.1 Error analysis in the regular case

We apply Lemma 5 with $\boldsymbol{w}_h^n = \boldsymbol{s}_h(t_n)$, so that

$$\boldsymbol{t}_{h,0}^n = \boldsymbol{0}, \quad \boldsymbol{t}_{h,1}^n = \Pi_h^{\mathrm{div}}(\boldsymbol{\tau}_{1,1}^n + \boldsymbol{\tau}_{1,2}^n + \boldsymbol{\tau}_2(t_n)), \quad \boldsymbol{t}_{h,2}^n = \tau_3(t_n) + \tau_4(t_n),$$

where $\boldsymbol{\tau}_2$, τ_3 and τ_4 are those defined in (31), and

$$\boldsymbol{\tau}_{1,1}^n = \frac{(\boldsymbol{s}_h(t_n) - \boldsymbol{u}(t_h)) - (\boldsymbol{s}_h(t_{n-1}) - \boldsymbol{u}(t_{n-1}))}{\Delta t} = \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} \partial_t(\boldsymbol{s}_h(t) - \boldsymbol{u}(t))\, dt,$$
$$(80)$$

$$\boldsymbol{\tau}_{1,2}^n = \frac{\boldsymbol{u}(t_n) - \boldsymbol{u}(t_{n-1})}{\Delta t} - \partial_t \boldsymbol{u}(t_n) = -\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} (s - t_{n-1})\partial_{tt}\boldsymbol{u}(s)\, ds. \qquad (81)$$

We notice that in view of (32) we have that

$$g_h^j \leq \hat{L} = 1 + C \max_{0 \leq t \leq T}\left(2\|\nabla \boldsymbol{u}(t)\|_\infty + \frac{\|\boldsymbol{u}(t)\|_2^2}{2\mu}\right), \quad j = 1, \dots, N, \qquad (82)$$

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so that

$$\exp\left(\Delta t \sum_{j=1}^{n-1} g_j\right) \leq \exp\left(\hat{L}t_n\right). \tag{83}$$

We also have that $\boldsymbol{\tau}_2(t_n)$, $\tau_3(t_n)$, and $\tau_4(t_n)$ have already been bounded in (34) and (35). Furthermore, applying Cauchy-Schwarz inequality in (80) and the bound (19) with $j = k - 1$ applied to $\partial_t(\boldsymbol{s}_h(t) - \boldsymbol{u}(t))$ we have

$$\|\boldsymbol{\tau}_{1,1}^n\|_0^2 \leq C \frac{h^{2k}}{\Delta t} \int_{t_{n-1}}^{t_n} \|\partial_t \boldsymbol{u}\|_k^2 \, dt$$

and applying the Cauchy-Schwarz inequality in (81)

$$\|\boldsymbol{\tau}_{1,2}^n\|_0^2 \leq C \Delta t \int_{t_{n-1}}^{t_n} \|\partial_{tt} \boldsymbol{u}\|_0^2 \, dt.$$

Thus, we have the following result.

Theorem 5 *For $T > 0$ let us assume for the solution $(\boldsymbol{u}, p)$ of (2) that*

$$\boldsymbol{u} \in L^\infty(0, T, H^{k+1}(\Omega)) \cap L^\infty(0, T, W^{1,\infty}(\Omega)),$$

$\boldsymbol{u}(0) \in H^{\max\{2,k\}}(\Omega)$, $\partial_t \boldsymbol{u} \in L^2(0, T, H^k(\Omega))$, $\partial_{tt} \boldsymbol{u} \in L^2(0, T, L^2(\Omega))$ *and* $p \in L^2(0, T, H^k(\Omega)/\mathbb{R})$ *with* $k \geq 1$. *Then, there exist positive constants* C_1 *and* C_2 *depending on*

$$\|\boldsymbol{u}(0)\|_k^2 + \int_0^T \|\partial_t \boldsymbol{u}(t)\|_k^2 \, dt + \max_{0 \leq t \leq T}\left(\frac{\|p(t)\|_{H^k/\mathbb{R}}^2}{\mu} + \left(\mu + \|\boldsymbol{u}(t)\|_2^2\right)\|\boldsymbol{u}(t)\|_{k+1}^2\right)$$

and

$$\int_0^T \|\partial_{tt} \boldsymbol{u}(t)\|_0^2 \, dt,$$

respectively, but none of them depending on inverse powers of ν, such that the following bound holds for $\boldsymbol{e}_h = \boldsymbol{U}_h^n - \boldsymbol{s}_h(t_n)$ and $1 \leq n \leq N$

$$\|\boldsymbol{e}_h^n\|_0^2 + \Delta t \sum_{j=1}^{n} \left(\nu\|\nabla \boldsymbol{e}_h^n\|_0^2 + \mu\|\nabla \cdot \boldsymbol{e}_h^n\|_0^2\right) \leq \exp\left(\hat{L}t_n\right)\left(C_1 h^{2k} + C_2(\Delta t)^2\right), \tag{84}$$

where $\hat{L}$ is defined in (82).

For the pressure, we can obtain error bounds by repeating the analysis in Section 3.2 with $\partial_t \boldsymbol{e}_h$ replaced by $D_t \boldsymbol{e}_h^n$ and the truncation error $\boldsymbol{\tau}_1$ by $\boldsymbol{\tau}_{1,1}^n + \boldsymbol{\tau}_{1,2}^n$. We observe, however, that instead of (48) we now have

$$\|B(\boldsymbol{U}_h^n, \boldsymbol{U}_h^n) - B(\boldsymbol{s}_h(t_n), \boldsymbol{s}_h(t_n))\|_{-1}$$
$$\leq C \|\boldsymbol{u}(t_n)\|_2 \left(\|\boldsymbol{e}_h^n\|_0 + \|\nabla \cdot \boldsymbol{e}_h^n\|_0\right) + C\left(h^{-d/2}\|\boldsymbol{e}_h^n\|_0 + h^{-1/2}\|\nabla \cdot \boldsymbol{e}_h^n\|_0\right)\|\boldsymbol{e}_h^n\|_0.$$

Now, the errors $\|\boldsymbol{e}_h^n\|_0$ and $\|\nabla \cdot \boldsymbol{e}_h^n\|_0$, as shown in (84) are bounded in terms of powers of h and Δt. Thus, we have the following result.

Theorem 6 *In the conditions of Theorem 5, there exists a positive constant C such that the following bound holds*

$$\beta_0^2 \Delta t \sum_{j=1}^{N} \|p_h^n - \pi_h(t_n)\|_0^2 \le C \left(1 + \frac{(\Delta t)^2}{h^d}\right) \left(h^{2k} + (\Delta t)^2\right). \tag{85}$$

where in the case $d = 3$ the bound is valid for $k \ge 2$.

Remark 11 Observe that in Theorem 6 the presence of negative powers of h in the error bound (85) does not affect the convergence rate in the pressure whenever $\Delta t \le C h^{d/2}$ for any positive constant C. This condition will be automatically satisfied if we try to balance spatial and temporal discretization errors, since in that case we would have to take $\Delta t \approx h^k$.

5.2 Error analysis without compatibility conditions

We now assume that (53) holds and that $M_3 < +\infty$ and

$$K_{3,0} = \int_0^T t \|\partial_{tt} \boldsymbol{u}(t)\|_0^2 \, dt < +\infty.$$

In the case $k = 1$ and $d = 2$ we will also assume $K_3 < +\infty$. The cases $k = 1$ and $k \ge 2$ will be analyzed separately.

For $k = 1$, and taking into account that $\|\boldsymbol{u}(t)\|_\infty \le \|\boldsymbol{u}(t)\|_2$ and $\|\nabla \boldsymbol{u}(t)\|_\infty \le C \|\boldsymbol{u}(t)\|_3 \le C M_3 t^{-1/2}$, the analysis of the previous section is still valid with the following two changes. First we must replace (82) and (83) by $g_h^j \le \hat{L}_1 + 2M_3 t_j^{-1/2}$, for $j = 1, \ldots, N$ where

$$\hat{L}_1 = 1 + \frac{M_2^2}{2\mu}, \tag{86}$$

so that

$$\exp\left(\Delta t \sum_{j=1}^{n-1} g_h^j\right) \le \exp\left(\hat{L}_1 t_n + 2M_3 \sqrt{t_n}\right). \tag{87}$$

The second and more relevant change is the estimation of $\boldsymbol{\tau}_{1,2}$, which now is

$$\|\boldsymbol{\tau}_{1,2}^n\|_0^2 \le \frac{1}{\Delta t^2} \left(\int_{t_{n-1}}^{t_n} (t - t_n) \, dt\right) \left(\int_{t_{n-1}}^{t_n} (t - t_n) \|\partial_{tt} \boldsymbol{u}(t)\|_0^2 \, dt\right)$$

$$\le \frac{1}{2} \int_{t_{n-1}}^{t_n} t \|\partial_{tt} \boldsymbol{u}(t)\|_0^2 \, dt.$$

Thus, we have the following result.

Theorem 7 *Fix $T > 0$ and assume that the solution $(\boldsymbol{u}, p)$ of (1) satisfies (53), and that M_3, K_3 and $K_{3,0}$ are finite. Assume linear finite element approximations in the velocity are used. Then,*

i) *There exists a positive constant C_1 depending on M_2 and K_3 but not on inverse powers of ν, such that the following bound holds for the error $\boldsymbol{e}_h^n = \boldsymbol{s}_h(t_n) - \boldsymbol{U}_h^n$, $1 \le n \le N$:*

$$\|\boldsymbol{e}_h^n\|_0^2 + \sum_{j=1}^n \|\boldsymbol{e}_h^j - \boldsymbol{e}_h^{j-1}\|_0^2 + \Delta t \sum_{j=1}^n \left(\nu\|\nabla\boldsymbol{e}_h^n\|_0^2 + \mu\|\nabla \cdot \boldsymbol{e}_h^n\|_0^2\right)$$
$$\le \ \exp\left(\hat{L}_1 t_n + 2M_3\sqrt{t_n}\right)\left(C_1 h^2 + K_{3,0}\Delta t\right),$$

where $\hat{L}_1$ is defined in (86).

ii) *In the case $d = 2$ then there exists a positive constant C_2 depending on M_2, M_3, $\hat{L}_1$ and K_3 but not on inverse powers of ν, such that the following bound holds:*

$$\beta_0^2 \Delta t \sum_{j=1}^N \|p_h^n - \pi_h(t_n)\|_0^2 \le C_2\left(1 + \frac{\Delta t}{h}\right)\left(h^2 + \Delta t\right).$$

Remark 12 Let us observe that contrary to Theorem 5 we have found a limitation in the rate of convergence of order $\mathcal{O}((\Delta t)^{1/2})$ in the temporal error. To our knowledge this paper is the first one in which error bounds with constants independent on ν are obtained for the fully discrete case without assuming nonlocal compatibility conditions for the Navier-Stokes equations. At present it is an open problem to find out if this $\mathcal{O}((\Delta t)^{1/2})$ limitation could be avoided using a different technique of analysis.

For $k \ge 2$, we will apply Lemma 5 to the differences

$$\boldsymbol{e}_h^n = \boldsymbol{u}_h^n - \boldsymbol{U}_h^n,$$

where

$$\boldsymbol{u}_h^n = \Pi_h^{\mathrm{div}}\boldsymbol{u}(t_n) \quad n = 0, 1, \ldots, N.$$

By projecting (1) onto V_h^{div} and adding and subtracting terms, it is easy to check that the projections $\boldsymbol{u}_h^n$ satisfy for $n = 1, \ldots, N$,

$$D_t \boldsymbol{u}_h^n + \nu A_h \boldsymbol{u}_h^n + B_h(\boldsymbol{u}_h^n, \boldsymbol{u}_h^n) + \mu C_h \boldsymbol{u}_h^n = \nu A_h \tilde{\boldsymbol{t}}_{h,0}^n - \tilde{\boldsymbol{t}}_{1,h}^n - D_h \tilde{t}_2^n,$$

where, $\tilde{\boldsymbol{t}}_{h,0}^n = \boldsymbol{u}_h^n - \boldsymbol{s}_h(t_n)$, $\tilde{\boldsymbol{t}}_{1,h}^n = \Pi_h^{\mathrm{div}}\boldsymbol{\tau}_{1,2}^n + \tilde{\boldsymbol{\tau}}_2^n$, and $\tilde{t}_2^n = \tau_3(t_n) + \tilde{\tau}_4^n$, $\boldsymbol{\tau}_{1,2}^n$ and τ_3 being those defined in (81) and (31), and where

$$\tilde{\boldsymbol{\tau}}_2 = B_h(\boldsymbol{u}_h^n, \boldsymbol{u}_h^n) - B_h(\boldsymbol{u}(t_n), \boldsymbol{u}(t_n)), \quad \tilde{\tau}_4^n = \mu\nabla \cdot (\boldsymbol{u}_h^n - \boldsymbol{u}(t_n)).$$

We will need estimates in $L^\infty(\Omega)$ of $\boldsymbol{u}_h^n$, which are given by the following result.

Lemma 6 *There is a constant $C_D > 0$ such that the following bounds hold:*

$$\left\|\Pi_h^{\mathrm{div}}\boldsymbol{u}\right\|_\infty \le C_D \|\boldsymbol{u}\|_2, \quad \left\|\nabla\Pi_h^{\mathrm{div}}\boldsymbol{u}\right\|_\infty \le C_D \|\boldsymbol{u}\|_3.$$

Proof We prove the second inequality for $d = 3$, the case $d = 2$ and the first inequality being proved similarly. By adding $\pm I_h \boldsymbol{u}$, using (5) and (13) we have

$$\left\|\nabla\Pi_h^{\mathrm{div}}\boldsymbol{u}\right\|_\infty \le C\left(h^{-(2+d)/2}\left\|(\Pi_h^{\mathrm{div}} - I_h)\boldsymbol{u}\right\|_0 + \|\nabla\boldsymbol{u}\|_\infty\right).$$

For the first term, by writing $(\Pi_h^{\text{div}} - I_h) = (\Pi_h^{\text{div}} - I) + (I - I_h)$ and using (10) with $j = 2, 3$ and (13) with $m = 0$ and $n = 2, 3$ we have

$$h^{-(2+d)/2}\left\|(\Pi_h^{\text{div}} - I_h)\boldsymbol{u}(t)\right\|_0 \leq C(\|\boldsymbol{u}\|_2 \|\boldsymbol{u}\|_3)^{1/2} \leq C \|\boldsymbol{u}\|_3.$$

Then, the result follows by estimating $\|\nabla \boldsymbol{u}(t)\|_\infty$ with Sobolev's inequality (3). $\square$

Thus, as with the case $\boldsymbol{w}_h^n = \boldsymbol{s}_h(t_n)$, we also have that for $\boldsymbol{w}_h^n = \boldsymbol{u}_h^n$ the value g_h^n defined in (77) satisfies the bound (87). We also observe that $\tilde{\boldsymbol{\tau}}_2$ and $\tilde{\tau}_4$ can be estimated similarly to (38) so that we can write

$$\|\tilde{\boldsymbol{\tau}}_2^n\|_0 \leq C \|\boldsymbol{u}(t_n)\|_2 \|\boldsymbol{u}(t_n) - \boldsymbol{u}_h^n\|_1 \leq C \|\boldsymbol{u}\|_2 \|\boldsymbol{u}\|_3 h^2 \leq CM_2 \frac{M_3}{t_n^{1/2}} h^2,$$

and, similarly to (34),

$$\|\tau_3^n + \tilde{\tau}_4^n\|_0^2 \leq C(\|p(t_n)\|_2^2 + \mu^2 \|\boldsymbol{u}(t_n)\|_3^2)h^4 \leq C(1 + \mu^2)\frac{M_3^2}{t_n} h^4,$$

and, applying (10) and (19)

$$\|\nabla \tilde{\boldsymbol{t}}_{h,0}^n\|_0^2 \leq C\|\boldsymbol{u}(t_n)\|_3^2 h^4 \leq C\frac{M_3^2}{t_n} h^4.$$

Then, noticing that

$$\Delta t \sum_{j=1}^n \frac{1}{t_j} = 1 + \sum_{j=2}^{n-1} \frac{1}{j} \leq 1 + \log\left(\frac{t_{n+1}}{\Delta t}\right).$$

we have

$$\Delta t \sum_{j=1}^n \left(\nu\|\nabla \boldsymbol{t}_{h,0}^j\|_0^2 + \|\boldsymbol{t}_{h,1}^j\|_0^2 + \frac{1}{\mu}\|\boldsymbol{t}_{h,2}^j\|_0^2\right)$$
$$\leq C\left(\log\left(\frac{t_{n+1}}{\Delta t}\right) M_3^2 \left(\nu + M_2^2 + \frac{1}{\mu} + \mu\right) h^4 + K_{3,0}\Delta t\right).$$

We conclude with the following result.

Theorem 8 *For $T > 0$, assuming the solution $(\boldsymbol{u}, p)$ of (1) satisfies (53), and that M_3, K_3 and $K_{3,0}$ are finite. Assume that piecewise approximations in the velocity of degree $k \geq 2$ are used. Then, the following bound holds for the error $\boldsymbol{e}_h^n = \Pi_h^{\text{div}} \boldsymbol{u}(t_n) - \boldsymbol{U}_h^n$, $1 \leq n \leq N$*

$$\|\boldsymbol{e}_h^n\|_0^2 + \sum_{j=1}^n \|\boldsymbol{e}_h^j - \boldsymbol{e}_h^{j-1}\|_0^2 + \Delta t \sum_{j=1}^n \left(\nu\|\nabla \boldsymbol{e}_h^n\|_0^2 + \mu\|\nabla \cdot \boldsymbol{e}_h^n\|_0^2\right)$$
$$\leq \exp\left(\hat{L}_1 t_n + 2M_3\sqrt{t_n}\right)\left(C_3 \log\left(\frac{t_{n+1}}{\Delta t}\right) h^4 + K_{3,0}\Delta t\right), \quad (88)$$

where $\hat{L}_1$ is defined in (86), and C_3 depends on M_2 and M_3 but not on inverse powers of ν.

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For the pressure, we take inner product of the first equation in (1) with $\boldsymbol{v}_h \in V_h$, subtract (74), add $\pm(\pi_h(t_n), \nabla \cdot \boldsymbol{v}_h)$ and use the inf-sup condition, to obtain (after some rearrangements)

$$
\begin{aligned}
\beta_0 \| p_h^n - \pi_h(t_n) \|_0 \leq{}& \nu \| \nabla \boldsymbol{e}_h^n \|_0 + \| B(\boldsymbol{u}_h^n, \boldsymbol{u}_h^n) - B(\boldsymbol{U}_h^n, \boldsymbol{U}_h^n) \|_{-1} + \| D_t \boldsymbol{e}_h^n \|_{-1} \\
&+ \mu \| \nabla \cdot \boldsymbol{e}_h^n \|_0 + \| \Pi_h^{\mathrm{div}} \boldsymbol{\tau}_{1,2}^n \|_{-1} + \| (I - \Pi_h^{\mathrm{div}}) \partial_t \boldsymbol{u}(t_n) \|_{-1} \\
&+ \| \tilde{\boldsymbol{\tau}}_2 \|_{-1} + \| \tau_3 \|_0 + \| \tilde{\tau}_4 \|_0 + \| l_h(t_n) \|_0,
\end{aligned}
\tag{89}
$$

where $\boldsymbol{e}_h^n = \boldsymbol{u}_h^n - \boldsymbol{U}_h^n$. As in Section 3.2, we estimate $\| \Pi_h^{\mathrm{div}} \boldsymbol{\tau}_{1,2}^n \|_{-1} \leq C \| \Pi_h^{\mathrm{div}} \boldsymbol{\tau}_{1,2}^n \|_0$ and $\| \tilde{\boldsymbol{\tau}}_2 \|_{-1} \leq C \| \tilde{\boldsymbol{\tau}}_2 \|_0$. Using (20) with $j = 1$ we have $\| l_h(t_n) \|_0 \leq \nu h^2 \| \boldsymbol{u}(t_n) \|_2$. Also, repeating the arguments that lead from (45) to (48) with $\boldsymbol{s}_h$ replaced by $\Pi_h^{\mathrm{div}} \boldsymbol{u}$, and in view of Lemma 6, we have

$$
\begin{aligned}
\| B(\boldsymbol{u}_h^n, \boldsymbol{u}_h^n) - B(\boldsymbol{U}_h^n, \boldsymbol{U}_h^n) \|_{-1} \leq{}& C \| \boldsymbol{u}(t_n) \|_2 \left(\| \boldsymbol{e}_h^n \|_0 + \| \nabla \cdot \boldsymbol{e}_h^n \|_0 \right) \\
&+ C \left(h^{-d/2} \| \boldsymbol{e}_h^n \|_0 + h^{-1/2} \| \nabla \cdot \boldsymbol{e}_h^n \|_0 \right) \| \boldsymbol{e}_h^n \|_0.
\end{aligned}
$$

We now estimate $\| (I - \Pi_h^{\mathrm{div}}) \partial_t \boldsymbol{u}(t_n) \|_{-1}$. For $\boldsymbol{\phi} \in H_0^1(\Omega)^d$, we use the Leray decomposition $\boldsymbol{\phi} = \Pi \boldsymbol{\phi} + \nabla q$, and recall that $\| \Pi \boldsymbol{\phi} \|_1 \leq C \| \boldsymbol{\phi} \|_1$ and $\| q \|_2 \leq C \| \boldsymbol{\phi} \|_1$. We notice that

$$
\begin{aligned}
\left((I - \Pi_h^{\mathrm{div}}) \partial_t \boldsymbol{u}, \nabla q \right) &= -\left(\nabla \cdot (I - \Pi_h^{\mathrm{div}}) \partial_t \boldsymbol{u}, q \right) = \left(\nabla \cdot (I - \Pi_h^{\mathrm{div}}) \partial_t \boldsymbol{u}, q - \pi_h q \right) \\
&= \left((I - \Pi_h^{\mathrm{div}}) \partial_t \boldsymbol{u}, \nabla(q - \pi_h q) \right) \leq C h^2 \| \partial_t \boldsymbol{u} \|_1 \| \boldsymbol{\phi} \|_1,
\end{aligned}
\tag{90}
$$

where in the last inequality we have applied (10) with $j = 0$ and (11) with $m = 1$ and $j = 1$. We also have,

$$
\left((I - \Pi_h^{\mathrm{div}}) \partial_t \boldsymbol{u}, \Pi \boldsymbol{\phi} \right) = \left((I - \Pi_h^{\mathrm{div}}) \partial_t \boldsymbol{u}, (I - \Pi_h^{\mathrm{div}}) \Pi \boldsymbol{\phi} \right),
$$

so that applying (10) with $j = 0$, and together with (90), it easily follows that

$$
\| (I - \Pi_h^{\mathrm{div}}) \partial_t \boldsymbol{u}(t_n) \|_{-1} \leq C h^2 \| \partial_t \boldsymbol{u}(t_n) \|_1.
\tag{91}
$$

Finally, arguing as Section 3.2 the term $\| D_t \boldsymbol{e}_h^n \|_{-1}$ can be bounded by the terms on the the right-hand side of (89) except itself, so that we can conclude with the following result.

Theorem 9 *In the conditions of Theorem 8, there exists a constant $C > 0$ not depending on inverse powers of ν such the following bound holds,*

$$
\beta_0^2 \Delta t \sum_{j=1}^{N} \| p_h^n - \pi_h(t_n) \|_0^2 \leq C r(t_n, h, \Delta t) \left(1 + \frac{r(t_n, h, \Delta t)}{h^d} \right),
$$

where $r(t_n, h, \Delta t)$ is the right-hand side of (88).

Remark 13 As in Remark 11, we observe that if the two sources of error (temporal and spatial) are to be balanced in (88) at the final time $t_N = T$, then $\Delta t \approx h^4 \log(N)$. Thus, $h^{-d} \Delta t = \mathcal{O}(h^{4-d} \log(N))$, so that the presence of negative powers of h in the error bound in Theorem 9 does not alter the convergence rate, and the error is $\mathcal{O}(h^4 \log(N) + \Delta t)$.
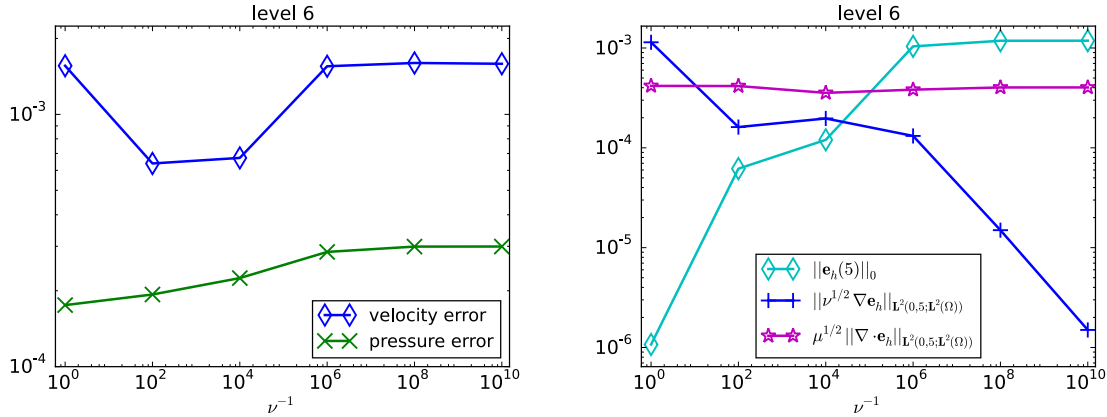
Figure 1: Left: numerical results for the velocity error, left-hand side of estimate (41), and the pressure error, integral term on the left-hand side of estimate (52) with $p - p_h$: different values of ν for a fixed spatial grid. Right: individual contributions of the left-hand side of (41).

6   Numerical studies

In this section, numerical studies will be presented that support the analytical results with respect to the order of convergence and the independence of the errors of ν. As usual for such purposes, an example with a known solution is considered.

Let $\Omega = (0,1)^2$ and $T = 5$, then the Navier–Stokes equations (1) were considered with the prescribed solution

$$
\begin{aligned}
\boldsymbol{u} &= \cos(t) \begin{pmatrix} \sin(\pi x - 0.7)\sin(\pi y + 0.2) \\ \cos(\pi x - 0.7)\cos(\pi y + 0.2) \end{pmatrix}, \\
p &= \cos(t)(\sin(x)\cos(y) + (\cos(1) - 1)\sin(1)).
\end{aligned}
$$

It is clear that examples constructed in this way satisfy the nonlocal compatibility condition. The simulations were performed for the P_2/P_1 pair of finite element spaces on a regular triangular grid consisting on the coarsest level 0 of two mesh cells (diagonal from lower left to upper right). The number of degrees of freedom for velocity/pressure on level 3 is 578/81 and on level 8 it is 526338/66049. As temporal discretization, the Crank–Nicolson scheme was used. The grad-div stabilization parameter was chosen to be $\mu = 0.25$ in all simulations, see [15] for a motivation of this specific choice. In each discrete time, the fully nonlinear problem was solved. The simulations were performed with the code MooNMD [25].

Results of the numerical studies are presented in Figs. 1 and 2. For the simulations on level 6 with different values of ν, Fig. 1, the equidistant time step 0.0625 was used in the Crank–Nicolson scheme. It can be clearly seen that the velocity and pressure errors, which were bounded in the analysis, are independent of ν. Considering the individual contributions of the velocity error, one can observe that in particular the norm of the divergence is almost the same for all values of ν.

For the simulations with constant ν on a sequence of grids, the smaller time steps 0.002 and 0.001 were used. Because the curves for both time steps are almost on top
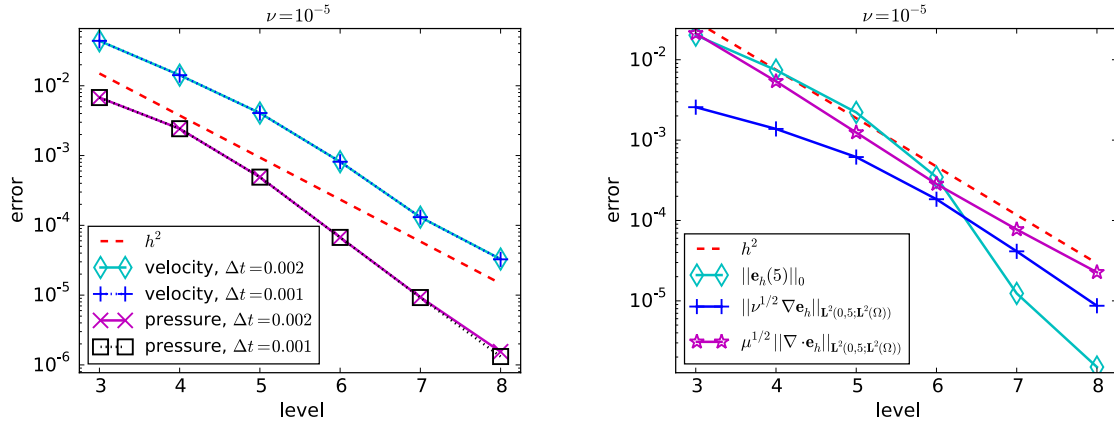
Figure 2: Numerical results for the velocity error, left-hand side of estimate (41), and the pressure error, integral term on the left-hand side of estimate (52) with $p - p_h$. Left: different grid levels and different lengths of the time step for fixed ν (the lines for the time steps 0.001 and 0.002 are almost on top of each other). Right: the individual terms of the left-hand side of (41).

of each other, see Fig. 2, it can be concluded that the temporal error is negligible. The pressure error decreases somewhat faster than predicted by the theory with an order of nearly 2.9. Also the velocity error decreases faster on coarse grids because on these grids the contribution $\|\boldsymbol{e}_h(5)\|_0$ dominates which is reduced by a higher order than two, compare the right picture of Fig. 2. But on finer grids, the predicted second order convergence can be seen.

7 Summary

Inf-sup stable finite element discretizations are considered to approximate the evolutionary Navier–Stokes equations. The Galerkin finite element method is augmented with a grad-div stabilization term. It had been reported in the literature [23, 29] that stable simulations were obtained in the computation of turbulent flows using exclusively grad-div stabilization. This observation is the motivation of the present paper. Error bounds for the Galerkin plus grad-div stabilization method were derived, both for the continuous-in-time case and a fully discrete scheme. The error constants do not depend on inverse powers of ν, although they depend on norms of the solution that are assumed to be bounded. The paper extends a previous work by the same authors [15], where the evolutionary Oseen equations were considered. The analysis covers both the case in which the solution is assumed to be smooth and the practically relevant situation in which nonlocal compatibility conditions are not satisfied and, hence, the derivatives of the solution cannot be assumed to be bounded up to $t = 0$. To the best of our knowledge, this paper is the first one where this breakdown of regularity at $t = 0$ has been taking into account to analyze the effect of the grad-div stabilization. Related works like [4, 8, 7] assume that the solution satisfies nonlocal compatibility conditions. The present paper also seems to be the first one where error bounds with constants independent of ν are obtained for

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a fully discrete method for the Navier–Stokes equations without assuming nonlocal compatibility conditions.

References

[1] Robert A. Adams. *Sobolev spaces*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[2] S. Agmon, *Lectures on Elliptic Boundary Value Problems*. Prepared for publication by B. Frank Jones, Jr. with the assistance of George W. Batten, Jr. Revised edition of the 1965 original. AMS Chelsea Publishing, Providence, RI, 2010.

[3] Naveed Ahmed, Tomás Chacón Rebollo, Volker John and Samuele Rubino. Analysis of Full Space-Time DIscretization of the Navier–Stokes Equations by a Local Projection Stabilization Method. *IMA J. Numer. Anal.* (to appear).

[4] Daniel Arndt, Helene Dallmann, and Gert Lube. Local projection FEM stabilization for the time-dependent incompressible Navier-Stokes problem. *Numer. Methods Partial Differential Equations*, 31(4):1224–1250, 2015.

[5] Blanca Ayuso, Bosco García-Archilla, and Julia Novo. The postprocessed mixed finite-element method for the Navier-Stokes equations. *SIAM J. Numer. Anal.*, 43(3):1091–1111, 2005.

[6] Susanne C. Brenner and L. Ridgway Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, third edition, 2008.

[7] Erik Burman. Robust error estimates for stabilized finite element approximations of the two dimensional Navier-Stokes' equations at high Reynolds number. *Comput. Methods Appl. Mech. Engrg.*, 288:2–23, 2015.

[8] Erik Burman and Miguel A. Fernández. Continuous interior penalty finite element method for the time-dependent Navier-Stokes equations: space discretization and convergence. *Numer. Math.*, 107(1):39–77, 2007.

[9] Hongsen Chen. Pointwise error estimates for finite element solutions of the Stokes problem. *SIAM J. Numer. Anal.*, 44(1):1–28 (electronic), 2006.

[10] Philippe G. Ciarlet. *The finite element method for elliptic problems*. North-Holland Publishing Co., Amsterdam, 1978. Studies in Mathematics and its Applications, Vol. 4.

[11] P. Constantin and C. Foias, *Navier–Stokes Equations*, The Unviersity of Chicago Press, Chicago, 1988.

[12] Helene Dallmann, Daniel Arndt, and Gert Lube. Local projection stabilization for the Oseen problem. *IMA J. Numer. Anal.*, 36(2):796–823, 2016.

[13] Javier de Frutos, Bosco García-Archilla, and Julia Novo. The postprocessed mixed finite-element method for the Navier-Stokes equations: refined error bounds. *SIAM J. Numer. Anal.*, 46(1):201–230, 2007/08.

[14] Javier de Frutos, Bosco García-Archilla, and Julia Novo. Postprocessing finite-element methods for the Navier-Stokes equations: the fully discrete case. *SIAM J. Numer. Anal.*, 47(1):596–621, 2008/09.

[15] Javier de Frutos, Bosco García-Archilla, Volker John, and Julia Novo. Grad-div stabilization for the evolutionary Oseen problem with inf-sup stable finite elements. *Journal of Scientific Computing*, 66(3):991–1024, 2016.

[16] Leopoldo P. Franca and Thomas J. R. Hughes. Two classes of mixed finite element methods. *Comput. Methods Appl. Mech. Engrg.*, 69(1):89–129, 1988.

[17] Vivette Girault and Pierre-Arnaud Raviart. *Finite element methods for Navier-Stokes equations*, volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986. Theory and algorithms.

[18] John G. Heywood and Rolf Rannacher. Finite element approximation of the nonstationary Navier-Stokes problem. I. Regularity of solutions and second-order error estimates for spatial discretization. *SIAM J. Numer. Anal.*, 19(2):275–311, 1982.

[19] John G. Heywood and Rolf Rannacher. Finite element approximation of the nonstationary Navier-Stokes problem. III. Smoothing property and higher order error estimates for spatial discretization. *SIAM J. Numer. Anal.*, 25(3):489–512, 1988.

[20] John G. Heywood and Rolf Rannacher. Finite element approximation of the nonstationary Navier-Stokes problem. IV. Error analysis for second order time discretization. *SIAM J. Numer. Anal.*, 27(2):353–384, 1990.

[21] Eleanor W. Jenkins, Volker John, Alexander Linke, and Leo. G. Rebholz. On the parameter choice in grad-div stabilization for the Stokes equations. *Adv. Comput. Math.*, 40, 491–516, 2014.

[22] Volker John. *Finite Element Methods for Incompressible Flow Problems*, volume 51 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2016.

[23] Volker John and Adela Kindl. Numerical studies of finite element variational multiscale methods for turbulent flow simulations. *Comput. Methods Appl. Mech. Engrg.*, 199(13-16):841–852, 2010.

[24] Volker John, Alexander Linke, Christian Merdon, Michael Neilan, and Leo G. Rebholz. On the divergence constraint in mixed finite element methods for incompressible flows. *SIAM Review*, 2016 (to appear).

[25] Volker John and Gunar Matthies. MooNMD—a program package based on mapped finite element methods. *Comput. Vis. Sci.*, 6(2-3):163–169, 2004.

[26] Gert Lube, Daniel Arndt, and Helene Dallmann. Understanding the Limits of Inf-Sup Stable Galerkin-FEM for Incompressible Flows. in *Boundary and Interior Layers, Computational and Asymptotic Methods - BAIL 2014*, Petr Knobloch (ed.), 147–169, 2016.

[27] Maxim A. Olshanskii. A low order Galerkin finite element method for the Navier-Stokes equations of steady incompressible flow: a stabilization issue and iterative methods. *Comput. Methods Appl. Mech. Engrg.*, 191, 5515–5536, 2002.

[28] Maxim A. Olshanskii, and Arnold Reusken. Grad-div stabilization for Stokes equations. *Math. Comp.*, 73, 1699–1718, 2004

[29] Lars Röhe and Gert Lube. Analysis of a variational multiscale method for large-eddy simulation and its application to homogeneous isotropic turbulence. *Comput. Methods Appl. Mech. Engrg.*, 199(37-40):2331–2342, 2010.

[30] Philipp W. Schoroeder and Gert Lube. Pressure-robust analysis of divergence-free and conforming FEM for evolutionary incompressible Navier–Stokes flows *J. Num. Anal.*, in press, 2017.

[31] R. Temam. Navier–Stokes equations. Theory and numerical analysis- Studies in Mathematics and its Applications. Volume 2. North-Holland. 1979.