Incidence structures and Stone-Priestley duality

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Résumé

We observe that if $R := (I, \rho, J)$ is an incidence structure, viewed as a matrix, then the topological closure of the set of columns is the Stone space of the Boolean algebra generated by the rows. As a consequence, we obtain that the topological closure of the collection of principal initial segments of a poset P is the Stone space of the Boolean algebra Tailalg(P) generated by the collection of principal final segments of P, the so-called tail-algebra of P. Similar results concerning Priestley spaces and distributive lattices are given. A generalization to incidence structures valued by abstract algebras is considered.

Key words: Incidence structure, Galois lattice, Boolean algebra, Distributive lattice.

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1 Introduction

Basic objects of data analysis are matrices filled with 0-1 coefficients. The analysis of their structural properties single out an important class of properties : those for which the orders on the rows and on the columns are irrelevant, that is properties which, once true for a given $m \times n$ -matrix A, are also true for all matrices obtained from A by arbitrary permutations of its rows and columns. Each row $A_{i,-}$ of A determines a subset of the index set of colums, and dually for the columns. Simple manipulations of rows and columns translate to set theoretic operations on the corresponding subsets. For example, the set made of intersections of columns $A_{-,j}$ $(j := 1, \ldots, n)$ is a lattice, every matrix obtained from A by permuting its rows and columns yields an isomorphic lattice, and moreover, this lattice is dually isomorphic to the lattice made of intersections of rows. In terms of incidence relations, this later statement says that the dual of the Galois lattice of an incidence relation is isomorphic to the Galois lattice of the dual incidence relation, a result at the very heart of formal concept analysis. An other result of the same flavor, is the following :

Theorem 1.1 The number of distinct Boolean combinations made with the rows of a finite matrix consisting of 0 and 1 is 2^c where c is the number of distinct columns of the matrix.

The proof is immediate. For, let A be such a $m \times n$ matrix. Set $I := \{1, \ldots, m\}$, $J := \{1, \ldots, n\}, \mathcal{R} := \{k(A_{i,-}) : i \in I\}$, where $k(A_{i,-}) := \{j \in J : A(i,j) = 1\}$ and let \mathcal{B} be the set of Boolean combinations made of members of \mathcal{R} . Say that two indices $j', j'' \in J$ are *equivalent* if the corresponding columns are identical. There are c equivalence classes. Moreover, if U is an equivalence class then clearly for every $j' \in J \setminus U$ and $j'' \in U$ there is some $i \in I$ such that $A(i, j') \neq A(i, j')$. It follows then that U is the intersection of subsets X such that $U \subseteq X \subseteq J$ and either X or $J \setminus X$ belongs to \mathcal{R} . Thus, $U \in \mathcal{B}$ and, hence, \mathcal{B} consists of all unions of equivalence classes. The result follows.

This result, as basic as it may seem, might be as old as the notion of Boolean algebra. In this paper, we first point out that the extension of this result to infinite matrices, captures the essence of Stone and Priestley dualities alike. Next, we give an illustration with the notion of Tail algebra. Finally, we consider an extension to matrices with coefficients into an abstract algebra.

For convenience, we present our results in terms of incidence structures, rather than matrices.

1.1 Stone-Priestley duality for incidence structures

An *incidence structure* is a triple $R := (I, \rho, J)$ where ρ is a relation from a set I to a set J, identified with a subset of the cartesian product $I \times J$. For (i, j) in $I \times J$, set $R^{-1}(j) := \{i \in I : (i, j) \in \rho\}$ and $R(i) := \{j \in J : (i, j) \in \rho\}$; also set $\mathcal{C}_R := \{R^{-1}(j) : j \in J\}$ and $\mathcal{R}_R := \{R(i) : i \in I\}$.

Throughout this paper, J will denote a non-empty set and $\mathfrak{P}(J)$ its power set. Viewing $\mathfrak{P}(J)$ as a bounded lattice (resp. a Boolean algebra), we denote by $\mathcal{L}(R)$ (resp. $\mathcal{B}(R)$), the bounded sublattice (resp. the Boolean subalgebra) of $\mathfrak{P}(J)$ generated by \mathcal{R}_R .

Hence, $\mathcal{L}(R)$ is the smallest collection of subsets of J such that (1) \emptyset , J and every member of \mathcal{R}_R belongs to $\mathcal{L}(R)$; (2) $I' \cup I''$ and $I' \cap I''$ belong to $\mathcal{L}(R)$ whenever I' and I'' belong to $\mathcal{L}(R)$. Similarly, $\mathcal{B}(R)$ is the smallest collection of subsets of J such that (1) every member of \mathcal{L}_R belongs to $\mathcal{B}(R)$; (2) $I' \cup I''$ and $I \setminus I'$ belong to $\mathcal{B}(R)$ whenever I' and I'' belong to $\mathcal{B}(R)$.

Identifying $\mathfrak{P}(I)$ with 2^{I} , we may view it as a topological space. A basis of open sets consists of subsets of the form $O(F,G) := \{X \in \mathfrak{P}(I) : F \subseteq X$ and $G \cap X = \emptyset\}$, where F, G are finite subsets of I. Let $\overline{\mathcal{C}}_{R}$ denotes the topological closure of \mathcal{C}_{R} in $\mathfrak{P}(I)$. Recall that a compact totally disconnected space is called a *Stone space*, whereas a *Priestley space* is a set together with a topology and an ordering which is compact and totally order disconnected. We will use only the fact that closed subspaces of $\mathfrak{P}(I)$, with the inclusion order possibly added, are of this form[17].

Example

- (1) Let $R := (E, \in, \mathfrak{P}(E))$ where E is a set. Then $\mathcal{C}_R = \mathfrak{P}(E)$ (thus, it is closed), whereas $\mathcal{R}_R := \{\{X : x \in X \in \mathfrak{P}(E)\} : x \in E\}$. One may show that $\mathcal{B}(R)$ is the free Boolean algebra generated by E and \mathcal{C}_R is its Stone space.
- (2) Replacing R by its dual $R^{-1} := (\mathfrak{P}(E), \ni, E), \mathcal{B}(R)$ is then the power set $\mathfrak{P}(E)$ and $\overline{\mathcal{C}_R}$ is the set $\beta(E)$ of ultrafilters on E, the Cech-Stone compactification of E.

The content of these examples holds in a more general setting.

Theorem 1.2 The set $\overline{\mathcal{C}}_R$, endowed with the topology induced by the powerset $\mathfrak{P}(I)$, is homeomorphic to the Stone space of $\mathcal{B}(R)$. With the order of inclusion added, $\overline{\mathcal{C}}_R$ is isomorphic to the Priestley space of $\mathcal{L}(R)$.

Proof. Let $\varphi : \mathfrak{P}(\mathfrak{P}(J)) \to \mathfrak{P}(I)$ be defined by $\varphi(\mathcal{B}) := \{i \in I : R(i) \in \mathcal{B}\}$ for all $\mathcal{B} \subseteq \mathfrak{P}(J)$. Looking at $\mathfrak{P}(\mathfrak{P}(J))$ and $\mathfrak{P}(I)$ as topological spaces, we

can see that φ is continuous, whereas viewing these sets as Boolean algebras, φ is a Boolean homomorphism and, in particular, it preserves the ordering. Let Spec(L) be the collection of prime filters of L, the spectrum of L, for $L \in \{\mathcal{L}(R), \mathcal{B}(R)\}$. We claim that φ induces an isomorphism from Spec(L)onto $\overline{\mathcal{C}}_R$ (this isomorphism being a topological one if $L = \mathcal{B}(R)$, a topological and ordered one if $L = \mathcal{L}(R)$. To this end, let $e : J \to \mathfrak{P}(L)$ be the map defined by setting $e(j) := \{X \in L : j \in X\}$. Clearly, $e(j) \in Spec(L)$ and $\varphi(e(j)) = R^{-1}(j)$ for every $j \in J$. It follows that $\mathcal{C}_R \subseteq \varphi'(\mathcal{S}pec(L))$ (where $\varphi'(\mathcal{S}pec(L)) := \{\varphi(\mathcal{U}) : \mathcal{U} \in \mathcal{S}pec(L)\}\}; \text{ hence } \overline{\mathcal{C}}_R \subseteq \varphi'(\mathcal{S}pec(L)), \text{ since } \varphi$ is continuous. Next, $\varphi'(\mathcal{S}pec(L)) \subseteq \overline{\mathcal{C}}_R$. Indeed, let $\mathcal{U} \in \mathcal{S}pec(L)$ and let \mathcal{O} be an open set in $\mathfrak{P}(I)$ containing $\varphi(\mathcal{U})$; without loss of generality, we may assume that $\mathcal{O} = \mathcal{O}(F, G) := \{ X \in \mathcal{P}(I) : F \subseteq X \text{ and } G \cap X = \emptyset \},\$ where F, G are finite subsets of I. Then, since \mathcal{U} is a prime filter, the set $H := \cap \{R(i) : i \in F\} \setminus \cup \{R(i) : i \in G\}$ belongs to \mathcal{U} and hence H is nonempty. For $j \in H$, $R^{-1}(j) \in \mathcal{O}$; this proves that $\varphi(\mathcal{U}) \in \overline{\mathcal{C}}_R$. To finish up the proof, note that φ is 1-1 on Spec(L).

Every Boolean algebra is of the form $\mathcal{B}(R)$. Indeed, if B is a Boleean algebra, set $R := (B, \in, S)$ where S is the Stone space made of ultrafilters of B. In the next section, we introduce a proper class of Boolean algebras. Later on, in Section 3, we shall delineate the exact content of Theorem 1.2 in terms of abstract algebras.

2 Tail algebras and Tail lattices

Let P be a poset. For $x \in P$, the principal final (resp. initial) segment generated by x is $\uparrow x := \{y \in P : x \leq y\}$ (resp. $\downarrow x := \{y \in P : y \leq x\}$). Set $up(P) := \{\uparrow x : x \in P\}$ and $down(P) := \{\downarrow x : x \in P\}$.

The tail algebra of P is the subalgebra Tailalg(P) of the Boolean algebra $(\mathfrak{P}(P), \cap, \cup, \backslash, \emptyset, P)$ generated by up(P). According to J.D.Monk ([13]Chap. 2, p.40), this notion is due to G.Brenner. Denote by Taillat(P) the bounded sublattice of $(\mathfrak{P}(P), \cup, \cap, \emptyset, P)$, generated by up(P). Taking $R := (P, \leq, P)$ in Theorem 1.2 we have :

Theorem 2.1 The topological closure $\overline{down(P)}$ of down(P) in $\mathfrak{P}(P)$ is homeomorphic to the Stone space of Tailalg(P). With the order of inclusion added, $\overline{down(P)}$ it is isomorphic to the Priestley space of Taillat(P).

The topological closure of down(P) points to interesting collections of subsets of P.

A subset I of P is an *initial segment* (or is *closed downward*) if $x \leq y$ and $y \in I$ imply $x \in I$; if in addition I is non-empty and up-directed (that is every pair $x, y \in I$ has an upper bound $z \in I$), this is an *ideal*. For example,

each principal initial segment is an ideal. Let X be a subset of P. We set $\downarrow X := \{y \in P : y \leq x \text{ for some } x \in X\}$; this set is the least initial segment containing X, we say that it is generated by X; if X contains only one element x, we will continue to denote it by $\downarrow x$ instead of $\downarrow \{x\}$. We set $X^- := \bigcap \{\downarrow x : x \in X\}$, the set of lower bounds of X. An initial segment I of P is finitely generated if $I = \downarrow X$ for some finite subset X of P. We denote respectively by $\mathbf{I}(P), \mathbf{I}_{<\omega}(P)$ and $\mathcal{J}(P)$ the collection of initial segments, finitely generated initial segments, and ideals of P ordered by inclusion. The dual de P is the poset obtained from P by reversing the order; we denote it by P^* (instead of P^{-1}). Using the above definitions, a subset which is respectively an initial segment, a finitely generated final segment or an ideal of P. We denote by $\mathbf{F}(P), \mathbf{F}_{<\omega}(P)$, and $\mathcal{F}(P)$ respectively, the collections of these sets, ordered by inclusion. Also, denote respectively by $\uparrow X$ and X^+ the least final segment containing X and the set of upper bounds of X in P.

The topological condition of closedness translates in an order theoretic one as shown by the following lemma.

Lemma 2.2 Let X be a subset of P. Then $X \in \overline{down(P)}$ if and only if $F^+ \setminus \uparrow G \neq \emptyset$ for every finite subsets $F \subseteq X, G \subseteq P \setminus X$.

Proof. Clearly, for every $F, G \in \mathfrak{P}(P)$, we have $O(F,G) \cap down(P) \neq \emptyset$ if and only if $F^+ \uparrow G \neq \emptyset$. Let $X \in \mathfrak{P}(P)$. Since the O(F,G) 's, for all finite subsets $F \subseteq X, G \subseteq P \setminus X$, form a basis of neighborhoods of X, the lemma follows.

As an immediate corollary, we have :

Corollary 2.3 $\emptyset \notin \overline{down(P)} \iff P \in \mathbf{F}_{<\omega}(P)$

We also have :

Corollary 2.4 $down(P) \subseteq \mathcal{J}(P) \subseteq \overline{down(P)} \setminus \{\emptyset\}$. In particular, the topological closures in $\mathfrak{P}(P)$ of down(P) and $\mathcal{J}(P)$ are the same.

Proof. Trivially $down(P) \subseteq \mathcal{J}(P)$; thus, it suffices to check that $\mathcal{J}(P) \subseteq \overline{down(P)}$. Let $I \in \mathcal{J}(P)$. Let F, G be two finite subsets of I and $P \setminus I$ respectively. Since F is finite and I is an ideal, there is some $x \in I$ such that $F \subseteq \downarrow x$, and since I is an initial segment, $\uparrow G \subseteq P \setminus I$. Hence $x \in F^+ \setminus \uparrow G$. According to Lemma 2.2, $I \in \overline{down(P)}$. This finishes the proof of Corollary 2.4.

Remark 2.5 If $P \notin \mathbf{F}_{<\omega}(\mathbf{P})$, $\overline{down(P)}$ is isomorphic (as a Priestley space) to $\overline{down(P')}$, where P' is the poset obtained from P by adding a least element. A poset P is up-closed if every intersection of two members of up(P) is a finite union (possibly empty) of members of up(P).

Proposition 2.6 The following properties for a poset P are equivalent :

- (a) J(P) ∪ {Ø} is closed for the product topology;
 (b) J(P) = down(P) \ {Ø};
 (c) P is up-closed;
 (d) F_{<ω}(P) is a meet-semilattice;
- (e) $Taillat(P) = \mathbf{F}_{<\omega}(P) \cup \{P\}.$

Proof. (a) \Rightarrow (c) Suppose that $\mathcal{J}(P) \cup \{\emptyset\}$ is closed. Let $x, y \in P$. We prove that $\uparrow x \cap \uparrow y$ is finitely generated. We may suppose $\uparrow x \cap \uparrow y \neq \emptyset$ (otherwise the condition is satisfied). The set $O(\{x, y\}, \emptyset) \cap \mathcal{J}(P)$ is included into $\cup \{O(\{z\}, \emptyset) : z \in \uparrow x \cap \uparrow y\}$. Being compact, from our hypothesis, it is included into a finite union $\cup \{O(\{z\}, \emptyset) : z \in G\}$. Hence $\uparrow x \cap \uparrow y = \cup \{\uparrow z : z \in G\}$ as required.

- $(c) \Rightarrow (e)$ Trivial.
- $(e) \Rightarrow (d)$ Trivial.

 $(d) \Rightarrow (c)$ Let $x, y \in P$. Assuming that $\mathbf{F}_{<\omega}(P)$ is a meet-semilattice, it contains an element $Z := \uparrow x \land \uparrow y$. This element is equal to $Z' := \uparrow x \cap \uparrow y$. Indeed, we have obviously $Z \subseteq Z'$. If the inclusion was strict, then $Z \cup \uparrow z$, for $z \in Z' \setminus Z$, would be an element $\mathbf{F}_{<\omega}(P)$ in between, which is impossible. Hence $Z' \in \mathbf{F}_{<\omega}(P)$, proving (c).

 $(c) \Rightarrow (b)$ Suppose that P is up-closed. Let $I \in \overline{down(P)} \setminus \{\emptyset\}$. Since $\mathbf{I}(P)$ is closed in $\mathfrak{P}(P), I \in \mathbf{I}(P)$. Let $x, y \in I$; according to our hypothesis $\uparrow x \cap \uparrow y = \cup\{\uparrow z : z \in G\}$ for some finite subset G of P. Necessarily $G \cap I \neq \emptyset$, otherwise $I \in O(\{x, y\}, G)$, whereas $O(\{x, y\}, G) \cap \downarrow (P) = \emptyset$, contradicting $I \in \overline{down(P)}$. Every $z \in O(\{x, y\}, G) \cap I$ is an upper bound of x, y in I, proving $I \in \mathcal{J}(P)$.

 $(b) \Rightarrow (a)$ Trivial.

Corollary 2.7 The following properties for a poset P are equivalent :

- (1) $\mathcal{J}(P)$ is closed in $\mathfrak{P}(P)$;
- (2) $P \in \mathbf{F}_{<\omega}(P)$ and P is up-closed.

As an immediate consequence :

Fact 1 If P is a join-semilattice with a least element then $\mathcal{J}(P)$ is closed for the product topology.

Let us recall that if L is a join-semilattice, an element $x \in L$ is *join-irreducible* (resp. *join-prime*) if it is distinct from the least element 0, if any, and if $x = a \lor b$ implies x = a or x = b (resp. $x \le a \lor b$ implies $x \le a$ or $x \le b$) see [9] where 0

is allowed. We denote $\mathbb{J}_{irr}(L)$ (resp. $\mathbb{J}_{pri}(L)$) the set of join-irreducible (resp. join-prime) members of P. We recall that $\mathbb{J}_{pri}(L) \subseteq \mathbb{J}_{irr}(L)$; the equality holds provided that L is a distributive lattice. It also holds if $L = \mathbf{I}_{<\omega}(P)$. Indeed :

Fact 2 For an arbitrary poset P, we have :

$$\mathbb{J}_{irr}(\mathbf{I}_{<\omega}(P)) = \mathbb{J}_{pri}(\mathbf{I}_{<\omega}(P)) = down(P)$$
(1)

$$\mathbb{J}_{irr}(\mathbf{I}(P)) = \mathcal{J}(P) \tag{2}$$

Fact 3 For a poset L, the following properties are equivalent.

- L is isomorphic to $\mathbf{I}_{<\omega}(P)$ for some poset P;
- L is a join-semilattice with a least element in which every element is a finite join of primes.

We have the following characterization.

Proposition 2.8 Let L be a bounded distributive lattice. The following properties are equivalent :

- (a) Every element of L is a finite join of join-irreducible elements;
- (b) L is isomorphic to $\mathbf{F}_{<\omega}(P)$ for some poset P with P up-closed and $P \in \mathbf{F}_{<\omega}(P)$;
- (c) The Priestley space Spec(L) is isomorphic to $\mathcal{J}(P)$, for some poset P with P up-closed and $P \in \mathbf{F}_{<\omega}(P)$.

Proof.

 $(a) \Rightarrow (b)$ Set $Q := \mathbb{J}_{irr}(L)$ and let $\varphi : L \to \mathfrak{P}(Q)$ be defined by $\varphi(x) := \{y \in Q : y \leq x\}$. It is well known that φ is an homomorphism of bounded lattices, provided that L is itself a bounded distributive lattice. Moreover, φ is an isomorphism from L onto $\mathbf{I}_{<\omega}(Q)$ if and only if L satisfies hypothesis (a). Hence, under this condition, L is isomorphic to $\mathbf{I}_{<\omega}(Q)$, that is to $\mathbf{F}_{<\omega}(P)$, where $P := Q^*$. To conclude, it suffices to check that P is up-closed and $P \in \mathbf{F}_{<\omega}(P)$. As for the proof of $(d) \Rightarrow (c)$ in Proposition 2.6, this simply follows from the fact that $\mathbf{F}_{<\omega}(P)$ is a meet-semilattice, with a top element.

 $(b) \Rightarrow (c)$ Assuming that (b) holds, <u>L</u> is isomorphic to Taillat(P). From Theorem 2.1, Spec(L) is isomorphic to $\overline{down(P)}$, which according to Corollary 2.7, is isomorphic to $\mathcal{J}(P)$.

 $(c) \Rightarrow (a)$ This implication follows in the same way that $(b) \Rightarrow (c)$ from Theorem 2.1 and Corollary 2.7.

Examples

(1) An exemple from the theory of relations illustrates Theorem 2.1 and Corollary 2.7.

First, recall that a relational structure is a pair $R := (E, (\rho_i)_i \in I)$, where for each $i \in I$, ρ_i is a n_i -ary relation on E (that is a subset of E^{n_i}) and n_i is a non-negative integer; the family $\mu := (n_i)_i \in I$ is called the signature of R. An induced substructure of R is any relational structure of the form $R_{\uparrow F} := (F, (\rho_i \cap F^{n_i})_{i \in I})$. One may define the notion of relational isomorphism and then the notion of embeddability between relational structures with the same signature (e.g. R embeds into R' if Ris isomorphic to an induced substructure of R'). According to Fraïssé [7], the age of a relational structure R is the collection $\mathcal{A}(R)$ of its finite induced substructures, considered up to isomorphism. A first order sentence (in the language associated with the signature μ) is universal whenever it is equivalent to a sentence of the form $\forall x_1 \cdots \forall x_n \varphi(x_1, \ldots, x_n)$ where $\varphi(x_1, \ldots, x_n)$ is a formula built with the variable x_1, \ldots, x_n , the logical connectives \neg , \lor , \land and predicates =, $\rho_i, i \in I$.

Let Ω_{μ} be the set of finite relational structures with signature μ , these structures being considered up to isomorphism and ordered by embeddability. If μ is finite, then Ω_{μ} is a ranked poset with a least element which is up-closed, hence from Corollary 2.7, the set $\mathcal{J}(\Omega_{\mu})$ is a closed subset of $((\Omega_{\mu})$. The use of $\mathcal{J}(\Omega_{\mu})$ is justified by the following :

Proposition 2.9 $\mathcal{J}(\Omega_{\mu})$ is the set of ages of relational structures with signature μ and its dual is the Boolean algebra made of Boolean combinations of universal sentences, considered up to elementary equivalence.

Thus from Theorem 2.1 the tail algebra $Tailalg(\Omega_{\mu})$ provides an alternative description of the algebra made of Boolean combinations of universal sentences (see [14] [16] for more details on this correspondence).

If μ is constant and equal to 1 $(I := \{0, \ldots, k-1\})$ and $n_i = 1$ for $i \in I$ then Ω_{μ} can be identified to the direct product of 2^k copies of the chain ω of non-negative integers. From this it follows that $\mathcal{J}(\Omega_{\mu})$ is homeomorphic to the ordinal $\omega^{2^k} + 1$, equipped with the interval topology. If μ contains some integer larger than 1, then Ω_{μ} embeds as a subposet the set $[\omega]^{<\omega}$ of finite subsets of ω hence $\mathcal{J}(\Omega_{\mu})$ embeds the Cantor space $\mathfrak{P}(\omega)$.

(2) Algebraic lattices provide an other variety of examples. Indeed, if P is a join-semilattice with 0, then $\mathcal{J}(P)$ is an algebraic lattice, in fact the lattice of closed sets of an algebraic closure system, and every algebraic lattice has this form [9]. The fact that algebraic lattices are Priestley spaces leads to interesting results, see eg [12], and questions. Typical examples of algebraic lattices are $\mathbf{I}(P)$ and $\mathbf{F}(P)$, ordered by inclusion. For an example, the compact elements of $\mathbf{F}(P)$ are the finitely generated final segments of P, from which follows that $\mathbf{F}(P)$ is isomorphic to $\mathcal{J}(\mathbf{F}_{<\omega}(P))$. The importance of $\mathbf{F}(P)$ steems from the following notion.

The free Boolean algebra generated by a poset P is a Boolean algebra containing P in such a way that every order-preserving map from P into a Boolean algebra B extends to a homomorphism from this Boolean algebra into B; up-to an isomorphism fixing P pointwise, this Boolean algebra is unique, it is denoted by FB(P). An explicit description may be given in terms of tail algebra.

Proposition 2.10 The free-Boolean algebra generated by P is isomorphic to the tail algebra $Tailalg(\mathbf{F}_{<\omega}(P))$.

For this, note that the incidence structures $(P, \in, \mathbf{F}_{<\omega}(P))$ and $(\mathbf{F}_{<\omega}(P), \subseteq, \mathbf{F}_{<\omega}(P))$ yield the same Boolean algebra.

Tail algebras generated by join-semilattices with 0 are studied in [1], [20].

3 A generalization to valued incidence structures

Let E be a set. For $n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$, a map $f : E^n \to E$ is an *n*-ary operation on E, whereas a subset $\rho \subseteq E^n$ is an *n*-ary relation on E. Denote by $\mathcal{O}^{(n)}$ (resp. \mathcal{R}^n) the set of *n*-ary operations (resp. relations) on E and set $\mathcal{O} := \bigcup \{\mathcal{O}^{(n)} : n \in \mathbb{N}^*\}$ (resp $\mathcal{R} := \bigcup \{\mathcal{R}^{(n)} : n \in \mathbb{N}^*\}$). For $n, i \in \mathbb{N}^*$ with $i \leq n$, define the *i*th *n*-ary projection e_i^n by setting $e_i^n(x_1, \ldots, x_n) := x_i$ for all $x_1, \ldots, x_n \in E$ and set $\mathcal{P} := \{e_i^n : i, n \in \mathbb{N}^*\}$. An operation $f \in \mathcal{O}$ is constant if it takes a single value, it is *idempotent* provided $f(x, \ldots, x) = x$ for all $x \in E$. We denote by \mathcal{C} (resp. \mathcal{I}) the set of constant, (resp. idempotent) operations on E.

Let $m, n \in \mathbb{N}^*$, $f \in \mathcal{O}^{(m)}$ and $\rho \in \mathcal{R}^{(n)}$. Then f preserves ρ if :

$$(x_{1,1}, \dots, x_{1,n}) \in \rho, \dots, (x_{m,1}, \dots, x_{m,n}) \in \rho \Longrightarrow (f(x_{1,1}, \dots, x_{m,1}), \dots, f(x_{1,n}, \dots, x_{m,n})) \in \rho$$
(3)

for every $m \times n$ matrix $X := (x_{i,j})_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$ of elements of E. Let $g \in \mathcal{O}^{(n)}$, then f commutes with g if f preserves the n + 1-ary relation

$$\rho_g := \{ (x_1, \dots, x_n, g(x_1, \dots, x_n)) : (x_1, \dots, x_n) \in E^n \}.$$

A universal algebra (resp. a relational structure) on E is a pair (E, \mathcal{F}) where \mathcal{F} is a subset of \mathcal{O} (resp. of \mathcal{R}^1 . Powers of such structures, subalgebras, and homomorphisms are easy to define. For example, if $\mathbb{K} := (E, \mathcal{F})$ is an algebra,

¹ Contrarily to the definition given in the previous section, we do not require that \mathcal{F} is a family of members of \mathcal{R}

then a subset $\rho \subseteq E^n$ induces a subalgebra of \mathbb{K}^n if every $g \in \mathcal{F}$ preserves ρ ; also $f : \mathbb{K}^m \to \mathbb{K}$ is an homomorphism if f commutes with ρ_g for all $g \in \mathcal{F}$; equivalently, ρ_f is a subalgebra of \mathbb{K}^{m+1} .

Let $\mathbb{K} := (E, \mathcal{F})$ be a universal algebra, I, J be two sets and A be a map from the direct product $I \times J$ into \mathbb{K} . Given $i \in I$, let $A_{i,-} : J \to \mathbb{K}$ be defined by $A_{i,-}(j) := A(i, j)$; similarly, given $j \in J$ let $A_{-,j} : I \to \mathbb{K}$ be defined by $A_{-,j}(i) := A(i, j)$. Set $\mathcal{R}_A := \{A_{i,-} : i \in I\}$ and $\mathcal{C}_A := \{A_{-,j} : j \in J\}$. Looking at A as a matrix, \mathcal{R}_A and \mathcal{C}_A are the sets of rows and columns of A. Let B be the subalgebra of \mathbb{K}^J generated by \mathcal{R}_A , let $Hom(B, \mathbb{K})$ be the set of homomorphisms from B into \mathbb{K} and let $\overline{\mathcal{C}_A}$ be the topological closure of \mathcal{C}_A in \mathbb{K}^I equipped with the product topology, \mathbb{K} being equipped with the discrete topology.

We say that \mathbb{K} is projectively trivial if every homomorphism f from every subalgebra L of a finite power \mathbb{K}^n into \mathbb{K} is induced by a projection, that is $f(x_1, \ldots, x_n) = x_j$ for some j and all $(x_1, \ldots, x_n) \in L$. Theorem 1.2 is a consequence of the following statement.

- **Proposition 3.1** (a) If \mathbb{K} is a finite projectively trivial algebra, then $Hom(B, \mathbb{K})$ equipped with the topology induced by the product topology on \mathbb{K}^B is home-omorphic to the closure $\overline{\mathcal{C}_A}$ of \mathcal{C}_A .
- (b) The 2-element Boolean algebra and the 2-element bounded lattice are projectively trivial.

Proof. (a) We start with \mathbb{K} arbitrary. Let Y be such that $\mathcal{R}_A \subseteq Y \subseteq \mathbb{K}^J$. Let $\varphi : \mathbb{K}^Y \to \mathbb{K}^I$, setting $\varphi(h)(i) := h(A_{i,-})$ for all $h \in \mathbb{K}^Y$, $i \in I$. Let $e : J \to \mathbb{K}^Y$ be defined by e(j)(h) := h(j) for all $j \in J$, $h \in \mathbb{K}^J$.

Claim 1
$$\varphi(e(j)) = A_{-,}$$

Let $D := \{g \in \mathbb{K}^I \text{ such that } g(i) = g(i') \text{ whenever } A_{i,-} = A_{i',-} \}$. Let Z be a subset of \mathbb{K}^Y containing the image of I by e and let $\varphi'(Z)$ be the image of Z by φ .

Claim 2 $\mathcal{C}_A \subseteq \varphi'(Z) \subseteq D$

Next, we set Y := B and $Z := Hom(B, \mathbb{K})$.

Claim 3 $Ime \subseteq Hom(B, \mathbb{K})$ and φ is 1 - 1 on $Hom(B, \mathbb{K})$.

Proof. Let $h_1, h_2 \in Hom(B, \mathbb{K})$ so that $\varphi(h_1) = \varphi(h_2)$. This says $h_1(A_{i,-}) = h_2(A_{i,-})$ for all $i \in I$, meaning that h_1, h_2 coincide on \mathcal{R}_A . Being morphisms, h_1, h_2 coincide on the algebra generated by \mathcal{R}_A , this algebra is B hence $h_1 = h_2$ proving that φ is 1 - 1.

Let $Im\varphi_B$ be the image of $Hom(B, \mathbb{K})$ under φ .

Claim 4 If \mathbb{K} is finite then $\overline{\mathcal{C}_A} \subseteq Im\varphi_B$.

Proof. First, $Hom(B, \mathbb{K})$ is closed in \mathbb{K}^B ; next φ is continuous. Since \mathbb{K} is finite, \mathbb{K}^B is compact, thus $Im\varphi_B$ is closed, proving that it contains $\overline{\mathcal{C}_A}$.

Claim 5 The above inclusion is an equality if in addition \mathbb{K} is projectively trivial.

Proof. Let $g \in Im\varphi_B$ and I' be a finite subset of I. We claim that there is some $j \in J$ such that $A_{-,j}$ and g coincide on I'. Say that two indices $j', j'' \in J$ are equivalent if the restriction to I' of the columns $A_{-,j'}, A_{-,j''}$ are identical. Since I' and \mathbb{K} are finite, this equivalence relation has only finitely many classes. Let J' be a finite subset of J containing an element of each class. The projection map $p : \mathbb{K}^J \to \mathbb{K}^{J'}$ being 1 - 1 on $A_{I',-} := \{A_{i,-} : i \in I'\}$, it is 1 - 1 on B', the subalgebra of K^J generated by $A_{I',-}$. Let $h \in Hom(B,\mathbb{K})$ such that $\varphi(h) = g$. We have $g(i) = h(A_{i,-})$ for every $i \in I$. Since B' is a subalgebra of B, h induces an homomorphism from B' into \mathbb{K} and, since p is 1 - 1 on B', it induces an homomorphism h' from B'', the image of B' under p, into \mathbb{K} . Since \mathbb{K} is projectively trivial, h' is a projection, and since, from our construction, $h'(p(\overline{x})) = h(\overline{x})$ for every $\overline{x} \in B'$, there is some $j \in J$ such that $h(A_{i,-}) = A(i,j)$ for all $i \in I'$. It follows that g(i) = A(i,j) as required. \square

(b) Let $f: L \to \mathbb{K}$ be a homomorphism from a subalgebra L of a finite power \mathbb{K}^n where $\mathbb{K} := \{0, 1\}$ is the 2-element Boolean algebra or the 2-element lattice as well. In both cases, $f^{-1}(0)$ and $f^{-1}(1)$ have respectively a largest element $a := (a_1, \ldots, a_n)$ and a least element $b := (b_1, \ldots, b_n)$. Since $b \not\leq a$ there is an indice i, i < n, such that $a_i = 0$ and $b_i = 1$. Clearly, f is the restriction to L of the *i*-th projection p_i . Hence \mathbb{K} is projectively trivial.

3.1 From projectively trivial algebras to algebras with the projection property

Projectively trivial algebras seem to be interesting objects to consider, particularly in view of general studies about duality (as developped in [3]). These algebras fall into a general class of structures, which has attracted some attention recently, those with the *projection property*. Let \mathbb{K} be a structure (e.g. an algebra or a relational structure). Let n be a non-negative integer, \mathbb{K} has the *n*-projection property if every idempotent homomorphism from \mathbb{K}^n , the *n*-th power of \mathbb{K} , into \mathbb{K} is a projection. If this holds for every n, we say that \mathbb{K} has the projection property.

This notion was introduced for posets by E.Corominas [4] in 1990. Several papers have followed (e.g. see [15] for reflexive relational structures, [5][6] for reflexive graphs and [11] for irreflexive graphs).

Between the projectively trivial algebras and algebras with the projection property, are algebras \mathbb{K} which have no other homomorphisms from their finite powers into \mathbb{K} than the projections. For the ease of the discussion, we will name these algebras "*projective*" (despite the fact that this word has an other meaning in algebra).

Clearly, an algebra \mathbb{K} is projective if and only if it has the projection property and it is *rigid* in the sense that there is no other endomorphism from \mathbb{K} into itself other than the identity. A further discussion about these algebras belongs to the theory of clones.

Let us recall that a *clone* on E is a composition closed subset of \mathcal{O} containing \mathcal{P} . Equivalently, a clone is the set of term operations of some algebra on E. For example, the clone associated with the Boolean algebra on the 2-element set $E := \{0, 1\}$ is \mathcal{O} , whereas the clone associated with 2-element lattice is the set of all monotone operations. The structural properties of an algebra and its finite powers, namely the homomorphisms and subalgebras are entirely determined by the clone of its term operations. This is readily seen in terms of the Galois correspondence between operations and relations defined by the relation " f preserves ρ ". The polymorph of a set \mathcal{G} of relations is the set $Pol(\mathcal{G})$ of operations which preserve each ρ in \mathcal{G} . Polymorphs can be characterized as locally closed clones. If E is finite, there are just clones. The *invariant* of a set \mathcal{F} of operations is the set $Inv(\mathcal{F})$ of relations which are preserved by each f in \mathcal{F} . This is the set of subalgebras of finite powers of $(\mathcal{F}, \mathcal{F})$. The collection of polymorphs, once ordered by inclusion, form a complete lattice, namely the Galois lattice of the above correspondence. If E is finite, it coincides with the lattice of clones. The relation "f commutes with q" also defines a Galois correspondence: The *centralizer* of a set \mathcal{F} of operations is the set $Z(\mathcal{F})$ of operations which commute with every f in \mathcal{F} . The corresponding Galois lattice is a subset of the lattice of clones. On a finite set E, this lattice is finite [19], contrarily to the lattice of clones.

We will note the following well known fact :

Fact 4 An operation f is a projection (resp. is idempotent) if and only if it commutes with all operations (resp all constant unary operations). That is $Z(\mathcal{O}) = \mathcal{P}$ and $Z(\mathcal{C}) = \mathcal{I}$.

With the notion of centralizer, we immediately have

Fact 5 An algebra $\mathbb{K} := (E, \mathcal{F})$ is projective if and only if $Z(\mathcal{F}) = \mathcal{P}$.

Hence, the classification of these algebras amounts to the classification of clones whose the centralizer consists only of projections. Trivially, this collection of clones is a final segment of the lattice of clones which, from Fact 4, is non-empty. Hence, beyond \mathcal{O} , maximal clones are natural candidates. On a finite universe, there are only finitely many maximal clones, and they have been entirely determined [18]. A search is then possible. We do hope to report on it in the near future.

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