A Supernodal Formulation of Vertex Colouring with Applications in Course Timetabling

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Abstract For many problems in Scheduling and Timetabling the choice of an mathematical programming formulation is determined by the formulation of the graph colouring component. This paper briefly surveys seven known integer programming formulations of vertex colouring and introduces a new formulation using "supernodes". In the definition of George and McIntyre [SIAM J. Numer. Anal. 15 (1978), no. 1, 90–112], "supernode" is a complete subgraph, where each two vertices have the same neighbourhood outside of the subgraph. Seen another way, the algorithm for obtaining the best possible partition of an arbitrary graph into supernodes, which we give and show to be polynomial-time, makes it possible to use any formulation of vertex multicolouring to encode vertex colouring. The power of this approach is shown on the benchmark problem of Udine Course Timetabling. Results from empirical tests on DIMACS colouring instances, in addition to instances from other timetabling applications, are also provided and discussed.

Key words vertex colouring, graph colouring, multicolouring, supernode, integer programming

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1 Introduction

Graph colouring ("proper vertex colouring") is a well-known \mathcal{NP} -Complete problem (Karp, 1972; Garey & Johnson, 1976). It can be formulated as follows: Given a simple undirected, but not necessarily connected graph G = (V, E) and an integer k, decide if it is possible to assign k colours to vertices $v \in V$ such that no two adjacent vertices $\{u, v\} \in E$ are assigned the same colour. Graph colouring has a number of applications, ranging from university timetabling (Carter & Laporte, 1997; Schaerf, 1999; Petrovic & Burke, 2004) and frequency assignment in cellular networks (Aardal, Hoesel, Koster, & Mannino, 2007), to registry allocation in compilers (Springer & Thomas, 1994) and automating differentiation (Gebremedhin, Manne, & Pothen, 2005).

Graph colouring is a challenging problem: As well as being \mathcal{NP} -hard to solve exactly, the minimum number of colours needed to colour a graph is also \mathcal{NP} -Hard to approximate within a factor of $|V|^{1-\epsilon}$ for any $\epsilon > 0$, unless $\mathcal{NP} = \mathcal{P}$ (Krajíček, 1997; Feige & Kilian, 1998; Zuckerman, 2007). Also, there are still dense random instances on 125 vertices from the Second DIMACS Implementation Challenge announced in 1992 (Johnson & Trick, 1996), for which the decision problem cannot be solved within reasonable time limits (Méndez-Díaz & Zabala, 2008), However, it is often possible to solve considerably larger instances in practice, by exploiting applicationspecific structure of the graphs. Springer and Thomas (1994) have, for instance, shown that graph colouring in special cases of register allocation in compilers is polynomially solvable.

In cases that are not polynomially solvable, exact solvers introduced in the past twenty years have predominantly been based on a branch and bound/cut procedure with linear programming relaxations. There are a wide variety of such integer linear programming approaches to modelling graph colouring. A number of authors, including Zabala and Méndez-Díaz (2002; 2006; 2008), have used a natural assignment-type formulation. Williams and Yan (2001) have studied a formulation with precedence constraints. Lee (2002) and Lee and Margot (2007) have studied a binary encoded formulation. Mehrotra and Trick (1996) and more recently (Schindl, 2004; Hansen, Labbé, & Schindl, 2005) have been using formulations based on independent sets. Barbosa et al. (2004) have been experimenting with encodings based on acyclic orientations. Finally, the most recent formulation by Campêlo, Campos, and Corrêa (2008) is based on asymmetric representatives. These seven encodings of graph colouring, often together with the corresponding integer programming formulations, are surveyed in Section 2. In Section 3, we first review the concept of a "supernode", a complete subset of vertices of a graph, where each two vertices have the same neighbourhoods outside of the subset; this concept has been described many times previously (George & McIntyre, 1978; Duff & Reid, 1983; Eisenstat, Elman, Schultz, & Sherman, 1984). See Figure 1 for a simple illustration. Next, we show that the partition of a graph into supernodes, obtainable in polynomial time, proFig. 1: Example of a graph and a partition of its vertex-set into supernodes. Notice supernodes B' and C' need to be assigned two distinct colours each, distinct from the colour(s) assigned to A' and D'. Within each supernode, colours can be interchanged freely. For a more complex example, see Figure 5.

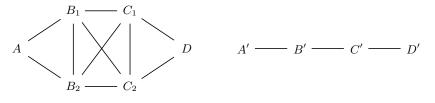


Table 1: Integer programming formulations of graph colouring:

Based on	Variables	Constraints	Selected references
Vertices	k V	V + k E	Méndez-Díaz and Zabala
(Standard)			(2002, 2006, 2008)
Binary Encoding	$\left\lceil \log_2 k \right\rceil V $	Exp. many	Lee (2002)
Max. Independent Sets	Exp. many	V + 1	Mehrotra and Trick (1996)
Any Independent Sets	Exp. many	V + 1	Hansen et al. (2005)
Precedencies	$\mathcal{O}(V ^2)$	E	Williams and Yan (2001)
Acyclic Orientations	E	Exp. many	Barbosa et al. (2004)
Asymmetric Represent.	$\mathcal{O}(E)$	$\mathcal{O}(V E)$	Campêlo et al. (2008)
Supernodes	k Q	Q + k E'	This paper

vides a transformation of graph colouring to graph multicolouring. Hence, we can use the standard binary integer formulation of multicolouring, with binary decision variable x_{ij} is set to one, if any member of supernode *i* is assigned colour *j*, for graph colouring. This translates to new formulations for numerous problems in Scheduling and Timetabling. An illustrative example of formulations of Udine Course Timetabling (Gaspero & Schaerf, 2003, 2006) is given in Section 4. The paper is concluded with a discussion of the empirical tests we carried out in Section 5.

2 Known Formulations of Graph Colouring

In graph colouring, we assume we are given a simple undirected, but not necessarily connected graph G = (V, E) and an integer k. Integer programming formulations of the decision version of the graph colouring problem have feasible integer solutions if and only if it is possible to assign colours $K = \{1, \ldots, k\}$ to vertices $v \in V$ of G such that no two adjacent vertices $\{u, v\} \in E$ are assigned the same colour. Although the minimum value of k is generally hard to approximate, it is of course always possible to pick k = |V|, and for real-life graphs, heuristics based on local search with suitable pre-processing often perform well (Galinier & Hertz, 2006). Estimators of the minimal k are also available for some classes of random graphs (Achlioptas & Naor, 2005). Notice that the decision version of the problem with fixed k, rather than the optimisation version looking for minimal k, is used in many applications. For instance in school timetabling (Schaerf, 1999), k is usually fixed to the number of periods per week.

Although there are at least seven possible encodings of feasible solutions and hence seven different integer programming formulations of graph colouring, as far as we are aware, there is no survey article or empirical comparison available in the literature. Méndez-Díaz and Zabala (2008) compare four classes of cuts using the standard formulation and Prestwich (2003) compares five encodings of graph colouring into propositional satisfiability testing. This section elaborates on the brief overview provided in Table 1.

Unless stated otherwise, we consider the decision version of the problem. In some cases, constraints necessary to reaching optimality are also mentioned. Notice, however, there have often been described many classes of additional constraints, which can be added dynamically in a branch and cut procedure.

2.1 The Standard Formulation

The natural assignment-type formulation of graph colouring uses k |V| binary variables:

$$x_{v,c} = \begin{cases} 1 & \text{if vertex } v \text{ is coloured with colour } c \\ 0 & \text{otherwise} \end{cases}$$
(1)

subject to k |E| constraints:

$$\sum_{c=1}^{k} x_{v,c} = 1 \quad \forall \text{ vertices } v \in V \tag{2}$$

$$x_{u,c} + x_{v,c} \le 1 \quad \forall \text{ colours } c \in K \quad \forall \text{ edges } \{u, v\} \in E \tag{3}$$

This formulation alone produces provably poor linear programming relaxations (Caprara, 1998). Mehrotra and Trick (1996) give the example of $x_{v,c} = 1/k$ for all vertices $v \in V$ and for all colours c, which is feasible when $k \geq 2$. However, a number of classes of strong valid inequalities have been described for this for this formulation, most notably by Zabala and Méndez-Díaz (2002; 2006; 2008), and (Campêlo, Corrêa, & Frota, 2003), either supplanting or replacing per-edge constraints (3) . Branch-and-cut codes using suitable implementations of separation routines have produced a number of optimal values and present-best bounds for the benchmark established by Johnson and Trick (1996) (Zabala & Méndez-Díaz, 2006).

2.2 Extension: Synchronisation with General Integer Variables

Williams and Yan (2001) have noted that the standard formulation could be extended with |V| additional general integer variables X, where $X_v = c$ if colour c is used to colour vertex v, subject to |V| additional constraints:

$$\sum_{c=1}^{k} cx_{v,c} = X_v \quad \forall \text{ vertices } v \in V$$
(4)

This extension can be applied together with custom branching rules with some success in some timetabling problems where, for instance, lectures should be timetabled before laboratory sessions.

2.3 The Independent Set Formulation

One of the first alternative formulations was proposed by Mehrotra and Trick (1996). It is based on set I of maximal independent sets. (Subset $S \subseteq V$ of graph G = (V, E) is defined to be independent, if no two $u, v \in S$ form an edge $\{u, v\} \in E$.) There are an exponential number of binary variables:

$$x_i = \begin{cases} 1 & \text{if independent set } i \text{ is assigned a single colour} \\ 0 & \text{otherwise} \end{cases}$$
(5)

subject to |V| + 1 constraints:

$$\sum_{i \in I} x_i \le k \tag{6}$$

$$\sum_{i \in I, \text{ s.t. } v \in i} x_i \ge 1 \quad \forall \text{ vertices } v \in V \tag{7}$$

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For processing any but the smallest of instances, such a formulation obviously requires very good routines for finding maximal independent sets and for adding them to the linear programming subproblems on-the-fly by the means of column generation. It should also be noted that solutions obtained using this formulation require a certain amount of post-processing, if constraints (7) remain inequalities. Alternatively, the problem could be reformulated so that I comprises all independent sets, not only maximal independent sets. In the per-vertex constraints (7), inequality can then be replaced with equality (Mehrotra & Trick, 1996). The original implementation of Mehrotra and Trick produced exceptionally good results (Mehrotra & Trick, 1996), but later reimplementation of Schindl (2004) and Hansen et al. (2005) failed to match the exceptional performance. It seems also rather difficult to adapt this formulation to extensions of vertex colouring such as the Udine Course Timetabling, which will be introduced in Section 4.
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2.4 The Scheduling Formulation (with Precedence Constraints)

Many researchers from a constraint programming background deal with graph colouring in terms of multiple simultaneously applied all_different constraints. In an assignment $A: V \to D$ of values from a finite domain D to variables V, applying the all_different constraint on a subset $W \subset V$ stipulates that there have to be |W| distinct values assigned to elements of W. Setting all_different (V) then makes assignment A injective. The case of a single all_different constraint is easy to solve, as it represents bipartite matching. The case of two simultaneously applied all_different constraints was studied by Appa, Magos, and Mourtos (2005). The general case of multiple simultaneously applied all_different constraints is, in some sense, equivalent to graph colouring. If we take, for example, the set of variables X defined in Section 2.2, constraints (3) implement |E| all_different constraints to pairs of elements of X. Williams and Yan (2001) have compared this standard integer programming formulation of the all_different constraint (of Section 2.1) with a formulation using precedence constraints. Their work leads to a formulation of vertex colouring using |V| integer variables, where $X_v = c$ if colour c is used to colour vertex v, and $\frac{1}{2}|V|(|V|-1)$ additional binary variables $x_{u,v}$, defined for u < v:

$$x_{u,v} = \begin{cases} 1 & \text{if for vertices } u, v \text{ holds } X_u < X_v \\ 0 & \text{otherwise} \end{cases}$$
(8)

subject to |E| precedence constraints:

$$x_{u,v} + x_{v,u} = 1 \qquad \forall \text{ edges } \{u, v\} \in E \tag{9}$$

(10)

However, in the experience of both Williams and Yan (2001) and the authors, this formulation does not offer particularly strong relaxations.

Tobias Achterberg (personal communication) suggested using another encoding inspired by scheduling:

$$x_{u,m} = \begin{cases} 1 & \text{if vertex } v \text{ is coloured by } c \le m \\ 0 & \text{otherwise} \end{cases}$$
(11)

This encoding is, as far as we know, also untested.

2.5 The Binary Encoded Formulation

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In his studies of the all_different polyhedron, Lee (2002) and Lee and Margot (2007) have introduced a formulation of binary encoding using $\lfloor \log_2 k \rfloor |V|$ binary variables:

Independent set	Used?	Vertex		Colour	
$\{ Math_1 \}$	0		Bit 1	Bit 2	Bit 3
$\{ Math_2 \}$	1	$Math_1$	1	0	0
$\{ Math_3 \}$	1	$Math_2$	0	1	0
$\{ Math_4 \}$	1	$Math_3$	1	1	0
$\{ Algo_1 \}$	1	$Math_4$	0	0	1
$\{ Algo_2 \}$	1	$Algo_1$	1	0	1
$\{ Algo_3 \}$	1	$Algo_2$	0	1	1
$\{ Phy \}$	0	$Algo_3$	1	1	1
$\{ Math_1, Phy \}$	1	Phy	1	0	0
$\{ Math_2, Phy \}$	0	(b) T	he Bina	ry Enco	ding
$\{ Math_3, Phy \}$	0	(0) 1	ne Dina	iy Enco	anig
$\{ Math_4, Phy \}$	0				

Fig. 2: Two encodings of a particular colouring of the graph from Figure 5:

(a) An Encoding Using Independent Sets

 $x_{v,b} = \begin{cases} 1 & \text{if vertex } v \text{ is assigned colour having bit } b \text{ set to } 1 \\ 0 & \text{otherwise} \end{cases}$ (12)

Lee and Margot (2007) also described three broad classes of applicable inequalities ("general block inequalities", "matching inequalities", "switched walk inequalities"), each exponentially large in |V|. We conjecture, but cannot prove, these include all inequalities introduced by Zabala and Méndez-Díaz (2006), when projected to the appropriate space. However, the development of separation routines for such general inequalities is by no means straightforward (Lee & Margot, 2007). In the context of edge colouring of graphs, it only remains to decide if a graph requires more colours than the maximum degree of vertices in the graph. The computationally expensive separation of general block inequalities could thus perhaps be offset by having to eliminate substantially fewer variables in the branch-and-cut procedure (Lee & Margot, 2007). In theory, such an argument could perhaps also apply to colouring of dense random graphs (Bollobás, 2001), where the chromatic number was shown to be almost surely one of two known values (Achlioptas & Naor, 2005). However, experimental results do not seem to be conclusive; not even in the case of edge colouring (Lee & Margot, 2007).

2.6 Encoding Using Acyclic Orientations

In the context of experimental formulations of graph colouring, we also mention acyclic orientations, an encoding based the Gallai-Roy-Vitaver theorem (Gallai, 1968; Roy, 1967; Vitaver, 1962): directed graphs, which contain no directed simple path of length $\geq k, k \geq 1$, are k-colorable. An acyclic orientation G' = (V, E') of an undirected G = (V, E) is then obviously a directed graph such that for each $\{u, v\} \in E$, there is either $(u, v) \in E'$ or $(v, u) \in E'$, and there is no directed cycle in G'. For further references, see also Werra and Hansen (2003) and Nešetřil and Tardif (2008). Together with an algorithm enumerating all possible acyclic orientations (Barbosa & Szwarcfiter, 1999), this could provide a basis for a column generation algorithm for graph colouring. There are some experiments with metaheuristics using this encoding (Barbosa et al., 2004). The only implementation using the linear programming relaxations with this encoding the authors are aware of, however, is an unpublished work of Rosa Maria Videira de Figueiredo.

2.7 Formulation Using Asymmetric Representatives

Finally, the most recently published alternative formulation of graph colouring is by Campêlo et al. (2008), although it does stem from their previous studies of graph colouring (Campêlo et al., 2003). There are $|V| + |V|^2 - |E|$ binary variables $x_{u,v}$, where $x_{u,v}$ is defined for $u, v \in V$, $u \neq v$, and $\{u, v\} \notin E$:

$$x_{u,v} = \begin{cases} 1 & \text{if vertices } u, v \text{ share one colour and } u \text{ represents } v \\ 0 & \text{otherwise} \end{cases}$$
(13)

Each independent set, which is assigned a unique colour, is assigned a unique vertex ("representative") representing the independent set. This can be done using a number of constraints cubic in |V|. Campêlo et al. (2008) then establish an order on the vertex set V, which induces an acyclic orientation introduced in Section 2.6. This enables addition of a number of symmetry-breaking constraints. No empirical results are available, though, as Campêlo et al. (2008) reportedly have problems designing separation routines for the cutting planes they propose.

3 The Main Result

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In this section, we propose another formulation, based on a particular type of clique partition. Let us reiterate, however, the definition of a clique partition first:

Definition 1 The clique partition of graph G = (V, E) is a partition Q of vertices V, such that for all sets $q \in Q$, all $v \in q$ are pairwise adjacent in G.

Notice we use $v \in q \in Q$ only to denote that vertex v in the original vertex-set V is an element of a clique represented by q in the clique partition Q. Hence, there is no need to interpret this as the use of hyper-graphs.

As is well known, the problem of finding the minimum cardinality of a clique partition, $\bar{\chi}(G)$, is \mathcal{NP} -Hard in general graphs and as hard to Fig. 3: Two more encodings of a particular colouring of the graph from Figure 5. Identical row headings are not repeated twice.

V_1		-		Vert	$ex V_2$	2		
	$Math_1$	$Math_2$	$Math_3$	$Math_4$	Algo_1	Algo_2	Algo_3	Phy
$Math_1$		1	1	1	1	1	1	0
$Math_2$	0		1	1	1	1	1	0
$Math_3$	0	0		1	1	1	1	0
$Math_4$	0	0	0		1	1	1	0
Algo_1	0	0	0	0		1	1	0
Algo_2	0	0	0	0	0		1	0
$Algo_3$	0	0	0	0	0	0		0
Phy	0	1	1	1	1	1	1	
0 Math ₁	$Math_2$ (e)	Math ₃ 941		$\frac{1}{1}$	-	Algo_{3}	Phy	
0								
0	1						1	
0	1	1					0	
0	1	1	1				0 0	
0	1	1	1	1			0	
0	1	1	1	1	1		0 0	
0	1	1	1	1	1	1	0 0	

(b) The Encoding Using Asymmetric Representatives

approximate as graph colouring itself (see MINIMUM-CLIQUE-PARTITION in Crescenzi, Kann, Halldórsson, Karpinski, & Woeginger, 2005). Indeed, $\bar{\chi}(G) = \chi(\bar{G})$, where $\chi(\bar{G})$ is the minimum number of colours needed to colour the complement graph. Another direction of arriving at probabilistic bounds on $\bar{\chi}(G)$ could, perhaps, follow from probabilistic results of Molloy and Reed (2002, Chapter 11) for maximal cliques. Notice, however, we do not require minimality in the definition, and hence V is the trivial clique partition of graph G.

Next, we introduce the *indistinguishability relation* between vertices of a graph:

Definition 2 Two vertices $u, v \in V$ of a graph G = (V, E) are indistinguishable, if and only if they are adjacent and have identical closed neighbourhoods; that is: $\{w \mid \{u, w\} \in E\} \cup \{u\}$ is the same as $\{w \mid \{v, w\} \in E\} \cup \{v\}$.

This relation has been studied previously in the context of pivoting in matrix factorisation, in connection with mass elimination (George & McIntyre, 1978), supervariables (Duff & Reid, 1983), and prototype vertices (Eisenstat et al., 1984). It is easy to observe the indistinguishability relation is reflexive, symmetric, and transitive. Hence:

Lemma 1 The indistinguishability relation is an equivalence.

Next, we define the particular type of clique partition we are interested in:

Definition 3 The reversible clique partition Q of a graph G = (V, E) is the clique partition of minimum cardinality such that each supernode $q \in Q$ represents a class of equivalence in a indistinguishability relation on G.

This means that for each supernode $q \in Q$ of the reversible clique partition (Q, E'), each two vertices $u, v \in q$ are indistinguishable. As usual, we will be interested also in the graph induced by the clique partition:

Definition 4 The graph induced by reversible clique partition Q of graph G = (V, E) is the graph G' = (Q, E'), where $E' = \{\{q_u, q_v\} | \{u, v\} \in E, q_u, q_v \in Q, q_u \neq q_v, u \in q_u, v \in q_v, \}$.

The use of the word *induced* in this context is reasonable, because it corresponds to a subgraph induced by taking a subset of the original vertex set with a single (arbitrary) representative of each supernode. The "reversibility" of the clique partition is, indeed, rather a strict requirement, which enables us to formulate the following:

Definition 5 Algorithm A Input: Graph G = (V, E)Output: Reversible clique partition Q of G

- 1. Construct an auxiliary graph H = (V, F), where there is an edge $\{u, v\} \in F$, if and only if there is an edge $\{u, v\} \in E$ and vertices u and v are indistinguishable in G
- 2. Run depth-first search on H to obtain collection Q of connected components of H
- 3. Return Q

We can easily deduce that:

Lemma 2 Algorithm A produces a reversible clique partition.

From Step 1, it is clear each element of the collection we return corresponds to a class of equivalence in the indistinguishability relation on G. By transitivity of the indistinguishability relation, it is clear the algorithm produces a clique partition. Now imagine there is a smaller clique partition R corresponds the indistinguishability relation on G. It is easy to see the contradiction. Hence, the algorithm obtains a reversible clique partition. Furthermore:

Lemma 3 Algorithm A runs in time $\mathcal{O}(|V||E|)$.

Given Algorithm A, we can straightforwardly reformulate the problem of vertex colouring as the problem of multicolouring of the corresponding reversible clique partition, where by multicolouring, we mean:

Definition 6 The problem of multicolouring of a graph G = (V, E) with a finite set of colours $K = \{1, \ldots, k\}$, which is given together with demand function $f: V \to \mathbb{N}$, is to obtain is a mapping $c: V \to 2^K$, such that for all $v \in V: |c(v)| = f(v)$ and for all $\{u, v\} \in E$, $c(u) \cap c(v) = \emptyset$. It makes sense to require $\bigcup_{v \in V} c(v) = K$.

Notice that multicolouring with sets of uniform cardinality has been studied under the name of set colouring, for example by Stahl (1976), Bollobas and Thomason (1979), and more recently used also by Duran et al. (2002; 2006). Other variants of the problems are surveyed by Halldórsson and Kortsarz (2004) and Aardal et al. (2007). Mehrotra and Trick (2007) seem to have the present-best solver for multicolouring.

From Lemma 2, it is easy to observe that Algorithm A provides a transformation of vertex colouring to vertex multicolouring. Hence, the standard formulation of vertex multicolouring can also be used as a formulation of vertex colouring. Given the graph G' = (Q, E') induced by reversible clique partition Q of graph G = (V, E) together with the demand function $f: V \to \mathbb{N}$, specifying the number f(q) of colours to attach to each vertex $q \in Q$ out of the set $K = \{1, \ldots, k\}$, we can use an integer programming formulation with k |Q| binary variables:

$$x_{q,c} = \begin{cases} 1 & \text{if colour } c \text{ is included in the set assigned to } q \in Q \\ 0 & \text{otherwise} \end{cases}$$
(14)

subject to |Q| + k |E'| constraints:

$$\sum_{c=1}^{k} x_{q,c} = f(q) \qquad \qquad \forall q \in Q \tag{15}$$

$$x_{u',c} + x_{v',c} \le 1 \qquad \qquad \forall c \in K \quad \forall \{u',v'\} \in E'$$
(16)

See Figure 4 for an example. It is easy to see that there exists a proper vertex colouring of G = (V, E) with k colours, if and only if there exists a multicolouring of a reversible clique partition (Q, E') of G with k colours, which exists if and only if the integer programming formulation has a feasible solution for the given instance. When a graph has only a trivial reversible clique partition, this formulation is reduced to the standard formulation. It thus remains \mathcal{NP} -Complete to decide, if there exists a multicolouring of G' with f(q) using k colours. Nevertheless, the proposed formulation breaks some symmetries inherent in the standard vertex colouring formulation,

Fig. 4: The standard and the proposed encoding of a particular colouring of the graph from Figure 5:

Vertex			С	olou	r		
	1	2	3	4	5	6	$\overline{7}$
$Math_1$	1	0	0	0	0	0	0
$Math_2$	0	1	0	0	0	0	0
$Math_3$	0	0	1	0	0	0	0
$Math_4$	0	0	0	1	0	0	0
$Algo_1$	0	0	0	0	1	0	0
$Algo_2$	0	0	0	0	0	1	0
$Algo_3$	0	0	0	0	0	0	1
Phy	1	0	0	0	0	0	0
(a) 7	The	Stan	dare	d En	codi	ng	
Partition			(Color	ur		
	1	2	3	4	5	6	7
Math'	1	1	1	1	0	0	0
Algo'	0	0	0	0	1	1	1
Phy'	1	0	0	0	0	0	0

(b) The Proposed Encoding

which assigns unique colours (or "labels") to individual vertices. If there was a trivial integer programming solver, using neither bounding, nor cuts, this formulation should reduce its search space and run time by the factor of:

 $\prod_{q\in Q} |q|!$

when compared to the standard formulation of Section 2.1. Although it is much more difficult to predict run times in modern integer programming solvers, it is obvious that there are k(|V| - |Q|) fewer variables, in the proposed formulation than in the standard one. It seems that the number of constraints is also reduced, often by more than k(|V| - |Q|), without making the constraint matrix considerably denser. Hence, reduction in run time of the order of |Q| / |V| could perhaps be expected. For empirical results, see Section 5.

4 An Application in Course Timetabling

In general, a comparison of formulations of graph colouring is non-trivial. Both encodings based on independent sets and representatives introduce less symmetry¹ than the standard formulation introduced in Section 2.1 or

¹ When we address the question of reducing or breaking symmetry below, the statements hold, when symmetry is thought of as the number of solutions of the instance of integer programming with the best possible cost, corresponding to, in some sense,

binary encoding. Although they neatly partition the set of vertices, without assigning unique labels to individual partitions, their merits are hard to quantify, as any empirical results are dependent on a particular implementation of separation and pricing routines, which have not been extensively studied thus far. Another important aspect is extensibility of the various formulations of graph colouring. Many real-world applications necessitate formulation of complex measures of the quality of feasible solutions ("key performance indicators"), which seem to be hard to formulate using an exponential number of variables representing independent sets (Mehrotra & Trick, 1996; Hansen et al., 2005) or using the binary encoding of Lee (2002). One such application arises in a number of universities (Burke, Werra, & Kingston, 2004): course timetabling.

In educational timetabling, considerable resources can be wasted by low utilisation of teaching space (Beyrouthy et al., 2008). Specific timetabling problems vary widely from institution to institution. Most problems, however, share a common model:

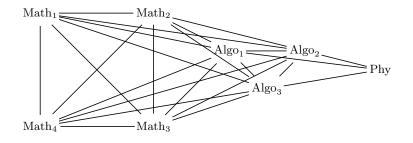
- set E of events is given, together with a subset of its powerset A, where for all distinct "enrolments" (or "conflict groups" or "curricula') $a \in A$, events $e \in a$ cannot take place at the same time
- assignment of events to |P| time periods is desired, such that all distinct enrolments are honoured and there are at most |R| events taking place at one period, where |P| is the number of periods per week and |R| is the number of available rooms.

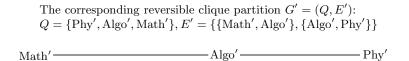
This model is, indeed, a straightforward application of |R|-bounded |P|colouring. In the graph to be coloured (the "conflict graph"), vertices represent events, two vertices are adjacent if the corresponding events are included in a single enrolment, and assignment of periods to events is represented by assignment of |P| colours to |E| vertices, such that adjacent vertices are assigned different colours and each colour is used at most |R| times. For an illustrative example, see Figure 5. For further graph-theoretical foundations, see Handbook of Graph Theory (Gross & Yellen, 2004), especially Section 5.6 (Burke et al., 2004). The most rigorous studies of integer programming formulations of this model, including competitive branch-and-cut implementations, are by Avella and Vasil'ev (2005) and Méndez-Díaz and Zabala (2008). For other recent research directions, see Burke and Petrovic (2002). However, it seems obvious that this model is rather removed from the needs of real-life applications, although given the complexity of vertex colouring, where the present-best solvers have difficulties with dense instances on 125 vertices (Zabala & Méndez-Díaz, 2006), it also presents an interesting challenge.

a single configuration. The assignment of colours is irrelevant, for example, as long as we are given the appropriate vertex-partition. Presumably, the statements could also hold for other definitions of symmetry as well.

Fig. 5: An example from timetabling. Imagine one student takes Algorithms and Mathematics (with three and four lectures per week), and another one takes Algorithms and Physics (with a single lecture per week); no two lectures attended by one student can take place at the same time.

The original conflict graph G = (V, E): $V = \{Phy, Algo_1, Algo_2, Algo_3, Math_1, Math_2, Math_3, Math_4\}$ $E = \{\{u, v\} \mid u, v \in V, u \neq v\} \setminus \{\{Phy, Math_1\}, \{Phy, Math_2\}, \{Phy, Math_3\}, \{Phy, Math_4\}\}$





In this paper, we use a model of course timetabling proposed by Schaerf and Di Gaspero (2003, 2006) at the University of Udine. In Udine Course Timetabling, the basic model is extended so that:

- events are grouped into disjoint sets, called courses, with events of one course taking place at different times and being freely interchangeable
- only important distinct enrolments, or non-disjoint sets of courses prescribed to various groups of students, are identified
- capacities of individual classrooms and enrolments in individual courses are also given, and assignment of events to rooms as well as periods is desired, minimising value of an objective function

What makes the extension more difficult (by orders of magnitude) than the basic model, however, is the objective function, consisting of a linear combination of three key performance indicators:

- the number of students left without a seat at an event, summed across all events
- the difference between the prescribed minimum number of distinct days of instruction for a course and the actual number of distinct days, when events of the course are held, summed across all courses, where the difference is positive

 the number of events occurring outside of a continuous block of two or more events in a timetable for an important distinct enrolment, summed across all important distinct enrolments

Notice that the third key performance indicator essentially consists of the sum of the number of breaks in individual timetables of individual students or groups of students, plus the number of single courses on a single day in the timetables. Its modelling proves to be very difficult (Burke, Mareček, Parkes, & Rudová, 2008) and the present best solvers yield "poor results" (Avella & Vasil'ev, 2005). See also Schimmelpfeng and Helber (2007) for another example of a timetabling problem with a number of soft-constraints, together with an interesting integer programming formulation.

In a further extension of the basic model, not studied in this paper, one relaxes also the colouring component. Vertices of an edge-weighted conflict graph then have to be partitioned into |P| disjoint subsets such that the sum of weights attached to edges with both end-points in a single subset is minimised (Kiaer & Yellen, 1992). The weight of an edge $\{e_1, e_2\} \in E$ can be determined, for instance, by the number of students enrolled in both events e_1 and e_2 . Obviously, if the conflict graph is |P|-colourable, a proper colouring is found. Such a model is employed, for instance, at Purdue University (Rudová & Murray, 2003; Murray, Müller, & Rudová, 2007).

4.1 Notation for Course Timetabling

In order to present timetabling applications of the proposed formulation of graph colouring, we have to introduce some notation. In the context of course timetabling, it is customary to refer to vertices as events and colours as periods. In addition to a period, each event is assigned also a room, and there can be, at most, a given number of events taking place at each period. Using this convention and the notation presented in Table 2, the standard integer programming formulation of course timetabling is written as:

$$T_{p,r,e} = \begin{cases} 1 & \text{if event } e \text{ is taught in room } r \text{ at period } p \\ 0 & \text{otherwise} \end{cases}$$
(17)

Course Timetabiling.	
R	set of rooms
$Capacity_r$	the subset of periods pertaining to day d
P	set of periods
D	set of days
$\operatorname{Periods}_d$	the subset of periods pertaining to day d
C	set of courses
$\operatorname{MinDays}_{c}$	the recommended minimum number of days
	of instruction for course c
$Students_c$	number of students enrolled in course c
E	set of events
E_c	the subset of events pertaining to course c
T	set of teachers
$Teaches_t$	the subset of courses taught by teacher t
U	set of identifiers of distinct enrolments
HasC_{u}	the subset of courses pertaining to curriculum u

Table 2: The notation used in our integer programming formulation of Udine Course Timetabling.

$$\sum_{r} \sum_{p} T_{p,r,e} = 1 \quad \forall \text{ events } e \in E$$

$$\sum_{r} T_{p,r,e} \leq 1 \quad \forall \text{ periods } p \in P \quad \forall \text{ rooms } r \in R$$

$$\sum_{r} \sum_{e \in c} T_{p,r,e} \leq 1 \quad \forall \text{ periods } p \in P \quad \forall \text{ courses } c \in C$$

$$\sum_{c \in \text{Teaches}_{t}} \sum_{e \in c} T_{p,r,e} \leq 1 \quad \forall \text{ periods } p \in P \quad \forall \text{ teachers } t \in T$$

$$(20)$$

$$\sum_{c \in \text{Teaches}_{t}} \sum_{e \in c} T_{p,r,e} \leq 1 \quad \forall \text{ periods } p \in P \quad \forall \text{ teachers } t \in T$$

$$(21)$$

$$\sum_{r} \sum_{c \in \text{HasC}_{u}} \sum_{e \in c} \mathbf{T}_{p,r,e} \qquad \leq 1 \quad \forall \text{ periods } p \in P \quad \forall \text{ curricula } u \in \mathbf{U}$$
(22)

This corresponds to the standard formulation of graph colouring introduced in Section 2.1. Constraints (18) ensure each event is assigned a time-place slot and constraints (19) ensure there is at most one event taking place in a given room at a period. Finally, the packing-type constraints (20)-(22) stipulate there should be no conflicts. Notice that constraints (22) make constraints (20) redundant, unless there are courses not included in any enrolment. In a similar spirit, the formulation introduced in Section 3 can be written, with courses as supernodes, as follows:

 \sum_{r}

$$\Gamma_{p,r,c} = \begin{cases} 1 & \text{if some event of course } c \text{ is taught in room } r \text{ at period } p \\ 0 & \text{otherwise} \end{cases}$$
(23)

$$\sum_{r} \sum_{p} \operatorname{T}_{p,r,c} = |E_c| \qquad \forall \text{ courses } c \in C$$
(24)

$$\sum_{c} \mathbf{T}_{p,r,c} \leq 1 \qquad \forall \text{ periods } p \in P \quad \forall \text{ rooms } r \in R \qquad (25)$$

$$\sum_{r} \mathbf{T}_{p,r,c} \leq 1 \qquad \forall \text{ periods } p \in P \quad \forall \text{ courses } c \in C \qquad (26)$$

$$\sum_{r} \sum_{c \in \text{Teaches}_{t}} \mathbf{T}_{p,r,c} \leq 1 \qquad \forall \text{ periods } p \in P \quad \forall \text{ teachers } t \in T \qquad (27)$$

$$\sum_{r} \sum_{c \in \text{HasC}_{u}} \mathbf{T}_{p,r,e} \leq 1 \qquad \forall \text{ periods } p \in P \quad \forall \text{ curricula } u \in \mathbf{U} \ (28)$$

What makes real-life course timetabling vastly more difficult than this formulation of graph colouring, are complex measures of the quality of feasible timetables, which are best illustrated by considering an example.

4.2 Formulation of Udine Course Timetabling

Udine Course Timetabling, introduced in Section 4, is an established benchmark in the field of course timetabling with complex performance indicators. Out of the three key performance indicators in Udine Course Timetabling, the minimisation of the number of students left without a seat can be formulated using a single term in the objective function:

$$\sum_{\substack{r \in \mathbb{R} \\ p \in \mathbb{P} \\ \text{Students}_c > \\ \text{Capacity}_r}} \sum_{\substack{c \in \mathbb{C} \\ \text{Students}_c > \\ \text{Capacity}_r}} \mathcal{T}_{p,r,c} \left(\text{Students}_c - \text{Capacity}_r \right).$$
(29)

The second key performance indicator, the number of missing days of instruction summed across all courses, can be formulated using two auxiliary arrays of decision variables. The first binary array, U, is indexed with courses and days. $U_{c,d}$ is set to one, if and only if there are some events of course c held on day d. The other array of integers, V, is indexed with courses. Value V_c is bounded below by zero and above by the number of days in a week and represents the number of days course c is short of its recommended days

of instruction. This enables addition of the constraints:

$$\sum_{r \in \mathbf{R}} \mathbf{T}_{p,r,c} \le \mathbf{U}_{c,d} \quad \forall c \in \mathbf{C} \quad \forall d \in \mathbf{D} \quad \forall p \in \mathrm{Periods}_d \tag{30}$$

$$\sum_{r \in \mathbf{R}} \sum_{p \in \operatorname{Periods}_d} \mathbf{T}_{p,r,c} \ge \mathbf{U}_{c,d} \quad \forall c \in \mathbf{C} \quad \forall d \in \mathbf{D}$$
(31)

$$V_c + \sum_{d \in D} U_{c,d} \ge MinDays_c \quad \forall c \in C$$
. (32)

The term $\sum_{c \in C} V_c$ can then easily be added to the objective function.

However, it is only the formulation of the third key performance indicator, the penalty incurred by patterns of distinct daily timetables of individual or groups of students, that proves to have a decisive impact on the performance of formulations of Udine Course Timetabling (Burke et al., 2008). The penalisation of patterns in timetables was traditionally formulated "by feature" (Avella & Vasil'ev, 2005). In an auxiliary binary array S indexed with curricula, days and features, $S_{u,d,f}$ is set to one, if and only if feature f is present in the timetable for curriculum u and day d. In the case of the penalisation of events timetabled for a curriculum outside of a single consecutive block of two or more events per day of four periods, there are four constraints:

 $\forall u \in \mathbf{U}, d \in \mathbf{D}, \forall \langle p_1, p_2, p_3, p_4 \rangle \in \mathbf{Periods}_d$

$$\sum_{c \in \operatorname{HasC}_{u}} \sum_{r \in \operatorname{R}} (\operatorname{T}_{p1,r,c} - \operatorname{T}_{p2,r,c}) \le \operatorname{S}_{u,d,1} \quad (33)$$

 $\forall u \in U, d \in D, \forall \langle p_1, p_2, p_3, p_4 \rangle \in \text{Periods}_d$

$$\sum_{c \in \text{HasC}_u} \sum_{r \in \mathbf{R}} (\mathbf{T}_{p2,r,c} - \mathbf{T}_{p1,r,c} - \mathbf{T}_{p3,r,c}) \le \mathbf{S}_{u,d,2} \quad (34)$$

 $\forall u \in U, d \in D, \forall \langle p_1, p_2, p_3, p_4 \rangle \in \text{Periods}_d$

$$\sum_{c \in \text{HasC}_u} \sum_{r \in \mathbf{R}} (\mathbf{T}_{p3,r,c} - \mathbf{T}_{p2,r,c} - \mathbf{T}_{p4,r,c}) \le \mathbf{S}_{u,d,3} \quad (35)$$

 $\forall u \! \in \! \mathbf{U}, d \! \in \! \mathbf{D}, \forall \langle p_1, p_2, p_3, p_4 \rangle \! \in \! \mathbf{Periods}_d$

$$\sum_{c \in \operatorname{HasC}_{u}} \sum_{r \in \mathbb{R}} (\mathrm{T}_{p4,r,c} - \mathrm{T}_{p3,r,c}) \leq \mathrm{S}_{u,d,4} \quad (36)$$

However, considerable improvement in the performance of pattern penalisation can be gained by introducing the concept of the enumeration of patterns. It is obviously possible to pre-compute a set B of n + 2 tuples w, x_1, \ldots, x_n, m , where n is the number of periods per day, x_i is one if there is instruction in period i of the daily pattern and minus one otherwise, w is the penalty attached to the pattern, and m is the sum of positive values x_i in the patterns decremented by one. Burke et al. (2008) have studied a number of possible applications of this concept, with one of the best performing being the addition of constraints, such as in the case of four periods per day:

 $\forall \langle w, x_1, x_2, x_3, x_4, m \rangle {\in} B \ \forall u {\in} \mathbf{U} \ \forall d {\in} \mathbf{D} \ \forall \langle p_1, p_2, p_3, p_4 \rangle {\in} \mathbf{Periods}_d$

$$w \left(x_{1} \sum_{c \in \operatorname{HasC}_{u}} \sum_{r \in \operatorname{R}} \operatorname{T}_{p_{1},r,c} + x_{2} \sum_{c \in \operatorname{HasC}_{u}} \sum_{r \in \operatorname{R}} \operatorname{T}_{p_{2},r,c} + x_{3} \sum_{c \in \operatorname{HasC}_{u}} \sum_{r \in \operatorname{R}} \operatorname{T}_{p_{3},r,c} + x_{4} \sum_{c \in \operatorname{HasC}_{u}} \sum_{r \in \operatorname{R}} \operatorname{T}_{p_{4},r,c} - m\right) \leq \sum_{s \in \operatorname{Checks}} \operatorname{S}_{u,d,s} .$$

$$(37)$$

The third term in the objective function is $\sum_{u \in U} \sum_{d \in D} \sum_{s \in Checks} S_{u,d,s}$. For further details on formulations of these soft constraints and their impact on the overall performance, see Burke et al. (2008).

5 Computational Experience

In order to evaluate performance of the new formulation, we have conducted a number of experiments. We report:

- 1. the dimensions of reversible clique partitions obtained from graphs in the standard DIMACS benchmark
- 2. performance gains on graph colouring instances originating from timetabling, both from real-life and randomly generated instances of the Udine Course Timetabling problem
- 3. performance gains on the the complete instances of Udine Course Timetabling problem, as compared to the effects of symmetry breaking built into CPLEX.

All reported results were measured on a desktop PC running Linux, equipped with two Intel Pentium 4 processors clocked at 3.20 GHz. ILOG CPLEX version 10.0 integer programming solver was restricted to use only a single thread on a single processor. Default parameter settings were used, outside of settings for symmetry breaking described below and settings imposing the time limit of one hour on run time per instance. DIMACS instances descibed by Johnson and Trick (1996) were downloaded from the on-line repository². Four real-life timetabling instances were taken from the benchmark used by (Gaspero & Schaerf, 2003, 2006) and eighteen more instances were obtained using a random instance generator³ developed by the authors. Their dimensions are listed in Table 4. In all instances, each course has one to six events per week, with the average being three, each teacher teaches one or

² Available at http://mat.gsia.cmu.edu/COLOR/ (Nov 7, 2007)

³ Available at http://cs.nott.ac.uk/~jxm/timetabling/generator/ (Nov 7, 2007)

two courses totalling at one to six hours per week, and enrolments consist of less than ten events per week, on average. All instances were passed to CPLEX in LP format as generated from sources in Zimpl, the free algebraic modelling language (Koch, 2004), and are available on-line in Zimpl format. Instances in LP format, whose total size exceeds 1.3 GB, can be also made available upon request. Verification of the results is thus possible with freely available solvers, such as SCIP (Achterberg, 2007).

First, we have obtained reversible clique partions from DIMACS graphs. To illustrate the effects of pre-processing of the original graph on the size of the reversible clique partition, in Table 3, we list the sizes first without using any preprocessing (under Q), as well as after some pre-processing specific to graph colouring, but not specific to the transformation (under Q'). This preprocessing included:

- Removal of vertices of degree less than a lower bound on the chromatic number
- Removal of vertices connected to all other vertices in the graph
- Removal of each vertex u whose neighboughood is a subset of the neighbourhood of another non-adjacent vertex v.

For details of the pre-processing and the source code used, please see the authors' website⁴.

Second, we evaluated performance of the standard formulation of graph colouring introduced in Section 2.1 and performance of the formulation proposed in Section 3 on the graph colouring component of instances of Udine Course Timetabling. (The complete constraint set was used, but no objective function.) Notice (in Section 4.1) that both formulations use the same amount of information on cliques found in the conflict graph, only expressed in terms of different decision variables. From the results reported in Table 5, it seems that with the exception of a single random instance (rand16) and one heavily constrained real-life instance (udine4), the proposed formulation performs considerably better.

Next, we compared performance of the formulations of Udine Course Timetabling, differing only in the formulation of the underlying graph colouring component. Notice that the CPLEX run time necessary to reach optimality was two orders of magnitude higher than in the previous experiment looking for feasible colouring. Whether the performance gains observed in the graph colouring component alone would still be manifested, was thus not clear. As is summarised in Table 6, however, the new formulation again seems to perform considerably better, reducing CPLEX run times approximately by factor of four, where it is possible to reach optimum within one hour using both formulations.

We have also studied effects of symmetry breaking implemented in CPLEX on performance of both formulations. In all previous experiments, both formulations were run using no built-in symmetry breaking in CPLEX. Table 7 compares these results (denoted -SB) with results obtained with symmetry

⁴ Available at http://cs.nott.ac.uk/~jxm/colouring/supernodal/ (Nov 7, 2008)

breaking built-in in CPLEX 10.0 set to aggressive (denoted +SB). Again, the new formulation using no built-in symmetry breaking performs better than the standard formulation using aggressive built-in symmetry breaking.

These results are rather encouraging, although the performance gains are limited only to graphs, where it is possible to obtain a reversible clique partition of V, whose cardinality is considerably less than |V|. This is not the case in triangle-free graphs and many dense random graphs, often used in benchmarking general graph colouring procedures. In many real-world applications, the graphs seem to be, however, highly structured, and the structure is worth exploiting.

6 Conclusions and Further Work

We have presented a transformation of graph colouring to graph multicolouring, making it possible to use the standard formulation of graph multicolouring as a formulation of graph colouring. This can also be viewed as the supernodal integer programming formulation of graph colouring, where supernode of George and McIntyre (1978) is the complete subset of vertices of a graph where each two vertices have the same neighbours outside of the subset. It remains to be seen, if the transformation could be used in conjuction with other formulations of multicolouring.

This transformation can be seen as an example of symmetry breaking. Although there has been recently a considerable interest (Margot, 2002, 2003, 2007; Ostrowski, Linderoth, Rossi, & Smriglio, 2007; Kaibel, Peinhardt, & Pfetsch, 2007; Kaibel & Pfetsch, 2008) in the development of methods for automated symmetry breaking, these methods have so far not been competitive in solving graph colouring problems (Kaibel & Margot, 2007). Compared to the standard formulation with symmetry breaking embedded in ILOG CPLEX 10.0, the industrial standard in integer programming solvers, our reformulation without the embedded symmetry breaking enabled offers performance, which is improved by a factor of three. It would appear that application-specific formulations breaking symmetries will be necessary, at least until performance of automated symmetry breaking improves.

Additionally, we have briefly surveyed seven other integer programming formulations of vertex colouring, proposed in the literature. This seems to be the first time such a survey has been attempted. Generally speaking, in nontrivial applications of graph colouring, the performance of various integer programming formulations of the underlying graph colouring components seems to be highly dependent on their suitability for application-specific key performance indicators. Nevertheless, a proper computational comparison of integer programming formulations of graph colouring would be most interesting – and remains to be conducted. Another interesting research direction might explore hybridisation, using one encoding in an integer programming formulation, but multiple encodings for cut generation. Finally, the proposed formulation seems very convenient in timetabling applications. Compared to many formulations necessitating column generation, it is easy to extend this formulation to accommodate complex key performance indicators ("soft constraints"). We have demonstrated its performance on the example of Udine Course Timetabling, a benchmark problem in timetabling with soft constraints proposed by Gaspero and Schaerf (2003). Using ILOG CPLEX 10.0, we have been able to arrive at the previously unknown optimum for instance Udine1 within 143 seconds on a single processor. Such results give a new hope that real-life instances of course timetabling could be solved within provably small bounds of optimality.

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Table 3: Dimensions of graphs induced by reversible clique partitions obtained from DIMACS instances (G), with (Q') and without (Q) preprocessing. Empty spaces indicate graphs trivial to colour.

Instance	Original	Graph G	Rev. Cli	iq. Part. Q	Rev. Cli	q. Part. Q'
	Vert.	Edges	Vert.	Edges	Vert.	Edges
1-FullIns_3	30	100	29	89		0
1-FullIns_4	93	593	92	561	25	85
1-FullIns_5	282	3247	281	3152	61	358
1-Insertions_4	67	232	67	232	60	208
1-Insertions_5	202	1227	202	1227	202	1227
1-Insertions_6	607	6337	607	6337	600	6301
2-FullIns_3	52	201	51	186		
2-FullIns_4	212	1621	211	1566	16	65
2-FullIns_5	852	12201	851	11986	93	582
2-Insertions_3	37	72	37	72		
2-Insertions_4	149	541	149	541	149	541
2-Insertions_5	597	3936	597	3936	597	3936
3-FullIns_3	80	346	79	327	17	65
3-FullIns_4	405	3524	404	3440	22	114
3-FullIns_5	2030	33751	2029	33342	94	768
3-Insertions_3	56	110	56	110		
3-Insertions_4	281	1046	281	1046	281	1046
3-Insertions_5	1406	9695	1406	9695	1395	9642
4-FullIns_3	114	541	113	518		
4-FullIns_4	690	6650	689	6531		
4-FullIns_5	4146	77305	4145	76610	195	1769
4-Insertions_3	79	156	79	156		
4-Insertions_4	475	1795	475	1795	475	1795
5-FullIns_3	154	792	153	765	39	229
5-FullIns_4	1085	11395	1084	11235	121	1037
abb313GPIA	1557	53356	1557	53356	853	16093
anna	138	493	125	437		
ash331GPIA	662	4181	662	4181	661	4180
ash608GPIA	1216	7844	1216	7844	1215	7843
ash958GPIA	1916	12506	1916	12506	1915	12505
david	87	406	74	322		
DSJC1000.1	1000	49629	1000	49629	1000	49629
DSJC1000.5	1000	249826	1000	249826	1000	249826
DSJC1000.9	1000	449449	1000	449449	1000	449449
DSJC125.1	125	736	125	736	125	736
DSJC125.5	125	3891	125	3891	125	3891
DSJC125.9	125	6961	125	6961	125	6961
DSJC250.1	250	3218	250	3218	250	3218
DSJC250.5	250	15668	250	15668	250	15668
DSJC250.9	250	27897	250	27897	250	27897

Instance	Original	Graph G	Rev. Cli	iq. Part. Q	Rev. Clie	q. Part. Q'
	Vert.	Edges	Vert.	Edges	Vert.	Edges
DSJC500.1	500	12458	500	12458	500	12458
DSJC500.5	500	62624	500	62624	500	62624
DSJC500.9	500	112437	500	112437	500	112437
DSJR500.1	500	3555	480	3341		
DSJR500.1c	500	121275	500	121275	281	38166
DSJR500.5	500	58862	497	58218	483	56618
ear	190	4793	185	4758	172	4636
fpsol2.i.1	496	11654	427	5108	107	2454
fpsol2.i.2	451	8691	395	5657	154	2705
fpsol2.i.3	425	8688	369	5658	153	2665
games120	120	638	119	629		
hec	81	1363	81	1363	75	1277
homer	561	1628	503	1376		
huck	74	301	54	179		
inithx.i.1	864	18707	732	11140		
inithx.i.2	645	13979	539	9317	50	544
inithx.i.3	621	13969	521	9427	49	474
jean	80	254	67	177		
latin_square_10	900	307350	900	307350	900	307350
le450_15a	450	8168	450	8168	407	7802
$le450_{15b}$	450	8169	450	8169	410	7824
$le450_{-15c}$	450	16680	450	16680	450	16680
$le450_{-}15d$	450	16750	450	16750	450	16750
le450_25a	450	8260	450	8260	264	5840
$le450_{25b}$	450	8263	450	8263	294	6240
$le450_{25c}$	450	17343	450	17343	435	17096
$le450_{25d}$	450	17425	450	17425	433	17106
le450_5a	450	5714	450	5714	450	5714
le450_5b	450	5734	450	5734	450	5734
le450_5c	450	9803	450	9803	450	9803
le450_5d	450	9757	450	9757	450	9757
miles1000	128	3216	123	3049		
miles1500	128	5198	104	3486		
miles250	128	387	117	341		
miles500	128	1170	115	1065		
miles750	128	2113	122	2011		
mug100_1	100	166	84	118		
mug100_25	100	166	83	115		
mug88_1	88	146	75	107		
mug88_25	88	146	72	98		
mulsol.i.1	197	3925	166	2274		

Instance	Original	Graph G	Rev. Cl	iq. Part. Q	Rev. Cli	q. Part. Q'
	Vert.	Edges	Vert.	Edges	Vert.	Edges
mulsol.i.2	188	3885	158	2458	35	337
mulsol.i.3	184	3916	155	2504	35	336
mulsol.i.4	185	3946	155	2504	36	360
mulsol.i.5	186	3973	157	2549	36	356
myciel2						
myciel3	11	20	11	20	11	20
myciel4	23	71	23	71	23	71
myciel5	47	236	47	236	47	236
myciel6	95	755	95	755	95	755
myciel7	191	2360	191	2360	191	2360
qg.order100	10000	990000	10000	990000	10000	990000
qg.order30	900	26100	900	26100	900	26100
qg.order40	1600	62400	1600	62400	1600	62400
qg.order60	3600	212400	3600	212400	3600	212400
queen10_10	100	1470	100	1470	100	1470
queen11_11	121	1980	121	1980	121	1980
queen12_12	144	2596	144	2596	144	2596
queen13_13	169	3328	169	3328	169	3328
queen14_14	196	4186	196	4186	196	4186
queen15_15	225	5180	225	5180	225	5180
queen16_16	256	6320	256	6320	256	6320
queen5_5	25	160	25	160	25	160
queen6_6	36	290	36	290	36	290
queen7_7	49	476	49	476	49	476
queen8_12	96	1368	96	1368	96	1368
queen8_8	64	728	64	728	64	728
queen9_9	81	1056	81	1056	81	1056
school1	385	19095	376	18937	353	18799
school1_nsh	352	14612	344	14486	322	14343
wap01a	2368	110871	1594	73666	1594	73666
wap02a	2464	111742	1594	72498	1594	72498
wap03a	4730	286722	3716	224640	3716	224640
wap04a	5231	294902	3814	221704	3814	221704
wap05a	905	43081	749	35116	746	35102
wap06a	947	43571	741	34012	735	33760
wap07a	1809	103368	1611	91746	1609	91698
wap08a	1870	104176	1628	91140	1627	91122
will199GPIA	701	6772	701	6772	660	5836
zeroin.i.1	211	4100	182	2131		
zeroin.i.2	211	3541	188	2187		
zeroin.i.3	206	3540	183	2186		

Table 3: Dimensions of graphs induced by reversible clique partitions obtained from DIMACS instances. (Continued.)

Table 4: The dimensions of test instances: numbers of events, occupancy measured as the number of events divided by the number of available timeplace slots, and dimensions of the constraint matrices produced by formulations of Udine Course Timetabling (variables \times constraints, non-zeros in constaint matrix).

Instance	Ev.	Occ.	Standard	(Non-zero)	New	(Non-zero)
rand01	100	70%	15415×3194	469.35k	5398×4176	188.34k
rand02	100	70	15415×3197	508.63k	5398×4179	188.38k
rand03	100	70	15415×3197	522.44k	5398×4179	199.47k
rand04	200	70	60835×6447	$2.03 \mathrm{M}$	21002×8444	794.63k
rand05	200	70	60830×6416	$1.94 \mathrm{M}$	20696×8381	754.97k
rand06	200	70	60830×6417	2.16M	20696×8382	814.10k
rand07	300	70	136270×9799	$4.29 \mathrm{M}$	48174×12907	1.76M
rand08	300	70	136260×9729	$4.19 \mathrm{M}$	47262×12773	$1.69 \mathrm{M}$
rand09	300	70	136255×9698	4.46M	46806×12710	1.74M
rand11	100	80	12935×3296	356.88k	5097×4406	159.66k
rand12	100	80	12925×3233	380.59k	4835×4279	160.43k
rand13	200	80	50835×6402	$1.71 \mathrm{M}$	17652×8399	664.51k
rand14	200	80	50840×6427	1.56M	17908×8456	623.57k
rand15	200	80	50830×6371	$1.49 \mathrm{M}$	17396×8336	606.71k
rand16	300	80	113755×9627	$3.92 \mathrm{M}$	39231×12639	1.49M
rand17	300	80	113770×9726	$3.64 \mathrm{M}$	40374×12834	1.48M
rand18	300	80	113760×9650	3.66 M	39612×12694	1.46M
udine1	207	86	50350×4297	963.38k	11756×5393	280.62k
udine2	223	93	54440×5626	1.30M	13452×6889	378.48k
udine3	252	97	66940×7883	2.20M	16036×9252	579.15k
udine4	250	100	64200×12060	3.70M	15505×13678	915.37k

Table 5: The performance of the standard and the proposed (New) formulation of vertex colouring, measured in run times of CPLEX and numbers of iterations performed with no built-in symmetry breaking (-0). The last column lists ratios of CPLEX run times.

Instance	Std-0	(Its.)	New-0	(Its.)	$\frac{\text{Std-0}}{\text{New-0}}$
rand01	2.85s	1635	0.90s	931	3.16
rand02	2.99s	1666	0.94s	1106	3.18
rand03	9.92s	5792	1.05s	1045	9.45
rand04	99.48s	26317	5.18s	2802	19.20
rand05	73.72s	19802	33.49s	17467	2.20
rand06	83.78s	22537	40.35s	19836	2.08
rand07	216.08s	35821	86.44s	25541	2.50
rand08	59.70s	10760	43.45s	13342	1.37
rand09	127.19s	22155	98.32s	25782	1.29
rand11	3.80s	1761	1.51s	1194	2.52
rand12	4.55s	2005	2.31s	1377	1.97
rand13	95.67s	22851	47.94s	18957	2.00
rand14	45.25s	10544	6.64s	2629	6.81
rand15	30.77s	6799	6.89s	2685	4.47
rand16	114.32s	11603	275.44s	51518	0.42
rand17	251.15s	33185	144.93s	36949	1.73
rand18	160.25s	21686	138.04s	34461	1.16
udine1	23.23s	8082	4.45s	3370	5.22
udine2	14.51s	4749	10.04s	4826	1.45
udine3	83.41s	16807	17.25s	11698	4.84
udine4	144.49s	30655	$145.99 \mathrm{s}$	30655	0.99

Table 6: The performance of two formulations of Udine Course Timetabling, differing only in the formulation of the underlying graph colouring component: run times of CPLEX or gaps remaining after 1 hour of solving and numbers of iterations performed with no built-in symmetry breaking (-0). The last column lists ratios of CPLEX run times, where optimality was reached within 1 hour using both formulations.

Instance	Std-0	(Its.)	New-0	(Its.)	$\frac{\text{Std-0}}{\text{New-0}}$
rand01	$385.59 \mathrm{s}$	180854	84.42s	43737	4.57
rand02	290.09s	71537	72.42s	34296	4.01
rand03	443.95s	148961	59.99s	23310	7.40
rand04	gap 0.24%	419910	1242.50s	210104	
rand05	gap 4.15%	360868	1194.71s	250148	
rand06	${\rm gap}\ 8.33\%$	299998	1257.72s	247075	
rand07	gap 89.71%	234087	gap 90.11%	242978	
rand08	gap 99.85%	237243	gap 99.90%	312158	
rand09	gap 93.97%	199619	gap 95.44%	263820	
rand10	285.91s	66842	70.17s	27416	4.07
rand11	211.71s	68244	61.32s	31738	3.45
rand12	337.31s	129788	84.16s	48401	4.01
rand13	gap 0.24%	431148	884.60s	175513	
rand14	gap 6.47%	322073	1356.97s	320129	
rand15	gap 1.74%	303518	1166.50s	280722	
rand16	gap 66.44%	175766	gap 67.19%	417706	
rand17	gap 94.15%	239576	gap 94.06%	293519	
rand18	gap 90.57%	251822	gap 49.34%	345817	
udine1	1175.40s	166539	237.12s	104221	4.96
udine2	gap 100.00\%	639068	gap 100.00\%	3318838	
udine3	gap 99.31%	367505	gap 59.59\%	2000062	
udine4	gap 99.69%	220364	gap infinite	962856	

Table 7: The performance of two formulations of Udine Course Timetabling, differing only in the formulation of the underlying graph colouring component, and effects of disabling (+0) the built-in symmetry breaking in CPLEX, or setting it to very aggressive (+3): run times of CPLEX or gaps remaining after 1 hour of solving.

Instance	Std+0	New+0	Std+3	New+3	$\frac{\text{Std}+3}{\text{New-0}}$
rand01	$385.59 \mathrm{s}$	84.42s	165.52s	76.45s	1.96
rand02	290.09s	72.42s	343.33s	65.51s	4.74
rand03	443.95s	59.99s	298.52s	72.06s	4.98
rand04	gap 0.24%	1242.50s	gap 0.24%	1356.63s	
rand05	gap 4.15%	1194.71s	gap 4.15%	1107.12s	
rand06	${\rm gap}\ 8.33\%$	1257.72s	${\rm gap}\ 8.33\%$	1162.52s	
rand07	gap 89.71%	gap 90.11%	gap 89.71%	gap 90.11%	
rand08	gap 99.85%	gap 99.90%	gap 99.85%	gap 99.90%	
rand09	gap 93.97%	gap 95.44%	gap 93.97%	gap 95.44%	
rand10	285.91s	70.17s	321.51s	81.12s	4.58
rand11	211.71s	61.32s	207.41s	56.79s	3.38
rand12	337.31s	84.16s	253.75s	84.64s	3.02
rand13	gap 0.24%	884.60s	gap 1.85%	795.50s	
rand14	gap 6.47%	$1356.97 \mathrm{s}$	gap 6.47%	1197.39s	
rand15	gap 1.74%	1166.50s	gap 30.43%	1051.74s	
rand16	gap 66.44%	gap 67.19%	gap 66.44%	gap 67.19%	
rand17	gap 94.15%	gap 94.06%	gap 94.15%	gap 94.06%	
rand18	gap 90.57%	gap 49.34%	gap 90.57%	gap 92.25%	
udine1	1175.40s	237.12s	1247.33s	142.84s	5.26
udine2	gap 100.00\%	gap 100.00\%	gap 100.00\%	gap 100.00\%	
udine3	gap 99.31%	gap 59.59%	gap 99.33%	gap 70.04%	
udine4	gap 99.69%	gap infinite	gap infinite	gap infinite	