# Redividing the Cake * 

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#### Abstract

The paper considers fair allocation of resources that are already allocated in an unfair way. This setting requires a careful balance between the fairness considerations and the rights of the present owners.

The paper presents re-division algorithms that attain various trade-off points between fairness and ownership rights, in various settings differing in the geometric constraints on the allotments: (a) no geometric constraints; (b) connectivity - the cake is a one-dimensional interval and each piece must be a contiguous interval; (c) rectangularity - the cake is a two-dimensional rectangle or rectilinear polygon and the pieces should be rectangles; (d) convexity - the cake is a two-dimensional convex polygon and the pieces should be convex.

These re-division algorithms have implications on another problem: the price-of-fairness-the loss of social welfare caused by fairness requirements. Each algorithm implies an upper bound on the price-of-fairness with the respective geometric constraints.


Keywords Cake-cutting • Land reform • Dynamic fair division • Computational Geometry • Two-dimensional resource allocation

[^0]
## 1 Introduction

Most theoretical works on fair resource allocation consider a one-shot division: the resource is divided once and for all, like a cake that is divided and eaten soon after it comes out of the oven. But in practice, it is often required to redivide an already-divided resource (see subsection 7.1). One example is a cloudcomputing environment, where new agents come and require resources held by other agents. A second example is fair allocation of radio spectrum among several broadcasting agencies: it may be required to re-divide the frequencies to accommodate new broadcasters. A third example is land-reform: large landestates are held by a small number of landlords, and the government may want to re-divide them to landless citizens.

In the classic one-shot division setting, there are $n$ agents with equal rights, and the goal is to give each agent a fair share of the cake. A common definition of a "fair share" is a piece worth at least $1 / n$ of the total cake value, according to the agent's personal valuation function. This fairness requirement is usually termed proportionality. When proportionality cannot be attained, it is often (see subsection 7.2 relaxed to $r$-proportionality, where $r \in(0,1)$ is a constant independent of $n$, which means that each agent receives at least a fraction $r / n$ of the total.

In contrast, in the re-division setting, there is an existing allocation of the cake among the $n$ agents. This allocation is not necessarily fair; in particular, there may be some agents who do not have any cake. When the cake is re-divided, it may be required to give extra rights to current holders. In particular, it may be required to give each agent the opportunity to keep a substantial fraction of their current value. This may be due either to efficiency reasons (in the cloud computing scenario) or economic reasons (in the radio spectrum scenario) or political reasons (in the land-reform scenario). This requirement will be called ownership. Given a constant $w \in(0,1)$, $w$-ownership means that each agent receives at least $w$ times their old value. What levels of proportionality and ownership can be attained simultaneously?

### 1.1 Results: Redivision

The first two results (in Section 3) provide a tight answer to this question.

Proposition 1 For every constants $r, w \in[0,1]$ where $r+w>1$, it may be impossible to simultaneously guarantee r-proportionality and w-ownership.

Theorem 1 For every constants $r, w \in[0,1]$ where $r+w \leq 1$, and for every existing allocation of the cake, there exists a division that simultaneously satisfies $r$-proportionality and $w$-ownership. Moreover, when $r, w$ are rational numbers, such a division can be found using $O\left(n^{2} \operatorname{len}(r)\right)$ queries, where len $(r)$ denotes binary representation length.

As an example, taking $r=w=1 / 2$, it is possible to re-divide the cake, giving each agent at least half their previous value, while simultaneously giving each agent at least $1 /(2 n)$ of the total cake value.

The parameters $r, w$ represent the level of balance between two principles: large $r$ means more emphasis on fairness while large $w$ means more emphasis on ownership rights. The above theorems imply that the re-dividers (e.g. the government) may choose any level of fairness and ownership-rights that fit their ideological, political or economic goals, as long as the sum of these fractions is at most 1 .

The balance parameters can also be given probabilistic interpretation. Suppose the government wants to do a land reform and needs the agreement of the current landowners. Naturally, the current landowners do not want to give away their lands. However, they may fear that, without land-reform, the landless citizens might revolt and they might lose all their lands. If the landowners believe that the probability of a successful revolt is $1-w$, then they may agree to a land-reform that guarantees $w$-ownership. Theorem 1 implies that, in this case, it is possible to carry out a land-reform that guarantees $(1-w)$ proportionality.

While Theorem 1 is encouraging, it ignores an important aspect of practical division problems: geometry. The division it guarantees may be highly fractioned, giving each agent a large number of disconnected pieces. In many practical division problems, e.g. when the resource to divide is time, the agents may need to receive a single connected piece rather than a large number of disconnected ones. Can partial-proportionality and partial-ownership be attained simultaneously with a connectivity constraint? The following proposition (proved in Section 4) answers this question negatively.

Proposition 2 When the cake is a 1-dimensional interval and each piece must be an interval, for every positive constants $r, w \in(0,1)$, it may be impossible to simultaneously satisfy r-proportionality and w-ownership.

Moreover, for every $r>0$ and every integer $d \in[n]$, there might be d agents who, in any r-proportional division, receive at most a fraction $1 /\left\lfloor\frac{n}{d}\right\rfloor$ of their old value.

The latter part of the proposition involves a fairness property much weaker than proportionality, that can be termed positivity - guaranteeing each agent a piece with a positive value. With the connectivity constraint, even this weak fairness requirement is incompatible with $w$-ownership for every constant $w>$ 0 : a positive division might require to give one agent at most $1 / n$ of their previous value, give two agents at most $2 / n$ of their previous value, give $n / 3$ agents at most $1 / 3$ of their previous value, etc.

Proposition 2 motivates the following weaker ownership requirement: for every $d$, at least $n-d$ agents receive at least a fraction $1 /\left\lfloor\frac{n}{d}\right\rfloor$ of their old value. For example (taking $d=n / 3$ and assuming all quotients are integers), at least $2 n / 3$ agents should receive at least $1 / 3$ of their old value. This criterion is inspired by the " 90 th percentile" criterion common in Service-LevelAgreements and Quality-of-Service analysis, e.g. (Zhang et al., 2014 Delim-
itrou and Kozyrakis, 2014). It can also be justified by political reasoning: in a democratic country, it may be sufficient to win the support of a sufficiently large majority.

The following results almost match this relaxed ownership criterion. Formally, let us define the democratic ownership property as follows: for every integer $d \in\{1, \ldots, n-1\}$, at least $n-d$ agents receive more than a fraction $1 /\left\lceil\frac{n}{d}\right\rceil$ of their previous value. Democratic-ownership corresponds to the best guarantee one could hope for given Proposition 2 the only difference is that in the upper bound the fraction is rounded down $\left(1 /\left\lfloor\frac{n}{d}\right\rfloor\right)$ while in democraticownership the fraction is rounded up.

Theorem 2 When the cake is a 1-dimensional interval and each piece must be an interval, for every existing allocation of the cake, it is possible to find in time $O\left(n^{2} \log n\right)$ a division simultaneously satisfying democratic-ownership and $1 / 2$-proportionality.

It is an open question whether democratic-ownership is compatible with $r$ proportionality for some constant $r>1 / 2$.

Theorem 2, like most works in cake-cutting, assumes that the cake is $1-$ dimensional. In realistic division scenarios, the cake is often 2-dimensional and the pieces should have a pre-specified geometric shape, such as a rectangle or a convex polygon. Rectangularity and convexity requirements are sensible when dividing land, exhibition space in museums, advertisement space in newspapers and even virtual space in web-pages. Moreover, in the frequency-range allocation problem, it is possible to allocate frequency ranges for a limited time-period; the frequency-time space is two-dimensional and it makes sense to require that the "pieces" are rectangles in this space (Iyer and Huhns, 2009).

2-dimensional cake-cutting introduces new challenges over the traditional 1-dimensional setting. As an example, in one dimension, it can be assumed that the initial allocation is a partition of the entire cake; this is without loss of generality, since any "blank" (unallocated part) can be attached to a neighboring allocated interval without harming its shape or value. However, in two dimensions, the initial allocation might contain blanks that cannot be attached to any allocated piece due to the rectangularity or convexity constraints. For example, suppose the cake is the large rectangle in Figure 1. There are 4 agents and each agent $i$ has positive value-density only inside the rectangle $Z_{i}$. The most reasonable division (e.g. the only Pareto-efficient division) is to give each $Z_{i}$ entirely to agent $i$. But, this allocation leaves a blank in the center of the cake, and this blank cannot be attached to any allocated piece due to the rectangularity constraint. This counter-intuitive scenario cannot happen in a one-dimensional cake. Handling such cases requires new geometry-based tools. With such tools, the redivision problem can be solved in two common 2-dimensional settings (Section 5):
Theorem 3 When the cake is a rectangle and each piece must be a parallel rectangle, for every existing allocation of the cake, it is possible to find in time $O\left(n^{2} \log n\right)$ a division simultaneously satisfying democratic-ownership and 1/3proportionality.


Fig. 1 With geometric constraints, an efficient allocation might leave some cake unallocated. All figures were made with GeoGebra 5 (Hohenwarter et al. 2013).


Fig. 2 A rectilinear polygon with $T=4$ reflex vertices (circled)

Theorem 4 When the cake is a 2-dimensional convex polygon and each piece must be convex, for every existing allocation of the cake, there exists a division simultaneously satisfying democratic-ownership and 1/4-proportionality.

Remark 1 In the interval, rectangle and convex settings, the geometric constraints are mostly harmless without the ownership requirement: when the cake is an interval/rectangle/convex, classic algorithms for proportional cakecutting, such as Even and Paz (1984), can be easily made to return interval/rectangle/convex pieces by ensuring that the cuts are parallel. Similarly, the ownership requirement is easy to satisfy without the geometric constraints, as shown by Theorem 1. It is the combination of these two requirements that leads to interesting challenges.

Most land-estates are not exact rectangles, but they can be approximated by a rectilinear polygon-a polygon in which all angles are $90^{\circ}$ or $270^{\circ}$. The next result generalizes Theorem 3 to a rectilinear polygonal cake. The complexity of a rectilinear polygon is characterized by the number of its reflex vertices - vertices with a $270^{\circ}$ angle. Denote this number by $T$. A rectangle the simplest rectilinear polygon-has $T=0$. The cake in Figure 2 has $T=4$.

Theorem 5 When the cake is a rectilinear polygon with $T$ reflex vertices, and each piece must be a rectangle, for every existing allocation of the cake, it is
possible to find in time $O\left(n^{2} \log n+\operatorname{poly}(T)\right)$ a division satisfying democraticownership, in which each agent receives at least $1 /(3 n+T)$ of the total cake value.

The dependence on $T$ is necessary: even without ownership requirements, there are instances in which it is impossible to guarantee a fraction of more than $1 /(n+T)$ to all $n$ agents (Segal-Halevi, 2021).

### 1.2 Results: Price of Fairness

Redivision algorithms can be used not only to compromise between old and new agents, but also to compromise between fairness and efficiency. Often, the most economically-efficient allocation is not fair, while a fair allocation is not economically-efficient. The trade-off between fairness and efficiency is quantified by the price-of-fairness (Bertsimas et al., 2011, 2012; Caragiannis et al. 2012, Aumann and Dombb, 2015). It is defined as the worst-case ratio of the maximum attainable social-welfare to the maximum attainable social-welfare of a fair allocation. The social welfare is usually defined as the arithmetic mean of the agents' values (also called utilitarian welfare) or their geometric mean (also called Nash welfare; see Moulin (2004)).

A redivision algorithm can be used to calculate an upper bound on the price of fairness in the following way. Take a welfare-maximizing allocation as the initial allocation; use a redivision algorithm to produce a partiallyproportional allocation in which the utility of each agent is close to their initial utility; conclude that the new welfare is close to the initial (maximal) welfare.

Without geometric constraints, the following is an upper bound on the price-of-fairness w.r.t. utilitarian welfare ${ }^{1}$

Theorem 6 For every $r \in[0,1]$, the utilitarian price of $r$-proportionality is at most $1 /(1-r)$.

When $r=1$ the above bound is infinite, and indeed, the price of 1-proportionality in this setting is $\Theta(\sqrt{n})$, which is not bounded by any constant (Caragiannis et al., 2012). Theorem 6 shows that a small compromise on the level of proportionality allows a constant (independent of $n$ ) bound on the utilitarian-price. The parameter $r$ sets the level of trade-off between fairness and efficiency.

With geometric constraints, the following upper bounds are proved:
Theorem 7 When the cake is an interval and each piece must be an interval, for every $r \leq 1 / 2$ :

- The utilitarian-price of r-proportionality is $O(\sqrt{n})$;
- The Nash-price of r-proportionality is at most 5.6.

[^1]Theorem 8 When the cake is a rectangle and each piece must be a rectangle, for every $r \leq 1 / 3$ :

- The utilitarian-price of r-proportionality is $O(\sqrt{n})$;
- The Nash-price of r-proportionality is at most 8.4.

Theorem 9 When the cake is convex polygon and each piece must be convex, for every $r \leq 1 / 4$ :

- The utilitarian-price of $r$-proportionality is $O(\sqrt{n})$;
- The Nash-price of r-proportionality is at most 11.2.

Note that the first claim in Theorem 7 is subsumed by Aumann and Dombb (2015), who prove that the utilitarian-price of 1-proportionality in this setting is $\Theta(\sqrt{n})$. It is brought here only for completeness. The second claim in that theorem, as well as the following theorems regarding two-dimensional constraints, are not implied by previous results.

Appendix Apartially complements the above results by showing some lower bounds on the price of fairness with interval cake and interval pieces:

- With two agents, for all $r \in[0,1]$, the utilitarian price of $r$-proportionality is $1+r / 2$ and the Nash price of $r$-proportionality is $\max (1, \sqrt{2 r})$.
- With $n$ agents, there is a lower bound on the Nash price of proportionality, which approaches 2 as $n \rightarrow \infty$.
Computing the exact utilitarian price and Nash price of $r$-proportionality for any $n \geq 2$ and $r \leq 1$ in this setting remains an open question.

Remark 2 A third measure of welfare is the egalitarian welfare, defined as the minimum of the agents' values (normalized such that the total cake value is the same for all agents). The egalitarian price of $r$-proportionality is 1 for all $r \leq 1$ whenever an $r$-proportional allocation exists. This is because, whenever an $r$-proportional allocation exists, its egalitarian welfare is at least $r / n$ of the total cake value. Therefore, in any egalitarian-optimal allocation, the egalitarian welfare is at least $r / n$ of the total cake value. By definition, any such allocation is $r$-proportional. So the maximum attainable egalitarian welfare in an $r$-proportional allocation equals the maximum attainable egalitarian welfare overall.

## 2 Model

### 2.1 Cake Division

The cake $C$ is a polytope in the $d$-dimensional Euclidean plane $\mathbb{R}^{d}$. This paper focuses on the common cases in which $d=1$ and $C$ is an interval, or $d=2$ and $C$ is a polygon. A piece is a Borel subset of $C$; usually an interval or a polygon.
$C$ has to be divided among $n \geq 1$ agents. We denote by $[n$ ] the set of integers $\{1, \ldots, n\}$. Each agent $i \in[n]$ has a value-density function $v_{i}$, which
is an integrable, non-negative and bounded function on $C$. The value of a piece $X_{i}$ to agent $i$ is marked by $V_{i}\left(X_{i}\right)$ and it is the integral of its value-density: $V_{i}\left(X_{i}\right)=\int_{x \in X_{i}} v_{i}(x) d x$. The definition implies that the $V_{i}$ are finite measures and are absolutely-continuous with respect to the Lebesgue measure, i.e., any piece with zero area has zero value to all agents.

Division algorithms access the value measures via queries (Robertson and Webb, 1998, Woeginger and Sgall, 2007): an eval query asks an agent to report the value of a specified piece of cake; a mark query asks an agent to mark a piece of cake with a specified value $\int^{2}$ The present paper ignores strategic considerations and assumes that agents answer truthfully. Indeed, in general it may be impossible to build a cake-cutting algorithm that is both fair and strategy-proof (Brânzei and Miltersen, 2015).

The geometric constraints, if any, are represented by a pre-specified family $S$ of usable pieces. In this paper, $S$ will either be the set of all pieces (which means that there are no geometric constraints), or the set of all intervals, or the set of all rectangles, or the set of all convex pieces. It is assumed that each agent can use only a single piece from the family $S$.

An allocation is a vector of $n$ pieces, $\mathbf{X}=\left(X_{1}, \ldots, X_{n}\right)$, one piece per agent, such that the $X_{i}$ are pairwise-disjoint and $X_{1} \sqcup \cdots \sqcup X_{n} \subseteq C{ }^{3}$ Note that some cake may remain unallocated, i.e, free disposal is assumed. As illustrated in the introduction, free disposal may be necessary when there are geometric constraints. An $S$-allocation is an allocation in which all pieces are usable, i.e, $X_{i} \in S$ for each agent $i$.

For every constant $r \in[0,1]$, an allocation $\mathbf{X}$ is called $r$-proportional if every agent receives at least $r / n$ of the total cake value:

$$
\text { For all } i \in[n]: \quad V_{i}\left(X_{i}\right) \geq(r / n) \cdot V_{i}(C)
$$

A 1-proportional division is also known as proportional.

### 2.2 Cake Redivision

There is an existing $S$-allocation of the cake, $Z_{1} \sqcup \cdots \sqcup Z_{n} \subseteq C$. It is assumed that the old pieces $Z_{j}$ are pairwise-disjoint and that $Z_{j} \in S$ for all $j$, but nothing else is assumed on the division. In particular, the initial division is not necessarily proportional, and some of $C$ may be unallocated.

It is required to construct a new $S$-allocation $X_{1} \sqcup \cdots \sqcup X_{n} \subseteq C$. The re-allocation satisfies the $w$-ownership property, for some constant $w \in(0,1)$, if every agent receives at least a fraction $w$ of their old value:

$$
\text { For all } j \in[n]: \quad V_{j}\left(X_{j}\right) \geq w \cdot V_{j}\left(Z_{j}\right)
$$

[^2]Since $w$-ownership is not always compatible with $r$-proportionality for any constant $r>0$, the following weaker property is defined. A re-allocation $\mathbf{X}$ satisfies the democratic-ownership property if, for every $d \in\{1, \ldots, n-1\}$, there are at least $n-d$ agents $j \in[n]$ for whom

$$
V_{j}\left(X_{j}\right)>\frac{1}{\lceil n / d\rceil} \cdot V_{j}\left(Z_{j}\right)
$$

### 2.3 Social Welfare and Price-of-Fairness

In addition to fairness, it is often required that a division has a high social welfare. The social welfare of an allocation is a certain aggregate function of the normalized values of the agents (the normalized value is the piece value divided by the total cake value). Common social welfare functions are sum (utilitarian) and product (Nash), see Moulin (2004). When calculating the welfare, it is convenient to normalize the values such that the proportional share of an agent corresponds to a value of 1 (so receiving the entire cake corresponds to a value of $n$ ). This way, when all agents receive exactly their proportional share, the welfare is 1 .

- Utilitarian welfare - the arithmetic mean of the agents' normalized values

$$
W_{u t i l}(\mathbf{X})=\frac{1}{n} \sum_{i=1}^{n} \frac{V_{i}\left(X_{i}\right)}{V_{i}(C) / n}
$$

- Nash welfare - the geometric mean of the agents' normalized values:

$$
W_{N a s h}(\mathbf{X})=\left(\prod_{i=1}^{n} \frac{V_{i}\left(X_{i}\right)}{V_{i}(C) / n}\right)^{1 / n}
$$

The goal of maximizing the social welfare is not always compatible with the goal of guaranteeing a fair share to every agent. For example, Caragiannis et al. (2012) describe a simple example in which the maximum utilitarian welfare of a proportional allocation is in 1 while the maximum utilitarian welfare of an arbitrary (unfair) allocation is in $\Omega(\sqrt{n})$. This means that society has to pay a price, in terms of social-welfare, for insisting on fairness. This is called the price of fairness. Formally, given a social welfare function $W$ and a fairness criterion $F$, the price-of-fairness relative to $W$ and $F$ (also called: "the $W$-price-of- $F$ ") is the ratio:

$$
\begin{equation*}
\frac{\sup _{\mathbf{X}} W(\mathbf{X})}{\sup _{\mathbf{Y} \in F} W(\mathbf{Y})} \tag{*}
\end{equation*}
$$

where the supremum at the numerator is over all allocations $\mathbf{X}$ and the supremum at the denominator is over all allocations $\mathbf{Y}$ that also satisfy the fairness
criterion $F$. The cited example shows that the utilitarian-price-of-proportionality is in $\Omega(\sqrt{n})$.

When there are geometric constraints, they affect both the numerator and the denominator of $\left({ }^{*}\right)$, i.e, the suprema are taken only on $S$-allocations. Therefore, it is not a-priori clear whether the price-of-fairness with constraints is higher or lower than without constraints.

## 3 Arbitrary Cake and Arbitrary Pieces

In this section there are no geometric constraints on the cake or its pieces. Consider the negative result first.

Proposition 1 For every constants $r, w \in[0,1]$ where $r+w>1$, it may be impossible to simultaneously guarantee $r$-proportionality and $w$-ownership.

Proof Here is a scenario in which no $r$-proportional division satisfies $w$-ownership. In the initial allocation, a single agent owns the entire cake. All $n$ agents have the same value-density and they value the entire cake at 1 . In any $r$ proportional division, the $n-1$ landless citizens must receive a total value of $(n-1) r / n=r-r / n$. Therefore the old landlord receives at most $1-r+r / n$. By assumption, $1-r<w$. Hence, if $n$ is sufficiently large, the old landlord receives less than $w$ of his/her previous value, contradicting $w$-ownership.

The proof of the matching positive result requires a lemma.
Lemma 1 Given cake-allocations $\mathbf{Z}$ and $\mathbf{Y}$ and a constant $r \in[0,1]$, there exists an allocation $\mathbf{X}$ such that, for every agent $i \in[n]: \quad V_{i}\left(X_{i}\right) \geq r V_{i}\left(Y_{i}\right)+$ $(1-r) V_{i}\left(Z_{i}\right)$. Moreover, when $r$ is a constant rational number, $\mathbf{X}$ can be found using $O\left(n^{2} \cdot \operatorname{len}(r)\right)$ queries, where $\operatorname{len}(r)$ is the length of the binary representation of $r$.

Proof Let us begin with an existential proof. Consider the set of all possible cake-partitions. For each cake-partition, consider the $n \times 1$ vector of utilities of the agents. The Dubins-Spanier theorem (Dubins and Spanier, 1961) implies that the set of all such vectors is convex. So there is an allocation $\mathbf{X}$ satisfying the requirement as an equality: $\forall i \in[n]: V_{i}\left(X_{i}\right)=r V_{i}\left(Y_{i}\right)+(1-r) V_{i}\left(Z_{i}\right)$.

The Dubins-Spanier theorem is not constructive. But when $r$ is a rational number, $r=p / q$ with $p<q$ some positive integers, an allocation $\mathbf{X}$ satisfying the lemma requirements can be constructed in polynomial time using an algorithm for a different problem: fair division with different entitlements. In this problem, each agent $i \in[n]$ is entitled to a share $d_{i} / D$ of the entire cake, where $d_{i}$ is a positive integer and $D=\sum_{i} d_{i}$. Recently, Cseh and Fleiner (2020) presented an algorithm that finds, using $2(n-1)\left\lceil\log _{2}(D)\right\rceil$ queries, an allocation in which the value of each agent $i$ is at least $d_{i} / D$ of the total cake value. This algorithm should be applied to all pairs of agents. For every pair $i, j \in[n]$ with $i \neq j$, partition $Y_{i} \cap Z_{j}$ between $i$ and $j$ with $d_{i}:=p$ and $d_{j}:=q-p$. The

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Algorithm 1 Cake allocation with partial proportionality and ownership.
Input:
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    - A cake \(C\) and an existing allocation \(Z_{1} \sqcup \cdots \sqcup Z_{n} \subseteq C\);
    - A rational number \(r=p / q\).
    Output: A new allocation $X_{1} \sqcup \cdots \sqcup X_{n} \subseteq C$ satisfying the following:
- Partial proportionality: for all $i \in[n], V_{i}\left(X_{i}\right) \geq r \cdot V_{i}(C) / n$.
- Partial ownership: for all $i \in[n], V_{i}\left(X_{i}\right) \geq(1-r) \cdot V_{i}\left(Z_{i}\right)$.
Find a proportional allocation $Y_{1} \sqcup \cdots \sqcup Y_{n}=C$.
for $i:=1, \ldots, n$ and $j:=1, \ldots, n$ (when $i \neq j$ ) do
Using an algorithm for fair cake cutting with different entitlements (Cseh and Fleiner
2020, divide $Y_{i} \cap Z_{j}$ between $i$ and $j$ such that $i$ is entitled to $p / q$ and $j$ is entitled
to $(q-p) / q$.
end for
Allocate to each agent $i \in[n]$ the union of the following pieces:
- The piece $Y_{i} \cap Z_{i}$;
- $i$ 's share from $Y_{i} \cap Z_{j}$ for all $j \neq i$;
- $i$ 's share from $Y_{j} \cap Z_{i}$ for all $j \neq i$.
pairs $i, j$ can be processed in any order, even in parallel. Finally, agent $i$ also gets the entire piece $Y_{i} \cap Z_{i}$.

For the sake of the proof, divide this latter piece arbitrarily into two subsets: one is worth $\frac{p}{q} V_{i}\left(Y_{i} \cap Z_{i}\right)$ and the other $\frac{q-p}{q} V_{i}\left(Y_{i} \cap Z_{i}\right)$ for agent $i$. Now, each agent $i$ is allocated a piece $X_{i}$ which can be written as a union of $2 n$ disjoint subsets: some $n$ subsets of $Y_{i} \cap Z_{1}, \ldots, Y_{i} \cap Z_{n}$, and some $n$ subsets of $Y_{1} \cap Z_{i}, \ldots, Y_{n} \cap Z_{i}$. The former subsets are worth for $i$ at least $\frac{p}{q} V_{i}\left(Y_{i} \cap Z_{1}\right)+$ $\cdots+\frac{p}{q} V_{i}\left(Y_{i} \cap Z_{n}\right)=\frac{p}{q} V_{i}\left(Y_{i} \cap C\right)=\frac{p}{q} V_{i}\left(Y_{i}\right)=r V_{i}\left(Y_{i}\right)$, and the latter subsets are worth for $i$ at least $\frac{q-p}{q} V_{i}\left(Y_{1} \cap Z_{i}\right)+\cdots+\frac{q-p}{q} V_{i}\left(Y_{n} \cap Z_{i}\right)=\frac{q-p}{q} V_{i}\left(Z_{i} \cap C\right)=$ $\frac{q-p}{q} V_{i}\left(Z_{i}\right)=(1-r) V_{i}\left(Z_{i}\right)$.

The algorithm requires $O(\log q)$ steps for every pair and $O\left(n^{2} \log q\right)$ steps overall. Since len $(r)=\log _{2} p+\log _{2} q \geq \log _{2} q$, the run-time is in $O\left(n^{2} \operatorname{len}(r)\right)$ as claimed.

Theorem 1 For every constants $r, w \in[0,1]$ where $r+w \leq 1$, and for every existing division of the cake, there exists a division that simultaneously satisfies $r$-proportionality and $w$-ownership. Moreover, when $r, w$ are rational numbers, such a division can be found using $O\left(n^{2} \operatorname{len}(r)\right)$ queries, where len(•) denotes binary representation length.

Proof Given a pair $r, w$ where $r+w \leq 1$, apply Lemma 1, with the initial allocation as $\mathbf{Z}$, and any proportional allocation as $\mathbf{Y}$ (a proportional allocation can be found efficiently by classic algorithms such as Steinhaus (1948), Even and Paz (1984)). By Lemma 1, the new division satisfies $r$-proportionality and $(1-r)$-ownership, and $1-r \geq w$. The process is summarized as Algorithm 1.

Remark 3 (a) The redivision algorithm gives each agent a piece that is not only worth at least $(1-r) V_{i}\left(Z_{i}\right)$, but also a subset of $Z_{i}$ (in addition to a subset of $Y_{i}$ ). This may be desirable in some cases. E.g. in land division, old landlords may want not only a high value but also a subset of their old plot.
(b) Cseh and Fleiner (2020) present an algorithm for cake-cutting even when the entitlements are irrational. The number of queries is finite (but unbounded). This algorithm can be used in Algorithm 1 to attain $r$-proportionality even when $r$ is irrational, though it is unclear why any government would be interested in such a strange fairness condition.

## 4 Interval Cake and Interval Pieces

In this section the cake is an interval and each piece must be an interval. Consider the negative result first.

Proposition 2 When the cake is a 1-dimensional interval and each piece must be an interval, for every positive constants $r, w \in(0,1)$, it may be impossible to simultaneously satisfy $r$-proportionality and $w$-ownership.

Moreover, for every $r \in(0,1]$ and every integer $d \in[n]$, there might be $d$ agents who, in any r-proportional division, receive at most a fraction $1 /\left\lfloor\frac{n}{d}\right\rfloor$ of their old value.

Proof Consider an existing allocation $\mathbf{Z}$, a positive constant $r \in(0,1]$, and an integer $d \leq n$. Here is a scenario in which, in every $r$-proportional allocation, there are $d$ agents $j$ who receive a value of at most $V_{j}\left(Z_{j}\right) /\left\lfloor\frac{n}{d}\right\rfloor$. Partition the set of $n$ agents into $d$ groups:

1. $(n \bmod d)$ groups containing $\left\lceil\frac{n}{d}\right\rceil$ agents; these groups exist only when $d$ does not divide $n$.
2. $(d-n \bmod d)$ groups containing $\left\lfloor\frac{n}{d}\right\rfloor$ agents.

Note that the total number of agents in these groups is indeed $(n \bmod d)$. $\left\lceil\frac{n}{d}\right\rceil+(d-n \bmod d) \cdot\left\lfloor\frac{n}{d}\right\rfloor=n$.

In each group of type 1 , a single agent $j$ has a nonempty share $Z_{j}$ in the initial allocation. Agent $j$ values $Z_{j}$ at $\left\lceil\frac{n}{d}\right\rceil$, and the rest of the cake at 0 . The value-density inside $Z_{j}$ is piecewise-uniform: it has $\left\lceil\frac{n}{d}\right\rceil$ regions with a value of 1 and $\left\lceil\frac{n}{d}\right\rceil-1$ "gaps"-regions with a value of 0 . Each of the other $\left\lceil\frac{n}{d}\right\rceil-1$ agents in the same group assigns a positive value only to a unique gap in $Z_{j}$; Figure 3 illustrates the value-densities that are positive in one such $Z_{j}$.

In each group of type 2 , the valuations are defined similarly to the groups of type 1 , except that $j$ 's total value is $\left\lfloor\frac{n}{d}\right\rfloor$ and there are $\left\lfloor\frac{n}{d}\right\rfloor-1$ other agents.

In any $r$-proportional division, each gap in $Z_{j}$ must be at least partially allocated to an agent in group $j$. Hence, the interval allocated to agent $j$ must contain at most a single positive region in $Z_{j}$-it cannot overlap any gap. Therefore the value of agent $j$ is at most 1 . For agents in groups of type 2, this value is at most $V_{j}\left(Z_{j}\right) /\left\lfloor\frac{n}{d}\right\rfloor$; for agents in groups of type 1 , it is at most $V_{j}\left(Z_{j}\right) /\left\lceil\frac{n}{d}\right\rceil$, which is even smaller.


Fig. 3 Solid boxes represent the value-density of agent $j$ within $Z_{j}$; each dotted box represents a value-density of some other agent in the same group as agent $j$. In this example, $\left\lceil\frac{n}{d}\right\rceil=5$.

```
Algorithm 2 Auctioning a cake.
Input: A cake \(Z_{0}\) and a set \(N \subseteq[n]\) of agents.
Output: A subset of agents \(W \subseteq N\), possibly empty, such that
    (a) For each agent \(i^{\prime} \in W, \quad V_{i^{\prime}}\left(Z_{0}\right) \geq|W|\);
    (b) For each agent \(i^{\prime \prime} \in N \backslash W, \quad V_{i^{\prime \prime}}\left(Z_{0}\right)<|W|+1\).
Choose an ordering \(\sigma\) on \(N\) such that \(V_{\sigma[1]}\left(Z_{0}\right) \geq V_{\sigma[2]}\left(Z_{0}\right) \geq \cdots V_{\sigma[|N|]}\left(Z_{0}\right)\).
Initialize \(W:=\emptyset\)
for \(j:=1, \ldots,|N|\) do
    if \(V_{\sigma[j]}\left(Z_{0}\right) \geq j\) then
        Add agent \(\sigma[j]\) to \(W\).
    else
        return \(W\).
    end if
end for
```

The corresponding positive result (Theorem 2) uses an algorithm for a different problem: fair multicake cutting. In this problem, there is a multicake $C$, which is a union of $m$ pairwise-disjoint subcakes, $C=Z_{1} \sqcup \cdots \sqcup Z_{m}$. The goal is to give each agent a piece contained in a single subcake. It is easy to see that a proportional allocation might not exist even for a single agent. However, there always exists an allocation $\left(X_{1}, \ldots, X_{n}\right)$ such that

$$
\begin{equation*}
V_{i}\left(X_{i}\right) \geq \frac{1}{m+n-1} \cdot V_{i}(C) \tag{1}
\end{equation*}
$$

and this is the largest fraction that can be guaranteed (Segal-Halevi, 2021), ${ }^{4}$ Below, a different algorithm is presented, that attains the same value guarantee (1), and simultaneously guarantees democratic ownership.

The algorithm uses as a subroutine Algorithm 2, which is called an "auction". It accepts as input a subcake $Z_{0}$ and a set $N$ of agents. Each agent $i$

[^3]"bids" by evaluating $Z_{0}$. The auction then chooses a subset $W \subseteq N$ of "winners". The criterion for selecting the set of winners is specified by the following lemma.

Lemma 2 Given a subcake $Z_{0}$ and a set $N$ of agents, Algorithm 2 returns a subset $W \subseteq N$ of winners such that (a) each winner values $Z_{0}$ at least $|W|$, and (b) each loser values $Z_{0}$ at less than $|W|+1$.

Proof The set of winners contains the first $|W|$ agents in the ordering $\sigma$. Now:

- Let $i^{\prime}:=\sigma[|W|]=$ the last agent added to $W$. Step 4 implies that $V_{i^{\prime}}\left(Z_{0}\right) \geq$ $|W|$. The same is true for all preceding agents in the ordering $\sigma$. Hence, condition (a) is satisfied for all winners.
- Let $i^{\prime \prime}:=\sigma[|W|+1]=$ the first agent not added to $W$. Step 4 implies that $V_{i^{\prime \prime}}\left(Z_{0}\right)<|W|+1$. The same is true for all following agents in the ordering $\sigma$. Hence, (b) is satisfied for all losers.

Note that Lemma 2 allows the set of winners $W$ to be empty, if all agents in $N$ value $Z_{0}$ at less than 1.

Before proving Theorem 2, let us consider a simpler warm-up algorithm that attains only the partial-proportionality guarantee (1). It uses Algorithm 3. Its input is a multicake and a set of $n$ agents. By repeatedly applying the auction algorithm, it assigns the agents to the subcakes such that all agents assigned to a subcake value it sufficiently high, as formalized below.

Lemma 3 Given a multicake $C=Z_{1} \sqcup \cdots \sqcup Z_{m}$, a positive integer $n \leq m$, and some $n$ agents who value $C$ at least $m+n-1$, Algorithm 3 returns a partitioning of the set of agents $[n]=W_{1} \sqcup \cdots \sqcup W_{m}$ such that for all $j \in[m]$ and for each agent $i^{\prime} \in W_{j}, V_{i^{\prime}}\left(Z_{j}\right) \geq\left|W_{j}\right|$.

Proof Lemma 2(a) ensures that all agents assigned to $W_{j}$ in step 3 value $Z_{j}$ at least $\left|W_{j}\right|$. It remains to prove that, by the end of the algorithm, every agent $i \in[n]$ is assigned to some $W_{j}$. Suppose by contradiction that some $i \in[n]$ is not in any $W_{j}$. Lemma $2(\mathrm{~b})$ ensures that $V_{i}\left(Z_{j}\right)<\left|W_{j}\right|+1$ for all $j \in[m]$. Summing over all $j \in[m]$ gives $V_{i}(C)=V_{i}\left(Z_{1}\right) \cup \cdots \cup V_{i}\left(Z_{m}\right)<\sum_{j=1}^{m}\left(\left|W_{j}\right|+\right.$ $1) \leq(n-1)+m$, which contradicts the assumption $V_{i}(C) \geq(n-1)+m$.

Once the agents are partitioned using Algorithm 3, for each $j \in[m], Z_{j}$ can be divided among the agents in $W_{j}$ using any proportional cake-cutting algorithm. Since all these agents value $Z_{j}$ at least $\left|W_{j}\right|$, each agent gets a piece valued at least 1 , which is at least $\frac{1}{m+n-1} \cdot V_{i}(C)$ as in condition (1).

To prove Theorem 2 it is required to guarantee, in addition to (1), also the democratic ownership condition. To this end, Algorithm 3 is replaced with a modified assignment algorithm, presented as Algorithm 4. The main difference is that Algorithm 4 allows each agent $j$ to participate in the auction on the subcake with the same index $Z_{j}$, even if $j$ was already assigned to a previous subcake $Z_{j^{\prime}}$ for some $j^{\prime}<j$. If $j$ is one of the winners for $Z_{j}$ (that is, $j \in W_{j}$ ), then $j$ is removed from the previous assignment $W_{j^{\prime}}$. This creates a "vacancy" in $W_{j^{\prime}}$; this vacancy is filled by running a single step of the auction on $Z_{j^{\prime}}$. Let

```
Algorithm 3 Assigning agents to a multicake (warm-up algorithm).
Input: A multicake \(C=Z_{1} \sqcup \cdots \sqcup Z_{m}\) and a set \([n]\) of agents, where \(m \geq n\).
    - Valuations are normalized such that \(V_{i}(C)=m+n-1\) for all \(i \in[n]\).
Output: A partitioning of the agents \([n]=W_{1} \sqcup \cdots \sqcup W_{m}\) such that:
    - For all \(j \in[m]\) and for each agent \(i \in W_{j}, V_{i}\left(Z_{j}\right) \geq\left|W_{j}\right|\).
    Initialize \(N:=[n]=\) the set of all agents.
    for \(j:=1, \ldots, m\) do
        Using Algorithm 2 auction the subcake \(Z_{j}\) among the agents in \(N\).
        Let \(W_{j}\) be the set of winners.
        Remove \(W_{j}\) from \(N\).
    end for
```

$i^{\prime}$ be the first unassigned agent who did not win the first auction on $Z_{j^{\prime}}$ ("first" by the $\sigma$ ordering in that auction). Recall that, by condition (b) of the auction algorithm, $V_{i^{\prime}}\left(Z_{j^{\prime}}\right)<\left|W_{j^{\prime}}\right|+1$ held before $j$ was removed from $W_{j^{\prime}}$. If the condition does not hold after $j$ is removed (that is: if $V_{i^{\prime}}\left(Z_{j^{\prime}}\right) \geq\left|W_{j^{\prime}}\right|+1$ after the removal), then $i^{\prime}$ is added to $W_{j^{\prime}}$. This step guarantees that both conditions (a) and (b) still hold for $W_{j^{\prime}}$, that is: $V_{i^{\prime}}\left(Z_{j^{\prime}}\right) \geq\left|W_{j^{\prime}}\right|$ for all $i^{\prime} \in W_{j^{\prime}}$, and $V_{i^{\prime \prime}}\left(Z_{j^{\prime}}\right)<\left|W_{j^{\prime}}\right|+1$ for all $i^{\prime \prime}$ who are not assigned yet.

This new winner $i^{\prime}$, who is added to $W_{j^{\prime}}$, might be the agent $j^{\prime}$ itself, who is already assigned to another set $W_{j^{\prime \prime}}$. In this case, moving the agent $j^{\prime}$ from $W_{j^{\prime \prime}}$ to $W_{j^{\prime}}$ creates a vacancy in $W_{j^{\prime \prime}}$, which has to be filled in the same way. This chain reaction must eventually end, since whenever a vacancy is created, the number of agents $j$ who are assigned to the subset with the same index $Z_{j}$ increases by one, and this number never decreases as no agent $j$ is ever removed from $W_{j}$.

The correctness of Algorithm 4 is proved formally below.
Lemma 4 Given a multicake $C=Z_{1} \sqcup \cdots \sqcup Z_{m}$, a positive integer $n \leq m$, and some $n$ agents who value $C$ at least $m+n-1$, Algorithm 4 returns a partitioning of the set of agents $[n]=W_{1} \sqcup \cdots \sqcup W_{m}$ such that (a) For all $j \in[m]$ and for each agent $i^{\prime} \in W_{j}, V_{i^{\prime}}\left(Z_{j}\right) \geq\left|W_{j}\right|$; (b) For all $j \in[n]$, either $j \in W_{j}$ or $V_{j}\left(Z_{j}\right)<\left|W_{j}\right|+1$.

Proof (a) Lemma 2(a) ensures that, in step 4 in iteration $j$, all agents assigned to $W_{j}$ value $Z_{j}$ at least $\left|W_{j}\right|$. The same holds for agents added to $W_{j}$ in step 10 in a later iteration. The size of $W_{j}$ never increases above its initial level: it can only increase by one in step 10 after it has decreased by one in step 7 Therefore, the condition still holds when the algorithm ends.
(b) Agent $j$ always participates in the auction in step 4 of iteration $j$. There are two possible cases.

- If $j$ is assigned to $W_{j}$, then he remains in $W_{j}$ until the end of the algorithm, since $j$ is never removed from $W_{j}$.
- Otherwise, Lemma2(b) ensures that $V_{j}\left(Z_{j}\right)<\left|W_{j}\right|+1$. It remains to show that the condition still holds in later iterations.

```
Algorithm 4 Assigning agents to a multicake, with ownership.
Input: A multicake \(C=Z_{1} \sqcup \cdots \sqcup Z_{m}\) and a set \([n]\) of agents, where \(m \geq n\).
    - Valuations are normalized such that \(V_{i}(C)=m+n-1\) for all \(i \in[n]\).
Output: A partitioning of the agents \([n]=W_{1} \sqcup \cdots \sqcup W_{m}\) such that
    - (a) For all \(j \in[m]\) and for each agent \(i^{\prime} \in W_{j}, V_{i^{\prime}}\left(Z_{j}\right) \geq\left|W_{j}\right|\).
    - (b) For each agent \(j \in[n]\), either \(j \in W_{j}\) or \(V_{j}\left(Z_{j}\right)<\left|\bar{W}_{j}\right|+1\).
    Initialize \(N:=[n]=\) the set of all agents.
    for \(j:=1, \ldots, m\) do
        If \(j \leq n\) then let \(N_{j}:=N \cup\{j\}\); else let \(N_{j}:=N\).
        Using Algorithm 2 auction the subcake \(Z_{j}\) among the agents in \(N_{j}\).
    Let \(W_{j}\) be the set of winners.
    Remove \(W_{j}\) from \(N\).
    if \(j \in W_{j}\) and also \(j \in W_{j^{\prime}}\) for some \(j^{\prime}<j\) then
        Remove \(j\) from \(W_{j^{\prime}}\).
        Let \(i^{\prime}\) be the first loser in the auction on \(Z_{j^{\prime}}\), who is still in \(N \cup\left\{j^{\prime}\right\}\).
            if \(V_{i^{\prime}}\left(Z_{j^{\prime}}\right) \geq\left|W_{j^{\prime}}\right|+1\) then
            Add \(i^{\prime}\) to \(W_{j^{\prime}}\);
            Remove \(i^{\prime}\) from \(N\).
            If \(i^{\prime}=j^{\prime}\) and also \(i^{\prime} \in W_{j^{\prime \prime}}\) for some \(j^{\prime \prime}<j\), then repeat steps 712 with \(j^{\prime}\).
        end if
    end if
    end for
```

For clarity, focus on a specific agent $j^{\prime}$, and suppose that in iteration $j^{\prime}$, agent $j^{\prime}$ did not win the auction, so $V_{j^{\prime}}\left(Z_{j^{\prime}}\right)<\left|W_{j^{\prime}}\right|+1$. In later iterations, the size of $W_{j^{\prime}}$ may decrease by one in step 7 . In this case, lines $8 \sqrt{10}$ guarantee that, either $j^{\prime}$ is added to $W_{j^{\prime}}$, or another agent is added to $W_{j^{\prime}}$; in the latter case, $W_{j^{\prime}}$ returns to its original size, so $V_{j^{\prime}}\left(Z_{j^{\prime}}\right)<\left|W_{j^{\prime}}\right|+1$ still holds.

It remains to prove that, by the end of the algorithm, every agent $i \in[n]$ is assigned to some $W_{j}$. Suppose by contradiction that some $i \in[n]$ is not in any $W_{j}$. This means that $i$ did not win the auction in step 4 in any iteration $j$, so Condition (b) of the auction algorithm ensures that $V_{i}\left(Z_{j}\right)<\left|W_{j}\right|+1$ for all $j \in[m]$.

For any $j^{\prime}$, the size of $W_{j^{\prime}}$ may decrease by one in step 7 in later iterations. In this case, lines $8-10$ guarantee that, if $i$ is not added to $W_{j^{\prime}}$, then either $V_{i}\left(Z_{j^{\prime}}\right)<\left|W_{j^{\prime}}\right|+1$ still holds, or another agent is added to $W_{j^{\prime}}$; in the latter case, $W_{j^{\prime}}$ returns to its original size, so $V_{i}\left(Z_{j^{\prime}}\right)<\left|W_{j^{\prime}}\right|+1$ still holds.

Summing over all $j \in[m]$ gives $V_{i}(C)<\sum_{j=1}^{m}\left(\left|W_{j}\right|+1\right) \leq(n-1)+m$, contradicting the assumption $V_{i}(C) \geq(n-1)+m$.

The above lemmas and algorithms are used to prove the following theorem.

Theorem 2 When the cake is a 1-dimensional interval and each piece must be an interval, it is possible to find in time $O\left(n^{2} \log n\right)$ a division simultaneously satisfying democratic-ownership and 1/2-proportionality.

Algorithm 5 Allocation of an interval cake, with partial proportionality and democratic ownership.

## Input:

- An interval $C$ and an existing allocation into $n$ intervals, $Z_{1} \sqcup \cdots \sqcup Z_{n} \subseteq C$.

Output: A new allocation $X_{1} \sqcup \cdots \sqcup X_{n} \subseteq C$ into $n$ intervals, satisfying the following:

- 1/2-proportionality: for all $i \in[n], V_{i}\left(X_{i}\right) \geq V_{i}(C) /(2 n)$.
- Democratic ownership: for all $d<n$, for at least $n-d$ agents $i, V_{i}\left(X_{i}\right)>V_{i}\left(Z_{i}\right) /\left\lceil\frac{n}{d}\right\rceil$.

Normalize the valuations such that $V_{i}(C)=2 n-1$ for all $i \in[n]$.
Given the original partial allocation $Z_{1} \sqcup \cdots \sqcup Z_{n} \subseteq C$, expand it to a complete partition $Z_{1}^{\prime} \sqcup \cdots \sqcup Z_{n}^{\prime}=C$, by attaching each "blank" (unallocated interval in $C$ ) arbitrarily to one of the two adjacent allocated intervals, to its left or to its right.
3: Considering the intervals $Z_{1}^{\prime}, \ldots, Z_{n}^{\prime}$ as subcakes in a multicake, use Algorithm 4 (with $m=n$ ) to partition the agents into $n$ subsets $W_{1}, \ldots, W_{n}$.
4: Divide each interval $Z_{j}^{\prime}$ among the agents in $W_{j}$ using any algorithm for connected proportional cake-cutting, e.g. Even and Paz (1984).

Proof The proof is constructive and uses Algorithm 5. It is proved below that the output of this algorithm satisfies the requirements of the theorem.

Proof that the output of Algorithm 5 satisfies 1/2-proportionality. Each subcake $Z_{j}^{\prime}$ is divided proportionally among the agents in $W_{j}$. By Lemma 4(a), all these agents value $Z_{j}^{\prime}$ at least $\left|W_{j}\right|$. Hence, their piece has a value of at least 1. By the normalization step (with $m=n$ ), $1 \geq \frac{1}{2 n-1} V_{i}(C)>V_{i}(C) /(2 n)$.

Proof that the output of Algorithm 5 satisfies democratic-ownership. Applying the pigeonhole principle to the partition yielded by Algorithm 4 implies that, for every integer $d \in\{1, \ldots, n-1\}$, at most $d$ of the subsets $W_{j}$, for $j \in[n]$, are populated by at least $\left\lceil\frac{n}{d}\right\rceil$ agents. Hence, at least $n-d$ such subsets are populated by at most $\left\lceil\frac{n}{d}\right\rceil-1$ agents, that is, they satisfy $\left|W_{j}\right| \leq\left\lceil\frac{n}{d}\right\rceil-1$. For each $j \in[n]$, consider two cases:
Case \#1 $: j \in W_{j}$. Then agent $j$ receives a piece of the subcake $Z_{j}^{\prime}$. By the
proportionality of the subcake division (Algorithm 5 step 4 ):

$$
V_{j}\left(X_{j}\right) \geq \frac{V_{j}\left(Z_{j}^{\prime}\right)}{\left|W_{j}\right|} \geq \frac{V_{j}\left(Z_{j}^{\prime}\right)}{\left\lceil\frac{n}{d}\right\rceil-1}>\frac{V_{j}\left(Z_{j}^{\prime}\right)}{\left\lceil\frac{n}{d}\right\rceil}
$$

Case \#2 : $j \notin W_{j}$. Then, by Lemma 4 (b), $V_{j}\left(Z_{j}^{\prime}\right)<\left|W_{j}\right|+1 \leq\left\lceil\frac{n}{d}\right\rceil$. Therefore, $V_{j}\left(Z_{j}^{\prime}\right) /\left\lceil\frac{n}{d}\right\rceil<1$. As explained in the proof of $1 / 2$ proportionality, the value of each agent is at least 1 :

$$
V_{j}\left(X_{j}\right) \geq 1>\frac{V_{j}\left(Z_{j}^{\prime}\right)}{\left\lceil\frac{n}{d}\right\rceil}
$$

In both cases, agent $j$ receives a value greater than $V_{j}\left(Z_{j}^{\prime}\right) /\left\lceil\frac{n}{d}\right\rceil$. The latter ratio is at least $V_{j}\left(Z_{j}\right) /\left\lceil\frac{n}{d}\right\rceil$ since $Z_{j}^{\prime} \supseteq Z_{j}$.

Run-time complexity of Algorithm 5. The auction in Algorithm 2 requires $O(n \log n)$ queries. Algorithm 4 performs $m$ auctions. Each auction might lead to a sequence of at most $n$ vacancies which require one query each to be filled. Algorithm Even-Paz requires $O(n \log n)$ queries, and it is done $m$ times-once for each subcake. All in all, the run-time is in $O(m n \log n)=O\left(n^{2} \log n\right)$, since $m=n$.

### 4.1 Future Work

Crossing the boundary lines. Algorithm 5 treats each existing piece $Z_{j}$ as an isolated subcake, and insists that each new piece be entirely contained in an existing piece, i.e, it does not cross the existing division lines. This may be desirable in the context of land division, since it respects the Uti possidetis juris (Lalonde, 2002) - an international law principle saying that newly-formed sovereign states should retain the internal borders that their preceding dependent area had before their independence. However, it also implies that the resulting division can only be $1 / 2$-proportional and never fully proportional, as the fraction $\frac{1}{n+m-1}$ in (1) is tight.

Theoretically, it may be possible to improve the proportionality guarantee by devising a different redivision procedure that crosses the existing division lines. This raises the following open question: what is the highest level of proportionality that is compatible with democratic-ownership?

Several pieces per agent. Theorem 1 allows an unlimited number of pieces per agent $5^{5}$ while Theorem 2 allows only a single piece per agent. What happens between these extremes? In particular, if each agent can get $k$ intervals, for some fixed $k \geq 1$, then there is an algorithm for dividing a multicake with $m$ subcakes among $n$ agents such that each agent gets at least $\min \left(\frac{1}{n}, \frac{k}{m+n-1}\right)$ of the total cake value (Segal-Halevi, 2021). However, the algorithm does not guarantee democratic ownership. If a similar proportionality guarantee could be attained together with democratic ownership, it could be used in Section 4 with $m=k \cdot n$ subcakes (since for each agent there could be up to $k$ subcakes in the original division), to get a bound of $\frac{k}{k n+n-1}$, which implies $\frac{k}{k+1}$ proportionality for any $k \geq 1$.

## 5 Polygonal Cake and Polygonal Pieces

In this section the cake is a polygon in $\mathbb{R}^{2}$. There is a set $S$ of usable pieces (e.g. rectangles), the initial allocation $Z_{1}, \ldots, Z_{n}$ is an $S$-allocation, and the output should be an $S$-allocation too.

[^4]The main obstacle in applying Algorithm 5 to such a cake is step 2 extending the initial partial allocation to a complete partition of the entire cake. It is not possible to simply attach each unallocated part of $C$ to an allocated $S$-piece, since the result might not be an $S$-piece. The initial partial allocation $Z_{1} \sqcup \cdots \sqcup Z_{n} \subseteq C$ still must be expanded to a complete partition of $C$, since Algorithm 5 uses Algorithm 4 which requires a complete partition. But the number of pieces in the complete partition might be larger than $n$, since there might be unattached "blanks" (holes).

The goal, then, is to find a partition of $C$ into $S$-pieces, $Z_{1}^{\prime} \sqcup \cdots \sqcup Z_{n+b}^{\prime}=C$, with $b \geq 0$, such that every input $S$-piece is contained in a unique output $S$ piece: $\forall j \in[n]: Z_{j} \subseteq Z_{j}^{\prime}$. The additional $b S$-pieces are called blanks. In Step 3 the multicake will contain $m=n+b$ subcakes. Hence, the fraction guaranteed to each agent will be $1 /(n+m-1)=1 /(2 n+b-1)$. A smaller value of $b$ translates to a better proportionality guarantee.

An example of the input and output of the allocation-completion step, when $S$ is the set of rectangles, is shown in Figure 4. In the partial allocation there are $n=4$ rectangles; in the complete partition there are $m=5$ rectangles.


Fig. 4 Allocation-completion with $n=4$ original pieces and $b=1$ blank, denoted $Z_{5}^{\prime}$.

This raises the question of what is the minimum number of blanks required for a complete partition? This geometric question has been studied in a different paper Akopyan and Segal-Halevi, 2018). The answers are summarized in Table $1^{6}$ Moreover, it is proved there that the worst-case optimal number of blanks is attained in any arrangement in which all pieces are maximal, that is, cannot be expanded without overlapping another piece. Formally:

Definition 1 Given a set $S$ of usable pieces, a cake $C$ and some $m$ pairwisedisjoint $S$-pieces $Z_{1}, \ldots, Z_{n} \subseteq C$ :
(a) An $S$-piece $Z^{\prime} \subseteq C$ is called maximal w.r.t. $C, Z_{1}, \ldots, Z_{n}$, if every superset $S$-piece $Z^{\prime \prime} \supsetneq Z^{\prime}$ that is contained in $C$ overlaps one of $Z_{1}, \ldots, Z_{n}$.

[^5]Table 1 Worst-case number of blanks in a maximal arrangement of pairwise-disjoint $S$ pieces contained in a cake $C$. From Akopyan and Segal-Halevi (2018).

| Cake $C$ | Usable pieces $S$ | Number of blanks $b$ |
| :---: | :---: | :---: |
| Polygon | Polygons | 0 |
| Simple polygon (without holes) | Simple polygons | 0 |
| Axes-parallel rectangle | Axes-parallel rectangles | $n-\lceil 2 \sqrt{n}-1\rceil$ |
| Convex figure | Convex figures | $2 n-5$ |
| Rectilinear polygon, $T$ reflex vertices | Axes-parallel rectangles | $T+n-\lceil 2 \sqrt{n}-1\rceil$ |

(b) An $S$-piece $Z^{\prime} \subseteq C$ is called a maximal expansion of $Z_{j}$ if $Z^{\prime} \supseteq Z_{j}$, and $Z^{\prime}$ is maximal w.r.t. $C, Z_{1}, \ldots, Z_{j-1}, Z_{j+1}, \ldots, Z_{n}$.
(c) A set of pairwise-disjoint $S$-pieces $Z_{1}^{\prime}, \ldots, Z_{n}^{\prime} \subseteq C$ is called a complete expansion of $Z_{1}, \ldots, Z_{n}$ if for each $j \in[m], Z_{j}^{\prime}$ is a maximal expansion of $Z_{j}$ w.r.t. $C, Z_{1}^{\prime}, \ldots, Z_{j-1}^{\prime}, Z_{j+1}^{\prime}, \ldots, Z_{n}^{\prime}$.

Complete expansions are used to prove Theorems 3, 4, and 5 below.
Theorem 3 When the cake is a rectangle and each piece must be a parallel rectangle, it is possible to find in time $O\left(n^{2} \log n\right)$ a division simultaneously satisfying democratic-ownership and 1/3-proportionality.

Proof Find a complete expansion of the initial allocation $Z_{1} \sqcup \cdots \sqcup Z_{n} \subseteq C$ in the following way:

```
for j:= 1,\ldots,n do
    Expand each of the four sides of Z}\mp@subsup{Z}{j}{}\mathrm{ until it touches the boundary
        of C or of another piece Z}\mp@subsup{Z}{i}{}\mathrm{ for some i}=j\mathrm{ .
    Denote the expanded piece by Z}\mp@subsup{Z}{j}{\prime}\mathrm{ ; note that it is a maximal expansion
        of Z}\mp@subsup{Z}{j}{}\mathrm{ w.r.t. }C,\mp@subsup{Z}{1}{},\ldots,\mp@subsup{Z}{j-1}{},\mp@subsup{Z}{j+1}{},\ldots,\mp@subsup{Z}{n}{}
    Replace }\mp@subsup{Z}{j}{}\mathrm{ with }\mp@subsup{Z}{j}{\prime}\mathrm{ .
end for
```

Akopyan and Segal-Halevi (2018) prove that the remaining "holes" (unfilled parts of $C$ ) are all rectangular, and their number $b$ satisfies $b \leq n-$ $\lceil 2 \sqrt{n}-1\rceil<n$. So there is a complete $S$-partition $Z_{1}^{\prime} \sqcup \cdots \sqcup Z_{n+b}^{\prime}=C$.

Considering the $S$-pieces $Z_{1}^{\prime}, \ldots, Z_{n+b}^{\prime}$ as subcakes, use Algorithm 4 with $m=n+b$ to partition the agents into subsets $W_{1} \sqcup \cdots \sqcup W_{n+b}$. Then, use the Even-Paz algorithm to partition each $Z_{j}^{\prime}$ among the agents in $W_{j}$. While the Even-Paz algorithm was originally presented for 1-dimensional intervals, it is easily applicable to axes-parallel rectangles, for example by ensuring that all cuts are parallel to the $y$ axis. The resulting allocation satisfies democratic ownership. In addition, for each agent $i \in[n]: V_{i}\left(X_{i}\right) \geq \frac{1}{n+m-1} V_{i}(C) \geq$ $\frac{1}{2 n+b-1} V_{i}(C)>V_{i}(C) /(3 n)$, so the allocation is $1 / 3$-proportional.

The run-time of finding a maximal expansion of a rectangle is in $O(n)$, since it requires to compare each side of the rectangle to the sides of the other
$n-1$ rectangles. Therefore, the run-time of finding a complete expansion is $O\left(n^{2}\right)$. The run-time of executing Algorithm 4 and the Even-Paz algorithms is $O\left(n^{2} \log n\right)$ as in Theorem 2, so the total run-time is $O\left(n^{2} \log n\right)$ too.

Theorem 4 When the cake is a 2-dimensional convex polygon and each piece must be convex, there exists a division simultaneously satisfying democraticownership and 1/4-proportionality.

Proof The proof is similar to that of the previous theorem, and relies on the existence of a complete expansion of the initial allocation. However, I do not have a constructive algorithm for finding a maximal expansion of a convex figure. Recently, Dmitry (2021) presented an algorithm for finding a maximal expansion of a convex polygon contained in an arbitrary polygon. However, since it is not formally published, Theorem 4 is stated only as an existence result.

To prove existence of a maximal expansion of $Z_{j} \square^{7}$ let $Y_{j}$ be the set of potential expansions of $Z_{j}$, i.e:

$$
Y_{j}:=\left\{X_{j} \mid X_{j} \supseteq Z_{j} \text { and } X_{j} \subseteq\left(C \backslash \cup_{i \neq j} Z_{i}\right) \text { and } X_{j} \text { is convex. }\right\}
$$

$Y_{j}$ is partially ordered by inclusion. The Kuratowski-Zorn lemma can be used to prove that it has a maximal element. To use this lemma, one has to prove that every chain in $Y_{j}$ has an upper bound in $Y_{j}$. Indeed, let $Y_{j}^{\prime} \subseteq Y_{j}$ be a chain. Let $\widehat{Y_{j}^{\prime}}$ be the union of all sets in $Y_{j}^{\prime}$. Then $\widehat{Y_{j}^{\prime}}$ is an upper bound on $Y_{j}^{\prime}$, and $\widehat{Y_{j}^{\prime}} \in Y_{j}$ because:
$-\widehat{Y_{j}^{\prime}} \supseteq Z_{j}$-since all sets in $Y_{j}$ contain $Z_{j}$.
$-\widehat{Y_{j}^{\prime}} \subseteq\left(C \backslash \cup_{i \neq j} Z_{i}\right)$-since all sets in $Y_{j}$ are contained in $\left(C \backslash \cup_{i \neq j} Z_{i}\right)$.
$-\widehat{Y_{j}^{\prime}}$ is convex - since for every two points in $\widehat{Y_{j}^{\prime}}$, there exists a set in the chain $Y_{j}^{\prime}$ that contains both of them. This set is convex so it contains the segment between them, so $\widehat{Y_{j}^{\prime}}$ contains this segment too.

Thus, by the Kuratowski-Zorn lemma, $Y_{j}$ has a maximal element. Denote this element by $Z_{j}^{\prime}$. By definition, it is a maximal expansion of $Z_{j}$ w.r.t $C, Z_{1}, \ldots, Z_{j-1}, Z_{j+1}, \ldots, Z_{n}$. As in the proof of Theorem 3, one can proceed iteratively for $j:=1, \ldots, n$, replacing each $Z_{j}$ with its maximal expansion $Z_{i}^{\prime}$. This yields a complete expansion of $Z_{1}, \ldots, Z_{n}$. Akopyan and Segal-Halevi (2018) prove that the remaining holes are all convex, and their number $b$ satisfies $b \leq 2 n-5<2 n$. So there is a complete $S$-partition $Z_{1}^{\prime} \sqcup \cdots \sqcup Z_{n+b}^{\prime}=C$.

Considering the $S$-pieces $Z_{1}^{\prime}, \ldots, Z_{n+b}^{\prime}$ as subcakes, use Algorithm 4 with $m=n+b$ to partition the agents into subsets $W_{1} \sqcup \cdots \sqcup W_{n+b}$. Then, use the Even-Paz algorithm to partition each $Z_{j}^{\prime}$ among the agents in $W_{j}$. Requiring that all cuts made by this algorithm are parallel to the $y$-axis guarantees

[^6] Soleimani-damaneh (2017). I am grateful to Ashkan for his help with this argument.
that the pieces are convex. The resulting allocation satisfies democratic ownership, and for each agent $i \in[n], V_{i}\left(X_{i}\right) \geq \frac{1}{n+m-1} V_{i}(C) \geq \frac{1}{2 n+b-1} V_{i}(C)>$ $V_{i}(C) /(4 n)$, so the allocation is $1 / 4$-proportional.

Theorem 5 When the cake is an axes-parallel rectilinear polygon with $T$ reflex vertices, and each piece must be an axes-parallel rectangle, it is possible to find in time $O\left(n^{2} \log n+\operatorname{poly}(T)\right)$ a division satisfying democratic-ownership, in which each agent receives at least $1 /(3 n+T)$ of the total cake value.

Proof The proof starts similarly to Theorem 3, by finding a complete expansion of the initial allocation $Z_{1} \sqcup \cdots \sqcup Z_{n} \subseteq C$. Akopyan and Segal-Halevi (2018) prove that the remaining "holes" are all simply-connected rectilinear polygons, and that they can be partitioned into at most $b$ rectangles, where $b \leq n-\lceil 2 \sqrt{n}-1+T\rceil<n+T$. The partitioning can be done in time poly $(T)$; see (Keil, 2000; Eppstein, 2010).

Proceeding as in the previous theorems, Algorithm 4 finds an allocation satisfying democratic-ownership in which, for each agent $i \in[n], V_{i}\left(X_{i}\right) \geq$ $\frac{1}{n+m-1} V_{i}(C) \geq \frac{1}{2 n+b-1} V_{i}(C)>V_{i}(C) /(3 n+T)$. The run-time of this part is in $O\left(n^{2} \log n\right)$ as explained in Theorem 3 .

### 5.1 Future Work

The results in this section raise several future work questions, which may be of interest to researchers in computational geometry.

Rectangle and convex pieces. While the bounds in Table 1 are worst-case optimal, in specific instances there may be a maximal expansion with fewer blanks. What is an efficient algorithm for finding a maximal expansion of a given allocation, which has the smallest number of blanks possible in the given instance?

General polygons. When the cake and the pieces are general polygons, or holefree (simply-connected) polygons, but not necessarily convex, there exists a maximal expansion with no blanks at all (Table 1). Using such an expansion, one could expect to find an allocation satisfying democratic ownership and $1 / 2$-proportionality. However, this requires to apply the Even-Paz algorithm to a non-convex polygon such that the pieces remain connected (or simplyconnected); cutting along the y axis (as in the rectangle and convex cases) might yield disconnected pieces. One way to partition a polygon into connected pieces is to map each point $p$ of the polygon to the point nearest to $p$ on the polygon perimeter, as in Figure 5. Then, the perimeter can be partitioned like a 1-dimensional interval. Har-Peled (2021) and Yagami (2021) present sketches of how this can be done, but again, they were not formally published so I do not claim any result for the cases in which $S$ is the family of connected polygons, or of simply-connected polygons.


Fig. 5 The points in the blue trapezoid are mapped to the points on the blue interval at the bottom side of the polygon; each point in the trapezoid is mapped to the point just below it, which is the point nearest to it on the polygon perimeter.

More than two dimensions. When the cake and pieces are three-dimensional (e.g. axes-parallel boxes), what is an upper bound on the number of blanks in a maximal expansion of a partial allocation?

## 6 Price-of-Fairness Bounds

This section uses the redivision theorems of previous sections to prove upper bounds on the price of partial-proportionality.

Theorem 6 For every $r \in[0,1]$, the utilitarian price of $r$-proportionality is at most $1 /(1-r)$.

Proof Let Z be a utilitarian-optimal allocation of a cake. Apply Theorem 1 with $\mathbf{Z}$ as the original allocation. The resulting division is $r$-proportional and satisfies $(1-r)$-ownership, so its utilitarian welfare is at least $1-r$ times the utilitarian welfare in $\mathbf{Z}$.

The proofs of Theorems 7, 8 and 9 are similar; only the constants are different. The proof of Theorem 8 is presented below; to get the proofs of the other theorems, replace the constant " 3 " with " 2 " or " 4 " respectively.

Theorem 8 When the cake is a rectangle and each piece must be a rectangle, for every $r \leq 1 / 3$ :

1. The utilitarian-price of r-proportionality is $O(\sqrt{n})$;
2. The Nash-price of $r$-proportionality is at most $(3 e) \cdot \exp (1 /(4 \pi e)) \approx 8.4$.

Proof Part 1 is proved by Lemma 5 below; part 2 is proved by Lemma 6 below.
Lemma 5 Let $\mathbf{Z}$ be a utilitarian-optimal rectangular allocation of a cake $C$ among $n$ agents, with $Z_{1} \sqcup \cdots \sqcup Z_{n} \subseteq C$. Assume, without loss of generality, that the valuations are normalized such that $V_{i}(C)=n$ for all $i \in[n]$, so the utilitarian welfare of $\mathbf{Z}$ is.

$$
U:=\frac{1}{n} \sum_{i=1}^{n} V_{i}\left(Z_{i}\right)
$$

Then, there exists a (1/3)-proportional rectangular allocation of $C$ to these same $n$ agents with utilitarian welfare $W$, such that $U / W \in O\left(n^{1 / 2}\right)$.

Proof Apply Theorem 3 to the allocation Z. The theorem ensures that the new division is ( $1 / 3$ )-proportional and satisfies democratic-ownership. The latter ensures that for every integer $d \in\{1, \ldots, n-1\}$, there is a set $S_{d}$ containing at least $n-d$ agents, such that the value of every $j \in S_{d}$ is larger than $V_{j}\left(Z_{j}\right) /\left\lceil\frac{n}{d}\right\rceil \geq V_{j}\left(Z_{j}\right) /((n+1) / d)=d \cdot V_{j}\left(Z_{j}\right) /(n+1)$.

Renumber the agents in the following way. Choose an agent from $S_{n-1}$ (which contains at least one agent) and number him/her $n-1$. Choose an agent from $S_{n-2}$ (which contains at least one other agent) and number him/her $n-2$. Continue this way to number the agents by $d=n-1, \ldots, 1$; renumber the remaining agent 0 . Now, the utilitarian welfare of the new division is lowerbounded by:

$$
\begin{aligned}
W & >\frac{1}{n} \sum_{d=0}^{n-1} \max \left(\frac{d \cdot V_{d}\left(Z_{d}\right)}{(n+1)}, \frac{1}{3}\right) \\
& \geq \frac{1}{n} \cdot \frac{1}{3} \cdot \sum_{d=0}^{n-1} \max \left(d \cdot V_{d}\left(Z_{d}\right) / n, 1\right)
\end{aligned}
$$

and the utilitarian welfare ratio is at most:

$$
\frac{U}{W}<3 n \cdot \frac{\sum_{d=0}^{n-1} V_{d}\left(Z_{d}\right) / n}{\sum_{d=0}^{n-1} \max \left(d \cdot V_{d}\left(Z_{d}\right) / n, 1\right)}
$$

Denote the right-hand side by $3 n \cdot \frac{\text { NUM }}{\mathrm{DEN}}$. Let $a_{d}=V_{d}\left(Z_{d}\right) / n$, so that

$$
\mathrm{NUM}=\sum_{d=0}^{n-1} a_{d} \quad \mathrm{DEN}=\sum_{d=0}^{n-1} \max \left(d \cdot a_{d}, 1\right) .
$$

Since $n$ is fixed, an upper bound on $U / W$ requires to find a sequence $a_{0}, \ldots, a_{n-1}$ that maximizes $\frac{\text { NUM }}{\text { DEN }}$ subject to $\forall d: 0 \leq a_{d} \leq 1$.

Observation 1 In a maximizing sequence, $a_{0}=1$ and there is no $d>0$ such that $a_{d}<1 / d$.

Proof If $a_{0}<1$, then setting $a_{0}$ to 1 strictly increases NUM and does not change DEN. Similarly, if $a_{d}<1 / d$ for some $d>0$, then setting it to $1 / d$ strictly increases Num and does not change Den.

Observation 2 A maximizing sequence must be weakly-decreasing (for all $d<$ $\left.d^{\prime}, a_{d} \geq a_{d^{\prime}}\right)$.

Proof If there exists $d<d^{\prime}$ such that $a_{d}<a_{d^{\prime}}$, then swapping $a_{d}$ with $a_{d^{\prime}}$ strictly decreases Den and does not change Num.

Observation 3 There exists at least one maximizing sequence in which there is no $d>0$ such that $1 / d<a_{d}<1$.

Proof ${ }^{8}$ Call an element $a_{d}$ "bad" if $1 / d<a_{d}<1$. Consider a maximizing sequence with the smallest number of bad elements. If the number of bad elements is 0 , then the proof is complete. Otherwise, pick one bad element $a_{d}$. Let $\epsilon:=\min \left(1-a_{d}, a_{d}-1 / d\right)$. Since $a_{d}$ is bad, $\epsilon>0$, and both $a_{d}+\epsilon$ and $a_{d}-\epsilon$ are in $[1 / d, 1]$. Replacing $a_{d}$ with $a_{d}+\epsilon$ or $a_{d}-\epsilon$ yields a new ratio, and it is at most the maximum ratio. In particular:

- Replacing $a_{d}$ with $a_{d}+\epsilon$ makes the ratio $\frac{\mathrm{NUM}+\epsilon}{\mathrm{DEN}+d \epsilon}$; this new ratio is at most $\frac{\text { NUM }}{\text { DEN }}$, so $\epsilon \cdot$ DEN $\leq d \epsilon \cdot$ NUM $\Longrightarrow$ DEN $\leq d \cdot$ NUM.
- Replacing $a_{d}$ with $a_{d}-\epsilon$ makes the ratio $\frac{\text { NUM }-\epsilon}{\text { DEN-d } \epsilon}$; that new ratio is at most $\frac{\text { Num }}{\text { Den }}$, so $-\epsilon \cdot$ Den $\leq-d \epsilon \cdot$ Num $\Longrightarrow$ Den $\geq d \cdot$ Num.

Moreover, at least one of these two inequalities is strict: if $\epsilon=1-a_{d}$, then $a_{d}+\epsilon=1$, so replacing $a_{d}$ with $a_{d}+\epsilon$ yields a sequence with strictly fewer bad elements. Similarly, if $\epsilon=a_{d}-1 / d$, then replacing $a_{d}$ with $a_{d}-\epsilon$ yields a sequence with strictly fewer bad elements. Since, by assumption, the maximizing sequence had a smallest number of bad elements, the new sequence must not be maximizing. So either Den $\leq d \cdot$ Num and Den $>d \cdot$ Num, or DEN $<d \cdot$ NUM and DEN $\geq d \cdot$ NUM. In both cases there is a contradiction. Therefore, there must exist a maximizing sequence with no bad elements.

Observations $1,2,3$ imply that a maximizing sequence has a very specific format. It is characterized by an integer $l \in\{0, \ldots, n-1\}$ such that, for all $d \leq l, a_{d}=1$ and for all $d \geq l+1, a_{d}=1 / d$. So:

$$
\begin{aligned}
\frac{\mathrm{NUM}}{\mathrm{DEN}} & =\frac{\sum_{d=0}^{n-1} a_{d}}{\sum_{d=0}^{n-1} \max \left(d \cdot a_{d}, 1\right)} \\
& =\frac{(l+1)+\left(H_{n-1}-H_{l}\right)}{\frac{1}{2} l(l+1)+(n-l-1)}<\frac{2\left(l+H_{n}+1\right)}{l^{2}-l+2(n-1)}
\end{aligned}
$$

where $H_{n}=\sum_{d=1}^{n}(1 / d)$ is the $n$-th harmonic number.
The number $l$ is an integer, but the expression is upper-bounded by the maximum attained when $l$ is allowed to be real. By standard calculus (taking the derivative of the expression w.r.t. $l$, comparing the derivative to 0 , and checking the second derivative), the real value of $l$ which maximizes the above expression is

$$
l=\sqrt{H_{n}^{2}+3 H_{n}+2 n}-\left(H_{n}+1\right)
$$

which is in $\Theta(\sqrt{n})$. Substituting into the above inequality gives:

[^7]\[

$$
\begin{aligned}
& \frac{\mathrm{NUM}}{\mathrm{DEN}}
\end{aligned}
$$ \leq \frac{\Theta\left(n^{1 / 2}\right)}{\Theta(n)} \in \Theta\left(n^{-1 / 2}\right)
\]

as claimed.
Lemma 6 Let $\mathbf{Z}$ be a Nash-optimal rectangular allocation of a cake $C$ among $n$ agents, with $Z_{1} \sqcup \cdots \sqcup Z_{n} \subseteq C$. Assume the valuations are normalized such that $V_{i}(C)=n$ for all $i \in[n]$. Let $U$ be the Nash welfare of $\mathbf{Z}$ (the geometric mean of the values), defined by

$$
U^{n}=\prod_{i=1}^{n} V_{i}\left(Z_{i}\right) .
$$

Then, there exists a (1/3)-proportional rectangular allocation of $C$ to these same $n$ agents with $N a s h$ welfare $W$, and $U / W \leq 8.4$.

Proof Apply Theorem 3 to the allocation Z. Renumber the agents as in Lemma 5. The Nash welfare of the new allocation, raised to the $n$-th power, can be bounded as:

$$
\begin{aligned}
W^{n} & >\prod_{d=0}^{n-1} \max \left(\frac{d \cdot V_{d}\left(Z_{d}\right)}{(n+1)}, \frac{1}{3}\right) \\
& \geq\left(\frac{1}{3}\right)^{n} \cdot \prod_{d=0}^{n-1} \max \left(d \cdot V_{d}\left(Z_{d}\right) / n, 1\right)
\end{aligned}
$$

and the ratio of the new welfare to the previous welfare can be bounded as:

$$
\begin{aligned}
\frac{U^{n}}{W^{n}} & <3^{n} \cdot \frac{\prod_{d=0}^{n-1} V_{d}\left(Z_{d}\right)}{\prod_{d=0}^{n-1} \max \left(d V_{d}\left(Z_{d}\right) / n, 1\right)} \\
& =\frac{(3 n)^{n}}{\prod_{d=0}^{n-1} \max \left(d, n / V_{d}\left(Z_{d}\right)\right)}
\end{aligned}
$$

The numerator does not depend on the valuations, so the ratio is maximized when the denominator is minimized. This happens when each factor in the product is minimized. The 0 -th factor is at least 1 , since $V_{d}\left(Z_{d}\right) \leq n$. The $d$-th factor, for $d \geq 1$, is at least $d$. Therefore,

$$
\begin{aligned}
\frac{U^{n}}{W^{n}} & <\frac{(3 n)^{n}}{\prod_{d=1}^{n-1} d}=\frac{(3 n)^{n}}{(n-1)!} \\
& =\frac{n(3 n)^{n}}{n!} \approx \frac{n(3 n)^{n}}{\sqrt{2 \pi n}(n / e)^{n}}=\sqrt{\frac{n}{2 \pi}} \cdot(3 e)^{n}
\end{aligned}
$$

where $e$ is the base of the natural logarithm. Taking the $n$-th root gives

$$
\frac{U}{W}<(3 e) \cdot \sqrt{n / 2 \pi}^{1 / n}=(3 e) \cdot \exp \frac{\ln n-\ln 2 \pi}{2 n} .
$$

Consider the expression $\frac{\ln x-\ln 2 \pi}{2 x}$. By taking its derivative, one finds that its global maximum over the positive real numbers is attained at $x=2 \pi e$, and this maximum equals $1 /(4 \pi e)$. Substituting in the above expression gives

$$
\frac{U}{W}<(3 e) \cdot \exp (1 /(4 \pi e)) \approx 8.4
$$

as claimed.

### 6.1 Future Work

Theorems $6 \sqrt{9}$ invoke the question of whether the upper bounds proved in them are tight.

Utilitarian price of fairness. There is a lower bound of $\Omega(\sqrt{n})$ on the utilitarian price of proportionality for a cake with no geometric constraints (Caragiannis et al., 2012), as well as for an interval cake and interval pieces (Aumann and Dombb, 2015). However, these lower bounds do not imply similar lower bounds for partial proportionality. In fact, without geometric constraints, our Theorem 6 shows that the price of partial-proportionality is $O(1)$. Therefore, it is interesting to know which of the following two options is correct for a cake with geometric constraints (e.g. interval cake and interval pieces):

1. There is a lower bound of $\Omega(\sqrt{n})$ matching Theorems $7 \sqrt{9}$, or -
2. The actual price of partial-proportionality is $o(\sqrt{n})$, maybe even $O(1)$.

The latter option is particularly attractive, since it may lead to a feasible and practical compromise between fairness and social welfare.

Nash price of fairness. It is known that without geometric constraints, every Nash-optimal allocation is envy-free; see e.g. Sziklai and Segal-Halevi (2019). Hence, such allocation is proportional, so the Nash price of $r$-proportionality is 1 for any $r \in[0,1]$. However, this is not true when the pieces must be connected (or rectangular, or convex). Appendix A shows several lower bounds on the Nash price of $r$-proportionality with connectivity constraints. However, there is a substantial gap between these lower bounds and the upper bounds proved above.

## 7 Related Work

### 7.1 Dynamic Fair Division

The cake redivision problem differs from several division problems studied recently.

1. Dynamic resource allocation (Kash et al., 2014; Friedman et al., 2015 2017, Huo et al. 2020) is a common problem in cloud-computing environments. The server has several resources, such as memory and disk-space. Agents (processes) come and depart. The server has to allocate the resources fairly among agents. When new agents come, the server may have to take some resources from existing agents. The goal is to do the re-allocation with minimal disruption to existing agents. In these problems, the resources are homogeneous, which means that the only thing that matters is what quantity of each resource is given to each agent. In contrast, the present paper considers a heterogeneous cake, so the algorithms must decide which parts of the cake should be given to which agent.
2. Population monotonicity (Thomson, 1983; Moulin, 1990, 2004, Thomson, 2011, Sziklai and Segal-Halevi, 2018, 2019) is an axiom that describes a desired property of allocation rules. When new agents arrive and the same division rule is re-activated, the value of all old agents should be weakly smaller than in the initial allocation. This axiom represents the virtue of solidarity: if sacrifices have to be made to support an additional agent, then everybody should contribute

Population monotonicity is related to a special case of the redivision model, in which the new agents have no share at all in the initial allocation. However, the redivision model differs in two important aspects. First, even in the special case of new agents with no initial share, there is no upper bound on the value allocated to the incumbent agents. On the contrary, the ownership requirements puts a lower bound on their value in the new allocation. Second, the redivision model is more general, and relates to settings in which all agents already have a (possibly unfair) share in the initial allocation.
3. Private endowment in economics resource allocation problems means that each agent is endowed with an initial bundle of resources. Then, agents exchange resources using a market mechanism. The classic problem in economics involves homogeneous resources, but it has also been studied in the cake-cutting framework (Berliant and Dunz, 2004, Aziz and Ye, 2014). A basic requirement in these works is individual rationality, which means that the final value allocated to each agent must be weakly larger than the value of the initial endowment (note the contrast with the population monotonicity axiom). This requirement is not made in the redivision problem as it is incompatible with fairness: since some agents may initially own no land, individual rationality would mean that they might not receive anything in the exchange.
4. Online division is a setting in which either the agents or the divided resources are not all available at the time of the division, but rather arrive at different times. Walsh (2011) studies the online division of a divisible resource.

The motivation is a birthday party in an office, in which some agents come or leave early while others come or leave late. It is required to give some cake to agents who come early while keeping a fair share to those who come late. Aleksandrov et al. (2015) and Benade et al. (2018) study the online division of indivisible items. The motivation is the food-bank problem, where a charity organization receives food donations and must decide on-line to whom each donation should be allocated. In these papers, in contrast to the present paper, it is impossible to re-divide allocated resources, since they are consumed by their receivers.
5. Land reform is the re-division of land among citizens. It has been attempted in numerous countries around the globe and in many periods throughout history. Some books on land reform are Powelson (1988); Bernstein (2002); Rosset et al. (2006); Lipton (2009). The earliest recorded land-reform was done in ancient Egypt in the times of King Bakenranef, 8th century BC. The most recent land-reform act has been legislated in Scotland in 2016 AD. Balancing fairness and ownership rights is a major concern in such reforms (Sellar, 2006 Hoffman, 2013, Wightman, 2015, MacInnes and Shields, 2015).

### 7.2 Partial Proportionality

While proportionality is the most common criterion of fair cake-cutting, it is often relaxed to partial-proportionality in order to achieve additional goals:

1. Speed: finding a proportional division takes $\Theta(n \log n)$ queries, but finding an $r$-proportional division takes only $\Theta(n)$ queries, for some sufficiently small $r \leq 0.1$ (Edmonds and Pruhs, 2006; Edmonds et al., 2008).
2. Improving social welfare: proportional allocations may be socially inefficient; efficiency can be improved by decreasing the value-guarantee per agent (Zivan, 2011, Arzi, 2012).
3. Minimum-size constraint: In some 1-dimensional settings, each agent may get several intervals but the length of each interval should be above a threshold. It is impossible to guarantee an $r$-proportional allocation for any $r>0$, but additive approximations exist (Caragiannis et al., 2011).
4. Geometric constraints: For example, when the cake is square and the pieces must be square, it is impossible to guarantee an $r$-proportional allocation for any $r>1 / 2$, but there is an algorithm that guarantees a $1 / 4-$ proportional allocation (Segal-Halevi et al., 2017, 2020). When the cake is a connected graph, and the pieces must be connected, there is an algorithm that guarantees each agent at least $1 /(2 n-1)$ of the total value, and it is impossible to guarantee more than that (Bei and Suksompong, 2021).

### 7.3 Democratic fairness

While most works on fair division aim to guarantee unanimous fairness, this is not always compatible with other requirements. Hence, some works explore the
possibility of guaranteeing fairness to a subset of the agents, which shouldideally - be as large as possible. Some examples are:

1. Envy-free allocation of multiple cakes (where each agent should receive a piece in each cake) to $(n+1) / 2$ agents (Nyman et al., 2020).
2. Maximin-share fair allocation of indivisible objects to $n-1$ or $2 n / 3$ agents (Searns, 2020, Hosseini and Searns, 2020).
3. Stable matching rules that guarantee resource-monotonicity to $n / 2$ agents (Ortega, 2018).
4. Pricing rules that are envy-free to a pre-selected subset of the buyers (Bilò et al., 2018).

### 7.4 Geometric Cake Models

The most prominent cake-model is a one-dimensional interval, in which case the pieces are often required to be contiguous sub-intervals. Some exceptions are:

1. The cake is a 1-dimensional circle ("pie") and the pieces are contiguous arcs (Thomson, 2007, Brams et al., 2008; Barbanel et al., 2009; Elkind et al. 2021).
2. The cake is the union of edges of a connected graph, and the pieces are contiguous sub-graphs (Bei and Suksompong, 2021).
3. The cake is a 2 -dimensional territory that lies among several countries. Each country should receive a piece adjacent to its border (Hill, 1983, Beck 1987).
4. The cake is 2-dimensional and the pieces are rectangles determined by the agents (Iyer and Huhns, 2009).
5. The cake is 2-dimensional and the pieces must be squares or fat polygons (Segal-Halevi et al., 2017, 2020).
6. The cake is 2-dimensional; the geometric constraints are connectivity or convexity (Devulapalli, 2014).
7. The cake is multi-dimensional and the pieces are simplices or polytopes (Berliant et al., 1992, Ichiishi and Idzik, 1999, Dall'Aglio and Maccheroni, 2009).

Very recently, geometric fair division problems have been studied based on real two-dimensional land-value data Aleskerov and Shvydun, 2019, Shtechman et al., 2021).

Many natural 2-dimensional settings have not been studied yet. For example, the setting studied in Section 5, where the cake is a rectilinear polygon and the pieces should be rectangles, has not been studied. As shown by Figure 1. there is a qualitative (not only quantitative) difference between 2-D and 1-D division. 2-D division introduces interesting paradoxes, that might be missed by the habit of assuming a one-dimensional cake.

It is important to distinguish geometric cake-cutting from the geometric knapsack problem (Arkin et al., 1993; Adamaszek and Wiese, 2015). In the latter there is a single value-function that should be optimized. In cake-cutting,
there are $n$ agents with different value-functions, and the goal is to guarantee each agent a value higher than some threshold.

### 7.5 Price of Fairness

The price-of-fairness in cake-cutting has been studied in two settings:

- The cake is a one-dimensional interval and the pieces must be intervals Aumann and Dombb (2015). The utilitarian-price-of-proportionality in this case is $\Theta(\sqrt{n})$.
- The cake is arbitrary and the pieces may be arbitrary Caragiannis et al. (2012). The utilitarian-price-of-proportionality in this case is $\Theta(\sqrt{n})$ too.

Both papers study the price of other fairness criteria such as envy-freeness and equitability, but do not study the price in Nash-welfare, and do not handle two-dimensional geometric constraints such as rectangularity or convexity.

The price of fairness was also studied in the context of allocating homogeneous resources (Bertsimas et al., 2011, 2012), fair subset sum (Nicosia et al., 2017), kidney exchange (Dickerson et al., 2014), connected chore cutting (Heydrich and van Stee, 2015), indivisible object allocation (Caragiannis et al. 2012; Kurz, 2016; Bei et al.| 2019b; Suksompong, 2019; Barman et al., 2020), budget division (Michorzewski et al., 2020; Tang et al., 2020) and machine scheduling (Agnetis et al.| 2019; Zhang et al., 2020).

A related notion-the price of connectivity-was studied both for cakecutting (Arunachaleswaran and Gopalakrishnan, 2018) and for indivisible objects (Bei et al., 2019a).

The Nash-price of fairness is related to results about approximating the maximum Nash welfare with indivisible goods. The approximation factors range from 2.89 (Cole and Gkatzelis, 2015) to $e($ Anari et al., 2017) to 2 (Cole et al., 2017; McGlaughlin and Garg, 2020 Caragiannis et al., 2019) to 1.45 (Barman et al., 2018).

Several authors study the algorithmic problem of finding a welfare-maximizing cake-allocation in various settings:

1. The cake is an interval and the pieces must be connected Aumann et al. 2013);
2. The cake is an interval and the pieces must be connected, and additionally, the division must be proportional (Bei et al., 2012);
3. The cake and pieces are arbitrary, and the division must be envy-free Cohler et al., 2011).
4. The cake and pieces are arbitrary, and the division must be equitable (Brams et al., 2012).

## 8 More Future Work

Besides the open questions mentioned in subsections 4.1 5.1 and 6.1 it may be interesting to study the redivision problem with other requirements besides proportionality.

Envy-freeness. Envy-freeness means that each agent values their piece at least as much as each of the other pieces. Similarly, $r$-envy-freeness means that each agent values their piece as at least $r$ times the value of each of the other pieces. For what pairs $r, w$ is $r$-envy-freeness compatible with $w$-ownership? With democratic-ownership?

One issue with envy-freeness is that the redivision problem is inherently asymmetric: agents whose initial piece is valuable are entitled to a higher final value than agents whose initial piece is empty. A potentially useful notion here is justified envy, which has been recently studied in the literature on two-sided matching (Abdulkadiroğlu et al. 2020). In two-sided matching (for example, between doctors and hospitals), "justified envy" means that doctor $d_{1}$, who is matched to hospital $h_{1}$, envies doctor $d_{2}$, who is matched to hospital $h_{2}$, and at the same time, $h_{2}$ prefers $d_{1}$ to $d_{2}$. Analogously, one can defined "justified envy" in our setting as some agent $i$ envying another agent $j$ in the final allocation, while $i$ 's initial piece was more valuable than $j$ 's.

Pareto-efficiency. From an existential point of view, Pareto-efficiency does not add much difficulty. Both $r$-proportionality and $w$-ownership are preserved by Pareto-improvements. Therefore, if there exists a division satisfying $r$-proportionality and $w$-ownership (or democratic-ownership), then there also exists a Pareto-optimal division satisfying these properties. However, it may not be easy to find such a division algorithmically.

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## A Lower bounds on price of $\boldsymbol{r}$-proportionality

This appendix presents several lower bounds on the price of $r$-proportionality with geometric constraints. As a warm-up, the following simple lower bound is proved (the bound for utilitarian price of proportionality was already proved by Aumann and Dombb (2015)):

Theorem 9 With $n=2$ agents, when the cake is an interval and the pieces must be intervals, the utilitarian price of proportionality is $3 / 2$ and the Nash-price of proportionality is $\sqrt{2}$.

Proof For the lower bound 9 consider an interval cake with four homogeneous regions, where the values of each of two agents for each of the four regions is given by the table below (where $\epsilon>0$ is an infinitesimally small positive constant):

| George's value: | $1-\epsilon$ | $\epsilon$ | $\epsilon$ | $1-\epsilon$ |
| :---: | :---: | :---: | :---: | :---: |
| Alice's value: | $\epsilon$ | $1-\epsilon$ | $1-\epsilon$ | $\epsilon$ |

Due to symmetry, the only connected proportional allocation divides the cake exactly in the middle and gives each agent exactly 1 , so both the utilitarian welfare and the Nash welfare are 1 . However, giving the leftmost region to George and the rightmost three regions to Alice gives Alice a value of almost 2 and George a value of almost 1 , so the utilitarian welfare is almost $3 / 2$ and Nash welfare is almost $\sqrt{2}$. When $\epsilon \rightarrow 0$, the utilitarian price of proportionality approaches $3 / 2$ and the Nash price of proportionality approaches $\sqrt{2}$.

For the matching upper bound, note that, in any proportional allocation, the utilitarian welfare and the Nash welfare are at least 1 (it is attained when all agents get exactly their proportional share). On the other hand, in any non-proportional allocation, the utilitarian welfare is less than $n-1+\frac{1}{n}$, and the Nash welfare is less than $\left(n^{n-1}\right)^{1 / n}$ (since one agent gets a value of less than 1 , and the other agents get a value of at most $n$ ). Therefore, the utilitarian price of proportionality is at most $n-1+\frac{1}{n}=3 / 2$ and the Nash price of proportionality is at most $n^{1-1 / n}=\sqrt{2}$.

Below, this lower bound is extended in two ways: two agents with $r$-proportionality and 2-dimensional cakes, and $n$ agents with 1-proportionality.

Theorem 10 With $n=2$ agents, interval cake and interval pieces, or rectangular cake and rectangular pieces, for any $r \in[0,1]$, the utilitarian price of $r$-proportionality is $1+r / 2$ and the Nash price of $r$-proportionality is $\max (1, \sqrt{2 r})$.

Proof For the lower bound, consider an interval cake with six homogeneous interval regions, or a rectangular cake of dimensions $6 \times 6$ with six homogeneous rectangular regions of dimensions $1 \times 6$ each, where the values of each of two agents for each region are given below (where $\epsilon>0$ is an infinitesimally small positive constant):

| George's value: | $r-\epsilon$ | $\epsilon$ | $1-r$ | $1-r$ | $\epsilon$ | $r-\epsilon$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Alice's value: | $\epsilon$ | $r-\epsilon$ | $1-r$ | $1-r$ | $r-\epsilon$ | $\epsilon$ |

Observation 4 In any r-proportional allocation, both the utilitarian and the Nash welfare are at most 1.

Proof Suppose first that the cake is an interval. In any $r$-proportional allocation, each agent must get a value of at least $r$. Hence, one agent must get the two leftmost regions and the other agent must get the two rightmost regions. The two central regions can be divided arbitrarily; regardless of how they are divided, the utilitarian welfare is 1 . To compute the maximum Nash welfare, suppose George receives the two leftmost regions and a fraction $x$ of the two central regions; so his value is $r+2 x(1-r)$, while Alice's value is $r+2(1-x)(1-r)$. By taking the derivative, one can find out that the product is maximized when $x=1 / 2$, where

[^8]the value of each agent is exactly 1 . Hence, the maximum Nash welfare in an $r$-proportional allocation is 1 too.

If the cake is rectangular, then there are two ways to divide it into two rectangles: vertically or horizontally. If it is divided by a vertical cut, then the situation is exactly as in the case of an interval. If it is divided by a horizontal cut, then - regardless of the cut location - the sum of the agents' values is 2 , so the utilitarian welfare is 1 and the maximum Nash welfare is attained when both values are 1. In both cases, the maximum utilitarian and Nash welfare in an $r$-proportional allocation is 1 .

Observation 5 There is an allocation in which, when $\epsilon \rightarrow 0$, the utilitarian welfare approaches $1+r / 2$ and the Nash welfare approaches $\sqrt{2 r}$.

Proof Cut the cake at the right of the leftmost region (if the cake is rectangular, use a vertical cut). Give George the leftmost region and Alice the other regions. George's value is $r-\epsilon$ while Alice's value is $2-\epsilon$, so when $\epsilon \rightarrow 0$, the utilitarian welfare approaches $1+r / 2$ and the Nash welfare approaches $\sqrt{2 r}$.

The above two observations imply a lower bound of $1+r / 2$ on the utilitarian price of $r$ proportionality and a lower bound of $\max (1, \sqrt{2 r})$ on the Nash price of $r$-proportionality 10

For the matching upper bound, note that in any proportional allocation (which is, in particular, $r$-proportional), both the utilitarian and the Nash welfare are at least 1, while in any non- $r$-proportional allocation, the utilitarian welfare is less than $1+r / 2$ the Nash welfare is less than $\sqrt{2 r}$ (since one agent gets less than $r$ and the other agent gets at most $2)$.

Theorem 11 When the cake is an interval and the pieces must be intervals, the Nash price of proportionality when there are $n$ agents is at least $2^{1-1 / n}$.

Proof Consider a piecewise-homogeneous cake consisting of $2 n$ regions. The agents are partitioned into two groups: odd-indexed agents and even-indexed agents. The agents in each group have the same valuation. The odd-indexed agents value the regions at $1-\epsilon, \epsilon, \epsilon, 1-$ $\epsilon, \ldots$, while the even-indexed agents value the regions at $\epsilon, 1-\epsilon, 1-\epsilon, \epsilon, \ldots$, An example is shown in the table below for $n=5$, where $1^{-}$is a shorthand for $1-\epsilon$ :

| Agents 1, 3, 5: | $1^{-}$ | $\epsilon$ | $\epsilon$ | $1^{-}$ | $1^{-}$ | $\epsilon$ | $\epsilon$ | $1^{-}$ | $1^{-}$ | $\epsilon$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Agents 2, 4: | $\epsilon$ | $1^{-}$ | $1^{-}$ | $\epsilon$ | $\epsilon$ | $1^{-}$ | $1^{-}$ | $\epsilon$ | $\epsilon$ | $1^{-}$ |

The total cake value for all agents is $n$, so in a proportional allocation, each agent should get a value of at least 1 . The following two observations imply the theorem.

Observation 6 In a proportional allocation, the value of every agent is exactly 1; hence the Nash welfare is 1 .

Proof In a proportional allocation, the agent who receives the leftmost piece must receive at least two adjacent regions in order to have a value of at least 1 .

The two leftmost agents must receive together at least four adjacent regions. This is because the value measure arrives at 2 only at the end of the fourth region, and the leftmost agent consumes at least two regions which are worth at least 1 , so to ensure the secondleftmost agent a value of at least 1 , their combined consumption must consist of at least the four leftmost regions.

Proceeding this way, it is possible to prove by induction that, for every integer $\ell \geq 1$, the $\ell$ leftmost agents consume together at least $2 \ell$ regions. By a symmetric argument, the same is true for the $\ell$ rightmost agents. But this implies that, in a proportional allocation, each agent must consume exactly 2 regions. The value of every two consecutive regions when starting from the left (or from the right) is exactly 1.

[^9]Observation 7 There is an allocation in which the Nash welfare approaches $2^{1-1 / n}$ as $\epsilon \rightarrow 0$.

Proof Give the leftmost region to agent 1. Then, give each of agents $2, \ldots, n-1$ two consecutive regions. Give agent 2 the last three consecutive regions.

The value of agent 1 is $1-\epsilon$; the value of each of $2, \ldots, n-1$ is $2-2 \epsilon$; and the value of $n$ is $2-\epsilon$. When $\epsilon \rightarrow 0$, the Nash welfare approaches $\left(2^{n-1}\right)^{1 / n}=2^{1-1 / n}$.

The above two observations imply that the Nash price of proportionality approaches $2^{1-1 / n}$ when $\epsilon \rightarrow 011$

So far, I could not extend Theorem 11 to rectangular cakes: the main difficulty is that there are many possible ways to cut a rectangle into $n$ rectangles, so it is hard to reason about what the possible $r$-proportional allocations can be. Similarly, I could not extend the theorem to $r$-proportionality: giving even a single agent a value of $r$ apparently allows a lot of freedom in allocating to the other agents, so again, it is hard to reason about what an $r$-proportional allocations can be. Thus, the exact utilitarian and Nash price of $r$-proportionality for all $n \geq 3$ and $r \in(0,1)$ remains open.

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[^0]:    * A preliminary version of this paper was presented in the 27th International Joint Conference on Artificial Intelligence, IJCAI (Segal-Halevi, 2018). The following are new in the present paper. (a) Theorem 5 which generalizes Theorem 3 from a rectangle to any rectilinear polygon. (b) Section 6 which applies the algorithms for cake redivision to obtain upper bounds on the price of fairness in cake-cutting with geometric constraints. (c) The proof of Lemma 1 now uses a recently-introduced algorithm by Cseh and Fleiner (2020) to obtain an algorithm with run-time polynomial in the binary representation of the input. (d) Theorem 2 now uses an improved algorithm, which attains $1 / 2$ proportionality instead of $1 / 3$-proportionality. Similarly, the constant in Theorem 3 is $1 / 3$ instead of $1 / 4$, and the constant in Theorem 4 is $1 / 4$ instead of $1 / 5$.
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[^1]:    1 The price-of-fairness of $r$-proportionality w.r.t. the Nash welfare is 1 for all $r \leq 1$, since any Nash-optimal cake allocation is proportional (Sziklai and Segal-Halevi 2019).

[^2]:    2 It is often called a cut query, but the term mark query better differentiates query answers from actual cuts through the cake.
    ${ }^{3}$ The symbol $\sqcup$ denotes disjoint union-it emphasizes that the pieces $X_{1}, \ldots, X_{n}$ are pairwise-disjoint.

[^3]:    ${ }^{4}$ Consider $m$ subcakes and $n$ agents with the same valuation, who value the entire multicake at $m+n-1$. Suppose the value of each subcake $j$ (for all agents) is some integer $u_{j} \geq 1$. If some subcake is not allocated to any agent, then the multicake can be reduced to a smaller one with $m^{\prime}:=m-1$ subcakes, which all agents value at most $m^{\prime}+n-1$. So suppose each subcake is allocated to at least one agent. Define the surplus of each subcake as $u_{j}$ minus the number of agents who are allocated a piece in that subcake. The total surplus is $(m+n-1)-n=m-1$, so at least one subcake $j_{0}$ must have a surplus of at most 0 . At least one of the $u_{j_{0}}$ agents allocated a piece in subcake $j_{0}$ has a value of at most 1 .

[^4]:    ${ }^{5}$ In fact, $4 n-3$ pieces per agent are sufficient. It is known that, for every pair of agents, an allocation with different entitlements can be attained with two cuts, e.g. (Segal-Halevi, 2019). So each agent receives at most 2 pieces. In Algorithm 1 each agent participates in $2 \cdot(n-1)$ such allocation instances, and gets one additional piece.

[^5]:    6 The expressions in Table 1 are tight in the worst case: there are partial allocations that require exactly this number of blanks.

[^6]:    7 The following proof is an adaptation of the proof of Lemma 3.3 in Mohammadi and

[^7]:    8 The proof idea is due to Varun Dubey in http://math.stackexchange.com/q/1609071/29780

[^8]:    ${ }^{9}$ It extends an example in Section 5 of Sziklai and Segal-Halevi (2018).

[^9]:    10 The max in the latter expression comes from the fact that, when $\sqrt{2 r}<1$, the maximum Nash welfare is not attained by the allocation of Observation 5 but rather by a proportional allocation.

[^10]:    11 The same example implies that the utilitarian price of proportionality approaches $2-$ $1 / n$. But this is subsumed by the $\Omega(\sqrt{n})$ lower bound of Aumann and Dombb (2015).

