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Complexity of shift bribery for iterative voting rules

Cynthia Maushagen[1] · Marc Neveling[1] · Jörg Rothe[1] · Ann-Kathrin Selker[1]

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Abstract

In iterative voting systems, candidates are eliminated in consecutive rounds until either a fixed number of rounds is reached or the set of remaining candidates does not change anymore. We focus on iterative voting systems based on the positional scoring rules plurality, veto, and Borda and study their resistance against shift bribery attacks introduced by Elkind et al. [1] and Kaczmarczyk and Faliszewski [2]. In constructive shift bribery (Elkind et al. [1]), an attacker seeks to make a designated candidate win the election by bribing voters to shift this candidate in their preferences; in destructive shift bribery (Kaczmarczyk and Faliszewski [2]), the briber's goal is to prevent this candidate's victory. We show that many iterative voting systems are resistant to these types of attack, i.e., the corresponding decision problems are NP-hard. These iterative voting systems include iterated plurality as well as the voting rules due to Hare, Coombs, Baldwin, and Nanson; variants of Hare voting are also known as single transferable vote, instant-runoff voting, and alternative vote.

Keywords Computational social choice · Voting · Shift bribery · Iterative voting rules

Mathematics Subject Classification (2010) 68Q17 · 91B12 · 91B14

1 Introduction

One of the main themes in computational social choice [3, 4] is the study of the computational complexity of manipulative attacks on voting systems. Besides *manipulation* [5, 6] itself (also referred to as strategic voting where voters cast insincere ballots instead of revealing their true preferences) and *electoral control* [7, 8] (where an election chairs seeks

✉ Jörg Rothe
rothe@hhu.de

Cynthia Maushagen
cynthia.maushagen@hhu.de

Marc Neveling
marc.neveling@hhu.de

Ann-Kathrin Selker
ann-kathrin.selker@hhu.de

[1] Institut für Informatik, Heinrich-Heine-Universität Düsseldorf, Universitätsstr. 1, Düsseldorf 40225, Germany

to influence the outcome of an election by structural changes such as adding, deleting, or partitioning either candidates or voters), much work has been done to study *bribery attacks*. For a comprehensive overview of the formal models and the related complexity results, we refer to the book chapters by Conitzer and Walsh [9] for manipulation, by Faliszewski and Rothe [10] for control and bribery, and by Baumeister and Rothe [11] for all three topics.

1.1 Shift bribery and other bribery attacks in voting

Bribery in voting was introduced by Faliszewski et al. [12] (see also the article by Faliszewski et al. [13]). In their model, a briber intends to change the outcome of an election to his or her own advantage by bribing certain voters without exceeding a given budget. Bribery shares some features with manipulation, as the briber (just like a strategic voter) has to find the right preference orders that the bribed voters are then requested to change their votes to. Bribery also shares some features with electoral control, as the briber (just like an election chair) has to pick the right voters to bribe so as to make the cost of bribing them low enough to stay within the allowed budget.

We will focus on *shift bribery*, which was introduced by Elkind et al. [1] for the constructive variant (where the briber's goal is to make a favorite candidate win the election) and was later studied by Kaczmarczyk and Faliszewski [2] in the destructive variant (where the briber's goal is to make sure that a despised candidate does not win the election). In *swap bribery* [1], which generalizes shift bribery, the briber has to pay for each swap of any two adjacent candidates in the votes. Shift bribery additionally requires that swaps always involve the designated candidate that the briber wants to see win (in the constructive case) or not win (in the destructive case).

A natural interpretation of swap bribery—and thus in particular of shift bribery—regards *campaign management*: A campaign manager organizing a political campaign for some candidate seeks to influence the public opinion about this candidate by legal activities such as, e.g., running targeted television ads. Those ads might influence voters to change their opinion (and consequently their vote) of the targeted candidate positively or negatively. Campaign managers are restricted by a budget and need to choose the right ads to run in order to increase their candidates' chances of winning. The *constructive* variant of shift bribery can be seen to model campaign management in a more ethical way than general (constructive) swap bribery, as campaign managers then always target their own candidates only and thus cannot change the voters' opinions over pairs of other candidates. On the other hand, negative campaigning can still be modeled as *destructive* shift bribery.

Another natural interpretation of swap bribery regards election fraud detection: If the winner of an election can be dethroned by only a few changes (by swapping candidates) to the votes then the election might have been tampered with or, from a more optimistic viewpoint, small errors in the counting of the votes might have influenced the election result. In that situation, a recounting would be required since for a close election result only few errors in the counting are needed to elect a candidate that is not the "true" winner of the election. This has been studied as the *margin of victory*, a critical robustness measure for voting systems. Specifically, Xia [14] has shown that the margin of victory is closely related to destructive bribery. Reisch et al. [15] add to this connection by showing that the former problem can be hard, even if the latter is easy. Furthermore, Baumeister and Hogrebe [16] and Boehmer et al. [17] consider swap bribery for counting problems to study election robustness. In this context, shift bribery models a more fine-grained search for election fraud which targets only a specific candidate.

Swap bribery generalizes the possible winner problem [18, 19], which itself is a generalization of unweighted coalitional manipulation.[1] Therefore, each of the many hardness results known for the possible winner problem is directly inherited by the swap bribery problem. This was the motivation for Elkind et al. [1] to look at restricted variants of swap bribery such as shift bribery.

1.2 Some related work for shift bribery

Even though shift bribery possesses a number of hardness results [1], it has also been shown to allow exact and approximate polynomial-time algorithms in a number of cases [1, 20, 21]. For example, Elkind et al. [1] provided a 2-approximation algorithm for shift bribery when using Borda voting.[2] This result was extended by Elkind and Faliszewski [20] to all positional scoring rules; they also obtained somewhat weaker approximations for Copeland and maximin voting. Very recently Faliszewski et al. [22] further extended this result to a polynomial-time approximation scheme. For Bucklin and fallback voting, the shift bribery problem is even exactly solvable in polynomial time [21]. Faliszewski et al. [23] have complemented these results on Bucklin and fallback voting. In particular, they studied a number of bribery problems for these rules, including a variant called "extension bribery," which was previously introduced by Baumeister et al. [24] in the context of campaign management when the voters' ballots are truncated.

In addition, Bredereck et al. [25] were the first to analyze shift bribery in terms of *parameterized* complexity, and only recently a long-standing open problem regarding the parameterized complexity of bribery (including shift bribery) with the number of candidates as the parameter (see the survey by Bredereck et al. [26] for a deeper discussion on this problem) was solved by Knop et al. [27] for a multitude of voting rules.

Bredereck et al. [28] introduced combinatorial shift bribery in which a single shift bribery action affects multiple voters and Bredereck et al. [29] studied shift bribery in the context of multiwinner elections for various committee selection rules.

[1] In the unweighted coalitional manipulation problem, we are given the votes of the nonmanipulators and a distinguished candidate c, whereas the manipulators' votes are as yet unspecified, and we ask whether they can be set so as to make c win the election. This is the special case of the possible winner problem where some votes (namely those of the nonmanipulators) are completely specified and some other votes (namely those of the manipulators) are completely unspecified; in general, all votes in an instance of the possible winner problem can be partially ordered, and we ask whether they can be extended to complete linear orders such that the given candidate c wins. The possible winner problem, in turn, is the special case of the swap bribery problem where swaps between any two candidates that already are linearly ordered in the instance of the possible winner problem are made so costly that they exceed the briber's budget, whereas any two candidates that are not linearly ordered can be swapped for free.

[2] In *Borda* with m candidates, each vote is a linear order of the candidates, the i th candidate in a vote scores $m - i$ points, and whoever has the most points wins. Borda is a very prominent positional scoring rule and can be described by the scoring vector $(m-1, m-2, \ldots, 0)$. Other prominent positional scoring rules are *plurality*, where only the top candidates in the votes score a point and no one else (i.e., plurality has the scoring vector $(1, 0, \ldots, 0)$), and *veto* (a.k.a. *antiplurality*), where all except the bottom candidates in the votes score a point (i.e., veto has the scoring vector $(1, \ldots, 1, 0)$); again, whoever has the most points wins in these rules.

1.3 Iterative voting rules

While the complexity of shift bribery has been comprehensively investigated for many standard voting rules, it has not been considered yet for *iterative* voting systems. To close this glaring gap, we study shift bribery for eight iterative voting systems that are based on any one of the Borda, plurality, and veto rules (see Footnote 2 for their definitions) and that each proceed in rounds, eliminating (from the election) after each except the last round the candidates performing worst in a certain sense:

- The *system of Baldwin* [30] eliminates the candidates with the lowest Borda score,
- the *system of Nanson* [31] eliminates the candidates whose scores are lower than the average Borda score,
- the *system of Hare* (see, e.g., the book by Taylor [32]) eliminates the candidates with the lowest plurality score,
- the system called *iterated plurality* (again see, e.g., the book by Taylor [32]) eliminates the candidates that do not have the highest plurality score,
- the system called *iterated veto* is defined analogously to iterated plurality, except based on the veto rather than the plurality score, and
- the *system of Coombs* (defined, e.g., in the paper by Levin and Nalebuff [33]) eliminates the candidates with the lowest veto score.

The last two systems that we consider differ from the above iterative voting systems because they always use exactly two rounds:

- *Plurality with runoff* (as defined, e.g., in the book by Taylor [32]) works as follows. In the first round, if there is a unique plurality winner, all candidates that do not have the highest or second-highest plurality score are eliminated; if there are two or more plurality winners, all candidates that do not have the highest plurality score are eliminated. In the second round (the *runoff*), all remaining candidates with the highest plurality score win.
- *Veto with runoff* is defined analogously, except that veto scores instead of plurality scores and veto winners instead of plurality winners are considered.

These voting systems have been thoroughly studied and are also used in the real world. Among the systems we consider, Hare voting and variants thereof (some of which are called single transferable vote, instant-runoff voting, or alternative vote) are most widely used, for example in Australia, India, Ireland, New Zealand, Pakistan, the UK, and the USA.

1.4 Our contribution and more related work

Table 1 gives an overview of our complexity results for constructive and destructive shift bribery in our eight voting systems, where the shorthand NP-c stands for "NP-complete." Our results complement results by Davies et al. [34] who have shown unweighted coalitional manipulation to be NP-complete for Baldwin and Nanson voting (even with just a single manipulator)—and also for the underlying Borda system (with two manipulators; for the latter result, see also the paper by Betzler et al. [35]). Davies et al. [34] also list various

Table 1 Summary of complexity results for shift bribery problems

	Hare	Coombs	Baldwin	Nanson
Constructive	NP-c (Thm. 1)	NP-c (Thm. 3)	NP-c (Thm. 5)	NP-c (Thm. 7)
Destructive	NP-c (Thm. 2)	NP-c (Thm. 4)	NP-c (Thm. 6)	NP-c (Thm. 8)
	Iterated Plurality	Plurality with Runoff	Iterated Veto	Veto with Runoff
Constructive	NP-c (Thm. 9)	NP-c (Thm. 9)	NP-c (Thm. 11)	NP-c (Thm. 11)
Destructive	NP-c (Thm. 10)	NP-c (Thm. 10)	NP-c (Thm. 12)	NP-c (Thm. 12)

appealing features of the systems by Baldwin and Nanson, including that they have been applied in practice (namely, in the State of Michigan in the 1920s, in the University of Melbourne from 1926 through 1982, and in the University of Adelaide since 1968) and that (unlike Borda itself) they both are Condorcet-consistent.[3] Axiomatic properties of iterative voting systems were also studied by Freeman, Brill, and Conitzer [36] who showed, in particular, that Hare is the only iterative voting system based on scoring rules that satisfies the independence-of-clones property. Further, it was shown by Bartholdi and Orlin [37] that Hare (which is called STV in their work) is NP-hard to manipulate even with only one manipulator. This result was complemented by Davies, Narodytska, and Walsh [38] who showed the same result for Coombs and a general class of iterative versions of scoring rules. For plurality with runoff, it was shown by Conitzer, Sandholm, and Lang [6] that unweighted coalitional manipulation is NP-hard. Finally, plurality with runoff and veto with runoff were also studied by Erdélyi et al. [39] with respect to electoral control.

This paper is organized as follows. In Section 2, we will provide the needed definitions regarding elections and voting systems (in particular, iterative voting systems), define the shift bribery problem, and give some background on computational complexity. We will then study the complexity of shift bribery for Hare and Coombs elections in Section 3, for Baldwin and Nanson elections in Section 4, for iterated plurality and plurality with runoff in Section 5, and for iterated veto and veto with runoff in Section 6. Further, in Section 7 we will discuss how the nonmonotonicity property of our iterative voting systems can be exploited in our reductions showing NP-hardness, exemplified for Hare voting and plurality with runoff. Finally, we will conclude in Section 8 by presenting some open problems related to our work.

2 Preliminaries

Let us start by providing the needed notions and notation.

Elections and voting systems An *election* is specified as a pair (C,V) with C being a set of candidates and V a profile of the voters' preferences over C, typically given by a list of linear orders of the candidates. A *voting system* is a function that maps each election (C,V) to a subset of C, the *winner(s) of the election*. An important class of voting systems is the

[3] A *Condorcet winner* is a candidate who defeats every other candidate in a pairwise comparison. Such a candidate does not always exist. A voting rule is *Condorcet-consistent* if it chooses only the Condorcet winner whenever there exists one.

family of positional scoring rules whose most prominent members are plurality, veto, and Borda count, see, e.g., the book chapters by Zwicker [40] and Baumeister and Rothe [11] and the survey by Rothe [41] on using Borda in collective decision making.

Recall from Footnote 2 in Section 1 that, in *plurality*, each voter gives her top-ranked candidate one point; in *veto* (a.k.a. *antiplurality*), each voter gives all except the bottom-ranked candidate one point; in *Borda* with m candidates, each candidate in position i of a voter's ranking scores $m - i$ points; and the winners in each case are those candidates scoring the most points.

Iterative voting systems The iterative voting systems we will study are based on plurality, veto, and Borda but, unlike those, their election winner(s) are determined in consecutive rounds. For all iterative voting systems considered here except for plurality with runoff and veto with runoff (which will be defined shortly afterwards), if in some round all remaining candidates have the same score (for instance, there may be only one candidate left), then all those candidates are proclaimed winners of the election. In each preceding round, however, all candidates with the lowest score are eliminated.[4]

Recall from Section 1 that the eight scoring methods we will use work as follows: The iterative voting systems due to *Hare*, *Coombs*, and *Baldwin* use, respectively, plurality, veto, and Borda scores in order to decide which candidates are the weakest and thus to be removed. The *Nanson* system eliminates in every (except the last) round all candidates that have less than the average Borda score. *Iterated plurality* eliminates all candidates that do not have the highest plurality score, and *iterated veto* eliminates all candidates that do not have the highest veto score.

Unlike the above multiple-round iterative voting systems, *plurality with runoff* (respectively, *veto with runoff*) always proceeds in two rounds: In the first round, it eliminates all candidates that do not have the highest plurality score (respectively, veto score), unless there is a unique plurality winner (respectively, veto winner) in which case all candidates are eliminated except those with the highest or second-highest plurality score (respectively, veto score); in the second round, all candidates with the highest plurality score (respectively, veto score) win.

Shift bribery For any given voting system $\mathcal{E}$, we now define the problem $\mathcal{E}$-Shift-Bribery, which is a special case of $\mathcal{E}$-Swap-Bribery, introduced by Faliszewski et al. [13] in the context of so-called irrational voters (i.e., voters whose preferences can be intransitive) for Copeland and then comprehensively studied by Elkind et al. [1]. Informally stated, given a profile of votes, a swap-bribery price function exacts a price for each swap of any two candidates in the votes, and in shift bribery only swaps involving the designated candidate are allowed.

Formally, for a list of price functions $\rho = (\rho_1, \dots, \rho_n)$ with $\rho_i : \mathbb{N} \to \mathbb{N}$, in the constructive case $\rho_i(k)$ indicates the price the briber has to pay in order to move the designated candidate p in vote i by k positions to the top (respectively, to the bottom in the destructive case). For all i, we require that ρ_i is nondecreasing ($\rho_i(\ell) \leq \rho_i(\ell+1)$), $\rho_i(0) = 0$, and if p is at position r in vote i then $\rho_i(\ell) = \rho_i(\ell - 1)$ whenever $\ell \geq r$ in the constructive

[4] In the original sources defining these iterative voting systems as stated in the Introduction, certain tie-breaking schemes are used whenever more than one candidate has the lowest score in some round. For the sake of convenience and uniformity, however, we prefer eliminating them all and will therefore disregard tie-breaking issues in such a case.

case (respectively, whenever $\ell \geq |C| - r + 1$ in the destructive case). The latter condition ensures that p can be shifted upward no farther than to the top (respectively, the bottom).[5] When the voter i in ρ_i is clear from the context, we omit the subscript and simply write ρ.

$\mathcal{E}$-Constructive-Shift-Bribery

Given: An election (C,V) with n votes, a designated candidate $p \in C$, a budget B, and a list of price functions $\rho = (\rho_1, \dots, \rho_n)$.

Question: Is it possible to make p the unique $\mathcal{E}$ winner of the election by shifting p in the votes so that the total price does not exceed B?

In the corresponding problem $\mathcal{E}$-Destructive-Shift-Bribery, given the same input, we ask whether it is possible to prevent p from being a unique winner.

These problems are here defined in the unique-winner model where a constructive (respectively, destructive) bribery action is considered successful only if the designated candidate can be made (respectively, can be prevented from being) the only winner of the election. We also consider these problems in the nonunique-winner model where for a constructive (respectively, destructive) bribery action to be considered successful it is required that the designated candidate is merely one among possibly several winners (respectively, does not win at all). Note that a yes-instance of $\mathcal{E}$-Constructive-Shift-Bribery in the unique-winner model is also a yes-instance of the same problem in the nonunique-winner model, whereas a yes-instance of $\mathcal{E}$-Destructive-Shift-Bribery in the nonunique-winner model is also a yes-instance of the same problem in the unique-winner model; analogous statements apply to the no-instances of these problems by swapping the unique-winner model with the nonunique-winner model. We will make use of these facts in our proofs, which all work in both winner models.

Membership in NP is obvious for all considered problems, so it will be enough to show only NP-hardness so as to prove in fact NP-completeness.

Our proofs use the following notation: A vote of the form $a\ b\ c$ indicates that the voter ranks candidate a on top position, then candidate b, and last candidate c. If a set $S \subseteq C$ of candidates appears in a vote as $\overrightarrow{S}$, its candidates are placed in this position in lexicographical order. By $\overleftarrow{S}$ we mean the reverse of the lexicographical order of the candidates in S. If S occurs in a vote without an arrow on top, the order in which the candidates from S are placed here does not matter for our argument. We use $\cdots$ in a vote to indicate that the remaining candidates may occur in any order.

Computational complexity We assume familiarity with the standard concepts of complexity theory, including the classes P and NP, polynomial-time many-one reducibility, and NP-hardness and NP-completeness. We will use the following NP-complete problem:

Exact-Cover-by-3-Sets (X3C)

Given: A set $X = \{x_1, \dots, x_{3m}\}$ and a family of sets $\mathcal{S} = \{S_1, \dots, S_n\}$ such that $S_i \subseteq X$ and $|S_i| = 3$ for all $S_i \in \mathcal{S}$.

Question: Does there exist an exact cover of X, i.e., a subset $\mathcal{S}' \subseteq \mathcal{S}$ such that $|\mathcal{S}'| = m$ and $\bigcup_{S_i \in \mathcal{S}'} S_i = X$?

[5] If p is in the first (respectively, the last) position of a vote, this voter cannot be bribed and we tacitly assume a price function of $\rho(t) = 0$ for each $t \geq 0$. We will disregard these voters when setting price functions for the other voters in our proofs.

In instances of X3C, we assume that each $x_j \in X$ is contained in exactly three sets $S_i \in \mathcal{S}$; thus $|X| = |\mathcal{S}|$. Gonzalez [42] shows that X3C under this restriction remains NP-hard. Note that if not stated otherwise, we will use $(X, \mathcal{S})$ to denote an X3C instance, where $X = \{x_1, \ldots, x_{3m}\}$, $\mathcal{S} = \{S_1, \ldots, S_{3m}\}$, and $S_i = \{x_{i,1}, x_{i,2}, x_{i,3}\}$. Also note that we assume $x_{i,1}$ to be the $x_j \in S_i$ with the smallest subscript and $x_{i,3}$ to be the $x_j \in S_i$ with the largest subscript.

One-In-Three-Positive-3SAT

Given: A set X of boolean variables, a set S of clauses over X, each containing exactly three unnegated literals.

Question: Does there exist a truth assignment to the variables in X such that exactly one literal is set to true for each clause in S?

In instances of One-In-Three-Positive-3SAT, we assume that each $x_j \in X$ is contained in exactly three clauses. Porschen et al. [43] show that this restricted problem remains NP-complete.

For more background on computational complexity, the reader is referred to, for instance, the textbooks by Garey and Johnson [44] and Papadimitriou [45].

3 Hare and Coombs

We start by showing NP-hardness of shift bribery for Hare elections. Right after the proof of Theorem 1, its construction is explained and illustrated in Example 1.

Theorem 1 In both the unique-winner and the nonunique-winner model, Hare-Constructive-Shift-Bribery is NP-hard.

Proof NP-hardness follows by a reduction from X3C. Given an X3C instance $(X, \mathcal{S})$, construct an instance $((C,V),p,B,\rho)$ of Hare-Constructive-Shift-Bribery with candidate set $C = X \cup \mathcal{S} \cup \{p\}$, designated candidate p, and the following list V of votes, with # denoting their number:

line	#	vote	for
1	1	$S_i\, x_{i,1}\, \overrightarrow{X \setminus \{x_{i,1}\}} \cdots$	$1 \le i \le 3m$
2	1	$S_i\, x_{i,2}\, \overrightarrow{X \setminus \{x_{i,2}\}} \cdots$	$1 \le i \le 3m$
3	1	$S_i\, x_{i,3}\, \overrightarrow{X \setminus \{x_{i,3}\}} \cdots$	$1 \le i \le 3m$
4	4	$x_i\, \overrightarrow{X \setminus \{x_i\}} \cdots$	$1 \le i \le 3m$
5	1	$S_i\, p \cdots$	$1 \le i \le 3m$
6	3	$p \cdots$	

Note that when S_i occurs in a vote here or in later proofs, we mean the candidate corresponding to the 3-element set $S_i \in \mathcal{S}$ (and not the three candidates corresponding to the three elements of this set).

For votes of the form $S_i\, p \cdots$, we use the price function $\rho(1) = 1$, and $\rho(t) = m + 1$ for all $t \geq 2$; and for every other vote, we use the price function ρ with $\rho(t) = m + 1$ for all $t \geq 1$. Finally, set the budget $B = m$. Without loss of generality, we assume that $m > 1$.

Note that p scores three points while the rest of the candidates score four points each, so p is eliminated in the first round and does not win the election without bribing voters.

We claim that $(X, \mathcal{S})$ is in X3C if and only if $((C,V),p,B,\rho)$ is in Hare-Constructive-Shift-Bribery, regardless of the winner model.

($\Rightarrow$) Suppose that $(X, \mathcal{S})$ is a yes-instance of X3C. Then there exists an exact cover $\mathcal{S}' \subseteq \mathcal{S}$ of size m. We now show that it is possible for p to become a unique Hare winner of an election obtained by shifting p in the votes without exceeding the budget B. For every $S_i \in \mathcal{S}'$, we bribe the voter with the vote of the form $S_i\, p \cdots$ by shifting p once, so her new vote is of the form $p\ S_i \cdots$; each such bribe action costs us only 1 from our budget, so the budget will not be exceeded. In the first round, p now has $m + 3$ points, every candidate from $\mathcal{S}'$ has 3 points, and every other candidate has 4 points. Therefore, all candidates in $\mathcal{S}'$ are eliminated. In the second round, all candidates in X now gain one point from the elimination of $\mathcal{S}'$, since it is an exact cover. Therefore, p and all candidates in X proceed to the next round and the remaining candidates $\mathcal{S} \setminus \mathcal{S}'$ are eliminated. In the next round with only p and the candidates from X remaining, p has $3m + 3$ points, while every candidate in X scores 7 points (recall that every $x_i \in X$ is contained in exactly three members of $\mathcal{S}$). Since all candidates from X have been eliminated now, p is the only remaining candidate and thus the unique Hare winner.

($\Leftarrow$) Suppose that $(X, \mathcal{S})$ is a no-instance of X3C. Then no subset $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| \leq m$ covers X. We now show that we cannot make p become a Hare winner of an election obtained by bribing voters without exceeding budget B. Note that we can only bribe at most m voters with votes of the form $S_i\, p \cdots$ without exceeding the budget. Let $\mathcal{S}' \subseteq \mathcal{S}$ be such that $S_i \in \mathcal{S}'$ exactly if the voter with the vote $S_i\, p \cdots$ has been bribed. Clearly, $|\mathcal{S}'| \leq m$ and in all those votes p has been shifted once to the left, so p is now ranked first in these votes. Therefore, p now has $3 + |\mathcal{S}'|$ points and every $S_i \in \mathcal{S}'$ scores 3 points. Since every other candidate scores as many points as before the bribery (namely, 4 points), the candidates in $\mathcal{S}'$ are eliminated in the first round. Let $X' = \{x_i \in X | x_i \notin \bigcup_{S_j \in \mathcal{S}'} S_j\}$ be the subset of candidates $x_i \in X$ that are not covered by $\mathcal{S}'$. We have $X' \neq \emptyset$ (otherwise, $\mathcal{S}'$ would have been an exact cover of X). In the second round, unlike the candidates from $X \setminus X'$, the candidates in X' will not gain additional points from eliminating the candidates in $\mathcal{S}'$. Thus, in the current situation, the candidates from X' and $\mathcal{S} \setminus \mathcal{S}'$ are trailing behind with 4 points each and are eliminated in this round.[6] Therefore, in the next round, only p and the candidates from $X \setminus X'$ are remaining in the election. Let $x_\ell \in X \setminus X'$ be the candidate from $X \setminus X'$ with the smallest subscript. Since all candidates from $\mathcal{S}$ are eliminated, p has $3m + 3$ points and every candidate from $X \setminus X'$ except x_ℓ has 7 points. On the other hand, x_ℓ gains additional points from eliminating the candidates from X'; therefore, x_ℓ survives this round by scoring more than 7 points. In the final round with only p and x_ℓ remaining, p is eliminated, since $3m \cdot 7 > 3m + 3$. $\square$

[6] Note that in the case that $|\mathcal{S}'| = 1$, i.e., only one voter was bribed, p also gets eliminated in this round and is consequently not a Hare winner, which is what we want to show. Therefore, we will now assume that at least two voters were bribed.

Example 1 Let $(X, \mathcal{S})$ be a yes-instance of X3C defined by

$$X = \{x_1, \ldots, x_6\} \text{ and}$$
$$\mathcal{S} = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_2, x_3, x_6\}, \{x_2, x_4, x_5\}, \{x_1, x_3, x_4\}, \{x_1, x_5, x_6\}\}.$$

Construct $((C,V),p,B,\rho)$ from $(X, \mathcal{S})$ as in the proof of Theorem 1; in particular, the budget is $B = 2$. If we bribe the voters with $S_1 p \cdots$ and $S_2 p \cdots$ so as to shift p to the top of their votes, p will be the unique winner of the election, which proceeds as follows (where the numbers in the columns below candidates give their scores):

Round	p	$x \in X$	S_1, S_2	S_3, S_4, S_5, S_6
1	5	4	3	4
2	5	5	out	4
3	9	7	out	out

Now consider a no-instance $(X, \mathcal{S})$ of X3C with

$$X = \{x_1, \ldots, x_6\} \text{ and}$$
$$\mathcal{S} = \{\{x_1, x_2, x_4\}, \{x_4, x_5, x_6\}, \{x_2, x_3, x_6\}, \{x_2, x_3, x_5\}, \{x_1, x_3, x_4\}, \{x_1, x_5, x_6\}\}.$$

If we bribe no voter, p gets eliminated in the first round and so does not win. If we bribe one voter, say the one with vote $S_1\ p \cdots$, then p gets eliminated in the second round:

Round	p	x_1	x_2, x_4	x_3, x_5, x_6	S_1	$S_i \in \mathcal{S} \setminus \{S_1\}$
1	4	4	4	4	3	4
2	4	5	5	4	out	4
3	out	≥ 28	≥ 7	out	out	out

Since $(X, \mathcal{S})$ is a no-instance of X3C, no matter which two subsets $S_i, S_j \in \mathcal{S}$ we choose, at least one x_k is in both subsets, so p loses the direct comparison in the last round. For example, if we bribe the voters with $S_1\ p \cdots$ and $S_2\ p \cdots$, the election proceeds as follows:

Round	p	x_1	x_3	x_4	x_2, x_5, x_6	S_1, S_2	S_3, S_4, S_5, S_6
1	5	4	4	4	4	3	4
2	5	5	4	6	5	out	4
3	9	14	out	7	7	out	out
4	9	42	out	out	out	out	out

This completes Example 1.

Next, we show that shift bribery is NP-hard for Hare also in the destructive case.

Theorem 2 In both the unique-winner and the nonunique-winner model, Hare-Destructive-Shift-Bribery is NP-hard.

Proof Again, we use a reduction from X3C. Construct from a given X3C instance $(X, \mathcal{S})$ a Hare-Destructive-Shift-Bribery instance $((C,V),p,B,\rho)$ as follows. Let $D = \{d_1, \ldots, d_{3m}\}$

be a set of $3m$ dummy candidates. The candidate set is $C = X \cup \mathcal{S} \cup D \cup \{p, w\}$ with designated candidate p. The list V of votes is constructed as follows:

line	#	vote	for
1	2	$S_i\, x_{i,1}\, \overrightarrow{X \setminus \{x_{i,1}\}}\; w\, p\, \cdots$	$1 \le i \le 3m$
2	2	$S_i\, x_{i,2}\, \overrightarrow{X \setminus \{x_{i,2}\}}\; w\, p\, \cdots$	$1 \le i \le 3m$
3	2	$S_i\, x_{i,3}\, \overrightarrow{X \setminus \{x_{i,3}\}}\; w\, p\, \cdots$	$1 \le i \le 3m$
4	7	$x_i\, \overrightarrow{X \setminus \{x_i\}}\; w\, p\, \cdots$	$1 \le i \le 3m$
5	1	$p\, S_i \cdots$	$1 \le i \le 3m$
6	12	$wp\cdots$	
7	$18m$	$p\cdots$	
8	6	$d_i\, S_i\, p\, \cdots$	$1 \le i \le 3m$

For votes of the form $p\ S_i \cdots$, we use the price function $\rho(1) = 1$, and $\rho(t) = m + 1$ for all $t \ge 2$; and for every other vote, we use the price function ρ with $\rho(t) = m + 1$ for all $t \ge 1$. Finally, set the budget $B = m$.

Without bribing, the election (C,V) proceeds as follows:

Round	p	w	$x_i \in X$	$S_i \in \mathcal{S}$	$d_i \in D$
1	$21m$	12	7	6	6
2	$39m$	12	13	out	out
3	$39m + 12$	out	13	out	out

It follows that p has won the election after three rounds.

We claim that $(X, \mathcal{S})$ is in X3C if and only if $((C,V),p,B,\rho)$ is in Hare-Destructive-Shift-Bribery, regardless of the winner model.

($\Rightarrow$) Suppose that $(X, \mathcal{S})$ is a yes-instance of X3C. Then there exists an exact cover $\mathcal{S}' \subseteq \mathcal{S}$ of size m. We now show that it is possible to eliminate p from an election obtained by shifting p in the votes without exceeding the budget B. For every $S_i \in \mathcal{S}'$, we bribe the voter with the vote of the form $p\ S_i \cdots$ by shifting p once, so her new vote is of the form $S_i\, p\ \ldots$; each such bribe action costs us only 1 from our budget, so the budget will not be exceeded. Now the election proceeds as follows:

Round	p	w	$x_i \in X$	$S_i \in \mathcal{S}'$	$S_i \in \mathcal{S} \setminus \mathcal{S}'$	$d_i \in D$
1	$20m$	12	7	7	6	6
2	$32m$	12	11	13	out	out
3	$32m$	$33m + 12$	out	13	out	out
4	$39m$	$39m + 12$	out	out	out	out

We see that p is eliminated in the fourth round and w wins.

($\Leftarrow$) Suppose that $(X, \mathcal{S})$ is a no-instance of X3C. Then no subset $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| \le m$ covers X. We now show that p will not be eliminated in any election obtained by bribing voters without exceeding budget B but will in fact become the only winner. Note that we can only bribe at most m voters with votes of the form $p\ S_i \cdots$ without exceeding the budget. Let $\mathcal{S}' \subseteq \mathcal{S}$ be such that for every $S_i \in \mathcal{S}'$ we have bribed the voter whose vote is

$p\ S_i\ \cdots$. We can assume that $|\mathcal{S}'| > 0$. Every candidate in $\mathcal{S}'$ will gain an additional point and therefore survives the first round. All candidates from D and $\mathcal{S} \setminus \mathcal{S}'$ will be eliminated, since p only loses at most m points.

In the second round, the remaining candidates from $\mathcal{S}$ will additionally gain six points from the elimination of candidates in D and will score 13 points in this round (and in all subsequent rounds with p still standing). If a candidate $S_i \in \mathcal{S}$ was eliminated in the previous round, every $x_i \in S_i$ gains two additional points in this round. Partition X into sets X_0, X_1, X_2, and X_3 so that $x_i \in X_k \Leftrightarrow |\{S_j \in \mathcal{S}' | x_i \in S_j\}| = k$ for $k \in \{0,1,2,3\}$. Note that X_0, X_1, X_2, and X_3 are disjoint and $|X_0| > 0$, but one or two of X_1, X_2, and X_3 may be empty. Then $x_i \in X_j$ scores $7 + (6 - 2j) \in \{7,9,11,13\}$ points depending on how many times x_i is covered by $\mathcal{S}'$. Therefore, every $x_i \in X_0$ scores more points than w who has 12 points. Thus there are candidates from X that survive this round and other candidates from X (more precisely, candidates from X_1, X_2, or X_3) who are eliminated.

In the third round, the candidate $x_\ell \in X$ with the smallest subscript who is still standing gains at least seven points from the eliminated candidates according to the votes of the form $x_i\ \overrightarrow{X \setminus \{x_i\}}\ w\ p\ \cdots$, so that x_ℓ scores at least 16 points.[7] All other candidates (i.e., still standing candidates from X except x_ℓ, all candidates from $\mathcal{S}'$, p, and w) still score the same number of points as in the last round. Therefore, p scores at least $20m$ points, w scores still 12 points, every $S_i \in \mathcal{S}'$ scores 13 points, and every still standing candidate from X except x_ℓ scores at most 13 points. Since w can only gain additional points when all candidates from X are eliminated and only x_ℓ gains points from the elimination of candidates from $X \setminus \{x_\ell\}$in the subsequent rounds, all candidates $X \setminus (\{x_\ell\} \cup X_0)$ and w are eliminated. Then all still standing candidates from $X_0 \setminus \{x_\ell\}$ and candidates from $\mathcal{S}'$ who each score 13 points are eliminated, which leaves p and x_ℓ in the last round. In this round, p scores $39m + 12$ points and x_ℓ scores $39m$ points, so p solely wins the election, no matter how we bribe voters within the budget, i.e., we have a no-instance of Hare-Destructive-Shift-Bribery in both winner models. □

Next, we turn to shift bribery for Coombs elections. While the idea of the reduction is similar, and perhaps even simpler than in the previous two proofs, the proof of correctness is way more involved. Again, we explain and illustrate the reduction right after the proof of Theorem 3 in Example 2.

Theorem 3 In both the unique-winner and the nonunique-winner model, Coombs-Constructive-Shift-Bribery is NP-hard.

Proof To prove NP-hardness, we now describe a reduction from X3C to Coombs-Constructive-Shift-Bribery. Given an X3C instance $(X, \mathcal{S})$, construct an election (C,V) with the set $C = \{p,w,d_1,d_2,d_3\} \cup X \cup Y$ of candidates, where p is the designated candidate and $Y = \{y_i | x_i \in X\}$. Construct the following list V of votes:

[7] Since this candidate x_ℓ is still in the election, x_ℓ cannot have been in X_3 and thus must have had at least nine points.

line	#	vote	for
1	1	$\cdots x_{i,1}\ x_{i,2}\ x_{i,3}\ p$	$1 \le i \le 3m$
2	$2m$	$\cdots p\ \overrightarrow{Y \setminus \{y_i\}}\ y_i\ x_i$	$1 \le i \le 3m$
3	$2m$	$\cdots p\ \overrightarrow{Y}\ w\ d_1\ d_2\ d_3$	
4	1	$\cdots p\ \overrightarrow{Y}\ w\ X\ d_1\ d_2\ d_3$	
5	m	$\cdots p\ \overrightarrow{Y}\ w$	

For votes of the form $\cdots x_{i,1}\ x_{i,2}\ x_{i,3}\ p$, we use the price function $\rho(1) = \rho(2) = \rho(3) = 1$, and $\rho(t) = m + 1$ for all $t \ge 4$; and for all the remaining votes, we use the price function $\rho(t) = m + 1$ for all $t \ge 1$. Furthermore, our budget is $B = m$.

The candidates have the following veto counts: p has $3m$ vetoes, each $x_i \in X$ has $2m$ vetoes, w has m vetoes, d_3 has $2m + 1$ vetoes, and the remaining candidates each have 0 vetoes. Therefore, p will be eliminated in the first round and thus does not win the election.

We claim that $(X, \mathcal{S})$ is in X3C if and only if $((C,V),p,B,\rho)$ is in Coombs-Constructive-Shift-Bribery, regardless of the winner model.

($\Rightarrow$) Assume that $(X, \mathcal{S})$ is in X3C. This means that there exists a subset $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| = m$ and $\bigcup_{S_i \in \mathcal{S}'} S_i = X$. So we have a partition of X into three sets, $X = X_1 \cup X_2 \cup X_3$, such that:

$$\begin{aligned} X_1 &= \{x_{i,1} | S_i \in \mathcal{S}'\}, \\ X_3 &= \{x_{i,3} | S_i \in \mathcal{S}'\}, \text{ and} \\ X_2 &= X \setminus (X_1 \cup X_3). \end{aligned}$$

Let $Y = Y_1 \cup Y_2 \cup Y_3$ be the corresponding partition of Y (i.e., $x_i \in X_j \Leftrightarrow y_i \in Y_j$).

We bribe the voters with votes of the form $\cdots x_{i,1}\ x_{i,2}\ x_{i,3}\ p$ for $S_i \in \mathcal{S}'$ so that they change their votes to $\cdots p\ x_{i,1}\ x_{i,2}\ x_{i,3}$. Since $\mathcal{S}'$ is an exact cover of X, it follows that p now has a total of $2m$ vetoes, whereas each $x \in X_3$ receives an additional veto for a total of $2m + 1$. The number of vetoes for the remaining candidates remain unchanged. If a candidate has the highest number of vetoes then she has the fewest number of points and cannot proceed to the next round (unless all candidates have the same score). Here, the candidates in X_3 and d_3 have the highest number of vetoes (and higher than the other candidates) and therefore are eliminated in the first round.

Without the candidates in X_3, each candidate in X_2 gets an additional veto and the candidates in Y_3 each take all but one of the vetoes of the eliminated candidates in X_3. Furthermore, d_2 receives the vetoes of d_3. As a consequence, in the second round the candidates in X_2 and d_2 have the highest number of vetoes (and higher than the remaining candidates) and are eliminated.

Similarly to the first round, vetoes from candidates in X_2 and d_2 are passed on to candidates in X_1 and Y_1 and to d_1. Thus the candidates have the following veto counts in the third round: p and each $y \in Y_2 \cup Y_3$ receive $2m$ vetoes, w receives m vetoes, each $y \in Y_1$ receives zero vetoes, and d_1 and each $x_i \in X_1$ receive $2m + 1$ vetoes. Consequently, all the candidates $x_i \in X_1$ and d_1 are eliminated in the third round, so in the next round there are no candidates from X and no d_i, $1 \le i \le 3$.

It follows that w receives $2m + 1$ additional vetoes in the fourth round, so w has the most vetoes in this round and is eliminated. We need $3m$ further rounds until p ends up as the last remaining candidate and sole winner of the election. In each of these rounds, the candidate in Y that is still alive and has the highest subscript has at least $2m+2m+1+m=5m+1$ vetoes, while p always has only $3m$ vetoes.

($\Leftarrow$) Suppose that $(X, \mathcal{S})$ is a no-instance of X3C. We will show that $((C,V),p,B,\rho)$ then is a no-instance of Coombs-Constructive-Shift-Bribery in the nonunique-winner (and thus also in the unique-winner) model. Observe that if we were going to make p a winner of the election, we would have to bribe at least m voters with a vote of the form $\cdots x_{i,1}\ x_{i,2}\ x_{i,3}\ p$.[8] Due to our budget, on the other hand, we can bribe no more than m (and thus would have to bribe exactly m) such voters and cannot bribe any further voters. Let $\mathcal{S}' \subseteq \mathcal{S}$ be such that $S_i \in \mathcal{S}'$ exactly if the voter with the vote of the form $\cdots x_{i,1}\ x_{i,2}\ x_{i,3}\ p$ has been bribed. Note that $|\mathcal{S}'| = m$ and $\mathcal{S}'$ does not cover X because we have a no-instance of X3C. Now p has only $2m$ vetoes and will not be eliminated in the first round.

Let X_1 be the set of candidates $x_i \in S_i$ for $S_i \in \mathcal{S}'$ with the smallest subscript in S_i, let X_2 be the set of candidates $x_i \in S_i$ for $S_i \in \mathcal{S}'$ with the second-smallest subscript in S_i, and let X_3 be the set of candidates $x_i \in S_i$ for $S_i \in \mathcal{S}'$ with the highest subscript in S_i. Note that $X_1 \cup X_2 \cup X_3 \neq X$, since $\mathcal{S}'$ does not cover X.

For w to have more vetoes than p, the candidates d_1, d_2, and d_3 need to be eliminated. For that to happen, there must be three rounds in which no candidate has more than $2m+1$ vetoes. In the round where d_i, $1 \leq i \leq 3$, is eliminated, all still standing candidates in X_i are eliminated as well. Assume there were three rounds in which $2m+1$ was the maximal number of vetoes for a candidate. Then d_1, d_2, d_3, and all candidates in $X_1 \cup X_2 \cup X_3$ are eliminated. Note that those candidates that are not covered by $\mathcal{S}'$ always have only $2m$ vetoes and are still participating in the election. Therefore, in the next round, p and w have $3m$ vetoes each, the remaining candidates from X have at most $2m+1$ vetoes, and the candidates from Y have at most $2m$ vetoes. So even if p survives the first rounds with the candidates d_1, d_2, and d_3 still present, p will then surely be eliminated in the following round. If there is at least one voter who shifts p only one or two positions upward, then p has to drop out with d_1 or even before d_1 drops out, because at the latest after two rounds (with $2m+1$ being the maximal number of vetoes for a candidate) p receives another veto and thus has at least the same number of vetoes as d_1. $\square$

Example 2 Let $(X, \mathcal{S})$ be a yes-instance of X3C defined by

$$\begin{aligned} X &= \{x_1, \ldots, x_6\} \text{ and} \\ \mathcal{S} &= \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_2, x_3, x_6\}, \{x_2, x_4, x_5\}, \{x_1, x_3, x_4\}, \{x_1, x_5, x_6\}\}. \end{aligned}$$

Construct $((C,V),p,B,\rho)$ from $(X, \mathcal{S})$ as in the proof of Theorem 3; in particular, the budget is $B=2$. If we bribe the voters that correspond to the sets in the exact cover, S_1 and S_2, to change their votes from $\cdots x_1\ x_2\ x_3\ p$ and $\cdots x_4\ x_5\ x_6\ p$ to $\cdots p\ x_1\ x_2\ x_3$ and $\cdots p\ x_4\ x_5\ x_6$, then p alone wins the election that proceeds as follows, where in order to make this example easier to follow, the numbers in the table count the candidates' vetoes, not their points, i.e., the candidates with the *highest* number in a round (row) get eliminated:

[8] Assume that we bribe fewer than m votes. Then either (a) p is still the candidate with the most vetoes with a total of more than $2m$ vetoes and would be eliminated in the first round, or (b) there is at least one candidate x_i that has at least $2m+2$ vetoes after the bribery action, but eliminating any such x_i only increases the number of vetoes for a corresponding candidate y_i (to a total of $2m$) so that eventually p will be eliminated.

Round	p	w	x_1,x_4	x_2,x_5	x_3,x_6	y_1	y_2	y_3	y_4	y_5	y_6	d_1	d_2	d_3
1	4	2	4	4	5	0	0	0	0	0	0	0	0	5
2	4	2	4	5	out	0	0	4	0	0	4	0	5	out
3	4	2	5	out	out	0	4	4	0	4	4	5	out	out
4	6	7	out	out	out	4	4	4	4	4	4	out	out	out
5	6	out	out	out	out	4	4	4	4	4	11	out	out	out
6	6	out	out	out	out	4	4	4	4	15	out	out	out	out
7	6	out	out	out	out	4	4	4	19	out	out	out	out	out
8	6	out	out	out	out	4	4	23	out	out	out	out	out	out
9	6	out	out	out	out	4	27	out	out	out	out	out	out	out
10	6	out	out	out	out	31	out	out	out	out	out	out	out	out

It follows that p is the sole winner of the election.

Now consider a no-instance $(X, \mathcal{S})$ with

$$X = \{x_1, \ldots, x_6\} \text{ and}$$
$$\mathcal{S} = \{\{x_1, x_2, x_4\}, \{x_4, x_5, x_6\}, \{x_2, x_3, x_6\}, \{x_2, x_3, x_5\}, \{x_1, x_3, x_4\}, \{x_1, x_5, x_6\}\}.$$

Recall that we can bribe at most two voters. If we bribe fewer than two voters, however, p will be eliminated in the first round. Since $(X, \mathcal{S})$ is a no-instance of X3C, no matter which two subsets $S_i, S_j \in \mathcal{S}$ we choose, at least one x_k is in both S_i and S_j. For example, if we bribe the voters that correspond to the sets S_1 and S_2, changing their votes from $\cdots\ x_1\ x_2\ x_4\ p$ and $\cdots\ x_4\ x_5\ x_6\ p$ to $\cdots p\ x_1\ x_2\ x_4$ and $\cdots p\ x_4\ x_5\ x_6$, then the election proceeds as follows:

Round	p	w	x_1	x_2,x_5	x_3	x_4,x_6	y_1	y_2,y_5	y_3	y_4,y_6	d_1	d_2	d_3
1	4	2	4	4	4	5	0	0	0	0	0	0	5
2	4	2	4	5	4	out	0	0	0	4	0	5	out
3	5	2	5	out	4	out	0	4	0	4	5	out	out
4	out	≥ 6	out	out	≥ 5	out	≥ 4	≥ 4	≥ 0	≥ 4	out	out	out
...	...	...	...	...	...	...	...	...	...	...	...	...	...

Since x_4 is in both S_1 and S_2, p gets an additional veto in round 3 and is subsequently eliminated. The same will happen for similar reasons in every other case.

This completes Example 2.

We now modify the previous reduction so as to work for the destructive case in Coombs elections.

Theorem 4 In both the unique-winner and the nonunique-winner model, Coombs-Destructive-Shift-Bribery is NP-hard.

Proof To prove NP-hardness, we again reduce from the NP-complete problem X3C to Coombs-Destructive-Shift-Bribery. Given an X3C instance $(X, \mathcal{S})$ where we may assume that $m > 2$ for $|X| = 3m$, we construct a Destructive-Shift-Bribery instance $((C,V),p,B,\rho)$ as follows. Let $C = X \cup \mathcal{S} \cup D \cup \{p, w, y\}$ be the candidate set with designated candidate p and a set $D = \{d_{i,j} | 1 \leq i \leq m-1, 1 \leq j \leq 4\}$ of dummy candidates. Let $D = D_1 \cup D_2 \cup D_3 \cup$

D_4 be a partition of D with $D_j = \{d_{i,j} | 1 \le i \le m-1\}$ for $1 \le j \le 4$. The list V of votes is then constructed as follows:

line	#	vote	for
1	1	$\cdots p\ S_i$	$1 \le i \le 3m$
2	$4m$	$p \cdots w\ x_{i,1}\ x_{i,2}\ x_{i,3}\ S_i$	$1 \le i \le 3m$
3	$4m+1$	$\cdots p\ X\ d_{i,1}\ d_{i,2}\ d_{i,3}\ d_{i,4}$	$1 \le i \le m-1$
4	1	$p \cdots y\ x_i$	$1 \le i \le 3m$
5	3	$\cdots p$	
6	2	$p \cdots w$	

Unlike in the previous proofs, it is here necessary that the candidates that are represented by "⋯" are placed in lexicographical order. For votes of the form $\cdots p\ S_i$, we use the price function $\rho(1) = 1$, and $\rho(t) = 2m+1$ for all $t \ge 2$; and for all the remaining voters, we use the price function $\rho(t) = 2m+1$ for all $t \ge 1$. Finally, we set the budget $B = 2m$.

Analyzing the constructed election without bribing voters, the candidates have the following veto counts: p has three vetoes, w has two vetoes, each $x \in X$ has one veto, each $S_i \in \mathcal{S}$ and each $d \in D_4$ has $4m+1$ vetoes, and the remaining candidates each have zero vetoes. It follows that all candidates from $\mathcal{S}$ and D_4 are eliminated. The candidates from D_4 transfer their vetoes to candidates in D_3 who each have $4m+1$ vetoes now; p gets $3m$ additional vetoes from the eliminated candidates in $\mathcal{S}$; and the remaining $12m^2$ vetoes (from the second group of voters) are shared among candidates from X. Since they are ordered lexicographically in those votes, there must be one candidate from X (now and in the following rounds) that obtains more than $4m+1$ vetoes leading to the elimination of all candidates from X in the following rounds. One after another, the candidates w, y, and the remaining candidates in D are eliminated, eventually leaving p as the last standing candidate and sole winner.

We claim that $(X, \mathcal{S})$ is in X3C if and only if $((C,V),p,B,\rho)$ is in Coombs-Destructive-Shift-Bribery, regardless of the winner model.

($\Rightarrow$) Assume that $(X, \mathcal{S})$ is in X3C. This means that there exists a subset $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| = m$ and $\bigcup_{S_i \in \mathcal{S}'} S_i = X$. So we have a partition of X into three sets, $X = X_1 \cup X_2 \cup X_3$, such that

$$\begin{aligned} X_1 &= \{x_{i,1} | S_i \in \mathcal{S}'\}, \\ X_3 &= \{x_{i,3} | S_i \in \mathcal{S}'\}, \text{ and} \\ X_2 &= X \setminus (X_1 \cup X_3). \end{aligned}$$

We bribe the voters with a vote of the form $\cdots p\ S_i$ with $S_i \in \mathcal{S} \setminus \mathcal{S}'$ such that they change their vote to $\cdots S_i\ p$. Now the election proceeds as follows, where we again count the vetoes and not the points:

Round	p	w	y	$S_i \in \mathcal{S}'$	$S_i \in \mathcal{S} \setminus \mathcal{S}'$	$x_i \in X_1$	$x_i \in X_2$	$x_i \in X_3$	$d_{i,1} \in D_1$	$d_{i,2} \in D_2$	$d_{i,3} \in D_3$	$d_{i,4} \in D_4$
1	$2m+3$	2	0	$4m+1$	$4m$	1	1	1	0	0	0	$4m+1$
2	$3m+3$	2	0	out	$4m$	1	1	$4m+1$	0	0	$4m+1$	out

Round	p	w	y	$S_i \in \mathcal{S}'$	$S_i \in \mathcal{S} \setminus \mathcal{S}'$	$x_i \in X_1$	$x_i \in X_2$	$x_i \in X_3$	$d_{i,1} \in D_1$	$d_{i,2} \in D_2$	$d_{i,3} \in D_3$	$d_{i,4} \in D_4$
3	$3m+3$	2	m	out	$4m$	1	$4m+1$	out	0	$4m+1$	out	out
4	$3m+3$	2	$2m$	out	$4m$	$4m+1$	out	out	$4m+1$	out	out	out
5	$4m^2+2$	$4m^2+2$	$3m$	out	$4m$	out	out	out	out	out	out	out

We see that p is eliminated in the fifth round, whereas y and some other candidates from $\mathcal{S} \setminus \mathcal{S}'$ are still in the election. Hence, p does not win.

($\Leftarrow$) Suppose that $(X, \mathcal{S})$ is a no-instance of X3C. Then no subset $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| \leq m$ covers X. We now show that p will not be eliminated in an election obtained by bribing voters without exceeding budget B but will in fact become the only winner. Note that we can only bribe at most $2m$ voters with votes of the form $\cdots\ p\ S_i$ without exceeding the budget. Let $\mathcal{S}' \subseteq \mathcal{S}$ be such that for every $S_i \in \mathcal{S} \setminus \mathcal{S}'$ we have bribed the voter whose vote was $\cdots\ p\ S_i$ and now is $\cdots\ S_i\ p$. We can assume that $|\mathcal{S} \setminus \mathcal{S}'| > 0$.

Every candidate in $\mathcal{S} \setminus \mathcal{S}'$ will gain an additional point and therefore survives the first round. All candidates in D_4 and $\mathcal{S}'$ will be eliminated in the first round. It follows that p has $3m+3$ vetoes in the second round. At this point, p is in each voter group other than the third voter group (with votes of the form $\cdots\ p\ X\ d_{i,1}\ d_{i,2}\ d_{i,3}\ d_{i,4}$) either the most (groups 2, 4, and 6) or the least preferred (groups 1 and 5) candidate; therefore, p does not receive any further vetoes before some candidate $d \in D_1$ is eliminated.

Note that $|\mathcal{S}'| \geq m$. Since $\mathcal{S}'$ is not an exact cover of X, we have at least one x in X that is in two sets $S, S' \in \mathcal{S}'$. Let $X' = \{x \in X | \exists\ S, S' \in\ \mathcal{S}',\ S \neq S',\ x \in S \cap S'\}$. In the next three rounds, it can happen that (1) a candidate $x \in X'$ receives at least $8m+1$ vetoes (e.g., because this x has the same subscript in different $S, S' \in \mathcal{S}'$) and is therefore eliminated without also eliminating the candidates in D_3 (respectively, D_2) with $4m+1$ vetoes, and (2) a candidate $x \in X'$ with a high subscript in $S \in \mathcal{S}'$ has a low subscript in $S' \in \mathcal{S}'$ and is therefore eliminated early, so that w has a total of at least $4m+2$ vetoes. Both cases lead to w having at least $4m+2$ vetoes while each $d \in D_1$ still has at most $4m+1$ vetoes. After w is eliminated, in each following round the candidate x with the highest subscript and later the candidate S with the highest subscript, y, and all possibly remaining $d \in D_2 \cup D_3$ will be eliminated. It follows that only p and the candidates $d \in D_1$ are still in the election. Now, in each round that follows, p has at most $4m^2-4m+1$ vetoes ($3m$ from line 1, $(4m+1)(m-2)$ from line 3 since there is at least one $d \in D_1$ still standing, and three vetoes from line 5) while the still standing candidate $d \in D_1$ with the highest subscript receives at least $12m^2+7m+3$ vetoes. Hence, eventually p alone wins the election. □

4 Baldwin and Nanson

We now show NP-hardness of shift bribery for Baldwin and Nanson elections. Note that our reductions are inspired by and similar to those used by Davies et al. [34] to show NP-hardness of the unweighted coalitional manipulation problem for these voting systems.

For a preference profile V over a set of candidates C, let $avg(V)$ be the average Borda score of the candidates in V (i.e., $avg(V) = \frac{1}{2}(|C|-1)|V|$). To conveniently construct votes, for a set of candidates C and $c_1, c_2 \in C$, let

$$W_{(c_1,c_2)} = (c_1\ c_2\ \overrightarrow{C \setminus \{c_1, c_2\}},\ \overleftarrow{C \setminus \{c_1, c_2\}}\ c_1\ c_2).$$

Under Borda, from the two votes in $W_{(c_1,c_2)}$ candidate c_1 scores $|C|$ points, c_2 scores $|C|-2$ points, and all other candidates score $|C|-1$ points. Also, observe that if a candidate $c^* \in C$ is eliminated in some round and $c^* \notin \{c_1,c_2\}$ then all other candidates lose one point due to the votes in $W_{(c_1,c_2)}$; if $c^* = c_1$ then c_2 loses no points but all other candidates lose one point; and if $c^* = c_2$ then c_1 loses two points and all other candidates lose one point. Therefore, if c^* is eliminated, the point difference caused by this elimination with respect to the votes in $W_{(c_1,c_2)}$ remains the same for all candidates, with two exceptions: (a) If $c^* = c_1$ then c_2 gains a point with respect to every other candidate, and (b) if $c^* = c_2$ then c_1 loses a point with respect to every other candidate. Note that this construction of votes makes determining the Borda score of a candidate c_i very convenient: We only need to look at pairs of votes $W_{(c_1,c_2)}$ in which c_i is c_1 or c_2, and the other candidate (c_2 and c_1, respectively) is still standing.

Furthermore, let $score_{(C,V)}(x)$ denote the number of points candidate x obtains in a Borda election (C,V), and let $dist_{(C,V)}(x,y) = score_{(C,V)}(x) - score_{(C,V)}(y)$.

We start with the complexity of shift bribery in Baldwin elections for the constructive case.

Theorem 5 In both the unique-winner and the nonunique-winner model, Baldwin-Constructive-Shift-Bribery is NP-hard.

Proof To prove NP-hardness, we reduce the NP-complete problem X3C to Baldwin-Constructive-Shift-Bribery. From a given X3C instance $(X, \mathcal{S})$, we construct an election (C,V) with the set of candidates $C = \{p, w, d\} \cup X \cup \mathcal{S}$ and designated candidate p and with V consisting of two lists of votes, V_1 and V_2, where V_1 contains the following votes:

#	votes	for	#	votes	for
1	$W_{(S_j,p)}$	$1 \le j \le 3m$	2	$W_{(x_{j,3},S_j)}$	$1 \le j \le 3m$
2	$W_{(x_{j,1},S_j)}$	$1 \le j \le 3m$	2	$W_{(w,x_i)}$	$1 \le i \le 3m$
2	$W_{(x_{j,2},S_j)}$	$1 \le j \le 3m$	7	$W_{(w,p)}$	

The votes in V_1 give the following scores to the candidates in C:

$$\begin{aligned}
score_{(C,V_1)}(x_i) &= avg(V_1) + 4 \text{ for every } x_i \in X,\\
score_{(C,V_1)}(S_j) &= avg(V_1) - 5 \text{ for every } S_j \in \mathcal{S},\\
score_{(C,V_1)}(p) &= avg(V_1) - 3m - 7,\\
score_{(C,V_1)}(w) &= avg(V_1) + 6m + 7,\\
score_{(C,V_1)}(d) &= avg(V_1).
\end{aligned}$$

Furthermore, V_2 contains the following votes:

#	votes	for	#	votes
$2m+1$	$W_{(d,S_j)}$	$1 \le j \le 3m$	1	$W_{(p,d)}$
$2m+9$	$W_{(d,x_i)}$	$1 \le i \le 3m$	$2m+14$	$W_{(d,w)}$

The votes in V_2 give the following scores to the candidates in C:

$$\begin{aligned}
score_{(C,V_2)}(x_i) &= avg(V_2) - (2m+9) \text{ for every } x_i \in X,\\
score_{(C,V_2)}(S_j) &= avg(V_2) - (2m+1) \text{ for every } S_j \in \mathcal{S},\\
score_{(C,V_2)}(p) &= avg(V_2) + 1,\\
score_{(C,V_2)}(w) &= avg(V_2) - (2m+14),\\
score_{(C,V_2)}(d) &= avg(V_2) + 12m^2 + 32m + 13.
\end{aligned}$$

Let $V = V_1 \cup V_2$ and $avg(V) = avg(V_1) + avg(V_2)$. Then we have the following Borda scores for the complete preference profile V over C:

$$\begin{aligned}
score_{(C,V)}(x_i) &= avg(V) - 2m - 5 \text{ for every } x_i \in X,\\
score_{(C,V)}(S_j) &= avg(V) - 2m - 6 \text{ for every } S_j \in \mathcal{S},\\
score_{(C,V)}(p) &= avg(V) - 3m - 6,\\
score_{(C,V)}(w) &= avg(V) + 4m - 7,\\
score_{(C,V)}(d) &= avg(V) + 12m^2 + 32m + 13.
\end{aligned}$$

Regarding the price function, for every first vote of $W_{(S_j,p)}$ (i.e., a vote of the form $S_j\ p\ \overrightarrow{C \setminus \{S_j,p\}}$), let $\rho(1) = 1$ and $\rho(t) = m + 1$ for every $t \geq 2$. For every other vote, let $\rho(t) = m + 1$ for every $t \geq 1$. Finally, we set the budget $B = m$.

It is easy to see that p is eliminated in the first round in the election (C,V) and thus does not win.

We claim that $(X, \mathcal{S})$ is in X3C if and only if $((C,V),p,B,\rho)$ is in Baldwin-Constructive-Shift-Bribery, regardless of the winner model.

($\Rightarrow$) Suppose there is an exact cover $\mathcal{S}' \subseteq \mathcal{S}$. Then we bribe the first votes of $W_{(S_j,p)}$ for every $S_j \in \mathcal{S}'$ by shifting p to the left once. Note that we won't exceed our budget, since shifting once costs 1 in those votes and $|\mathcal{S}'| = m$. After this bribery, for every $S_j \in \mathcal{S}'$, the two votes from $W_{(S_j,p)}$ result in two votes that are symmetric to each other (i.e., $p\ S_j\ \overrightarrow{C \setminus \{S_j,p\}}$ equals the vote $\overleftarrow{C \setminus \{S_j,p\}}\ S_j\ p$ in reverse order) and can thus be disregarded from now on, as all candidates gain the same number of points from those votes and all candidates lose the same number of points if a candidate is eliminated from the election. After those m votes have been bribed, only the scores of p and every $S_j \in \mathcal{S}'$ change. Let V' denote the correspondingly changed profile. With

$$score_{(C,V')}(p) = avg(V') - 2m - 6 \quad \text{and} \quad score_{(C,V')}(S_j) = avg(V') - 2m - 7,$$

all candidates in $\mathcal{S}'$ are tied for the last place. If any $S_j \in \mathcal{S}'$ is eliminated in a round (this might be the case for more than one of those candidates in a round), the three candidates $x_{j,1}$, $x_{j,2}$, and $x_{j,3}$ will lose two points more than the candidates from $\mathcal{S}' \setminus \{S_j\}$ that had the minimum score before S_j was eliminated. Therefore, those three candidates from X will then be in the last position in the next round. This means that all candidates $\mathcal{S}'$ and every $x_i \in X$ that is covered by $\mathcal{S}'$ will be eliminated in the subsequent rounds. Since $\mathcal{S}'$ is an exact cover, now there is no candidate from X left. Thus the point difference between p and w is 1 and between p and the remaining $S_j \in (\mathcal{S} \setminus \mathcal{S}')$ is -6. Note that p can beat d only if no candidate of $C \setminus \{p,d\}$ is still participating. So in the next round, w is eliminated. From this p gains seven points on all $S_j \in (\mathcal{S} \setminus \mathcal{S}')$, so these are tied for the last place. Therefore, the remaining candidates from $\mathcal{S}$ are eliminated, which leaves p and d for the next and final round, where d is eliminated and p wins the election alone.

($\Leftarrow$) Suppose there is no exact cover. It is obvious that at most m of the first votes of $W_{(S_j,p)}$ can be bribed without exceeding the budget. Without bribing, p is in the last place and the point difference to the second-to-last candidate(s) is $dist_{(C,V)}(p,S_j) = m$, $1 \le j \le 3m$. By bribing, as explained above, p gains m points while m candidates from $\mathcal{S}$ each lose a point and then will be eliminated from the election. This leads to the elimination of all $x_i \in X$ that are covered by the set $\mathcal{S}' \subseteq \mathcal{S}$ of candidates that were eliminated. Since there is no exact cover, $\mathcal{S}'$ doesn't cover X. So there are candidates $X' \subseteq X$, $|X'| \ge 1$, who were not eliminated before, as for every candidate $x_i \in X'$ all three candidates $S_j \in (\mathcal{S} \setminus \mathcal{S}')$ with $x_i \in S_j$ are still in the election. With the candidates $C_1 = \{p, w, d\} \cup (\mathcal{S} \setminus \mathcal{S}') \cup X'$ still standing, the point differences of p to the other remaining candidates are as follows:

$$\begin{aligned}
dist_{(C_1,V')}(p,d) &= -2m - 5 - 2m(2m+1) - |X'|(2m+9) - (2m+14) < 0,\\
dist_{(C_1,V')}(p,w) &= 1 - 2|X'| < 0,\\
dist_{(C_1,V')}(p,x_i) &= -1 \text{ for every } x_i \in X', \text{ and}\\
dist_{(C_1,V')}(p,S_j) &\le 0 \text{ for every } S_j \in \mathcal{S} \setminus \mathcal{S}'.
\end{aligned}$$

Therefore, p has the lowest score and is eliminated and thus does not win. □

The proof of the following theorem, which handles the destructive variant for Baldwin, uses a similar idea as the proof of Theorem 5. That is why we refrain from presenting all proof details in full; a proof sketch will suffice.

Theorem 6 In both the unique-winner and the nonunique-winner model, Baldwin-Destructive-Shift-Bribery is NP-hard.

Proof Sketch To prove NP-hardness, we reduce the NP-complete problem X3C to Baldwin-Destructive-Shift-Bribery. From a given X3C instance $(X, \mathcal{S})$, we construct an election (C,V), where $C = \{p, w, b, d\} \cup X \cup \mathcal{S}$ is the set of candidates, p is the designated candidate, and V consists of two lists of votes, V_1 and V_2, where V_1 contains the following votes:

#	votes	for	#	votes	for
1	$W_{(p,S_j)}$	$1 \le j \le 3m$	2	$W_{(w,x_i)}$	$1 \le i \le 3m$
2	$W_{(S_j,x_{j,1})}$	$1 \le j \le 3m$	$3m+7$	$W_{(w,d)}$	
2	$W_{(S_j,x_{j,2})}$	$1 \le j \le 3m$	$m+10$	$W_{(b,S_j)}$	$1 \le j \le 3m$
2	$W_{(S_j,x_{j,3})}$	$1 \le j \le 3m$			

Furthermore, V_2 contains the following votes:

#	votes	for	#	votes
1	$W_{(d,p)}$		$6m+14$	$W_{(p,w)}$
$2m+7$	$W_{(p,S_j)}$	$1 \le j \le 3m$	$3m^2+33m+12$	$W_{(p,b)}$
$3m+3$	$W_{(p,x_i)}$	$1 \le i \le 3m$		

Let $V = V_1 \cup V_2$. Then we have the following Borda scores for the complete profile V:

$$\begin{aligned}
score_{(C,V)}(x_i) &= avg(V) - 3m - 11 \text{ for every } x_i \in X,\\
score_{(C,V)}(S_j) &= avg(V) - 3m - 12 \text{ for every } S_j \in \mathcal{S},\\
score_{(C,V)}(d) &= avg(V) - 3m - 6,\\
score_{(C,V)}(w) &= avg(V) + 3m - 7,\\
score_{(C,V)}(b) &= avg(V) - 3m - 12,\\
score_{(C,V)}(p) &= avg(V) + 18m^2 + 72m + 25.
\end{aligned}$$

Regarding the price function, for every first vote of $W_{(p,S_j)}$ in V_1 (i.e., a vote of the form $p\ S_j\ \overrightarrow{C \setminus \{S_j, p\}}$), let $\rho(1) = 1$ and $\rho(t) = m + 1$ for every $t \geq 2$. For every other vote, let $\rho(t) = m + 1$ for every $t \geq 1$. Finally, we set the budget $B = m$.

It is easy to see that p wins the election (C,V). We claim that $(X, \mathcal{S})$ is in X3C if and only if $((C,V),p,B,\rho)$ is in Baldwin-Destructive-Shift-Bribery, regardless of the winner model.

($\Rightarrow$) Suppose there is an exact cover $\mathcal{S}' \subseteq \mathcal{S}$. Then we bribe the first votes of $W_{(p,S_j)}$ for every $S_j \in \mathcal{S}'$ by shifting p to the right once. With a similar argument as in the proof of Theorem 5, d alone wins the election, i.e., p is not among the winners.

($\Leftarrow$) Suppose there is no exact cover. Then, for every $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| \leq m$, there is at least one $x_i \in X$ that is not covered by $\mathcal{S}'$. It is obvious that at most m of the first votes of $W_{(p,S_j)}$ can be bribed without exceeding the budget. We can then show, similarly as in the proof of Theorem 5, that d will always be eliminated before w and therefore p cannot be prevented from winning the election alone. □

Finally, we turn to Nanson elections for which we again will show that shift bribery is NP-hard. The reduction below will only use pairs of votes of the form W_{c_1,c_2}. Although these types of votes have been discussed at the beginning of this section, we will now briefly explain how they behave in Nanson elections. The average Borda score for those two votes is $|C| - 1$. The candidate c_1 scores one point more than the average Borda score and c_2 scores one point fewer than the average Borda score. The other candidates score exactly the average Borda score. If a candidate is eliminated in a round, the average Borda score required to survive the next round decreases by one. Regardless of which candidate is eliminated, all remaining candidates that are not c_1 or c_2 lose one point and still have exactly the average Borda score. If c_2 is eliminated, c_1 loses its advantage with respect to the average Borda score and now scores exactly the average Borda score as well. If one of the other candidates is eliminated, c_1 continues to have one point more than the average Borda score. By symmetry, this holds analogously for c_2: If c_1 is eliminated, c_2 scores the average Borda score, and if one of the other candidates is eliminated, c_2 still has one point fewer than the average Borda score.

Theorem 7 In both the unique-winner and the nonunique-winner model, Nanson-Constructive-Shift-Bribery is NP-hard.

Proof To prove NP-hardness, we reduce the NP-complete problem X3C to Nanson-Constructive-Shift-Bribery. Again, starting from a given X3C instance $(X, \mathcal{S})$, we construct an election (C,V) with the set of candidates $C = \{p, w_1, w_2, d\} \cup X \cup \mathcal{S}$, where p is the designated candidate. Then we construct two sets of votes, V_1 and V_2, where V_1 contains the following votes:

#	votes	for	#	votes	for
1	$W_{(S_j,p)}$	$1 \le j \le 3m$	1	$W_{(x_{j,3},S_j)}$	$1 \le j \le 3m$
1	$W_{(x_i,p)}$	$1 \le i \le 3m$	4	$W_{(S_j,w_1)}$	$1 \le j \le 3m$
1	$W_{(x_{j,1},S_j)}$	$1 \le j \le 3m$	$15m$	$W_{(w_1,w_2)}$	
1	$W_{(x_{j,2},S_j)}$	$1 \le j \le 3m$	$3m$	$W_{(p,w_1)}$	

Furthermore, V_2 contains the following votes:

#	votes	for
$2m$	$W_{(p,d)}$	
2	$W_{(d,S_j)}$	$1 \le j \le 3m$
4	$W_{(d,x_i)}$	$1 \le i \le 3m$

Let $V = V_1 \cup V_2$. Then we have the following Borda scores for the complete profile V:

$$\begin{aligned} score_{(C,V)}(x_i) &= avg(V) \text{ for every } x_i \in X, \\ score_{(C,V)}(S_j) &= avg(V) \text{ for every } S_j \in \mathcal{S}, \\ score_{(C,V)}(p) &= avg(V) - m, \\ score_{(C,V)}(w_1) &= avg(V), \\ score_{(C,V)}(w_2) &= avg(V) - 15m, \\ score_{(C,V)}(d) &= avg(V) + 16m. \end{aligned}$$

The price function is again defined as follows. For every first vote of $W_{(S_j,p)}$ (i.e., a vote of the form $S_j\ p\ \overrightarrow{C \setminus \{S_j, p\}}$), let $\rho(1) = 1$ and $\rho(t) = m + 1$ for every $t \ge 2$. For every other vote, let $\rho(t) = m + 1$ for every $t \ge 1$. Finally, we set the budget $B = m$.

It is easy to see that p is eliminated in the first round of the election (C,V) and so does not win.

We claim that $(X, \mathcal{S})$ is in X3C if and only if $((C,V),p,B,\rho)$ is in Nanson-Constructive-Shift-Bribery, regardless of the winner model.

($\Rightarrow$) Suppose there is an exact cover $\mathcal{S}' \subseteq \mathcal{S}$. Then, for every $S_j \in \mathcal{S}'$, we bribe the first vote of $W_{(S_j,p)}$ by shifting p to the left once in all those votes. Note that we won't exceed our budget, since this bribe action costs 1 per vote and $|\mathcal{S}'| = m$. With the additional m points, p reaches the average Borda score and is not eliminated in the first round. However, all candidates in $\mathcal{S}'$ lose one point and are eliminated. Additionally, w_2 will be eliminated as well.

In the next round, w_1 will be eliminated, since she has $11m$ points fewer than the average Borda score required to survive this round. Since the candidates in $\mathcal{S}'$ were eliminated in the last round and $\mathcal{S}'$ is an exact cover, every candidate in X now has fewer points than the average Borda score and is eliminated.

In the third round, only p, d, and the candidates in $\mathcal{S} \setminus \mathcal{S}'$ are still standing. Therefore, the only pairs of votes that are not symmetric are $W_{(S_j,p)}$, twice $W_{(d,S_j)}$ for every $S_j \in (\mathcal{S} \setminus \mathcal{S}')$, and $2m$ pairs of $W_{(p,d)}$. Since $|\mathcal{S} \setminus \mathcal{S}'| = 2m$, we have that p scores exactly the average Borda score and survives this round, just as d. Every $S_j \in (\mathcal{S} \setminus \mathcal{S}')$ has one point fewer than the average Borda score and is eliminated. This leaves only p and d in the last round, which p alone wins.

($\Leftarrow$) Suppose there is no exact cover. Then, for every $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| = m$, there is at least one $x_i \in X$ that is not covered by $\mathcal{S}'$. Note that we can only bribe the first votes of any $W_{(S_j,p)}$ without exceeding the budget. For p to survive the first round, we need to bribe m of those votes by shifting p to the left once. Let $\mathcal{S}' \subseteq \mathcal{S}$ be such that $\mathcal{S}'$ contains S_j exactly if the first vote of $W_{(S_j,p)}$ has been bribed, and let V' be the changed profile. Then every $S_j \in \mathcal{S}'$ has a score of $avg(V') - 1$ and p has a score of $avg(V')$. Therefore, in the first round, every candidate from $\mathcal{S}'$ and w_2 are eliminated from the election.

In the second round, w_1 will be eliminated because of the $15m$ pairs of votes $W_{(w_1,w_2)}$ and the elimination of w_2. Furthermore, a candidate $x_i \in X$ reaches the average Borda score with p and d still standing only if all three $S_j \in \mathcal{S}$ with $x_i \in S_j$ are also not yet eliminated. Since the candidates in $\mathcal{S}'$ were eliminated in the previous round, for every $S_j \in \mathcal{S}'$, all three $x_i \in S_j$ will be eliminated in this round. Since $\mathcal{S}'$ is not an exact cover, there are candidates $X' \subseteq X$ that survive this round. d also reaches the average Borda score, as there are $2m$ candidates $\mathcal{S} \setminus \mathcal{S}'$ and those candidates $\mathcal{S} \setminus \mathcal{S}'$ survive due to w_1.

In the next round, the candidates still standing are p, d, X', and $\mathcal{S} \setminus \mathcal{S}'$. Because $|X'| \geq 1$, candidate p has $|X'|$ points fewer than the average Borda score and is eliminated in this round. Thus p does not win. $\square$

Our last result in this section shows that the destructive variant of shift bribery in Nanson elections is intractable as well.

Theorem 8 In both the unique-winner and the nonunique-winner model, Nanson-Destructive-Shift-Bribery is NP-hard.

Proof To prove NP-hardness, we reduce the NP-complete problem X3C to Nanson-Destructive-Shift-Bribery. Once more, given an X3C instance $(X, \mathcal{S})$, we construct an election (C,V) with the set of candidates $C = \{p, w_1, w_2, w_3, d\} \cup X \cup \mathcal{S}$, where p is the designated candidate and $(X, \mathcal{S})$ is the given X3C instance. Then we construct two sets of votes, V_1 and V_2, where V_1 contains the following votes:

#	votes	for	#	votes	for
1	$W_{(p,S_j)}$	$1 \leq j \leq 3m$	6	$W_{(S_j,w_3)}$	$1 \leq j \leq 3m$
1	$W_{(d,x_i)}$	$1 \leq i \leq 3m$	$20m$	$W_{(w_1,w_2)}$	
2	$W_{(x_{j,1},S_j)}$	$1 \leq j \leq 3m$	$19m$	$W_{(w_3,w_1)}$	
2	$W_{(x_{j,2},S_j)}$	$1 \leq j \leq 3m$	$3m+1$	$W_{(w_3,d)}$	
2	$W_{(x_{j,3},S_j)}$	$1 \leq j \leq 3m$			

Furthermore, V_2 contains the following votes:

#	votes	for	#	votes
1	$W_{(d,p)}$		$3m+1$	$W_{(p,w_3)}$
1	$W_{(p,x_i)}$	$1 \leq i \leq 3m$		

Let $V = V_1 \cup V_2$. Then we have the following Borda scores for the complete profile V:

$$\begin{aligned}
score_{(C,V)}(x_i) &= avg(V) + 4 \text{ for every } x_i \in X,\\
score_{(C,V)}(S_j) &= avg(V) - 1 \text{ for every } S_j \in \mathcal{S},\\
score_{(C,V)}(d) &= avg(V),\\
score_{(C,V)}(w_1) &= avg(V) + m,\\
score_{(C,V)}(w_2) &= avg(V) - 20m,\\
score_{(C,V)}(w_3) &= avg(V) + m,\\
score_{(C,V)}(p) &= avg(V) + 9m.
\end{aligned}$$

The price function is again defined as follows. For every first vote of $W_{(p,S_j)}$ (i.e., a vote of the form $p\ S_j\ \overrightarrow{C \setminus \{S_j, p\}}$), let $\rho(1) = 1$ and $\rho(t) = m + 1$ for every $t \geq 2$. For every other vote, let $\rho(t) = m + 1$ for every $t \geq 1$. Finally, we set the budget $B = m$.

It is easy to see that p will only have fewer points than the average Borda score if all candidates from $\mathcal{S}$, X, and the candidate w_3 are eliminated while d is still standing. Without bribing, d is eliminated in the third round while w_3 is still standing, and eventually p wins the election (C,V).

We claim that $(X, \mathcal{S})$ is in X3C if and only if $((C,V),p,B,\rho)$ is in Nanson-Destructive-Shift-Bribery, regardless of the winner model.

($\Rightarrow$) Suppose there is an exact cover $\mathcal{S}' \subseteq \mathcal{S}$. Then, for every $S_j \in \mathcal{S}'$, we bribe the first vote of $W_{(p,S_j)}$ by shifting p to the right once in all those votes. Note that we won't exceed our budget, since this bribe action costs 1 per vote and $|S'| = m$. After those m votes have been bribed, every $S_j \in \mathcal{S}'$ gains a point and therefore survives the first round. All other candidates $\mathcal{S} \setminus \mathcal{S}'$ and w_2 are eliminated.

Let $C_1 = \{p, d, w_1, w_3\} \cup X \cup \mathcal{S}'$ be the set of candidates present in the second round. w_1 loses $20m$ points on the average Borda score from the elimination of w_2 and is eliminated. Additionally, all candidates of X lose four points on the average Borda score but still survive this round, as they now have exactly the average Borda score.

Let $C_2 = \{p, d, w_3\} \cup X \cup \mathcal{S}'$ be the candidates in the third round. In this round, only w_3 is eliminated because w_3 lost $19m$ points on the average Borda score from the elimination of w_1.

Let $C_3 = \{p, d\} \cup X \cup \mathcal{S}'$ be the candidates in the fourth round and let V' be the changed profile. The scores are as follows:

$$\begin{aligned}
score_{(C_3,V')}(x_i) &= avg(V') \text{ for every } x_i \in X,\\
score_{(C_3,V')}(S_j) &= avg(V') - 7 \text{ for every } S_j \in \mathcal{S}',\\
score_{(C_3,V')}(d) &= avg(V') + 3m + 1,\\
score_{(C_3,V')}(p) &= avg(V') + 4m - 1.
\end{aligned}$$

Therefore all candidates in $\mathcal{S}'$ are eliminated. In the following round, all candidates in X are eliminated. This leaves only p and d in the final round in which p is eliminated and thus cannot win.

($\Leftarrow$) Suppose there is no exact cover. Then, for every $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| \leq m$, there is at least one $x_i \in X$ that is not covered by $\mathcal{S}'$. Note that we can only bribe the first votes of any $W_{(p,S_j)}$ without exceeding the budget.

We now show that, even with optimal bribing, d will be eliminated in the third round and, therefore, p alone wins the election. Within our budget, we can prevent at most m candidates from $\mathcal{S}$, say $\mathcal{S}'$, of being eliminated in the first round by bribing the corresponding vote of $W_{(p,S_j)}$. Since $\mathcal{S}'$ cannot be an exact cover of X, there is at least one $x_i \in X$ for which all $S_j \in \mathcal{S}$ with $x_i \in S_j$ are eliminated. This x_i is eliminated in the second round, as it has lost six points on the average Borda score from the eliminations of candidates in the previous round. In the third round, w_3 is still participating since w_1 was only eliminated in the previous round and gains most of its points from the votes $W_{(w_3,w_1)}$. Therefore, the score of d minus the average Borda score of this round is at most -1 (remember that at least one candidate from X was eliminated in the previous round), which means that d is eliminated in this round. Thus, there is no candidate left that can prevent p from winning the election. □

5 Iterated plurality and plurality with runoff

In this section, we show hardness of shift bribery for iterated plurality and plurality with runoff, handling both voting systems simultaneously and starting with the constructive case.

Theorem 9 In both the unique-winner and the nonunique-winner model, for iterated plurality and plurality with runoff, Constructive-Shift-Bribery is NP-hard.

Proof To prove NP-hardness, we reduce X3C to Constructive-Shift-Bribery for these two voting systems. Let $(X, \mathcal{S})$ be a given X3C instance. We construct the Constructive-Shift-Bribery instance $((C,V),p,B,\rho)$ as follows. Let $C = \{p, w\} \cup X \cup \mathcal{S} \cup D$ be the set of candidates, where p is the designated candidate and $D = \{d_{i,j} | 1 \leq i \leq 3m \text{ and } 1 \leq j \leq m-7\}$ is a set of dummy candidates. The list V of votes is constructed as follows:

line	#	vote	for
1	1	$S_i\, p \cdots$	$1 \leq i \leq 3m$
2	2	$S_i\, x_{i,1}\, \overrightarrow{X \setminus \{x_{i,1}\}} \cdots$	$1 \leq i \leq 3m$
3	2	$S_i\, x_{i,2}\, \overrightarrow{X \setminus \{x_{i,2}\}} \cdots$	$1 \leq i \leq 3m$
4	2	$S_i\, x_{i,3}\, \overrightarrow{X \setminus \{x_{i,3}\}} \cdots$	$1 \leq i \leq 3m$
5	1	$S_i\, d_{i,j}\, \overrightarrow{X \setminus \{x_i\}} \cdots$	$1 \leq i \leq 3m$, $1 \leq j \leq m-7$
6	m	$x_i\, \overrightarrow{X \setminus \{x_i\}} \cdots$	$1 \leq i \leq 3m$
7	m	$d_{i,j}\, \overrightarrow{X} \cdots$	$1 \leq i \leq 3m$, $1 \leq j \leq m-7$
8	3	$w\, p \cdots$	

For voters with votes of the form $S_i\, p \cdots$, we use the price function $\rho(1) = 1$, and $\rho(t) = m+1$ for all $t \geq 2$; and for every other voter, we use the price function $\rho(t) = m+1$ for $t \geq 1$. Finally, set the budget $B = m$.

Without bribing, p has a score of zero and is eliminated immediately in both voting systems.

We claim that $(X, \mathcal{S})$ is in X3C if and only if $((C,V),p,B,\rho)$ is in Constructive-Shift-Bribery for either of the two voting systems, regardless of the winner model.

($\Rightarrow$) Suppose that $(X, \mathcal{S})$ is a yes-instance of X3C. Then there exists an exact cover $\mathcal{S}' \subseteq \mathcal{S}$ of size m. We now show that it is possible for p to become a unique iterated-plurality (respectively, plurality-with-runoff) winner of an election obtained by shifting p in the votes without exceeding the budget. For every $S_i \in \mathcal{S}'$, we bribe the voter with the vote of the form $S_i\, p \cdots$, so her new vote is of the form $p\, S_i \cdots$. In the first round p, every $x_i \in X$, every $d_{i,j} \in D$, and every $S_i \in \mathcal{S} \setminus \mathcal{S}'$ is a plurality winner, so only these candidates participate in the next round. In the second round, p receives three further points from the three voters whose vote is $w\, p \cdots$. Every candidate $x_j \in X$ receives two further points from the votes of the form $S_i\, x_j \cdots$ with $x_j \in S_i$ and $S_i \in \mathcal{S}'$. Every $d_{i,j}$ with $S_i \in \mathcal{S}'$ and $1 \le j \le m - 7$ receives one additional point from the voters with vote $S_i\, d_{i,j} \cdots$. It follows that p has the most points and therefore p is the unique iterated-plurality (respectively, plurality-with-runoff) winner.

($\Leftarrow$) Suppose that $(X, \mathcal{S})$ is a no-instance of X3C. Then, for every $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| = m$, there is at least one candidate in X that is not covered and, therefore, at least one candidate in X occurring in at least two sets from $\mathcal{S}'$. We show that it is not possible for p to become a winner of the election obtained from the original election by bribing without exceeding the budget.

To become a winner of such an election with bribed voters, it is necessary for p to get at least m points in the first round. Due to the budget, it is also necessary to bribe m voters with a vote of the form $S_i\, p \cdots$ with $S_i \in \mathcal{S}'$. It follows that p, each $x \in X$, each $S_i \in \mathcal{S} \setminus \mathcal{S}'$, and each $d_{i,j} \in D$ participate in the second round. As mentioned above, at least one candidate in X receives at least four further points due to the fact that $\mathcal{S}'$ is not a cover of X. Thus p does not win. That means that $((C,V),p,B,\rho)$ is a no-instance of Constructive-Shift-Bribery for either of iterated plurality and plurality with runoff regardless of the winner model. □

We have the same result in the destructive case. This is the first proof where we use an NP-complete problem other than X3C to show NP-hardness, namely One-In-Three-Positive-3SAT, which was also defined in Section 2.

Theorem 10 In both the unique-winner and the nonunique-winner model, for iterated plurality and plurality with runoff, Destructive-Shift-Bribery is NP-hard.

Proof To prove NP-hardness, we reduce the NP-complete problem One-In-Three-Positive-3SAT to Destructive-Shift-Bribery for both voting systems. Let (X,S) be a given One-In-Three-Positive-3SAT instance, where $X = \{x_1, \dots, x_{3m}\}$ and $S = \{S_1, \dots S_{3m}\}$ with $S_i = \{x_{i,1}, x_{i,2}, x_{i,3}\} \subseteq X$ for each $1 \le i \le 3m$. Without loss of generality, we can assume that $m > 6$. We construct the Destructive-Shift-Bribery instance for both voting systems as follows. Let $C = \{p,w,e,f\} \cup D \cup Y \cup X$ with $D = \{d_{i,j} | 1 \le i \le 3m \text{ and } 1 \le j \le 2m-1\}$ and $Y = \{y_{i,j} | 1 \le i \le 3m \text{ and } 1 \le j \le 4\}$ and where p is the designated candidate. The list V of votes is constructed as follows:

line	#	votes	for
1	1	$p\, x_i \cdots$	$1 \le i \le 3m$
2	1	$y_{i,1}\, x_{i,1}\, x_{i,2}\, w\, p \cdots$	$1 \le i \le 3m$
3	1	$y_{i,2}\, x_{i,2}\, x_{i,3}\, w\, p \cdots$	$1 \le i \le 3m$
4	1	$y_{i,3}\, x_{i,1}\, x_{i,3}\, w\, p \cdots$	$1 \le i \le 3m$
5	4	$y_{i,4}\, x_{i,1}\, x_{i,2}\, x_{i,3}\, p \cdots$	$1 \le i \le 3m$
6	1	$x_i\, d_{i,j}\, p \cdots$	$1 \le i \le 3m$, $1 \le j \le 2m-1$

line	#	votes	for
7	$2m$	$d_{i,j}\, p \cdots$	$1 \le i \le 3m,\ 1 \le j \le 2m-1$
8	$2m$	$w\, p \cdots$	
9	$2m-1$	$e\, p \cdots$	
10	m	$f\, p \cdots$	

For votes of the form $p\ x_i \cdots$ we use the price function $\rho(1) = 1$ and $p(t) = m + 1$ for all $t \geq 2$. For every other vote, we use the price function $\rho(t) = m + 1$ for $t \geq 1$. Finally, set the budget $B = m$.

Without bribing, the election proceeds as follows. In the first round, p scores $3m$ points, w and every $d_{i,j} \in D$ scores $2m$ points, and each of the remaining candidates scores fewer than $2m$ points. In the second round, p scores $18m - 1$ points, w scores $11m$ points, and every $d_{i,j}$ scores $2m + 1$ points. It follows that p is the unique winner for either of iterated plurality (there would not be a second round here as p already uniquely wins in the first round) and plurality with runoff.

We claim that (X,S) is in One-in-Three-Positive-3SAT if and only if $((C,V),p,B,\rho)$ is in Destructive-Shift-Bribery for either of the two voting systems, regardless of the winner model.

($\Rightarrow$) Suppose that (X,S) is a yes-instance of One-in-Three-Positive-3SAT. Then there exists a subset $U \subseteq X$ such that for each clause S_j we have $|U \cap S_j| = 1$. We bribe the voters with the vote of the form $p\ x_i \cdots$ with $x_i \in U$ so that the new vote has the form $x_i\ p \cdots$. It follows that p, w, every $x_i \in U$, and every $d_{i,j} \in D$ reach the second round with $2m$ points each. In the second round, p gains $3m - 1$ additional points (since e and f were eliminated) while w gains $3m$ additional points (since candidates in Y and $X \setminus U$ were eliminated). It follows that p is not a winner of the election, so $((C,V),p,B,\rho)$ is a yes-instance of Destructive-Shift-Bribery for both voting systems, regardless of the winner model.

($\Leftarrow$) Suppose that (X,S) is a no-instance of One-in-Three-Positive-3SAT. We show that $((C,V),p,B,\rho)$ is also a no-instance of Destructive-Shift-Bribery for both voting systems. To ensure that p is not the only plurality winner in the first round, it is necessary to bribe m voters with votes of the form $p\ x_i \cdots$ to now vote $x_i\ p \cdots$. Note that we can only bribe at most m such voters without exceeding the budget. Let $U \subseteq X$ be the set of candidates that benefit from the bribery action. It follows that p, every $d_{i,j} \in D$, every $x_i \in U$, and w can move forward to the next round with $2m$ points each. In this round, the designated candidate p gains $3m - 1$ additional points from the votes of the form $e\ p \cdots$ and $f\ p \cdots$; every candidate $d_{i,j}$ with $x_i \notin U$ gains one additional point; every candidate $x_i \in U$ can receive at most 18 additional points (this is due to the fact that every $x_i \in U$ appears in exactly three sets of S); and w is discussed separately in the following paragraph.

To prevent the victory of p, it is necessary that w gains at least $3m$ points (since if w gains only $3m - 1$ points, it follows that w and p move forward to the final round, where p would achieve a clear victory). For w to gain at least one point from any one of the three votes of the form $y_{i,1}\ x_{i,1}\ x_{i,2}\ w\ p \cdots$, $y_{i,2}\ x_{i,2}\ x_{i,3}\ w\ p \cdots$, and $y_{i,3}\ x_{i,1}\ x_{i,3}\ w\ p \cdots$, it is necessary that at most one candidate $x_{i,j}$ participates in the second round. On the other hand, if no candidate $x_{i,j}$ participates in the second round, p gains four points from the voters of the fifth line, whose vote is $y_{i,4}\ x_{i,1}\ x_{i,2}\ x_{i,3}\ p \cdots$, i.e., this clause harms w. Only a clause S_i with $|S_i \cap U| = 1$ helps w to reduce the point difference to p. Since (X,S) is a no-instance of One-in-Three-Positive-3SAT, there are at most $3m - 2$ clauses with this property.

Due to those clauses the point difference of w to p reduces to (at most) one.[9] With the two remaining clauses the point difference is growing because either (a) $|S_i \cap U| = 0$ and p gains four points or (b) $|S_i \cap U| > 1$ and w gains no points for this clause. This implies that p is always a unique winner of the election, i.e., $((C,V),p,B,\rho)$ is a no-instance of Destructive-Shift-Bribery for both voting systems, regardless of the winner model. □

6 Iterated veto and veto with runoff

In this section, we show hardness of shift bribery for iterated veto and veto with runoff, again handling both voting systems simultaneously and starting with the constructive case.

Theorem 11 In both the unique-winner and the nonunique-winner model, for veto with runoff and iterated veto, Constructive-Shift-Bribery is NP-hard.

Proof To prove NP-hardness, we reduce X3C to Constructive-Shift-Bribery for veto with runoff and iterated veto at the same time. Let $(X, \mathcal{S})$ be a given X3C instance and construct the Constructive-Shift-Bribery instance $((C,V),p,B,\rho)$ as follows. Let $C = \{p, d_1, d_2\} \cup X \cup \mathcal{S}$ be the set of candidates, where p is the designated candidate, and construct the voter preferences in V as follows. Without loss of generality, we assume that $m > 3$.

line	#	votes	for
1	1	$\cdots\ S_i\ p$	$1 \le i \le 3m$
2	2	$\cdots\ x_{i,1}\ S_i$	$1 \le i \le 3m$
3	2	$\cdots\ x_{i,2}\ S_i$	$1 \le i \le 3m$
4	2	$\cdots\ x_{i,3}\ S_i$	$1 \le i \le 3m$
5	$2m-6$	$\cdots\ d_2\ S_i$	$1 \le i \le 3m$
6	$2m$	$\cdots\ x_i$	$1 \le i \le 3m$
7	m	$\cdots\ d_2\ x_i\ d_1$	$1 \le i \le 3m$
8	$m+2$	$\cdots\ d_2\ S_i\ d_1$	$1 \le i \le 3m$
9	$2m$	$\cdots\ d_2$	
10	1	$\cdots\ p\ d_1$	

For votes of the form $\cdots\ S_i\ p$, we use the price function $\rho(1) = 1$, and $\rho(t) = m + 1$ for all $t \ge 2$; and for every other voter, we use the price function $\rho(t) = m + 1$ for $t \ge 1$. Finally, set the budget $B = m$.

Note that for both voting rules, p is eliminated in the first round with $3m$ vetoes and therefore cannot be the winner without bribing voters.

We claim that $(X, \mathcal{S})$ is in X3C if and only if $((C,V),p,B,\rho)$ is in Constructive-Shift-Bribery for either of iterated veto and veto with runoff, regardless of the winner model.

[9] As explained above, due to the candidates in $X \setminus U$ being eliminated in the previous round and the candidates in U still present, for each of those clauses with $|S_i \cap U| = 1$, exactly one of the associated votes in lines 2–4 has w on top now.

($\Rightarrow$) Suppose that $(X, \mathcal{S})$ is a yes-instance of X3C. Then there exists an exact cover $\mathcal{S}' \subseteq \mathcal{S}$ of size m. Shift p one position forward in the votes of the form $\cdots\ S_i\ p$ for each $S_i \in \mathcal{S}'$, so that the new vote has the form $\cdots\ p\ S_i$. It follows that p, each $S \in \mathcal{S} \setminus \mathcal{S}'$, each x_i for $1 \leq i \leq 3m$, and d_2 are veto winners with $2m$ vetoes each and thus proceed to the second round. Since $\mathcal{S}'$ is an exact cover, each x_i receives two additional vetoes from the voters in lines 2–4 corresponding to the sets in the exact cover and m vetoes from the voters in line 7. Furthermore, each $S \in \mathcal{S} \setminus \mathcal{S}'$ receives $m+2$ vetoes from the voters in line 8, whereas p receives m vetoes from the voters in line 1 and only one additional veto from the voter in the last line. Since d_2 gains far more than $m+1$ vetoes in this round, it follows that p is the unique veto winner of the election with bribed voters. Thus $((C,V),p,B,\rho)$ is a yes-instance of Constructive-Shift-Bribery for either of iterated veto and veto with runoff, regardless of the winner model.

($\Leftarrow$) Suppose that $(X, \mathcal{S})$ is a no-instance of X3C. This means that for every $\mathcal{S}' \subseteq \mathcal{S}$, $|\mathcal{S}'| \leq m$\,there is an $x' \in X$ that is not covered by any $S \in \mathcal{S}'$.

In the first round, all candidates but those with the fewest vetoes—in this case $2m$—will be eliminated. Note that p has $3m$ vetoes, the budget is m and only voters in line 1 can be bribed without exceeding the budget. Therefore, exactly m voters in line 1 have to be bribed to change their vote to $\cdots\ p\ S_i$. Let $\mathcal{S}' \subseteq \mathcal{S}, |\mathcal{S}'| = m$, be the set that consists of the S_i corresponding to these bribed voters. After this bribery action, only the candidates p and d_2 as well as each $S \in \mathcal{S} \setminus \mathcal{S}'$ and each x_i, $1 \leq i \leq 3m$, reach the second round with $2m$ vetoes.

Since $\mathcal{S}'$ is not an exact cover, there is an $x' \in X$ that is not covered by $\mathcal{S}'$ and therefore does not receive a veto from the voters in lines 2–4. This x' only gains m vetoes from voters in line 7 for a total of $3m$ vetoes, whereas p receives m vetoes from voters in line 1 and one veto from the voter in line 10 for a total of $3m+1$ vetoes. It follows that p is not winning the election for either of the two voting rules. That means that $((C,V),p,B,\rho)$ is a no-instance of Constructive-Shift-Bribery for either of iterated veto and veto with runoff, regardless of the winner model. □

We now turn to the destructive variant of shift bribery for iterated veto and veto with runoff.

Theorem 12 In both the unique-winner and the nonunique-winner model, for veto with runoff and iterated veto, Destructive-Shift-Bribery is NP-hard.

Proof To prove NP-hardness, we reduce the NP-complete problem One-In-Three-Positive-3SAT to Destructive-Shift-Bribery for veto with runoff and iterated veto simultaneously. Given an instance (X,S) of One-In-Three-Positive-3SAT, where $X = \{x_1, \ldots, x_{3m}\}$ and $S = \{S_1, \ldots, S_{3m}\}$, with $S_i = \{x_{i,1}, x_{i,2}, x_{i,3}\} \subseteq X$ for each $1 \leq i \leq 3m$, we construct the election (C,V) with candidate set $C = \{p,w,d_1,d_2\} \cup X$, designated candidate p, and the following list V of votes:

line	#	votes	for
1	1	$\cdots\ p\ x_i$	$1 \leq i \leq 3m$
2	2	$\cdots\ p\ x_{i,1}\ x_{i,2}\ d_1$	$1 \leq i \leq 3m$
3	2	$\cdots\ p\ x_{i,2}\ x_{i,3}\ d_1$	$1 \leq i \leq 3m$
4	2	$\cdots\ p\ x_{i,1}\ x_{i,3}\ d_1$	$1 \leq i \leq 3m$

line	#	votes	for
5	7	$\cdots w\, x_{i,1}\, x_{i,2}\, x_{i,3}\, d_1$	$1 \le i \le 3m$
6	$2m$	$\cdots d_2\, x_i$	$1 \le i \le 3m$
7	$22m$	$\cdots d_2\, x_i\, d_1$	$1 \le i \le 3m$
8	$2m$	$\cdots d_2$	
9	m	$\cdots p$	
10	$2m$	$\cdots w$	
11	$8m-1$	$\cdots w\, d_1$	

For every vote of the form $\cdots\, p\, x_i$, let the price function be $\rho(1) = 1$ and $\rho(t) = m+1$ for every $t \ge 2$. For every other vote, define $\rho(t) = m+1$ for every $t \ge 1$. Finally, we set the budget $B = m$.

Note that p is the winner of the election for both voting rules: p has the fewest vetoes of all candidates and therefore wins under iterated veto after the first round. For veto with runoff, all candidates but p, w, and d_2, i.e., the candidates with the lowest and second-lowest number of vetoes, are eliminated after the first round, so that p is the unique winner with the fewest vetoes in the second round.

We claim that (X,S) is in One-In-Three-Positive-3SAT if and only if $((C,V),p,B,\rho)$ is in Destructive-Shift-Bribery for either of veto with runoff and iterated veto, regardless of the winner model.

($\Rightarrow$) Assume that (X,S) is in One-In-Three-Positive-3SAT. Then there is a subset $X' \subseteq X$ such that for each clause S_i we have $|X' \cap S_i| = 1$. Bribe the voters with votes of the form $\cdots\, p\, x_i$ with $x_i \in X'$ so that the new vote has the form $\cdots\, x_i\, p$. It follows that p, w, d_2, and each $x_i \in X'$ have the fewest vetoes (namely, $2m$) and therefore proceed to the second round. In the second round, p receives $2m$ vetoes from the votes in line 1 and for each of the $3m$ clauses two vetoes from the voters in lines 2–4 for a total of $8m$ additional vetoes, whereas w only receives a total of $8m-1$ vetoes. It follows that p is not a winner of the election for either of the two voting rules.

($\Leftarrow$) Let (X,S) be a yes-instance of Destructive-Shift-Bribery for veto with runoff (respectively, iterated veto), i.e., it is possible to bribe voters so that p does not win the election. Recall that it is only possible to bribe voters in line 1 without exceeding the budget. For the original election without bribed voters, in the first round, p receives m vetoes, i.e., the fewest vetoes of all candidates. Due to the votes in line 7, the only candidate capable of receiving fewer vetoes than p or the same number of vetoes as p in the second round is w.[10] However, this is only possible if p receives at least $9m-1$ additional vetoes since w has $10m-1$ vetoes in the second round from the last two lines alone. Note that p receives $3m$ of these additional vetoes from line 1—after bribing voters so that p is in the last position, or eliminating the x_i in the first round—leaving a gap of $6m-1$ vetoes. For each clause S_j such that no $x_i \in S_j$ is present in the second round, p receives six additional vetoes (lines 2–4), whereas w receives in this case seven additional vetoes from the voters in line 5, i.e., this widens the gap between p and w instead of closing it. That means that for each clause S_j, there has to be at least one $x_i \in S_j$ present in the second round, i.e., for each clause S_j, a voter with a vote of the form $\cdots\, p\, x_i$ with $x_i \in S_j$ needs to be bribed to cast a vote of the form $\cdots\, x_i\, p$ to bring the number of vetoes of x_i down to $2m$, the same as, e.g., d_2. However, if

[10] Note that d_1 and at least $2m$ candidates from X will definitely be eliminated in the first round. Due to the votes in line 7 the remaining candidates of X and d_2 will gain too many vetoes to be able beat p.

at least two literals, say x_i and x_k, in a clause S_j are present in the second round, p receives no additional veto from votes in lines 2–4, which does not help to close the gap between p and w. The only possibility remaining for p not to be a winner of the election after bribery is that the bribed voters correspond to the variables set to true in an assignment where in each clause there is exactly one literal true, i.e., we have a yes-instance of One-In-Three-Positive-3SAT. □

7 Using the Nonmonotonicity property

Informally stated, a voting rule is said to be *monotonic* if winners can never be turned into nonwinners by improving their position in some votes, everything else remaining the same.[11] Intuitively, this means that it is only beneficial to shift a candidate forward (closer to the top) and not backwards (closer to the bottom). In shift bribery under some monotonic voting rule, it thus only makes sense for the briber to shift the designated candidate forward in the constructive case (respectively, backward in the destructive case). However, all voting rules considered here except iterated plurality and iterated veto are *not* monotonic, and in nonmonotonic voting rules, shifting the designated candidate backward in the constructive case (respectively, forward in the destructive case) could also be beneficial for the briber.

It would therefore be interesting to find out whether the complexity of our problems changes when the nonmonotonicity of voting rules is specifically allowed, or even required, to be exploited in shift bribery actions. Indeed, with slight modifications to the proofs, we can show that Hare-Constructive-Shift-Bribery and plurality-with-runoff-Constructive-Shift-Bribery are still NP-hard if the designated candidate can *only* be shifted *backward*. We conjecture that all other proofs (except the proofs for the monotonic voting rules iterated plurality and iterated veto) can be adapted in such a way as well.

Note that in this section we can simply swap the definition of price functions for the constructive and destructive variants so that in all votes the designated candidate can only be moved forward in the constructive variant and can only be moved backward in the destructive variant.

We start with constructive shift bribery in Hare elections where the only allowed bribery action is to shift the designated candidate *backward*.

Theorem 13 In both the unique-winner and the nonunique-winner model, Hare-Constructive-Shift-Bribery is NP-hard even if the designated candidate can only be shifted backward.

Proof NP-hardness again follows by a reduction from X3C. Construct from a given X3C instance $(X, \mathcal{S})$ an instance $((C,V),p,B,\rho)$ of Hare-Constructive-Shift-Bribery with candidate set $C = X \cup \mathcal{S} \cup D \cup \{p, w\}$, where $D = \{d_1, \dots, d_{3m}\}$ is a set of dummy candidates and p the designated candidate, and the following list V of votes:

[11] This definition captures just one common notion of monotonicity, the one we will be using here; but note that there are also other notions of monotonicity for voting rules known in social choice theory (see, e.g., the papers by Felsenthal and Nurmi [46], Miller [47], and Brams and Fishburn [48]).

line	#	vote	for
1	1	$S_i\ x_{i,1}\ \overrightarrow{X \setminus \{x_{i,1}\}}\ w\ p\ \cdots$	$1 \leq i \leq 3m$
2	1	$S_i\ x_{i,2}\ \overrightarrow{X \setminus \{x_{i,2}\}}\ w\ p\ \cdots$	$1 \leq i \leq 3m$
3	1	$S_i\ x_{i,3}\ \overrightarrow{X \setminus \{x_{i,3}\}}\ w\ p\ \cdots$	$1 \leq i \leq 3m$
4	4	$x_i\ \overrightarrow{X \setminus \{x_i\}}\ w\ p\ \cdots$	$1 \leq i \leq 3m$
5	6	$w\ \vec{X}\ p\ \cdots$	
6	1	$p\ S_i \cdots$	$1 \leq i \leq 3m$
7	6	$p \cdots$	
8	3	$d_i\ S_i\ p\ w \cdots$	$1 \leq i \leq 3m$

For votes of the form $p\ S_i \cdots$, we use the price function $\rho(1) = 1$, and $\rho(t) = m + 1$ for all $t \geq 2$; and for every other vote, we use the price function ρ with $\rho(t) = m + 1$ for all $t \geq 1$. Finally, set the budget $B = m$.

Without bribing the voters the election proceeds as follows:

Round	p	w	x_1	$x_i \in X \setminus \{x_1\}$	$S_i \in \mathcal{S}$	$d_i \in D$
1	$3m + 6$	6	4	4	3	3
2	$12m + 6$	6	7	7	out	out
3	$12m + 6$	out	13	7	out	out
4	$12m + 6$	out	$21m + 6$	out	out	out

It follows that p is eliminated in the last round and does not win the election.

We claim that $(X, \mathcal{S})$ is in X3C if and only if $((C,V),p,B,\rho)$ is in Hare-Constructive-Shift-Bribery, regardless of the winner model, even if the designated candidate can only be shifted backward.

($\Rightarrow$) Suppose that $(X, \mathcal{S})$ is a yes-instance of X3C. Then there exists an exact cover $\mathcal{S}' \subseteq \mathcal{S}$ of size m. We now show that it is possible for p to become a unique Hare winner of an election obtained by shifting p in the votes without exceeding the budget B. For every $S_i \in \mathcal{S}'$, we bribe the voter with the vote of the form $p\ S_i \cdots$ by shifting p once, so her new vote is of the form $S_i\ p \cdots$; each such bribe action costs us only 1 from our budget, so the budget will not be exceeded. Now the election proceeds as follows:

Round	p	w	$x_i \in X$	$S_i \in \mathcal{S}'$	$S_i \in \mathcal{S} \setminus \mathcal{S}'$	$d_i \in D$
1	$2m + 6$	6	4	4	3	3
2	$8m + 6$	6	6	7	out	out
3	$26m + 12$	out	out	7	out	out

We see that p is the only candidate still standing in the fourth round and thus the only Hare winner of the election with bribed voters.

($\Leftarrow$) Suppose that $(X, \mathcal{S})$ is a no-instance of X3C. Then no subset $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| \leq m$ covers X. We now show that p will be eliminated in all elections obtained by bribing voters without exceeding budget B. Note that we can only bribe at most m voters with votes of the form $p\ S_i \cdots$ without exceeding the budget. Let $\mathcal{S}' \subseteq \mathcal{S}$ be such that for every $S_i \in \mathcal{S}'$ we have bribed the voter whose vote is $p\ S_i \cdots$. We can assume that $|\mathcal{S}'| > 0$.

Every candidate in $\mathcal{S}'$ will gain an additional point and therefore survives the first round. All candidates from D and $\mathcal{S} \setminus \mathcal{S}'$ will be eliminated, since p only loses at most m points.

In the second round, the remaining candidates from $\mathcal{S}$ will gain three additional points from the elimination of candidates in D (see line 8) and score seven points in this round (and in all subsequent rounds with p still standing). If a candidate $S_i \in \mathcal{S}$ was eliminated in the previous round, every $x_j \in S_i$ gains one additional point in this round (see lines 1–3). Partition X into sets X_0, X_1, X_2, and X_3 so that $x_i \in X_k \Leftrightarrow |\{S_j \in \mathcal{S}' | x_i \in S_j\}| = k$ for $k \in \{0,1,2,3\}$. Intuitively, the sets X_k count how many times a candidate x_j was covered by $\mathcal{S}'$, so x_j is contained in X_k if and only if x_j is present in k elements of $\mathcal{S}'$. Note that X_0, X_1, X_2, and X_3 are disjoint and $|X_0| > 0$ (recall that there is no exact covering of X), but one or two of X_1, X_2, and X_3 may be empty. Then $x_i \in X_j$ scores $4 + (3 - j) \in \{4,5,6,7\}$ points depending on how many times x_i is covered by $\mathcal{S}'$. Therefore, every $x_i \in X_0$ scores more points than w who has six points. So, there are candidates from X that survive this round and other candidates from X (i.e., candidates from X_1, X_2, or X_3), who are eliminated.

In the third round, the candidate $x_\ell \in X$ with the smallest subscript who is still standing gains at least four points from the eliminated candidates (from the votes in the fourth line), so that she scores at least nine points now (since no candidates from X_3 are left in the election). All other candidates still score the same number of points as in the previous round. Therefore, p scores $4|\mathcal{S} \setminus \mathcal{S}'| + 6$ points, w scores six points (if w was not already eliminated along with the candidates from X_1), every $S_i \in \mathcal{S}'$ scores seven points, and every still standing candidate from X except x_ℓ scores at most seven points. Since w can only gain additional points when all candidates from X are eliminated (see line 4) and only x_ℓ gains points from the elimination of w or candidates from $X \setminus \{x_\ell\}$ in the subsequent rounds, all candidates $X \setminus (\{x_\ell\} \cup X_0)$ and w are eliminated. Then all still standing candidates from $X_0 \setminus \{x_\ell\}$ and candidates from $\mathcal{S}'$, who score seven points each, are eliminated, which leaves p and x_ℓ in the last round. In this round, p scores $12m + 6$ points and x_ℓ scores $21m + 6$ points, so p is eliminated from the election and does not win. □

Next, we show the corresponding result for plurality with runoff.

Theorem 14 In both the unique-winner and the nonunique-winner model, plurality-with-runoff-Constructive-Shift-Bribery is NP-hard even if the designated candidate can only be shifted backward.

Proof To prove NP-hardness, we reduce X3C to Constructive-Shift-Bribery for plurality with runoff. Let $(X, \mathcal{S})$ be a given X3C instance, where $X = \{x_1, \ldots, x_{3m}\}$ and $\mathcal{S} = \{S_1, \ldots, S_{3m}\}$. Also, we require that $m > 3$. We construct the Constructive-Shift-Bribery instance $((C,V),p,B,\rho)$ as follows. Let $C = \{p\} \cup X \cup \mathcal{S} \cup D \cup Y$ with sets of dummy candidates $D = \{d_{i,j} | 1 \leq i \leq 3m \text{ and } 1 \leq j \leq 2m^2 - 5m - 4\}$ and $Y = \{y_i | 1 \leq i \leq 3m + 1\}$ and designated candidate p. The list V of votes is constructed as follows:

line	#	vote	for
1	1	$p\ S_i \cdots$	$1 \leq i \leq 3m$
2	2	$S_i\ x_{i,1}\ w\ \overrightarrow{X \setminus \{x_{i,1}\}} \cdots$	$1 \leq i \leq 3m$
3	2	$S_i\ x_{i,2}\ w\ \overrightarrow{X \setminus \{x_{i,2}\}} \cdots$	$1 \leq i \leq 3m$

line	#	vote	for
4	2	$S_i\ x_{i,3}\ w\ \overrightarrow{X \setminus \{x_{i,3}\}}\ \cdots$	$1 \le i \le 3m$
5	$3m$	$w\ p\ \cdots$	
6	1	$y_i\ p$	$1 \le i \le 3m + 1$
7	$m - 3$	$S_i\ w\ p$	$1 \le i \le 3m$
8	$m - 4$	$S_i\ p\ w$	$1 \le i \le 3m$
9	$2m$	$x_i\ w\ p$	$1 \le i \le 3m$
10	1	$d_{i,j}\ x_i\ w\ p\ \cdots$	$1 \le i \le 3m$, $1 \le j \le 2m^2 - 5m - 4$

For votes of the form $p\ S_i\ \cdots$, we use the price function $\rho(1) = 1$, and $\rho(t) = m + 1$ for all $t \ge 2$; and for every other vote, we use the price function $\rho(t) = m + 1$ for $t \ge 1$. Finally, set the budget $B = m$.

Without bribing, only p and w reach the second and final round with $3m$ points each. Clearly, w alone wins the election with only p and w present.

We claim that $(X, \mathcal{S})$ is in X3C if and only if $((C,V),p,B,\rho)$ is in Constructive-Shift-Bribery for plurality with runoff, regardless of the winner model.

($\Rightarrow$) Suppose that $(X, \mathcal{S})$ is a yes-instance of X3C. Then there exists an exact cover $\mathcal{S}' \subseteq \mathcal{S}$ of size m. We now show that it is possible for p to become a unique plurality-with-runoff winner of an election obtained by shifting p in the votes without exceeding the budget. For every $S_i \in \mathcal{S}'$, we bribe the voter with the vote of the form $p\ S_i\ \cdots$ once, so her new vote is of the form $S_i\ p\ \cdots$.

In the first round, w scores $3m$ points; p, every $x_i \in X$, and every $S_i \in \mathcal{S}'$ score $2m$ points each; every $S_i \in \mathcal{S} \setminus \mathcal{S}'$ scores $2m - 1$ points; and every candidate from D and Y scores only one point. Since w is the only plurality winner, all second-place candidates (namely, p, every $x_i \in X$, and every $S_i \in \mathcal{S}'$) proceed to the second round.

In the second round, every $S_i \in \mathcal{S}'$ still scores the same number of points as in the first round, w gains $2m(m - 3)$ additional points, p gains $(3m + 1) + 2m(m - 4)$ additional points, and every $x_i \in X$ gains $(2m^2 - 5m - 4) + 4$ additional points. Therefore, p alone wins the election with $2m^2 - 3m + 1$ points, ahead of w and every $x_i \in X$ with $2m^2 - 3m$ points each, and every $S_i \in \mathcal{S}'$ with $2m$ points each.

($\Leftarrow$) Suppose that $((C,V),p,B,\rho)$ is a yes-instance of Plurality-with-runoff-Constructive-Shift-Bribery. Notice that if no voters are bribed, p and w are leading in the election with $3m$ points each, so they both proceed to the final round. It is easy to see that w wins against p in a one-on-one election. To prevent w and p from being the only candidates in the second round, m voters with votes of the form $p\ S_i\ \cdots$ have to be bribed. Let $\mathcal{S}' \subseteq \mathcal{S}$ be such that $S_i \in \mathcal{S}'$ if the voter with vote $p\ S_i\ \cdots$ has been bribed. Then w, p, every $x_i \in X$, and every $S_i \in \mathcal{S}'$ survive the first round. Since every other candidate is deleted in the first round, p now scores $2m^2 - 5m + 1$ additional points and beats w by a margin of one point. Moreover, p beats every $S_i \in \mathcal{S}'$ since the candidates from $\mathcal{S}'$ did not gain any additional points in this round. Regarding the candidates from X, every $x_i \in X$ gains $2m^2 - 5m - 4$ points and two additional points for every $S_j \in \mathcal{S} \setminus \mathcal{S}'$ with $x_i \in S_j$ that was eliminated in the first round. Since there are exactly three $S_j \in \mathcal{S}$ with $x_i \in S_j$, every $x_i \in X$ can gain six points if all those candidates were eliminated in the last round, which would let x_i overtake p by one point. In order for p to beat all $x_i \in X$, at least one $S_j \in \mathcal{S}$ with $x_i \in S_j$ needs to be in $\mathcal{S}'$ and is therefore still standing in the second round. Since $|\mathcal{S}'| = m$ and there are $3m$

candidates in X, p can beat every $x_i \in X$ (and subsequently win the election) only if $\mathcal{S}'$ is an exact cover of X. □

8 Conclusions and open questions

We have shown that shift bribery is NP-complete for each of the iterative voting systems of Hare, Coombs, Baldwin, Nanson, iterated plurality, plurality with runoff, iterated veto, and veto with runoff, each for both the constructive and the destructive case and in both the unique-winner and the nonunique-winner model. This contrasts previous results due to Elkind et al. [1, 20, 21] and Schlotter et al. [21] showing that shift bribery can be solved efficiently by exact algorithms for many natural voting rules that do not proceed iteratively. Indeed, the iterative nature of the voting rules we have studied seems to be responsible for the NP-hardness of shift bribery. It would be interesting to investigate the approximability of shift bribery for iterative voting rules, for comparison with the known approximation results of shift bribery for noniterative voting rules shown in the papers mentioned above.

While these are interesting theoretical results complementing earlier work both on shift bribery and on these voting systems, NP-hardness of course has its limitations in terms of providing protection against shift bribery attacks in practice (see, e.g., [49, 50]). Therefore, it would be interesting to also study shift bribery for these voting systems in terms of approximation and parameterized complexity and to do a typical-case analysis. Based on our results in this article, Zhou and Guo [51] already obtained first results regarding the parameterized complexity of iterative voting systems with respect to a fixed number of shifts, votes, or candidates. Further, they have shown that the hardness of shift bribery for the Hare, Coombs, Baldwin, and Nanson rules also holds for unit price cost functions. It would be particularly interesting to determine the role of the cost function for the hardness of shift bribery. Furthermore, it would be interesting future work to study in detail the effect that specific tie-breaking models (such as the "parallel universes" model [52] and other models) may have on the complexity of shift bribery problems for iterative voting rules.

Elkind et al. [53] have proposed algorithms for swap and shift bribery regarding noniterative voting rules like plurality, Borda, and Condorcet-consistent rules when the electorate is domain-restricted, namely either single-peaked or single-crossing. An interesting task for future research would be to extend this study to iterative voting rules and to find out whether such domain restrictions can make these problems easier to solve.

A feature shared by most of the iterative voting rules we have studied is that many of them are not monotonic. This has the somewhat counterintuitive effect that shifting the designated candidate forward in some votes can actually hurt this candidate's chances to win, and shifting the designated candidate backward can increase these chances. We have discussed this feature in Section 7, showing that constructive shift bribery remains NP-hard even if we are allowed to only shift the designated candidate backward in some votes for two iterative voting systems: Hare voting and plurality with runoff. We leave the analogous question open for the remaining iterative voting systems studied here (except, of course, for the monotonic rules iterated plurality and iterated veto), and conjecture that they share this property. Even more interestingly, we pose as an open question whether there is a nonmonotonic voting system—a natural one or an artificially constructed one—for which unrestricted shift bribery is NP-hard but becomes efficiently solvable when restricted to shift bribery actions specifically exploiting their nonmonotonicity (i.e., allowing to shift the designated candidate only backward in the constructive case, or forward in the destructive case).

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Declarations

Author Jörg Rothe is or has been on the following editorial boards of scientific journals:
• *Annals of Mathematics and Artificial Intelligence* (AMAI), Associate Editor, since 01/2020,
• *Journal of Artificial Intelligence Research* (JAIR), Associate Editor, since 09/2017,
• *Journal of Universal Computer Science* (J.UCS), Editorial Board, since 01/2005,
• *Mathematical Logic Quarterly* (MLQ – Wiley), Editorial Board, 01/2008–12/2019, and
• *MDPI Algorithms*, Editorial Board, 04/2021–06/2022.

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