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A Multiple Search Operator Heuristic for the Max-k-cut Problem

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Abstract The max-k-cut problem is to partition the vertices of a weighted graph $G = (V, E)$ into $k \geq 2$ disjoint subsets such that the weight sum of the edges crossing the different subsets is maximized. The problem is referred as the max-cut problem when $k = 2$. In this work, we present a multiple operator heuristic (MOH) for the general max-k-cut problem. MOH employs five distinct search operators organized into three search phases to effectively explore the search space. Experiments on two sets of 91 well-known benchmark instances show that the proposed algorithm is highly effective on the max-k-cut problem and improves the current best known results (lower bounds) of most of the tested instances. For the popular special case $k = 2$ (i.e., the max-cut problem), MOH also performs remarkably well by discovering 6 improved best known results. We provide additional studies to shed light on the alternative combinations of the employed search operators.

Keywords Max-k-cut and max-cut · Graph partition · Multiple search strategies · Tabu list · Heuristics

1 Introduction

Let $G = (V, E)$ be an undirected graph with vertex set $V = \{1, \dots, n\}$ and edge set $E \subset V \times V$, each edge $(i, j) \in E$ being associated a weight $w_{ij} \in Z$. Given $k \in [2, n]$, the max-k-cut problem is to partition the vertex set V into k (k is given) disjoint subsets $\{S_1, S_2, \dots, S_k\}$, (i.e., $\overset{k}{\underset{i=1}{\cup}} S_i = V, S_i \neq \emptyset, S_i \cap S_j =$

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$\emptyset, \forall i \neq j$), such that the sum of weights of the edges from E whose endpoints belong to different subsets is maximized, i.e.,

$$\max \sum_{1 \leq p < q \leq k} \sum_{i \in S_p, j \in S_q} w_{ij}. \tag{1}$$

Particularly, when the number of partitions equals 2 (i.e., $k = 2$), the problem is referred as the max-cut problem. The max-k-cut is equivalent to the minimum k-partition (MkP) problem which aims to partition the vertex set of a graph into k disjoint subsets so as to minimize the total weight of the edges joining vertices in the same partition [13].

The max-k-cut problem is a classical NP-hard problem in combinatorial optimization and can not be solved exactly in polynomial time [16]. Moreover, when $k = 2$, the max-cut problem is one of the Karp's 21 NP-complete problems [17] which has been the subject of many studies in the literature.

In recent decades, the max-k-cut problem has attracted increasing attention for its applicability to numerous important applications in the area of data mining [9], VLSI layout design [2,6,7,8,25], frequency planning [10], sports team scheduling [24], and statistical physics [19] among others.

Given its theoretical significance and large application potential, a number of solution procedures for solving the max-k-cut problem (or its equivalent MkP) have been reported in the literature. In [13], the authors provide a review of several exact algorithms which are based on branch-and-cut and semidefinite programming approaches. But due to the high computational complexity of the problem, only instances of reduced size (i.e., $|V| < 100$) can be solved by these exact methods in a reasonable computing time.

For large instances, heuristic and metaheuristic methods are commonly used to find "good-enough" sub-optimal solutions. In particular, for the very popular max-cut problem, many heuristic algorithms have been proposed, including simulated annealing and tabu search [1], breakout local search [3], projected gradient approach [4], discrete dynamic convexized method[20], rank-2 relaxation heuristic [5], variable neighborhood search [11], greedy heuristics [15], scatter search [23], global equilibrium search [27] and its parallel version [26], memetic search [29,21], and unconstrained binary quadratic optimization [28]. Compared with max-cut, there are much fewer heuristics for the general max-k-cut problem or its equivalent MkP. Among the rare existing studies, we mention the very recent discrete dynamic convexized method of [31], which formulates the max-k-cut problem as an explicit mathematical model and uses an auxiliary function based local search to find satisfactory results.

In this paper, we partially fill the gap by presenting a new and effective heuristic algorithm for the general max-k-cut problem. The main originality of the proposed algorithm is its multi-phased multi-strategy approach which relies on five distinct local search operators for solution transformations. These operators are organized into three different search phases (descent-based improvement, diversified improvement, perturbation) to ensure an effective examination of the search space. The basic idea of our approach is as follows. The descent-based improvement procedure aims to locate a good local optimum from an initiating solution. This is achieved with two dedicated intensification operators. Then the diversified improvement phase discovers promising areas around the obtained local optimum by applying two additional operators. Once an improved solution is found, the search switches back to the descent-based improvement phase to make an intensive exploitation of the regional area. If the search is trapped in a deep local optimum, the perturbation phase applies a random search operator to definitively lead the search to a distant region from which a new round of the three-phased search procedure starts. This process is repeated until a stop condition is met.

We assess the performance of the proposed algorithm on two sets of well-known benchmarks with a total of 91 instances which are commonly used to test max-k-cut and max-cut algorithms in the literature. Computational results show that the proposed algorithm competes very favorably with respect to the existing max-k-cut heuristics, by improving the current best known results on most instances. Moreover, when the algorithm is applied to the very popular max-cut problem with $k = 2$, the results yielded by our algorithm remain highly competitive compared with the most effective and dedicated max-cut algorithms. In particular, for 6 (large) instances, our algorithm manages to improve the current best known solutions reported by any existing specific max-cut algorithms of the literature.

The rest of the paper is organized as follows. In Section 2, the proposed algorithm is fully presented. Section 3 provides computational results and comparisons with other state-of-the-art algorithms in the literature. Section 4 is dedicated to a analysis of several essential parts of the proposed algorithm. Concluding remarks are given in Section 5.

2 Multiple search operator heuristic for max-k-cut

2.1 General working scheme

The proposed multiple operator heuristic algorithm (MOH) for the general max-k-cut problem is described in Algorithm 1 whose components are explained in the following subsections. The algorithm explores the search space (Section 2.2) by alternately applying five distinct search operators (O_1 to O_5) to make transitions from the current solution to a neighboring solution (Section 2.4). Basically, from an initial solution, the algorithm makes, with two operators (O_1 and O_2), a descent local search to reach a local optimum I (Alg. 1, lines $11 - 21$, descent-based improvement phase, Section 2.6). Then the algorithm continues to the diversified improvement phase (Alg. 1, lines $30 - 40$, Section 2.7) which applies two other operators (O_3 and O_4) to locate new promising regions around the local optimum I. This second phase ends each time a better solution than the current local optimum I is discovered or when a maximum number of diversified moves ω is reached. In both cases, the search returns to the descent-based improvement phase with the best solution

Algorithm 1 General procedure for the max-k-cut problem

1: **Require**: Graph $G = (V, E)$, number of partitions k, max number ω of diversified moves, max number ξ of consecutive non-improvement rounds of the descent improvement and diversified improvement phases before the perturbation phase, probability ρ for applying operator O_3, γ the perturbation strength.
2: **Ensure**: the best solution I_{best} found so far
3: $I \leftarrow$ Generate_initial_solution(V, k) ▷ I is a partition of V into k subsets
4: $I_{best} \leftarrow I$ ▷ I_{best} Records the best solution found so far
5: $f_{lo} \leftarrow f(I)$ ▷ f_{lo} Records the objective value of the latest local optimum reached by $O_1 \cup O_2$
6: $f_{best} \leftarrow f(I)$ ▷ f_{best} Records the best objective value found so far
7: $c_{non\_impv} \leftarrow 0$ ▷ Counter of consecutive non-improvement rounds of descent and diversified search
8: $Iter \leftarrow 0$ ▷ Iteration counter
9: **while** stop condition not satisfied **do**
10:     /* lines 11 to 21: Descent-based improvement phase by applying O_1 and O_2, see Section 2.4*/
11:     **repeat**
12:         **while** $f(I \oplus O_1) > f(I)$ **do** ▷ Descent Phase by applying operator O_1
13:             $I \leftarrow I \oplus O_1$ ▷ Perform the move defined by O_1
14:             Update Δ ▷ Δ is the bucket structure recording move gains for vertices, see Section 2.5
15:             $Iter \leftarrow Iter + 1$
16:         **end while**
17:         **if** $f(I \oplus O_2) > f(I)$ **then** ▷ Descent Phase by applying operator O_2
18:             $I \leftarrow I \oplus O_2$
19:             Update Δ; $Iter \leftarrow Iter + 1$;
20:         **end if**
21:     **until** I can not be improved by operator O_1 and O_2
22:     $f_{lo} \leftarrow f(I)$
23:     **if** $f(I) > f_{best}$ **then**
24:         $f_{best} \leftarrow f(I)$; $I_{best} \leftarrow I$ ▷ Update the best solution found so far
25:         $c_{non\_impv} \leftarrow 0$ ▷ Reset counter c_{non_impv}
26:     **else**
27:         $c_{non\_impv} \leftarrow c_{non\_impv} + 1$
28:     **end if**
29:     /* lines 30 to 40: Diversified improv. phase by applying O_3 and O_4 at most ω times, see Section 2.4 */
30:     $c_{div} \leftarrow 0$ ▷ Counter c_{div} records number of diversified moves
31:     **repeat**
32:         **if** $Random(0, 1) < \rho$ **then** ▷ Random(0,1) returns a random real number between 0 to 1
33:             $I \leftarrow I \oplus O_3$
34:         **else**
35:             $I \leftarrow I \oplus O_4$
36:         **end if**
37:         Update H $(H, Iter, \lambda)$ ▷ Update tabu list H where λ is the tabu tenure, see Section 2.4
38:         Update Δ
39:         $Iter \leftarrow Iter + 1$; $c_{div} \leftarrow c_{div} + 1$
40:     **until** $c_{div} > \omega$ or $f(I) > f_{lo}$
41:     /* Perturbation phase by applying O_5 if f_{best} not improved for ξ rounds of phases 1-2, see Sect. 2.8 */
42:     **if** $c_{non\_impv} > \xi$ **then**
43:         $I \leftarrow I \oplus O_5$ ▷ Apply random perturbation γ times, see Section 2.8
44:         $c_{non\_impv} \leftarrow 0$
45:     **end if**
46: **end while**

found as its new starting point. If no improvement is obtained in ξ descent-based improvement and diversified improvement phases, the search is judged to be trapped in a deep local optimum. To escape this deep local optimum and jump to an unexplored region, the search turns into a perturbation-based diversification phase (Alg. 1, lines $42-45$), which uses a random operator (O_5) to strongly transform the current solution (Section 2.8). The perturbed solution serves then as the new starting solution of the next round of the descent-based improvement phase. This process is iterated until the stop criterion is met.

2.2 Search space and evaluation solution

Recall that the goal of max-k-cut is to partition the vertex set V into k subsets such that the sum of weights of the edges between the different subsets is maximized. As such, we define the search space Ω explored by our algorithm as the set of all possible partitions of V into k disjoint subsets, $\Omega = \{\{S_1, S_2, \ldots, S_k\} : \bigcup_{i=1}^{k} S_i = V, S_i \cap S_j = \emptyset, S_i \subset V, \forall i \neq j\}$, where each candidate solution is called a k-cut.

For a given partition or k-cut $I = \{S_1, S_2, \ldots, S_k\} \in \Omega$, its objective value $f(I)$ is the sum of weights of the edges connecting two different subsets:

$$f(I) = \sum_{1 \leq p < q \leq k} \sum_{i \in S_p, j \in S_q} w_{ij}. \tag{2}$$

Then, for two candidate solutions $I' \in \Omega$ and $I'' \in \Omega$, I' is better than I'' if and only if $f(I') > f(I'')$. The goal of our algorithm is to find a solution $I_{best} \in \Omega$ with $f(I_{best})$ as large as possible.

2.3 Initial solution

The MOH algorithm needs an initial solution to start its search. Generally, the initial solution can be provided by any means. In our case, we adopt a randomized two step procedure. First, from k empty subsets $S_i = \emptyset, \forall i \in \{1, \ldots, k\}$, we assign each vertex $v \in V$ to a random subset $S_i \in \{S_1, S_2, \ldots, S_k\}$. Then if some subsets are still empty, we repetitively move a vertex from its current subset to an empty subset until no empty subset exists.

2.4 Move operations and search operators

Our MOH algorithm iteratively transforms the incumbent solution to a neighboring solution by applying some *move* operations. Typically, a move operation (or simply a move) changes slightly the solution, e.g., by transferring a vertex to a new subset. Formally, let I be the incumbent solution and let mv be a move, we use $I' \leftarrow I \oplus mv$ to denote the neighboring solution I' obtained by applying mv to I.

Associated to a move operation mv, we define the notion of *move gain* Δ_{mv}, which indicates the objective change between the incumbent solution and the neighboring solution obtained by applying the move, i.e.,

$$\Delta_{mv} = f(I') - f(I) \tag{3}$$

where f is the optimization objective (see Formula (2)).

In order to efficiently evaluate the move gain of a move, we develop dedicated techniques which are described in Section 2.5. In this work, we employ two basic move operations: the *'single-transfer move'* and the *'double-transfer move'*. These two move operations form the basis of our five search operators.

– Single-transfer move (st): Given a k-cut $I = \{S_1, S_2, \ldots, S_k\}$, a vertex $v \in S_p$ and a target subset S_q with $p, q \in \{1, \ldots, k\}, p \neq q$, the 'single-transfer move' displaces a single vertex $v \in S_p$ from its current subset S_p to the target subset $S_q \neq S_p$. We denote this move by $st(v, S_p, S_q)$ or $v \rightarrow S_q$.
– Double-transfer move (dt): Given a k-cut $I = \{S_1, S_2, \ldots, S_k\}$, the 'double-transfer move' displaces vertex u from its subset S_{cu} to a target subset $S_{tu} \neq S_{cu}$, and displaces vertex v from its current subset S_{cv} to a target subset $S_{tv} \neq S_{cv}$. We denote this move by $dt(u, S_{cu}, S_{tu}; v, S_{cv}, S_{tv})$ or $dt(u, v)$, or still dt.

From these two basic move operations, we define five distinct *search operators* $O_1 - O_5$ which indicate precisely how these two basic move operations are applied to transform an incumbent solution to a new solution. After an application of any of these search operators, the move gains of the impacted moves are updated according to the dedicated techniques explained in Section 2.5.

– **The $\mathbf{O_1}$ search operator** applies the single-transfer move operation. Precisely, O_1 selects among the $(k-1)n$ single-transfer moves a best move $v \rightarrow S_q$ such that the induced move gain $\Delta_{(v \rightarrow S_q)}$ is maximum. If there are more than one such moves, one of them is selected at random. Since there are $(k-1)n$ candidate single-transfer moves from a given solution, the time complexity of O_1 is bounded by $O(kn)$. The proposed MOH algorithm employs this search operator as its main intensification operator which is complemented by the O_2 search operator to locate good local optima.

– **The $\mathbf{O_2}$ search operator** is based on the double transfer move operation and selects a best dt move with the largest move gain Δ_{dt}. If there are more than one such moves, one of them is selected at random.
Let $dt(u, S_{cu}, S_{tu}; v, S_{cv}, S_{tv})$ ($S_{cu} \neq S_{tu}$, $S_{cv} \neq S_{tv}$) be a double-transfer move, then the move gain Δ_{dt} of this double transfer move can be calculated by a combination of the move gains of its two underlying single-transfer moves ($\Delta_{u \rightarrow S_{tu}}$ and $\Delta_{v \rightarrow S_{tv}}$) as follows:

$$\Delta_{dt(u,v)} = \Delta_{u \rightarrow S_{tu}} + \Delta_{v \rightarrow S_{tv}} + \psi \omega_{uv} \tag{4}$$

where ω_{uv} is the weight of edge $e(u, v) \in E$ and ψ is a coefficient which is determined as follows:

$$\psi = \begin{cases} -2, & \text{if } S_{cu} = S_{cv}, S_{tu} = S_{tv} \\ -1, & \text{if } S_{cu} = S_{cv}, S_{tu} \neq S_{tv} \\ -1, & \text{if } S_{cu} \neq S_{cv}, S_{tu} = S_{tv} \\ 1, & \text{if } S_{cu} \neq S_{cv}, S_{tu} = S_{cv}, S_{tv} \neq S_{cu} \\ 1, & \text{if } S_{cu} \neq S_{cv}, S_{tu} \neq S_{cv}, S_{tv} = S_{cu} \\ 2, & \text{if } S_{cu} \neq S_{cv}, S_{tu} = S_{cv}, S_{tv} = S_{cu} \\ 0, & \text{if } S_{cu} \neq S_{cv}, S_{tu} \neq S_{cv}, S_{tv} \neq S_{cu}, S_{tu} \neq S_{tv} \end{cases} \tag{5}$$

It is clear that for a given incumbent solution, there are $(k-1)^2 n^2$ candidate double-transfer moves denoted as set DT. Seeking directly the move with the maximum Δ_{dt} among all these possible moves would just be too computationally expensive. In order mitigate this problem, we devise a strategy to accelerate the move evaluation process.
From Formula (4), one observes that among all the vertices in V, only the vertices verifying the condition $\omega_{uv} \neq 0$ and $\Delta_{dt(u,v)} > 0$ are of interest for the double-transfer moves. Thus, by examining all the endpoint vertices of edges in E, we shrink the move combinations by building a reduced subset: $DT^R = \{st(v, S_p, S_q) \in DT : \exists st(u, S_{p'}, S_{q'}) \in DT, \omega_{uv} \neq 0, \Delta_{dt(u,v)} > 0\}$. Based on DT^R, the complexity of examining all possible double-transfer moves drops to $O(|E|)$, which is not related to k. In practice, one can examine $\phi|E|$ endpoint vertices in case $|E|$ is too large. We empirically set $\phi = 0.1/d$, where d is the highest degree of the graph.
To summarize, the O_2 search operator selects two st moves $u \to S_{tu}$ and $v \to S_{tv}$ from the reduced set DT^R, such that the combined move gain $\Delta_{dt(u,v)}$ according to Formula (4) is maximum.
Operator O_2 is used when O_1 exhausts its improving moves and provides a first means to help the descent-based improvement phase to escape the current local optimum and discover solutions of increasing quality.

- Like O_1, **the $\mathbf{O_3}$ search operator** selects a best single-transfer move (i.e., with the largest move gain) while considering a tabu list H [14]. The tabu list is a memory which is used to keep track of the performed st moves to avoid revisiting previously encountered solutions. As such, each time a best st move is performed to move a vertex v from its original subset to a target subset, v becomes tabu and is forbidden to move back to its original subset for the next λ iterations (called tabu tenure), which is dynamically determined as follows.
$$\lambda = rand(3, n/10) \tag{6}$$
where $rand(3, n/10)$ denotes a random integer between 3 and $n/10$.
Based on the tabu list, O_3 considers all possible single-transfer moves except those forbidden by the tabu list H and selects the best st move with the largest move main Δ_{st}. Note that a forbidden move is always selected if the move leads to a solution better the best solution found so far. This is called aspiration in tabu search terminology [14].
Operator O_3 is jointly used with operator O_4 to ensure the diversified improvement search.

- Like O_2, **the $\mathbf{O_4}$ search operator** is based on the double-transfer operation. However, O_4 strongly constraints the considered candidate dt moves with respect to two target subsets which are randomly selected. Specifically, O_4 operates as follows. Select two target subsets S_p and S_q at random, and then select two single-transfer moves $u \to S_p$ and $v \to S_q$ such that the combined move gain $\Delta_{dt(u,v)}$ according to Formula (4) is maximum.

Operator O_4 is jointly used with operator O_3 to ensure the diversified improvement search phase.

- **The $\mathbf{O_5}$ search operator** is based on a randomized single-transfer move operation. O_5 first selects a random vertex $v \in V$ and a random target subset S_p, where $v \notin S_p$ and then moves v from its current subset to S_p. This operator is used to change randomly the incumbent solution for the purpose of (strong) diversification when the search is considered to be trapped in a deep local optimum.

Among the five search operators, four of them need to find a single-transfer move with the maximum move gain. To ensure a high computational efficiency of these operators, we develop below a streamlining technique for fast move gain evaluation and move gain updates.

2.5 Bucket sorting for fast move gain evaluation and updating

As mentioned in Section 2.4, to choose an appropriate move, it is crucial for the algorithm to be able to rapidly evaluate a number of candidate moves at each iteration. Since all the search operators basically rely on the single-transfer move operation, we devise a fast incremental evaluation technique based on a bucket data structure and a streamlining calculation to keep and update the move gains after each move application.

Our streamlining technique can be described as follows: let $v \rightarrow S_x$ be the move of transferring vertex v from its current subset S_{cu} to any other subset S_x, $x \in \{1, \dots, k\}, x \neq cu$. Then initially, each move gain can be determined by the following Formula:

$$\Delta_{v \rightarrow S_x} = \sum_{i \in S_{cu}, i \neq v} \omega_{vi} - \sum_{j \in S_x} \omega_{vj}, \ x \in \{1, \dots, k\}, x \neq cu \tag{7}$$

where ω_{vi} and ω_{vj} are respectively the weights of edges $e(v, i)$ and $e(v, j)$.

Suppose the move $v \rightarrow S_{tv}$, i.e., displacing v from its current subset S_{cv} to target subset S_{tv}, is performed, the algorithm needs to update the move gains by performing the following calculation:

1. $\Delta_{v \rightarrow S_{cv}} = -\Delta_{v \rightarrow S_{dv}}$
2. $\Delta_{v \rightarrow S_{dv}} = 0$
3. for each $S_x \neq S_{cv}$, $S_x \neq S_{dv}$,
   $\Delta_{v \rightarrow S_x} = \Delta_{v \rightarrow S_x}$

4. for each $u \in V - \{v\}$, moving $u \in S_{cu}$ to each other subset $S_y \in S - \{S_{cu}\}$,

$$\Delta_{u \to S_y} = \begin{cases} \Delta_{u \to S_y} - 2\omega_{uv}, & \text{if } S_{cu} = S_{cv}, S_y = S_{dv} \\ \Delta_{u \to S_y}, & \text{if } S_{cu} = S_{cv}, S_y \neq S_{dv}, S_y \neq S_{cv} \\ \Delta_{u \to S_y} + 2\omega_{uv}, & \text{if } S_{cu} = S_{dv}, S_y = S_{cv} \\ \Delta_{u \to S_y}, & \text{if } S_{cu} = S_{dv}, S_y \neq S_{cv} \\ \Delta_{u \to S_y} + \omega_{uv}, & \text{if } S_{cu} \neq S_{cv}, S_{cu} \neq S_{dv}, S_y = S_{cv} \\ \Delta_{u \to S_y} - \omega_{uv}, & \text{if } S_{cu} \neq S_{cv}, S_{cu} \neq S_{dv}, S_y = S_{dv} \end{cases} \tag{8}$$

It is easy to see that only the move gains of vertices affected by this move (i.e., the displaced vertex and its adjacent vertices) will be updated, which reduces the computation time significantly.

Normally, the move gains can be stored in an array, with which the time for finding the best move with maximum move gain grows linearly with the number of vertices ($O(n)$). For large problem instances (very large n), the required time can still be quite high. To avoid unnecessary searching for the vertex for the best move, we adopts a bucket structure which keeps vertices ordered by their move gains in decreasing order. The bucket sorting was proposed by Fiduccia and Mattheyes to improve the Kerninghan-Lin heuristic for the network partitioning problem [12]. In this work, we adapt for the first time the idea of bucket sorting for the max-k-cut problem. This is done by using k arrays of buckets, one for each partition subset $S_i \in \{S_1, S_2, \ldots, S_k\}$. In each bucket array i, $1 \leq i \leq k$, the j^{th} entry stores the vertices with the move gain $\Delta_{v \to S_i}$ currently equaling j, where those vertices are maintained by a doubly linked list. To ensure a direct access to the vertex in the doubly linked lists, as suggested in [12], the algorithm also maintains another array for all vertices, where each element points to its corresponding vertex in the doubly linked lists.

Fig. 1 shows an illustrative example of the bucket structure for max-k-cut for $k = 3$. In the graph of the example (Fig. 1, left), there are a total of 8 vertices belonging to the 3 subsets S_1, S_2 and S_3. The bucket structure for this graph is shown Fig. 1 (right). From the graph, one observes that the move gains of moving vertices e, g, h to subset S_1 equal -1, then those three vertices are stored in the entry of B_1 with index of -1. Notice that vertices e, g, h are managed as a doubly linked list. The array AI shown at the bottom of Fig. 1 manages position indexes of for all vertices.

For each array of buckets, finding the best vertex with maximum move gain is equivalent to finding the first non-empty bucket from top of the array and then selecting a vertex in its doubly linked list. If there are more than one vertices in the doubly linked list, a random vertex in this list is selected. To further reduce the searching time, the algorithm memorizes the position of the first non-empty bucket (e.g., $gmax_1, gmax_2, gmax_3$ in Fig. 1).

After each move, the bucket structure is updated by recomputing the move gains (see Formula (8)) of the affected vertices which include the moved vertex and its adjacent vertices, and shifting them to appropriate buckets.

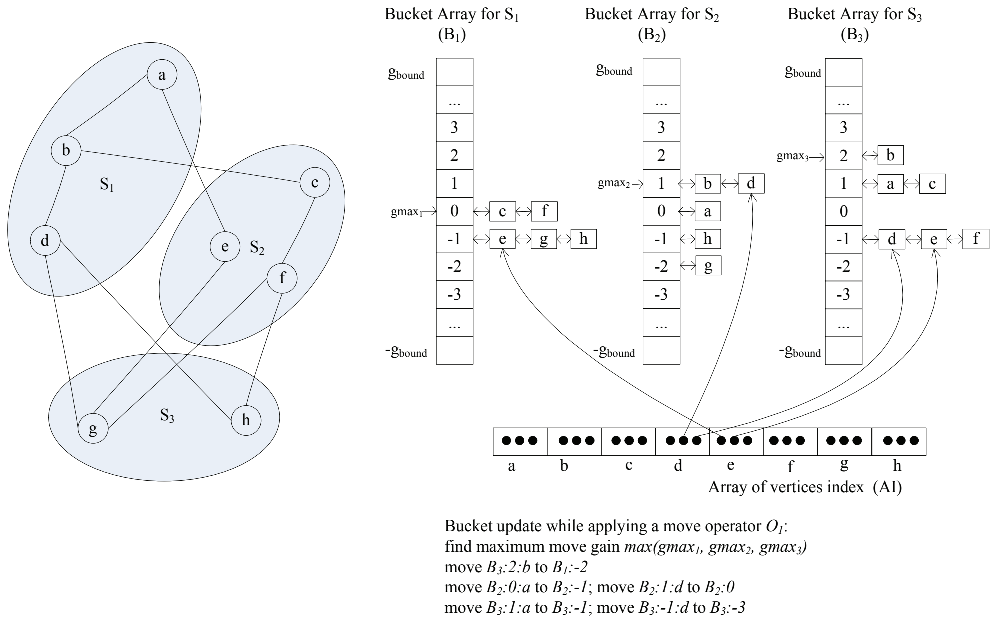


Fig. 1: An example of bucket structure for max-3-cut

For instance, the steps of performing an O_1 move based on Fig. 1 are shown as follows: First, obtain the index of maximum move gain in the bucket arrays by calculating $max(gmax_1, gmax_2, gmax_3)$, which equals $gmax_3$ in this case. Second, select randomly a vertex indexed by $gmax_3$, vertex b in this case. At last, update the positions of the affected vertices a, b, d.

The complexity of each move consists in searching for the vertex with maximum move gain, recomputing the move gain for the affected vertices and updating the bucket structure. The vertex with maximum move gain can be simply obtained in constant time. Recomputing move gain is in linear time relative to the number of affected vertices. The time of updating the bucket structure is also only related to the number of affected vertices. As a result, k has no influence on the performance of the proposed algorithm in terms of computing time. However, it does require a greater amount of memory as k increases.

2.6 Descent-based improvement phase for intensified search

The descent-based local search is used to obtain a local optimum from a given starting solution. As described in Algorithm 1 (lines 11 - 21), we alternatively uses two search operators O_1 and O_2 defined in Section 2.4 to improve a solution until reaching a local optimum. Starting from the given initial solution, the procedure progressively applies O_1 to the incumbent solution. According to the definition of O_1 in Section 2.4, at each step, the procedure examines all possible single-transfer moves and selects a move $v \rightarrow S_q$ with the largest move gain $\Delta_{v \rightarrow S_q}$ subject to $\Delta_{v \rightarrow S_q} > 0$, and then performs that move. After the move, the algorithm updates the bucket structure of move gains according to the technique described in Section 2.5.

When the incumbent solution can not be improved by the O_1 operator (i.e., $\forall v \in V, \forall S_q, \Delta_{v \rightarrow S_q} \leq 0$), the procedure turns to operator O_2 which makes one *best* double-transfer move. If an improved solution is discovered with respect to the local optimum reached by O_1, we are in a new promising area. We switch back to operator O_1 to resume an intensified search. The descent-based improvement phase stops when no better solution can be found with O_1 and O_2. This solution is a local optimum I_{lo} with respect to the single-transfer and double-transfer moves and serves as the input solution of the second search phase which is explained in the next section.

2.7 Diversified improvement phase for discovering promising region

The descent-based local phase described in Section 2.6 alone can not escape the best local optimum I_{lo} it encounters. The diversified improvement search phase is used 1) to jump out of this local optimum and 2) to intensify the search around this local optimum with the hope of discovering a solution better than the input local optimum I_{lo}.

The diversified improvement search procedure alternatively uses two search operators O_3 and O_4 defined in Section 2.4 to perform a move until the stop criterion is met. The application of O_3 or O_4 is determined probabilistically: with probability ρ, O_3 is applied; with $1 - \rho$, O_4 is applied.

When O_3 is selected, the algorithm searches for a best single transfer move $v \rightarrow S_q$ with maximum move gain $\Delta_{v \rightarrow S_q} > 0$ which is not forbidden by the tabu list or verifies the aspiration criterion. Each performed move is then recorded in the tabu list H and is classified tabu for the next λ (calculated by Formula (6)) iterations. The bucket structure is updated to actualize the impacted move gains accordingly. Note that the algorithm only keeps and updates the tabu list during the diversified improvement search phase. Once this second search phase terminates, the tabu list is cleared up.

Similarly, when O_4 is selected, two subsets are selected at random and a best double-transfer dt move with maximum move gain Δ_{dt} is determined from the bucket structure. After the move, the bucket structure is updated to actualize the impacted move gains. It is notated that in case of multiple best double-transfer moves, one of them is chosen at random.

The diversified improvement search procedure terminates once a solution better than the input local optimum I_{lo} is found, or a maximum number ω of diversified moves (O_3 or O_4) is reached. Then the algorithm returns to the descent-based search procedure and use the current solution I as a new starting point for the descent-based search. If the best solution founded so far (f_{best}) can not be improved over a maximum allowed number ξ of consecutive rounds of the descent-based improvement and diversified improvement phases, the perturbation phase (Section 2.8) is invoked to displace the search to a distant region.

2.8 Perturbation phase for strong diversification

The diversified improvement phase makes it possible for the search to escape some local optima. However, the algorithm may still get deeply stuck in a non-promising regional search area. This is the case when the best-found solution f_{best} can not be improved after ξ consecutive rounds of descent and diversified improvement phases. Thus the random perturbation is developed to displace the search into a more distant region.

The basic idea of the perturbation consists in applying the O_5 operator γ times. In other words, this perturbation phase moves γ randomly selected vertices from their original subset to a new and randomly selected subset. Here, γ is used to control the perturbation strength; a large (resp. small) γ value changes strongly (resp. weakly) the incumbent solution. In our case, we adopt $\gamma = 0.1|V|$, i.e., as a percent of the number of vertices. After the perturbation phase, the search returns to the descent-based improvement phase with the perturbed solution as its new starting solution.

3 Experimental results and comparisons

3.1 Benchmark instances

To evaluate the performance of the proposed MOH approach, we carry out computational experiments on two sets of well-known benchmarks with a total of 91 large instances of the literature[1]. The first set (G-set) is composed of 71 graphs with 800 to 20000 vertices and an edge density from 0.02% to 6%. These instances are generated by a machine-independent graph generator including toroidal, planar and random weighted graphs. These instances are available from: http://www.stanford.edu/yyye/yyye/Gset. The second set comes form [5], arising from 30 cubic lattices with randomly generated interaction magnitudes. Since the 10 small instances (with less than 1000 vertices) are very easy for our algorithm, only the results of the 20 larger instances with 1000 to 2744 vertices are reported. These well-known benchmarks are frequently used to evaluate the performance of max-bisection, max-cut and max-k-cut algorithms [3,11,27,26,28,29,30,31].

3.2 Experimental protocol

Our MOH algorithm is programmed in C++ and compiled with GNU g++ (optimization flag "-O2"). Our computer is equipped with a Xeon E5440/2.83GHz

[1] Our best results are available at: `http://www.info.univ-angers.fr/pub/hao/maxkcut/MOHResults.zip`.

CPU with 2GB RAM. When testing the DIMACS machine benchmark[2], our machine requires 0.43, 2.62 and 9.85 CPU time in seconds respectively for graphs r300.5, r400.5, and r500.5 compiled with g++ -O2.

3.3 Parameters

The proposed algorithm requires several parameters: tabu tenure λ, max allowed number ω of consecutive TS moves, max allowed number ξ of consecutive rounds of descent improvement and diversified improvement phases, probability ρ for selecting tabu-based move operator O_3, and number γ of perturbation moves. The parameter values were determined by performing a preliminary experiment on a selection of 10 representative and challenging instances from the G-set benchmark: G22, G23, G25, G29, G33, G35, G36, G37, G38, G40. For each parameter, we tested a range of different values, while keeping the rest of the parameters to their default values. To report our computational results, we adopt the set of parameter values ($\lambda = rand(3, |V|/10), \omega = 500, \xi = 1000, \rho = 0.5, \gamma = 0.1|V|$) for all our experiments throughout the paper, though it would be possible to attain better results by further fine-turning the parameters.

Considering the stochastic nature of our MOH algorithm, each instance is independently solved multiple times: 20 times for max-cut ($k = 2$), 10 times for max-k-cut ($k > 2$). For the purpose of fair comparisons reported in Sections 3.4 and 3.5, we follow the reference algorithms and use a timeout limit as the stop criterion of our MOH algorithm. The timeout limit is set to be 30 minutes for graphs with $|V| < 5000$, 120 minutes for graphs with $10000 \geq |V| \geq 5000$, 240 minutes for graphs with $|V| \geq 10000$.

To fully evaluate the performance of the proposed algorithm, we investigate two comparisons with the state of the art algorithms. First, we focus on the max-k-cut problem ($k = 2, 3, 4, 5$), where we thoroughly compare our algorithm with the recent discrete dynamic convexized algorithm [31] which provides the most competitive results for the general max-k-cut problem in the literature. Secondly, for the special max-cut case ($k = 2$), we compare our algorithm with six most recent max-cut algorithms [3,18,27,28,29,30]. It should be noted that those state of the art max-cut algorithms are specifically designed the particular max-cut problem while our algorithm is developed for the general max-k-cut problem. Normally, the dedicated algorithms are advantaged since they can better explore the particular features of the max-cut problem.

3.4 Comparison with state-of-the-art max-k-cut algorithms

In this section, we present the results attained by our MOH algorithm for the max-k-cut problem. As mentioned above, we compare the proposed algorithm

[2] dfmax:`ftp://dimacs.rutgers.edu/pub/dsj/clique/`

with the discrete dynamic convexized algorithm (DC) [31], which was published very recently. DC was tested on a computer with a 2.11 GHz AMD processor and 1 GB of RAM. According to the Standard Performance Evaluation Cooperation (SPEC) (www.spec.org), this computer is 1.4 times slower than the computer we used for our experiments. Note that DC is the only heuristic algorithm available in the literature, which published computational results for the general max-k-cut problem.

Tables 1 to 4 respectively show the computational results of our MOH algorithm ($k = 2, 3, 4, 5$) on the 2 sets of benchmarks in comparison with those of the DC algorithm. The first two columns of the tables indicate the name and the number of vertices of the graphs. Columns 3 to 7 present the results attained by our algorithm, where f_{best} and f_{avg} show the best objective value and the average objective value over 20 runs, *std* gives the standard deviation and $time(s)$ indicates the average CPU time in seconds required by our algorithm to reach the best objective value f_{best}. Columns 8 to 9 present the results (f_{best}, f_{avg}, $time(s)$) attained by DC. Considering the difference between our computer and the computer used by DC, we normalize the *times* of DC by dividing them by 1.4 according to the SPEC mentioned above. The entries marked as "-" in the tables indicate that the corresponding results are not available. The entries in bold indicate that those results are better than the results provided by the reference DC algorithm. The last column gives the gaps *gap* of the best objective value for each instance between our algorithm and DC. A positive gap implies an improved result.

From Table 1 on max-2-cut, one observes that our algorithm achieves a better f_{best} (best objective value) for 50 out of 74 instances reported by DC, while a better f_{avg} (average objective value) for 71 out of 74 instances. Our algorithm matches the results on other instances and there is no result worse than that obtained by DC. The average standard deviation for all 91 instances is only 2.82, which shows our algorithm is stable and robust.

From Table 2, 3, and 4, which respectively show the comparative results on max-3-cut, max-4-cut and max-5-cut. One observes that our algorithm achieves much higher solution quality on more than 90 percent of 44 instances reported by DC while getting 0 worse result. Moreover, even our *average* results (f_{avg}) are better than the *best* results reported by DC.

Our algorithm is also highly competitive in terms of computing time. It is not fully fair to directly compare the columns *times* for the two algorithms, because the *times* indicate the average time needed for the algorithm to attain its f_{best} value, while the f_{best} values obtained by the two algorithms are different. One observes that for most cases, our algorithm consumes significantly less time while obtaining better results, indicating that our algorithm can reach better solutions with less computing times. This is particularly true when $k > 2$.

We conclude that the proposed algorithm for the general max-k-cut probleme dominates the state of the art reference DC algorithm both in terms of solution quality and computing time.

3.5 Comparison with state-of-the-art max-cut algorithms

Our algorithm is designed for the general max-k-cut problem for $k \geq 2$. The assessment of the last section focuses on the general case. In this section, we further evaluate the performance of the proposed algorithm for the special max-cut problem ($k = 2$).

Recall that max-cut has been largely studied in the literature for a long time and there are many powerful heuristics which are specifically designed for the problem. These state-of-the-art max-cut algorithms constitute thus relevant references for our comparative study. In particular, we adopt the following 6 best performing sequential algorithms published since 2012.

1. Global equilibrium search (GES) [27] - an algorithm sharing ideas similar to simulated annealing and utilizing accumulated information of search space to generate new solutions for the subsequent stages. The reported results of GES were obtained on a PC with a 2.83GHz Intel Core QUAD Q9550 CPU and 8.0GB RAM.
2. Breakout local search (BLS) [3] - a heuristic algorithm integrating a local search and adaptive perturbation strategies. The reported results of BLS were obtained on a PC with 2.83GHz Intel Xeon E5440 CPU and 2GB RAM.
3. Two memetic algorithms respective for the max-cut problem (MACUT) [29] and the max-bisection problem (MAMBP) [30] - integrating a grouping crossover operator and a tabu search procedure. The results reported in the two papers were obtained on a PC with a 2.83GHz Intel Xeon E5440 CPU and 2GB RAM.
4. GRASP-Tabu search algorithm [28] - a method converting the max-cut problem to the UBQP problem and solving it by integrating GRASP and tabu search. The reported results were obtained on a PC with a 2.83GHz Intel Xeon E5440 CPU and 2GB RAM.
5. Tabu search (TS-UBQP) [18] - a tabu search algorithm designed for UBQP. The evaluation of TS-UBQP were performed on a PC with a 2.83GHz Intel Xeon E5440 CPU and 2GB RAM.

One notices that except GES, the other five reference algorithms were run on the same computer platform. Nevertheless, it is still difficult to make a fully fair comparison of the computing time, due to the differences on programming language, compiling options, and termination conditions, etc. Our comparison thus focuses on the best solution achieved by each algorithm. Recall that for our algorithm, the timeout limit is set to be 30 minutes for graphs with $|V| < 5000$, 120 minutes for graphs with $1000 \geq |V| \geq 5000$, 240 minutes for graphs with $|V| \geq 10000$. Our algorithm employs thus the same timeout limits as [29] on the graphs $|V| < 10000$, but for the graphs $|V| \geq 10000$, we use 240 minutes to compare with BLS [3].

Table 5 gives the comparative results on the 91 instances of the two benchmarks. Columns 1 and 2 respectively indicate the instance name and the number of vertices of the graphs. Columns 3 shows the current best known objective value f_{pre} reported by any existing max-cut algorithm in the literature including the latest *parallel* GES algorithm [26]. Columns 4 to 9 give the best objective value obtained by the 6 reference algorithms: GES [27], BLS [3], MACUT [29], TS-UBQP [18], GRASP-TS/PM [28], MAMBP [30]. Note that MAMBP is designed for the max-bisection problem (i.e., balanced max-cut), however it achieves some previous best known max-cut results. The last column 'MOH' recalls the best results of our algorithm from Table 1. The rows denoted by 'Better', 'Equal' and 'Worse' respectively indicate the number of instances for which our algorithm obtains a result of better, equal and worse quality relative to each reference algorithm. The entries are reported in the form of x/y/z, where z denotes the total number of the instances tested by our algorithm, y is the number of the instances tested by a reference algorithm and x indicates the number of instances where our algorithm achieved 'Better', 'Equal' or 'Worse' results. The results in bold mean that our algorithm has improved the best known results. The entries marked as "-" in the table indicate that the results are not available.

From Table 5, one observes that our algorithm is able to improve the current best known results in the literature for 6 instances, and match the best known results for 73 instances. For 12 cases (in italic), even if our results are worse than the current best known results achieved by the latest *parallel* GES algorithm [26], they are still better than the results of any other existing algorithms including TS-UBQP [28] and BLS [3]. Note that the results of the parallel GES algorithm are achieved on a more powerful computing platform (Intel CoreTM i7-3770 CPU @3.40 GHz and 8.0 GB RAM) and with longer time limits (4 parallel processes at the same time and 1 hour for each process).

Such a performance is remarkable given that we are comparing our more general algorithm designed for max-k-cut with the best performing specific max-cut algorithms. The experimental evaluations presented in this section and last section demonstrate that our algorithm not only performs well on the general max-k-cut problem, but also remains highly competitive for the special case of the popular max-cut problem.

Table 1: Comparative results for max-2-cut between the proposed MOH algorithm and DC [31].

Instance	$\|V\|$	MOH				DC			gap
		f_{best}	f_{avg}	std	$time(s)$	f_{best}	f_{avg}	$time(s)$	
G1	800	11624	**11624.00**	0.00	1.5	11624	11617.20	131.7	0
G2	800	11620	**11620.00**	0.00	4.6	11620	11610.00	131.4	0
G3	800	11622	**11622.00**	0.00	1.2	11622	11612.20	130.8	0
G4	800	11646	**11646.00**	0.00	5.2	11646	11633.90	133.8	0
G5	800	11631	**11631.00**	0.00	1.0	11631	11623.20	131.7	0
G6	800	2178	**2178.00**	0.00	3.0	2178	2175.90	132.1	0
G7	800	2006	**2006.00**	0.00	3.0	2006	1997.70	137.6	0
G8	800	2005	**2005.00**	0.00	5.7	2005	2000.00	139.2	0
G9	800	**2054**	**2054.00**	0.00	3.2	2049	2043.50	134.9	5
G10	800	**2000**	**2000.00**	0.00	68.1	1999	1998.40	133.3	1
G11	800	564	**564.00**	0.00	0.2	564	563.80	58.8	0
G12	800	556	**556.00**	0.00	3.5	556	555.40	58.7	0
G13	800	582	**582.00**	0.00	0.9	582	580.00	60.9	0
G14	800	**3064**	**3064.00**	0.00	251.3	3057	3054.30	82.7	7
G15	800	**3050**	**3050.00**	0.00	52.2	3044	3038.00	82.4	6
G16	800	3052	**3052.00**	0.00	93.7	3052	3039.60	81.1	0
G17	800	**3047**	**3047.00**	0.00	129.5	3043	3037.80	81.6	4
G18	800	**992**	**992.00**	0.00	112.6	989	984.00	89.1	3
G19	800	906	**906.00**	0.00	266.9	906	899.90	84.4	0
G20	800	941	**941.00**	0.00	43.7	941	938.20	86.3	0
G21	800	931	**931.00**	0.00	155.3	931	926.00	86.2	0
G22	2000	**13359**	**13357.00**	1.91	352.4	13339	13315.90	683.7	20
G23	2000	**13344**	**13344.00**	0.00	433.8	13323	13298.90	705.2	21
G24	2000	**13337**	**13336.70**	0.46	777.9	13314	13286.00	692.1	23
G25	2000	**13340**	**13335.50**	2.40	442.5	13324	13293.70	694.7	16
G26	2000	**13328**	**13325.50**	2.31	535.1	13313	13282.20	689.6	15
G27	2000	**3341**	**3341.00**	0.00	42.2	3326	3285.40	677.9	15
G28	2000	**3298**	**3298.00**	0.00	707.2	3292	3272.00	680.5	6
G29	2000	**3405**	**3397.85**	5.31	555.2	3390	3357.20	693.4	15
G30	2000	**3413**	**3412.15**	0.36	1427.0	3398	3369.50	676.5	15
G31	2000	**3310**	**3307.85**	0.91	592.6	3295	3273.90	696.4	15
G32	2000	**1410**	**1410.00**	0.00	65.7	1408	1402.70	514.9	2
G33	2000	**1382**	**1381.60**	0.80	504.1	1378	1373.70	508.8	4
G34	2000	**1384**	**1384.00**	0.00	84.2	1378	1376.70	531.5	6
G35	2000	**7687**	**7681.65**	1.59	796.7	7647	7632.20	614.5	40
G36	2000	**7680**	**7673.60**	1.62	1553.2	7625	7618.50	613.1	55
G37	2000	**7691**	**7685.75**	2.26	1195.1	7640	7627.70	623.7	51
G38	2000	**7688**	**7683.60**	2.27	30.6	7641	7614.40	632.9	47
G39	2000	**2408**	**2405.35**	1.85	787.7	2375	2352.50	659.3	33
G40	2000	**2400**	**2397.35**	2.43	472.5	2384	2371.70	656.8	16
G41	2000	**2405**	**2405.00**	0.00	377.3	2377	2357.40	666.8	28
G42	2000	**2481**	**2476.35**	2.01	65.1	2469	2441.30	657.1	12
G43	1000	**6660**	**6660.00**	0.00	1.2	6657	6648.90	156.7	3
G44	1000	6650	**6650.00**	0.00	5.3	6650	6643.70	155.8	0
G45	1000	**6654**	**6654.00**	0.00	6.9	6647	6640.70	155.3	7
G46	1000	**6649**	**6648.90**	0.30	67.3	6647	6637.90	157.0	2
G47	1000	6657	**6657.00**	0.00	43.3	6657	6648.50	157.8	0
G48	3000	6000	6000.00	0.00	0.0	6000	6000.00	420.1	0
G49	3000	6000	6000.00	0.00	0.0	6000	6000.00	440.3	0
G50	3000	5880	5880.00	0.00	532.1	5880	5880.00	552.5	0
G51	1000	**3848**	**3848.00**	0.00	189.2	3842	3831.50	137.6	6
G52	1000	**3851**	**3851.00**	0.00	209.7	3840	3830.50	132.7	11
G53	1000	**3850**	**3849.95**	0.22	299.3	3844	3835.00	136.3	6
G54	1000	**3852**	**3851.10**	0.30	190.4	3831	3824.40	136.0	21
G55	5000	10299	10283.40	7.13	1230.4	-	-	-	-
G56	5000	4016	4007.47	6.49	990.4	-	-	-	-
G57	5000	3494	3486.80	2.45	1528.3	-	-	-	-
G58	5000	19288	19275.40	4.58	1522.3	-	-	-	-
G59	5000	6087	6077.19	7.90	2498.8	-	-	-	-
G60	7000	14190	14173.00	6.98	2945.4	-	-	-	-
G61	7000	5798	5782.67	5.72	6603.3	-	-	-	-
G62	7000	4868	4851.73	7.10	5568.6	-	-	-	-
G63	7000	27033	27019.20	6.72	6492.1	-	-	-	-
G64	7000	8747	8700.87	17.28	4011.1	-	-	-	-
G65	8000	5560	5554.40	2.73	4709.5	-	-	-	-
G66	9000	6360	6354.53	2.37	6061.9	-	-	-	-
G67	10000	6942	6936.53	2.88	14214.3	-	-	-	-
G70	10000	9544	9527.80	9.93	6364.0	-	-	-	-
G72	10000	6998	6991.53	2.67	6586.6	-	-	-	-
G77	14000	9928	9920.00	3.08	9863.6	-	-	-	-
G81	20000	14036	14020.30	8.50	10422.0	-	-	-	-
3dl101000	1000	896	**896.00**	0.00	4.4	896	888.70	113.3	0
3dl102000	1000	900	**900.00**	0.00	6.8	900	898.50	111.5	0
3dl103000	1000	**892**	**892.00**	0.00	147.5	888	884.70	113.0	4
3dl104000	1000	898	**898.00**	0.00	2.7	898	895.00	112.2	0
3dl105000	1000	**886**	**886.00**	0.00	11.7	884	882.80	115.0	2
3dl106000	1000	888	**888.00**	0.00	2.1	888	883.70	114.7	0
3dl107000	1000	**900**	**899.60**	1.00	42.9	898	892.40	114.1	2
3dl108000	1000	**882**	**882.00**	0.00	8.0	880	877.70	120.0	2
3dl109000	1000	**902**	**902.00**	0.00	18.7	902	894.40	113.6	0
3dl1010000	1000	**894**	**894.00**	0.00	6.8	894	893.40	110.9	0
3dl141000	2744	**2446**	**2445.80**	1.00	298.7	2434	2416.40	1039.7	12
3dl142000	2744	**2458**	**2458.00**	0.00	223.3	2444	2431.00	1016.2	14
3dl143000	2744	**2444**	**2440.60**	1.55	376.1	2426	2415.00	1012.3	18
3dl144000	2744	**2450**	**2448.20**	1.55	619.6	2440	2425.30	997.5	10
3dl145000	2744	**2446**	**2445.50**	1.61	475.1	2432	2422.40	999.3	14
3dl146000	2744	**2452**	**2450.50**	1.84	565.9	2438	2430.00	1035.4	14
3dl147000	2744	**2444**	**2442.10**	1.84	172.4	2428	2413.40	1022.7	16
3dl148000	2744	**2448**	**2446.10**	1.73	265.9	2432	2424.40	1030.7	16
3dl149000	2744	**2428**	**2425.20**	1.48	64.5	2418	2403.70	1020.1	10
3dl1410000	2744	**2458**	**2456.80**	2.00	538.2	2438	2429.30	1018.1	20
Better		50/74/91	71/74/91						
Equal		24/74/91	3/74/91						
Worse		0/74/91	0/74/91						

Table 2: Comparative results for max-3-cut between the proposed MOH algorithm and DC [31]

Instance	$\|V\|$	MOH				DC		gap
		f_{best}	f_{avg}	std	$time(s)$	f_{best}	$time(s)$	
G1	800	**15165**	15164.90	0.36	605.4	15127	363.1	38
G2	800	**15172**	15171.20	0.99	539.2	15159	355.4	13
G3	800	**15173**	15173.00	0.00	227.4	15149	361.8	24
G4	800	15184	15181.40	2.46	657.0	-	-	-
G5	800	15193	15193.00	0.00	81.0	-	-	-
G6	800	2632	2631.95	0.22	269.6	-	-	-
G7	800	2409	2408.40	1.07	491.3	-	-	-
G8	800	2428	2427.55	0.67	682.5	-	-	-
G9	800	2478	2475.85	2.52	692.4	-	-	-
G10	800	2407	2406.40	0.86	930.9	-	-	-
G11	800	**669**	667.80	0.75	708.9	660	172.1	9
G12	800	**660**	658.95	0.50	992.9	655	151.8	5
G13	800	**686**	685.40	0.58	586.8	679	164.4	7
G14	800	**4012**	4009.45	1.88	45.7	3984	193.9	28
G15	800	**3984**	3982.40	0.58	282.0	3960	194.2	24
G16	800	**3991**	3986.30	1.87	10.8	3958	194.6	33
G17	800	3983	3981.00	1.05	79.9	-	-	-
G18	800	1207	1205.60	1.56	5.9	-	-	-
G19	800	1081	1078.05	2.38	3.0	-	-	-
G20	800	1122	1115.00	4.05	16.1	-	-	-
G21	800	1109	1106.75	2.30	90.9	-	-	-
G22	2000	**17167**	17157.80	7.62	561.0	17008	1515.3	159
G23	2000	**17168**	17156.70	6.40	888.4	17021	1564.5	147
G24	2000	**17162**	17152.10	4.98	321.4	17037	1592.9	125
G25	2000	17163	17155.20	3.44	1276.8	-	-	-
G26	2000	17154	17146.30	4.61	883.4	-	-	-
G27	2000	4020	4013.80	3.33	576.8	-	-	-
G28	2000	3973	3966.45	5.10	766.1	-	-	-
G29	2000	4106	4097.30	5.40	285.6	-	-	-
G30	2000	4119	4109.90	5.34	1482.9	-	-	-
G31	2000	4003	3999.20	6.69	819.7	-	-	-
G32	2000	**1653**	1651.85	0.73	522.3	1635	910.7	18
G33	2000	**1625**	1622.30	0.95	1233.4	1603	868.0	22
G34	2000	**1607**	1604.00	1.00	1752.1	1589	931.3	18
G35	2000	**10046**	10039.90	2.59	1304.4	9965	1280.9	81
G36	2000	**10039**	10034.40	3.81	1291.6	9945	1301.5	94
G37	2000	**10052**	10047.80	1.96	64.1	9952	1318.0	100
G38	2000	10040	10035.50	3.26	888.4	-	-	-
G39	2000	2903	2890.05	6.75	176.5	-	-	-
G40	2000	2870	2850.65	8.08	1632.8	-	-	-
G41	2000	2887	2862.90	9.77	1729.4	-	-	-
G42	2000	2980	2964.30	5.99	48.3	-	-	-
G43	1000	**8573**	8573.00	0.00	282.2	8510	366.1	63
G44	1000	**8571**	8569.60	2.35	705.5	8526	351.0	45
G45	1000	**8566**	8564.85	1.11	246.5	8515	360.1	51
G46	1000	8568	8564.60	2.01	1061.4	-	-	-
G47	1000	8572	8568.70	2.72	621.5	-	-	-
G48	3000	**6000**	6000.00	0.00	0.3	5998	1850.9	2
G49	3000	6000	6000.00	0.00	0.7	6000	1895.3	0
G50	3000	**6000**	6000.00	0.00	116.5	5998	1819.8	2
G51	1000	5037	5031.35	1.90	944.6	-	-	-
G52	1000	5040	5037.50	0.81	12.8	-	-	-
G53	1000	5039	5038.00	1.05	307.2	-	-	-
G54	1000	5036	5033.55	2.29	880.1	-	-	-
G55	5000	12429	12423.70	2.61	6573.0	-	-	-
G56	5000	4752	4741.90	7.84	1168.4	-	-	-
G57	5000	4083	4079.00	1.55	5457.3	-	-	-
G58	5000	25195	25182.10	8.89	397.3	-	-	-
G59	5000	7262	7246.70	9.20	3575.1	-	-	-
G60	7000	17076	17067.00	4.40	6745.0	-	-	-
G61	7000	6853	6842.10	5.26	3608.6	-	-	-
G62	7000	5685	5681.50	1.43	6250.1	-	-	-
G63	7000	35322	35301.60	10.35	6546.8	-	-	-
G64	7000	10443	10408.80	25.23	1563.7	-	-	-
G65	8000	6490	6485.80	2.04	3077.6	-	-	-
G66	9000	7416	7411.50	2.42	5126.0	-	-	-
G67	10000	8086	8083.50	2.29	1048.1	-	-	-
G70	10000	9999	9999.00	0.00	5.6	-	-	-
G72	10000	8192	8186.70	3.35	6393.0	-	-	-
G77	14000	11578	11568.90	4.01	1899.0	-	-	-
G81	20000	16321	16313.00	4.05	4821.4	-	-	-
3dl101000	1000	**1067**	1066.10	0.54	679.6	1043	238.2	24
3dl102000	1000	**1072**	1071.95	0.22	560.9	1044	242.4	28
3dl103000	1000	**1065**	1063.60	0.66	1303.4	1042	233.4	23
3dl104000	1000	**1071**	1070.30	0.46	526.5	1045	244.0	26
3dl105000	1000	**1064**	1061.90	0.77	71.0	1039	229.2	25
3dl106000	1000	**1063**	1061.80	0.60	882.4	1032	252.7	31
3dl107000	1000	**1075**	1074.40	0.58	467.2	1053	240.0	22
3dl108000	1000	**1071**	1069.95	0.38	178.5	1049	232.5	22
3dl109000	1000	**1079**	1078.20	0.81	510.1	1052	234.6	27
3dl1010000	1000	**1070**	1069.50	0.50	493.8	1044	247.2	26
3dl141000	2744	**2924**	2919.75	2.45	493.0	2845	1805.5	79
3dl142000	2744	**2935**	2929.25	2.53	1103.3	2856	1826.3	79
3dl143000	2744	**2912**	2909.50	1.40	1087.0	2829	1898.8	83
3dl144000	2744	**2924**	2919.90	2.41	458.5	2861	1779.2	63
3dl145000	2744	**2914**	2911.25	1.92	665.5	2839	1796.7	75
3dl146000	2744	**2913**	2909.00	2.00	331.3	2834	1815.3	79
3dl147000	2744	**2913**	2909.30	1.73	1381.3	2834	1824.4	79
3dl148000	2744	**2925**	2919.40	4.05	729.1	2845	1782.1	80
3dl149000	2744	**2906**	2901.50	2.62	125.2	2823	1768.9	83
3dl1410000	2744	**2933**	2927.65	2.22	589.6	2851	1799.4	82
Better		43/44/91						
Equal		1/44/91						
Worse		0/44/91						

Table 3: Comparative results for max-4-cut between the proposed MOH algorithm and DC [31]

Instance	$\|V\|$	MOH				DC		gap
		f_{best}	f_{avg}	std	$time(s)$	f_{best}	$time(s)$	
G1	800	**16803**	16801.00	0.87	522.1	16740	450.2	63
G2	800	**16809**	16808.00	1.12	694.2	16735	455.8	74
G3	800	**16806**	16804.70	1.05	909.6	16752	431.9	54
G4	800	16814	16811.20	1.50	967.7	-	-	-
G5	800	16816	16815.80	0.92	628.0	-	-	-
G6	800	2751	2748.45	1.16	1775.5	-	-	-
G7	800	2515	2513.75	0.92	1128.1	-	-	-
G8	800	2525	2523.35	0.74	1551.5	-	-	-
G9	800	2585	2583.35	1.02	324.7	-	-	-
G10	800	2510	2507.60	1.38	788.1	-	-	-
G11	800	**677**	676.00	0.32	400.7	675	171.3	2
G12	800	**664**	662.25	0.59	814.2	660	180.0	4
G13	800	**690**	689.10	0.45	689.2	685	187.5	5
G14	800	**4440**	4435.35	1.96	1095.5	4402	243.1	38
G15	800	**4406**	4403.40	0.89	1757.7	4373	249.7	33
G16	800	**4415**	4414.05	1.02	957.2	4378	246.1	37
G17	800	4411	4406.45	2.31	3.9	-	-	-
G18	800	1261	1253.90	3.19	5.9	-	-	-
G19	800	1121	1115.35	3.71	6.6	-	-	-
G20	800	1168	1160.95	3.26	7.9	-	-	-
G21	800	1155	1148.25	3.75	1079.7	-	-	-
G22	2000	**18776**	18765.70	5.71	1013.6	18615	1988.3	161
G23	2000	**18777**	18765.80	5.77	1454.7	18612	1941.9	165
G24	2000	**18769**	18763.60	3.79	521.1	18620	1822.8	149
G25	2000	18775	18767.60	4.40	1493.2	-	-	-
G26	2000	18767	18761.20	4.49	635.3	-	-	-
G27	2000	4201	4188.50	4.63	754.0	-	-	-
G28	2000	4150	4138.85	5.97	492.5	-	-	-
G29	2000	4293	4281.65	5.71	1725.1	-	-	-
G30	2000	4305	4296.40	4.14	661.2	-	-	-
G31	2000	4171	4164.40	6.47	1063.9	-	-	-
G32	2000	**1669**	1667.85	1.32	349.0	1659	1140.7	10
G33	2000	**1638**	1634.65	1.32	0.0	1629	1052.4	9
G34	2000	**1616**	1611.70	1.79	1.0	1604	1105.0	12
G35	2000	**11111**	11106.20	2.16	324.7	11007	1890.3	104
G36	2000	**11108**	11101.40	2.92	340.5	10993	1738.6	115
G37	2000	**11117**	11112.50	2.40	693.8	11023	1754.2	94
G38	2000	11108	11101.10	3.16	955.3	-	-	-
G39	2000	3006	2998.70	3.97	22.7	-	-	-
G40	2000	2976	2955.65	9.01	961.3	-	-	-
G41	2000	2983	2970.30	6.91	35.5	-	-	-
G42	2000	3092	3084.05	4.80	285.2	-	-	-
G43	1000	**9376**	9373.95	1.53	1656.1	9306	423.0	70
G44	1000	**9379**	9373.55	2.58	1340.2	9315	430.5	64
G45	1000	**9376**	9375.10	0.95	612.1	9312	463.5	64
G46	1000	9378	9375.35	1.99	639.0	-	-	-
G47	1000	9381	9377.05	2.04	1194.2	-	-	-
G48	3000	6000	6000.00	0.00	0.0	6000	1673.8	0
G49	3000	6000	6000.00	0.00	0.0	6000	1675.6	0
G50	3000	6000	6000.00	0.00	0.0	6000	1678.9	0
G51	1000	5571	5567.65	2.04	143.6	-	-	-
G52	1000	5584	5581.15	1.75	129.9	-	-	-
G53	1000	5574	5571.85	1.47	67.1	-	-	-
G54	1000	5579	5576.25	1.60	13.8	-	-	-
G55	5000	12498	12498.00	0.00	0.1	-	-	-
G56	5000	4931	4917.10	6.49	4190.5	-	-	-
G57	5000	4112	4110.50	1.22	2942.0	-	-	-
G58	5000	27885	27870.90	8.72	4297.1	-	-	-
G59	5000	7539	7515.10	15.09	4782.7	-	-	-
G60	7000	17148	17148.00	0.00	1.4	-	-	-
G61	7000	7110	7104.60	5.12	6440.2	-	-	-
G62	7000	5743	5738.70	2.77	3804.6	-	-	-
G63	7000	39083	39063.50	9.19	6515.7	-	-	-
G64	7000	10814	10797.40	13.29	4493.0	-	-	-
G65	8000	6534	6525.40	4.49	14.8	-	-	-
G66	9000	7474	7467.80	4.31	21.7	-	-	-
G67	10000	8155	8142.50	5.59	29.6	-	-	-
G70	10000	9999	9999.00	0.00	0.2	-	-	-
G72	10000	8264	8254.60	7.39	15.3	-	-	-
G77	14000	11674	11658.90	10.12	63.2	-	-	-
G81	20000	16470	16454.30	8.50	271.4	-	-	-
3dl101000	1000	**1103**	1100.60	1.05	1273.1	1073	304.4	30
3dl102000	1000	**1102**	1100.00	0.95	29.6	1070	351.3	32
3dl103000	1000	**1108**	1106.40	0.95	225.0	1072	341.0	36
3dl104000	1000	**1103**	1101.65	0.92	564.5	1076	323.5	27
3dl105000	1000	**1098**	1096.30	0.84	578.3	1074	334.4	24
3dl106000	1000	**1097**	1095.15	0.92	928.2	1063	358.3	34
3dl107000	1000	**1114**	1112.20	1.10	712.6	1093	308.3	21
3dl108000	1000	**1105**	1103.00	0.77	478.7	1079	276.1	26
3dl109000	1000	**1115**	1113.45	0.92	641.0	1086	271.3	29
3dl1010000	1000	**1109**	1106.10	0.89	1083.6	1088	277.2	21
3dl141000	2744	**3016**	3012.05	1.91	563.0	2893	1990.5	123
3dl142000	2744	**3026**	3019.80	2.19	364.2	2893	2007.3	133
3dl143000	2744	**3006**	3001.70	2.97	367.1	2892	1956.1	114
3dl144000	2744	**3012**	3007.85	2.04	943.5	2897	1980.3	115
3dl145000	2744	**3006**	3001.20	2.17	1146.8	2882	1972.2	124
3dl146000	2744	**3005**	3001.35	1.50	256.6	2888	1948.9	117
3dl147000	2744	**3007**	3001.95	2.50	301.0	2879	1995.7	128
3dl148000	2744	**3018**	3014.50	2.02	1632.9	2883	1982.7	135
3dl149000	2744	**2999**	2993.95	2.78	394.8	2877	2024.5	122
3dl1410000	2744	**3023**	3021.15	1.69	1075.8	2904	2007.4	119
Better		41/44/91						
Equal		3/44/91						
Worse		0/44/91						

Table 4: Comparative results for max-5-cut between the proposed MOH algorithm and DC [31]

Instance	$\|V\|$	MOH				DC		gap
		f_{best}	f_{avg}	std	$time(s)$	f_{best}	$time(s)$	
G1	800	**16803**	16801.00	0.87	522.1	16740	450.2	63
G2	800	**16809**	16808.00	1.12	694.2	16735	455.8	74
G3	800	**16806**	16804.70	1.05	909.6	16752	431.9	54
G4	800	16814	16811.20	1.50	967.7	-	-	-
G5	800	16816	16815.80	0.92	628.0	-	-	-
G6	800	2751	2748.45	1.16	1775.5	-	-	-
G7	800	2515	2513.75	0.92	1128.1	-	-	-
G8	800	2525	2523.35	0.74	1551.5	-	-	-
G9	800	2585	2583.35	1.02	324.7	-	-	-
G10	800	2510	2507.60	1.38	788.1	-	-	-
G11	800	**677**	676.00	0.32	400.7	675	171.3	2
G12	800	**664**	662.25	0.59	814.2	660	180.0	4
G13	800	**690**	689.10	0.45	689.2	685	187.5	5
G14	800	**4440**	4435.35	1.96	1095.5	4402	243.1	38
G15	800	**4406**	4403.40	0.89	1757.7	4373	249.7	33
G16	800	**4415**	4414.05	1.02	957.2	4378	246.1	37
G17	800	4411	4406.45	2.31	3.9	-	-	-
G18	800	1261	1253.90	3.19	5.9	-	-	-
G19	800	1121	1115.35	3.71	6.6	-	-	-
G20	800	1168	1160.95	3.26	7.9	-	-	-
G21	800	1155	1148.25	3.75	1079.7	-	-	-
G22	2000	**18776**	18765.70	5.71	1013.6	18615	1988.3	161
G23	2000	**18777**	18765.80	5.77	1454.7	18612	1941.9	165
G24	2000	**18769**	18763.60	3.79	521.1	18620	1822.8	149
G25	2000	18775	18767.60	4.40	1493.2	-	-	-
G26	2000	18767	18761.20	4.49	635.3	-	-	-
G27	2000	4201	4188.50	4.63	754.0	-	-	-
G28	2000	4150	4138.85	5.97	492.5	-	-	-
G29	2000	4293	4281.65	5.71	1725.1	-	-	-
G30	2000	4305	4296.40	4.14	661.2	-	-	-
G31	2000	4171	4164.40	6.47	1063.9	-	-	-
G32	2000	**1669**	1667.85	1.32	349.0	1659	1140.7	10
G33	2000	**1638**	1634.65	1.32	0.0	1629	1052.4	9
G34	2000	**1616**	1611.70	1.79	1.0	1604	1105.0	12
G35	2000	**11111**	11106.20	2.16	324.7	11007	1890.3	104
G36	2000	**11108**	11101.40	2.92	340.5	10993	1738.6	115
G37	2000	**11117**	11112.50	2.40	693.8	11023	1754.2	94
G38	2000	11108	11101.10	3.16	955.3	-	-	-
G39	2000	3006	2998.70	3.97	22.7	-	-	-
G40	2000	2976	2955.65	9.01	961.3	-	-	-
G41	2000	2983	2970.30	6.91	35.5	-	-	-
G42	2000	3092	3084.05	4.80	285.2	-	-	-
G43	1000	**9376**	9373.95	1.53	1656.1	9306	423.0	70
G44	1000	**9379**	9373.55	2.58	1340.2	9315	430.5	64
G45	1000	**9376**	9375.10	0.95	612.1	9312	463.5	64
G46	1000	9378	9375.35	1.99	639.0	-	-	-
G47	1000	9381	9377.05	2.04	1194.2	-	-	-
G48	3000	6000	6000.00	0.00	0.0	6000	1673.8	0
G49	3000	6000	6000.00	0.00	0.0	6000	1675.6	0
G50	3000	6000	6000.00	0.00	0.0	6000	1678.9	0
G51	1000	5571	5567.65	2.04	143.6	-	-	-
G52	1000	5584	5581.15	1.75	129.9	-	-	-
G53	1000	5574	5571.85	1.47	67.1	-	-	-
G54	1000	5579	5576.25	1.60	13.8	-	-	-
G55	5000	12498	12498.00	0.00	0.1	-	-	-
G56	5000	4931	4917.10	6.49	4190.5	-	-	-
G57	5000	4112	4110.50	1.22	2942.0	-	-	-
G58	5000	27885	27870.90	8.72	4297.1	-	-	-
G59	5000	7539	7515.10	15.09	4782.7	-	-	-
G60	7000	17148	17148.00	0.00	1.4	-	-	-
G61	7000	7110	7104.60	5.12	6440.2	-	-	-
G62	7000	5743	5738.70	2.77	3804.6	-	-	-
G63	7000	39083	39063.50	9.19	6515.7	-	-	-
G64	7000	10814	10797.40	13.29	4493.0	-	-	-
G65	8000	6534	6525.40	4.49	14.8	-	-	-
G66	9000	7474	7467.80	4.31	21.7	-	-	-
G67	10000	8155	8142.50	5.59	29.6	-	-	-
G70	10000	9999	9999.00	0.00	0.2	-	-	-
G72	10000	8264	8254.60	7.39	15.3	-	-	-
G77	14000	11674	11658.90	10.12	63.2	-	-	-
G81	20000	16470	16454.30	8.50	271.4	-	-	-
3dl101000	1000	**1103**	1100.60	1.05	1273.1	1073	304.4	30
3dl102000	1000	**1102**	1100.00	0.95	29.6	1070	351.3	32
3dl103000	1000	**1108**	1106.40	0.95	225.0	1072	341.0	36
3dl104000	1000	**1103**	1101.65	0.92	564.5	1076	323.5	27
3dl105000	1000	**1098**	1096.30	0.84	578.3	1074	334.4	24
3dl106000	1000	**1097**	1095.15	0.92	928.2	1063	358.3	34
3dl107000	1000	**1114**	1112.20	1.10	712.6	1093	308.3	21
3dl108000	1000	**1105**	1103.00	0.77	478.7	1079	276.1	26
3dl109000	1000	**1115**	1113.45	0.92	641.0	1086	271.3	29
3dl1010000	1000	**1109**	1106.10	0.89	1083.6	1088	277.2	21
3dl141000	2744	**3016**	3012.05	1.91	563.0	2893	1990.5	123
3dl142000	2744	**3026**	3019.80	2.19	364.2	2893	2007.3	133
3dl143000	2744	**3006**	3001.70	2.97	367.1	2892	1956.1	114
3dl144000	2744	**3012**	3007.85	2.04	943.5	2897	1980.3	115
3dl145000	2744	**3006**	3001.20	2.17	1146.8	2882	1972.2	124
3dl146000	2744	**3005**	3001.35	1.50	256.6	2888	1948.9	117
3dl147000	2744	**3007**	3001.95	2.50	301.0	2879	1995.7	128
3dl148000	2744	**3018**	3014.50	2.02	1632.9	2883	1982.7	135
3dl149000	2744	**2999**	2993.95	2.78	394.8	2877	2024.5	122
3dl1410000	2744	**3023**	3021.15	1.69	1075.8	2904	2007.4	119
Better		41/44/91						
Equal		3/44/91						
Worse		0/44/91						

Table 5: Comparative results of the proposed MOH algorithm with 6 state of the art max-cut algorithms

Instance	$\|V\|$	f_{pre}	GES [27]	BLS [3]	MACUT [29]	TS-UBQP [18]	TS/PM [28]	MAMBP [30]	MOH
G1	800	11624	11624	11624	11624	11624	11624	11624	11624
G2	800	11620	11620	11620	11620	11620	11620	11617	11620
G3	800	11622	11622	11622	11622	11620	11620	11621	11622
G4	800	11646	11646	11646	-	11646	11646	11646	11646
G5	800	11631	11631	11631	-	11631	11631	11631	11631
G6	800	2178	2178	2178	-	2178	2178	2177	2178
G7	800	2006	2006	2006	-	2006	2006	2002	2006
G8	800	2005	2005	2005	-	2005	2005	2004	2005
G9	800	2054	2054	2054	-	2054	2054	2052	2054
G10	800	2000	2000	2000	-	2000	2000	1998	2000
G11	800	564	564	564	564	564	564	564	564
G12	800	556	556	556	556	556	556	556	556
G13	800	582	582	582	582	580	582	582	582
G14	800	3064	3064	3064	3064	3061	3063	3062	3064
G15	800	3050	3050	3050	3050	3050	3050	3050	3050
G16	800	3052	3052	3052	3052	3052	3052	3052	3052
G17	800	3047	3047	3047	-	3046	3047	3047	3047
G18	800	992	992	992	-	991	992	992	992
G19	800	906	906	906	-	904	906	905	906
G20	800	941	941	941	-	941	941	941	941
G21	800	931	931	931	-	930	931	930	931
G22	2000	13359	13359	13359	13359	13359	13349	13359	13359
G23	2000	13344	13342	13344	13344	13342	13332	13344	13344
G24	2000	13337	13337	13337	13337	13337	13324	13336	13337
G25	2000	13340	13340	13340	-	13332	13326	13340	13340
G26	2000	13328	13328	13328	-	13328	13313	13328	13328
G27	2000	3341	3341	3341	-	3336	3325	3341	3341
G28	2000	3298	3298	3298	-	3295	3287	3298	3298
G29	2000	3405	3405	3405	-	3391	3394	3403	3405
G30	2000	3413	3413	3412	-	3403	3402	3412	3413
G31	2000	3310	3310	3309	-	3288	3299	3309	3310
G32	2000	1410	1410	1410	1410	1406	1406	1410	1410
G33	2000	1382	1382	1382	1382	1378	1374	1382	1382
G34	2000	1384	1384	1384	1384	1378	1376	1384	1384
G35	2000	7686	7686	7684	7686	7678	7661	7686	**7687**
G36	2000	7680	7680	7678	7679	7670	7660	7678	7680
G37	2000	7691	7691	7689	7690	7682	7670	7689	7691
G38	2000	7688	7687	7687	-	7683	7670	7688	7688
G39	2000	2408	2408	2408	-	2397	2397	2408	2408
G40	2000	2400	2400	2400	-	2390	2392	2400	2400
G41	2000	2405	2405	2405	-	2400	2398	2405	2405
G42	2000	2481	2481	2481	-	2469	2474	2481	2481
G43	1000	6660	6660	6660	6660	6660	6660	6659	6660
G44	1000	6650	6650	6650	6650	6639	6649	6650	6650
G45	1000	6654	6654	6654	6654	6652	6654	6654	6654
G46	1000	6649	6649	6649	-	6649	6649	6649	6649
G47	1000	6657	6657	6657	-	6656	6656	6657	6657
G48	3000	6000	6000	6000	6000	6000	6000	6000	6000
G49	3000	6000	6000	6000	6000	6000	6000	6000	6000
G50	3000	5880	5880	5880	5800	5880	5880	5880	5880
G51	1000	3848	3848	3848	-	3847	3847	3847	3848
G52	1000	3851	3851	3851	-	3849	3850	3851	3851
G53	1000	3850	3850	3850	-	3848	3848	3850	3850
G54	1000	3852	3852	3852	-	3851	3850	3851	3852
G55	5000	10299	-	10294	10299	10236	-	10299	10299
G56	5000	4017	-	4012	4016	3934	-	4016	*4016*
G57	5000	3494	-	3492	-	3460	-	3488	3494
G58	5000	19293	-	19263	-	19248	-	19276	*19288*
G59	5000	6086	-	6078	-	6019	-	6085	**6087**
G60	7000	14188	-	14176	14186	14057	-	14186	**14190**
G61	7000	5796	-	5789	-	5680	-	5796	**5798**
G62	7000	4870	-	4868	-	4822	-	4866	*4868*
G63	7000	27045	-	26997	-	26963	-	26754	*27033*
G64	7000	8751	-	8735	-	8610	-	8731	*8747*
G65	8000	5562	-	5558	5550	5518	-	5556	*5560*
G66	9000	6364	-	6360	6352	6304	-	6352	*6360*
G67	10000	6950	-	6940	6934	6894	-	6934	*6942*
G70	10000	9591	-	9541	-	9458	-	9580	*9544*
G72	10000	7006	-	6998	-	6922	-	6990	*6998*
G77	14000	9938	-	9926	-	-	-	9900	*9928*
G81	20000	14048	-	14030	-	-	-	13978	*14036*
3dl101000	1000	896	896	-	-	-	-	-	896
3dl102000	1000	900	900	-	-	-	-	-	900
3dl103000	1000	892	892	-	-	-	-	-	892
3dl104000	1000	898	898	-	-	-	-	-	898
3dl105000	1000	886	886	-	-	-	-	-	886
3dl106000	1000	888	888	-	-	-	-	-	888
3dl107000	1000	900	900	-	-	-	-	-	900
3dl108000	1000	882	882	-	-	-	-	-	882
3dl109000	1000	902	902	-	-	-	-	-	902
3dl1010000	1000	894	894	-	-	-	-	-	894
3dl141000	2744	2446	2446	-	-	-	-	-	2446
3dl142000	2744	2458	2458	-	-	-	-	-	2458
3dl143000	2744	2442	2442	-	-	-	-	-	**2444**
3dl144000	2744	2450	2450	-	-	-	-	-	2450
3dl145000	2744	2446	2446	-	-	-	-	-	2446
3dl146000	2744	2452	2452	-	-	-	-	-	2452
3dl147000	2744	2444	2444	-	-	-	-	-	2444
3dl148000	2744	2448	2448	-	-	-	-	-	2448
3dl149000	2744	2426	2426	-	-	-	-	-	**2428**
3dl1410000	2744	2458	2458	-	-	-	-	-	2458
Better		6/91/91	4/74/91	20/71/91	7/30/91	47/69/91	29/54/91	33/71/91	
Equal		73/91/91	70/74/91	51/71/91	23/30/91	22/69/91	25/54/91	37/71/91	
Worse		12/91/91	0/74/91	0/71/91	0/30/91	0/69/91	0/54/91	1/71/90	

4 Discussion

In this section, we investigate the role of several important ingredients of the proposed algorithm, including the descent improvement search operators O_1 and O_2 and the diversified improvement search operators O_3 and O_4. These studies are based on the same 10 challenging instances selected to determine the parameters (see Section 3.3). Only results for max-cut are presented in this section.

4.1 Impact of the descent improvement search operators

As described in Section 2.6, the proposed algorithm employs operators O_1 and O_2 for its descent improvement phase to obtain local optima. To analyze the impact of these two operators, we implement three variants of our algorithm, the first one using the operator O_1 alone, the second one using the union $O_1 \cup O_2$ such that the descent search procedure always chooses the best move among the O_1 *and* O_2 moves [22], the third one using operator $rand(O_1, O_2)$ where the descent procedure applies randomly and with equal probability O_1 or O_2, while keeping all the other ingredients and parameters fixed as described in Section 3.3. The strategy used by our original algorithm is denoted as $O_1 + O_2$, which is detailed in Section 2.6. Each selected instance is solved 10 times by each of these variants and our original algorithm. The stop criterion is a timeout limit of 30 minutes. The obtained results are presented in Table 6, including the best objective value f_{best}, the average objective value f_{avg} over the 10 independent runs, as well as the CPU times in seconds to reach f_{best}. To evaluate the performance, we calculate the gaps between the best objective values obtained by different strategies and the best objective values by our original algorithm, which is shown in Fig. 2(a). We also show in Fig. 2(b) the box and whisker plots which indicates, for different O_1, O_2 combination strategies, the distribution and the ranges of the obtained results for the 10 tested instances. The results are expressed as the additive inverse of percent deviation of the averages results from the best known objective values obtained by our original algorithm.

From Fig. 2(a), one observes that for the tested instances, other combination strategies obtain fewer best known results compared to the strategy $O_1 + O_2$, and produce large gaps to the best known results on some instances. From Fig. 2(b), we observe a clear difference in the distribution of the results with different strategies. For the results with the strategies of $O_1 + O_2$, the plot indicates a smaller mean value and significantly smaller variation compared to the results obtained by other strategies. We thus conclude that the strategy used by our algorithm ($O_1 + O_2$) performs better than other strategies.

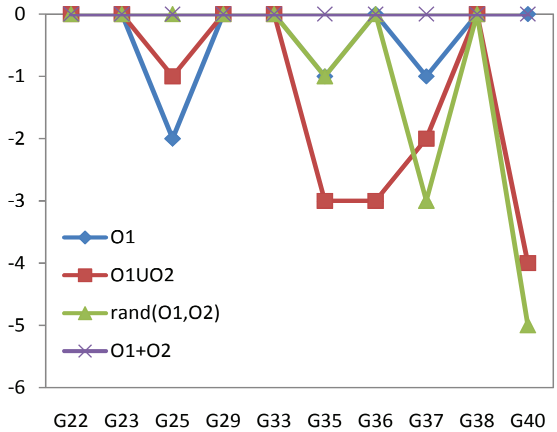
Table 6: Comparative results for max-cut with varying combination strategies of O_1 and O_2

Instance	O_1			$O_1 \cup O_2$		
	f_{best}	f_{avg}	$time(s)$	f_{best}	f_{avg}	$time(s)$
G22	13359	13357.6	381.6	13359	13355.8	357.3
G23	13344	13343.6	473.4	13344	13344	550.9
G25	13338	13334	442.8	13339	13335.8	690.4
G29	3405	3398.22	211.1	3405	3396.4	254.2
G33	1382	1381.4	553.5	1382	1382	716.5
G35	7686	7681.3	755.4	7684	7679.1	449.6
G36	7680	7672	1367.1	7677	7672.5	408.1
G37	7690	7685.5	1039.2	7689	7683.4	1099.0
G38	7688	7684	135.2	7688	7681.2	177.8
G40	2400	2384.7	453.5	2396	2381.6	427.2
Instance	$rand(O_1, O_2)$			$O_1 + O2$		
	f_{best}	f_{avg}	$time(s)$	f_{best}	f_{avg}	$time(s)$
G22	13359	13356	365.3	13359	13357	438.2
G23	13344	13343.9	584.9	13344	13344	302.1
G25	13340	13336.4	408.8	13340	13335.5	451.5
G29	3405	3398.4	403.9	3405	3398.1	569.9
G33	1382	1381.8	585.2	1382	1381.4	667.4
G35	7686	7683.1	628.0	7687	7684.3	968.3
G36	7680	7672	944.8	7680	7675.3	1075.6
G37	7688	7681.7	1078.3	7691	7687.5	1133.2
G38	7688	7680.8	153.6	7688	7685.7	333.0
G40	2395	2388.8	412.4	2400	2385.2	467.1

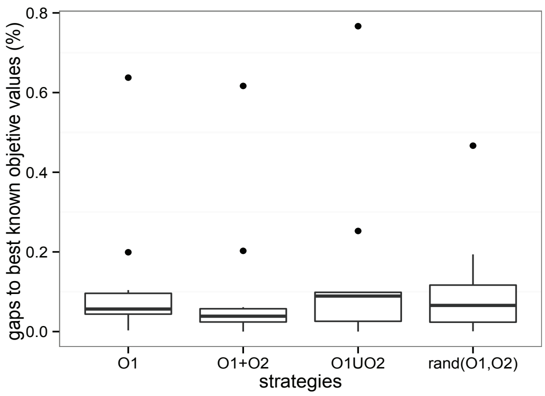
4.2 Impact of the diversified improvement search operators

As described in Section 2.7, the proposed algorithm employs two diversified operator O_3 and O_4 to enhance the search power of the algorithm and make it possible for the search to visit new promising regions. The diversified improvement procedure uses probability ρ to select O_3 or O_4. To analyze the impact of operators O_3 and O_4, we test our algorithm with $\rho = 1$ (using the operator O_3 alone), $\rho = 0.5$ (equal application of O_3 and O_4 used in our original MOH algorithm), $\rho = 0$ (using the operator O_4 alone), while keeping all the other ingredients and parameters fixed as described before. The stop criterion is a timeout limit of 30 minutes. We then independently solve each selected instance 10 times with those different values of ρ. The obtained results are presented in Table 7, including the best objective value f_{best}, the average objective value f_{avg} over the 10 independent runs, as well as the CPU times in seconds to reach f_{best}. To evaluate the performance, we again calculate the gaps between different best objective values shown in Fig. 3(a) and average objective values shown in Fig. 3(b), where the set of values f_{best}, f_{avg}, when $\rho = 0.5$, are set as the reference values.

As Section 4.1, to evaluate the performance, we calculate the gaps between the best objective values obtained with different values of ρ and the best objective values by our original MOH algorithm ($\rho = 0.5$), which is shown in Fig. 3(a). We also show in Fig. 3(b) the box and whisker plots which indicates, for different values of ρ, the distribution and the ranges of the obtained results



(a) $f_{best-strategy} - f_{bestknown}$, gaps to best known objective values



(b) $(f_{bestknown} - f_{avg-strategy})/f_{bestknown} * 100\%$, gaps to best known objective values

Fig. 2: Analysis of the move operators O_1, O_2

for the 10 tested instances. The results are expressed as the additive inverse of percent deviation of the averages results from the best known objective values obtained by our original algorithm.

Fig. 3(a) discloses that using O_3 or O_4 alone obtains fewer best known results than using them jointly and also achieves significantly worse results on some particular instances. From Fig. 3(b), we observes a visible difference in the distribution of the results with different strategies. For the results with the parameter $\rho = 0.5$, the plot indicates a smaller mean value and significantly smaller variation compared to the results obtained by other strategies. We thus

Table 7: Comparative results for max-cut with varying parameter ρ

Instance	$\rho = 1$			$\rho = 0$			$\rho = 0.5$		
	f_{best}	f_{avg}	$time(s)$	f_{best}	f_{avg}	$time(s)$	f_{best}	f_{avg}	$time(s)$
G22	13359	13350.1	352.7	13356	13355.2	440.6	13359	13357	438.2
G23	13344	13344	441.4	13338	13335.6	340.1	13344	13344	302.1
G25	13339	13335.1	426.1	13337	13333.5	412.9	13340	13335.5	451.5
G29	3405	3395.2	614.5	3402	3399.8	593.5	3405	3398.1	569.9
G33	1376	1373.6	519.9	1382	1382	609.2	1382	1381.4	667.7
G35	7686	7680.7	832.1	7680	7678.2	850.8	7687	7684.3	968.3
G36	7676	7669.2	1540.8	7671	7667.6	1304.8	7680	7675.3	1075.6
G37	7690	7681.2	1167.8	7685	7679.6	1053.8	7691	7687.5	1133.2
G38	7688	7681.4	275.1	7685	7679	257.3	7688	7685.7	333.0
G40	2394	2375.3	453.0	2399	2390.5	529.8	2400	2385.2	467.1

conclude that jointly using O_3 and O_4 with $\rho = 0.5$ is the best choice since it produces better results in terms of both best results and average results.

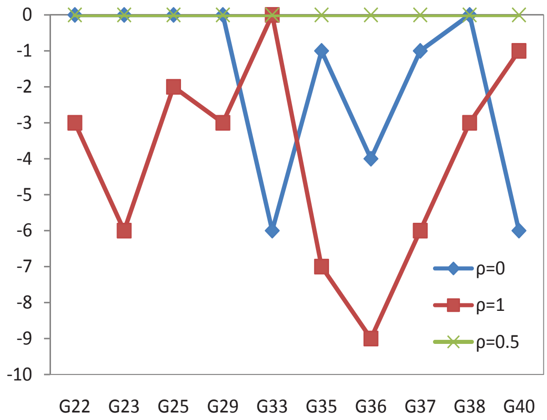
5 Conclusion

Our multiple search operator algorithm (MOH) for the general max-k-cut problem achieves a high level performance by including five distinct search operators which are applied in three search phases. The descent-based improvement phase aims to discover local optima of increasing quality with two intensification-oriented operators. The diversified improvement phase combines two other operators to escape local optima and discover promising new search regions. The perturbation phase is applied as a means of strong diversification to get out of deep local optimum traps. To obtain an efficient implementation of the proposed algorithm, we developed streamlining techniques based on bucket structures.

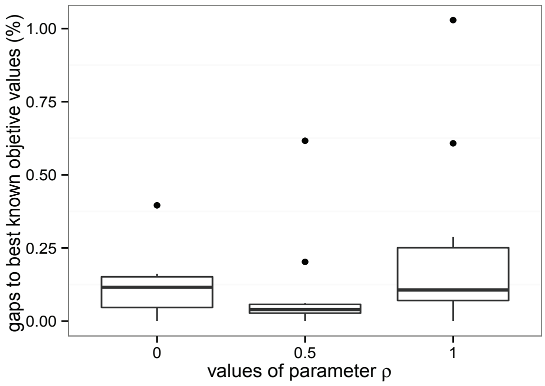
We demonstrated the effectiveness of the MOH algorithm both in terms of solution quality and computation efficiency by a computational study on the two sets of well-known benchmarks composed of 91 instances. For the general max-k-cut problem, the proposed algorithm is able to improve 90 percent of the current best known results available in the literature. Moreover, for the very popular special case with $k = 2$, i.e., the max-cut problem, MOH also performs extremely well by improving 6 best known results which were previously established by any max-cut algorithms of the literature including several recent algorithms published since 2012.

We also investigated alternative strategies for combing search operators and justified the combination adopted in the proposed MOH algorithm.

Given that most ideas of the proposed algorithm are general enough, it is expected that they can be useful to design effective heuristics for other graph partitioning problems.



(a) $f_{best-\rho} - f_{bestknown}$, gaps between f_{best} obtained with different ρ values to best known objective values



(b) $(f_{bestknown} - f_{avg-\rho})/f_{bestknown} * 100\%$, gaps to best known objective values

Fig. 3: Analysis of the move operators O_3, O_4

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