Re-evaluating natural resource investments under uncertainty: An alternative to limited traditional approaches

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Abstract

The need to evaluate natural resource investments under uncertainty has given rise to the development of real options valuation; however, the analysis of such investments has been restricted by the capabilities of existing valuation approaches. We re-visit the well-known example of a copper mine project under a one-factor and two multi-factor models using the influence diagram simulation-and-regression (IDSR) approach. The one-factor setting was originally proposed by Brennan and Schwartz (J Bus 58(2):135–157, 1985), who used partial differential equations (PDEs) and finite differences to approximately solve the valuation problem; extensions to two and three factors were later analysed by Tsekrekos et al. (Eur Financ Manag 18(4):543–575, 2012) using the least-squares Monte Carlo method. We apply the IDSR approach to perform a detailed portfolio analysis of the one-factor benchmark investment and find issues in both the definitions and values of the fixed-output-rate mine and closure option at the portfolio decomposition stage in Brennan and Schwartz (1985). We then apply the IDSR approach to re-evaluate two multi-factor extensions of Tsekrekos et al. (2012) and detect issues in their sensitivity analyses that impact on the reliability of some of their findings. To confirm this and validate the values we obtained, we integrate PDE-based analytical expressions that describe the volatilities implied by the multi-factor models into our IDSR-based analysis. Using the investment-uncertainty relationship, we are able to correctly analyse the impact of the complex multi-factor model parameters on investment value. We conclude that the limitations of PDE-based finite difference approaches may invalidate their use in portfolio situations, but analytical expressions obtained from PDE-based modelling may be profitably integrated into a simulation-based, numerical analysis to validate results and gain new insights.

Keywords Investment analysis · Natural resource investments · Real options portfolio · Investment-uncertainty relationship · Multi-factor models

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1 Introduction

The evaluation of natural resource investments under uncertainty has historically been one of the most popular areas of application of real options analysis (ROA). Considering their very capital intensive and largely irreversible nature as well as the high degrees of uncertainty involved, ROA is well-suited to evaluate natural resource investments. Given the wide applicability of ROA[1] in terms of scope and scale, applications in this area are numerous and encompass a wealth of renewable and non-renewable natural resources. Following the more general works of Tourinho (1979); Pindyck (1980), early studies on natural resource investments focused on minerals (Trigeorgis 1990) and metals including copper (Brennan and Schwartz 1985) and gold (Kelly 1998); agricultural commodities such as rubber/palm-oil (Bailey 1991) and lumber/timber (Clarke and Reed 1989; Morck et al. 1989); land (Quigg 1993); as well as conventional fossil energy such as oil and gas (Paddock et al. 1988; Ekern 1988; Bjerksund and Ekern 1990). In particular the study of Brennan and Schwartz (1985), which is widely regarded as a "pioneering article" (Dixit and Pindyck 1994) and "seminal paper" (Lambrecht 2017), has received considerably attention from academics.

Natural resource investments typically contain many interacting flexibilities and their performance is generally affected by multiple uncertainties. For example, the original copper mine example of Brennan and Schwartz (1985) considered a portfolio of options: to temporarily mothball and irreversibly abandon the mine. Relevant portfolio extensions include the option to delay/defer the mine's development (Gamba 2003); the option to expand production capacity (Cortazar and Casassus 1998); and a combined approach with operational, development and exploration options (Cortazar et al. 2001). The work of Brennan and Schwartz (1985) was not only one of the first to consider multiple options, but it also accounted for copper price uncertainty directly, rather than through a (single) risk-adjusted discount rate. Building upon their simple one-factor, constant-convenience model, where the price dynamics are described by a geometric Brownian motion, subsequent works aimed at accounting for the mean-reverting tendency of many commodities' spot prices. In the joint stochastic process of the two-factor model of Gibson and Schwartz (1990) the convenience yield evolves stochastically by following an Ornstein-Uhlenbeck process, and the three-factor model of Schwartz (1997) extends this two-factor model by assuming that the risk-free interest rate also follows such a simple mean-reverting process (see, e.g., Cortazar and Schwartz (2003); Casassus and Collin-Dufresne (2005) for related models).

To value the flexibility provided by real options inherent in natural resource investments whilst accounting for uncertainty, there are, broadly speaking, three types of valuation approaches based on partial differential equations (PDEs), lattice models, or simulation (i.e. Monte Carlo sampling). Considering a hypothetical copper mine, Brennan and Schwartz (1985) first demonstrated how the value of such a natural resource investment can be mathematically modelled by PDEs[2] and then numerically approximated using a finite-difference method. Subsequently, Cortazar and Schwartz (1997) applied such an approach in the context of an undeveloped oil field. Following the development of the binomial options pricing model by Cox et al. (1979), early applications of binomial lattice techniques evaluated petroleum projects (Ekern 1988) and a gold mine (Kelly 1998). See McCarthy and Monkhouse (2002) for a mine-related application of a trinominal lattice technique (Boyle 1988). Due to their ori-

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[1] For a recent review of real options from a finance and operational research perspective see Lambrecht (2017) and Trigeorgis and Tsekrekos (2018), respectively.

[2] Representing a stochastic optimal control problem, solving the related PDE, which is commonly known as the Hamilton-Jacobi-Bellman equation, gives rise to free-boundary problems.

gin in option-pricing theory, the application of the above two types of valuation approaches to real assets is sometimes referred to as contingent claims analysis (Bjerksund and Ekern 1990).

Unlike PDE-based finite differences and binomial techniques, simulation-based approaches are readily applicable in multi-factor situations. Initially developed for the pricing of European call options by Boyle (1977), subsequent works presented simulation-based approaches to value American-style options. These include the bundling algorithm of Tilley (1993); the partitioning algorithm of Barraquand and Martineau (1995), which was applied to on an oil field example by Cortazar and Schwartz (1998); and an extension of the latter algorithm by Raymar and Zwecher (1997), which Castillo-Ramirez (2000) tested using the mine example of Brennan and Schwartz (1985). By contrast, the popular works of Carriere (1996); Tsitsiklis and Van Roy (2001); Longstaff and Schwartz (2001) combine simulation with regression to approximately solve the optimal stopping problem embedded in the pricing of American options. Extending the least-squares Monte Carlo (LSM) method of Longstaff and Schwartz (2001) to optimal switching problems, Gamba (2003), Abdel Sabour and Poulin (2006), Cortazar et al. (2008) re-assessed the copper mine example of Brennan and Schwartz (1985). However, as noted by Abdel Sabour and Poulin (2006), the switching decisions obtained by Gamba (2003), Cortazar et al. (2008) are not in line with Brennan and Schwartz (1985) and are also inconsistent because the switching policy of Gamba (2003) is cyclic in the copper price rather than being linear, and the policy implied by Cortazar et al. (2008) indicates that it is optimal to open the mine at US$ 0.70/lbs. Multi-factor model extensions of the mine example were studied by Cortazar et al. (2008), who applied the three-factor model of Cortazar and Schwartz (2003), and Tsekrekos et al. (2012) considered both the two-factor model of Gibson and Schwartz (1990) and the three-factor models of Schwartz (1997); Casassus and Collin-Dufresne (2005).

The capabilities of existing valuation approaches restrict the analysis of natural resource investments and the insights that can be gained by such analysis in different ways. Continuous-time approaches such as those based on PDEs are arguably the most popular choice amongst academics conducting theoretical research. This is because although PDE-based modelling is intuitively unappealing and (mathematically) complex, the obtained solutions and exercise policies are sometimes available in closed-form and often more tractable than those obtained by mathematically simpler discrete-time approaches such as those based on binomial techniques and simulation (Glasserman 2003; Brandão et al. 2005a, b; Lambrecht 2017). However, PDE-based finite differences as well as binomial techniques quickly lose their tractability in multi-factor situations and when there is path-dependency[3] (Lambrecht 2017). In contrast, simulation-based approaches are readily applicable in such situations. To overcome the limitations of the LSM approach when it comes to valuing option portfolios (Smith 2005; Brandão et al. 2005a), several recent works introduced simulation-based alternatives. For example, Maier et al. (2020) proposed the influence diagram simulation-and-regression (IDSR) approach to value portfolios of interdependent real options and applied it to a complex natural resource investment that features four stochastic factors. Despite these successes, the key limitations of "black box" simulation-based valuation approaches are a lack of transparency and the difficulty to validate results (Brandão et al. 2005a; Lambrecht 2017).

In this work we re-evaluate a natural resource investment under three different models of the stochastic dynamics of commodity prices using the IDSR approach. The copper mine example under a simple one-factor model was initially evaluated in the seminal work of Brennan and Schwartz (1985), who mathematically modelled and numerically approximated

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[3] With regard to their model, Brennan and Schwartz (1985) noted "in general there exists no analytic solution to the valuation model, though it is straightforward to solve it numerically".

the valuation problem using PDEs and finite-differences, respectively. This example serves as our benchmark natural resource investment in which only the copper spot price evolves stochastically (see Sect. 2). While we obtained results for both the mine values and the switching decisions that are in line with the ones of Brennan and Schwartz (1985), we detected, quite unexpectedly, technical issues in both the definitions and values of their fixed-output-rate mine and closure option (see Sect. 3). We performed a comprehensive portfolio analysis enabled by the IDSR approach and show that their fixed-output-rate mine includes the option to abandon the mine, whereas a fixed-output-rate mine does not, in fact, include any options. We demonstrate that the definition of their closure option is flawed due to an inconsistency in the way it is calculated, and also that its value is incorrect as it corresponds to the value of mothballing within the options portfolio. To account for mean reversion in the copper spot price, we replace the one-factor model with the highly-cited two- and three-factor models of Gibson and Schwartz (1990) and Schwartz (1997), respectively.

These important multi-factor extensions of our benchmark investment—first to a stochastic convenience yield and then to stochastic interest rates—were previously investigated by Tsekrekos et al. (2012) applying the LSM approach. However, when re-examining the effects of the two- and three-factor model on the mine value using the parameters of Tsekrekos et al. (2012) and comparing values, we find values that are noticeably different from those of Tsekrekos et al. (2012), and note also that our mine values exhibit the opposite behaviour with respect to changes of the correlation between the copper price and convenience yield process (see Sects. 4 and 5). To validate our results, we use a key insight from options theory about the investment-uncertainty relationship: higher volatility in the underlying generally leads to a higher investment value. As a proxy measure of the investment's overall volatility we use analytical expressions that describe the volatilities implied by the multi-factor models and that are derived from solutions to futures prices' PDEs. This enabled us to confirm our IDSR-based results by demonstrating that an increase in this correlation coefficient actually reduces overall (or total) volatility and, as such, mine value; by contrast, Tsekrekos et al. (2012) found values to be increasing in that coefficient. In addition, applying this proxy measure enabled us to disprove other related findings of their analyses such as the impact of the short rate's volatility, thus providing further evidence of the benefit of implied volatilities in the analysis of natural resource investments in multi-factor situations.

Finally, in Sect. 6, we discuss existing option pricing and decision analysis approaches in the light of the findings obtained under the different stochastic models. We argue that while the limitations of PDE-based finite difference approaches may invalidate their use as practical and reliable methods for the valuation and analysis of portfolios of real options, analytical expressions obtained from PDE-based modelling may be profitably integrated into a simulation-based numerical analysis both to validate results and to provide new insights. These insights include demonstrating that the volatilities of futures returns implied by the considered multi-factor models may well be used as an adequate proxy measure for the copper mine project's actual overall (or total) volatility. The implied volatilities can then be used to analyse the effects of the complex models' parameters on investment values through their impact on overall volatility within the well-known investment-uncertainty relationship. Given the generally non-linear, non-monotonic dependency of overall volatility (and hence investment value) on models' individual parameters, our analysis demonstrates that even though an individual stochastic factor becomes more volatile, the investment value may decrease as a result of a decline in overall volatility. This highlights the importance of applying such a transparent and intuitive approach for investment analysis in multi-factor situations.

The rest of our article is organised as follows: Sect. 2 presents the re-evaluation of a natural resource investment in a one-factor setting. Results of this re-evaluation are presented and

discussed in Sect. 3. Section 4 extends this benchmark investment by considering two multi-factor models. Results of this extension are then presented and discussed in Sect. 5. In Sect. 6 we discuss existing option pricing and decision analysis approaches in the light of the findings obtained, and provide some concluding remarks.

2 Natural resource investments: the one-factor benchmark

In this section we re-evaluate the classical example of valuing a copper mine originally proposed[4] by Brennan and Schwartz (1985) applying the IDSR approach.

2.1 Problem setting

In this copper mine example, the decision maker has several possibilities to affect the mine's operation. While it only contains one stochastic factor (copper spot price), the operational flexibility available in this copper mine example represents a portfolio of interdependent real options, which we refer to as the option to switch. This and the portfolio's constituent real options are:

(a) Option to switch: In addition to extracting the copper immediately until the mine inventory, Q_0, is exhausted, the decision maker may decide to temporarily close the (operating) mine, to maintain or reopen the mine when it is closed, and/or to irreversibly abandon the copper mine before its inventory is fully exhausted, i.e. before Q_0/q years of operation at an annual output rate of q.

  (a-i) Option to temporarily mothball the mine: If the copper spot price at time t, X_t, is too low in relation to the mine's production costs, A_t, the decision maker can close down the opened (i.e. operating) mine at a cost of K_t^c, maintain the closed mine at an annual maintenance cost of M_t, and, if the copper price becomes favourable again, reopen the closed mine at a cost of K_t^o at time t.

  (a-ii) Option to irreversibly abandon the mine: Whether opened or closed, the decision maker retains the right to permanently abandon the copper mine at any time t without incurring any cost.

The value of this portfolio of real options is affected by the uncertainty surrounding future commodity prices, in other words by copper price uncertainty. The copper spot price at time t, X_t, is assumed to evolve according to a discretised version of the geometric Brownian motion used by Brennan and Schwartz (1985).

2.2 Modelling

The flexibilities inherent in the copper mine are illustrated by the ID in Fig. 1. It contains two decision nodes (*Opened* (1) and *Closed* (4)) and two terminal nodes (*Abandoned* (2) and *Exhausted* (3)), as well as seven transitions that link these nodes, resulting in $\mathcal{N} = \{1, 2, 3, 4\}$ and $\mathcal{H} = \{1, 2, \ldots, 7\}$. The duration (in years) of transition $h \in \mathcal{H}$ is Δ_h. When the mine

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[4] It should be noted that even though the numerical illustration of Brennan and Schwartz (1985) considered "a mine example based on the stylized facts for copper", the valuation model is general and not limited to copper, so can, in theory, be applied to any other natural resource.

Fig. 1 Influence diagram for the copper mine project – configuration (a)
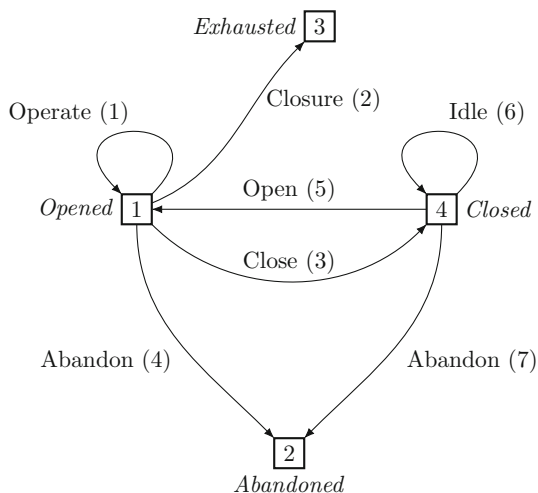
is *Opened*, the decision maker has to decide whether to Operate (1) for Δ_1 year(s) whilst extracting $q\Delta_1$ of copper, temporarily Close (3), or irreversibly Abandon (4) the mine project. On the other hand, if the mine is *Closed*, the available transitions are to keep the mine Idle (6), Open (5) it, or irreversibly Abandon (7) the project. However, the mine closures (2) if the inventory is fully depleted and, as such, becomes *Exhausted*. Also, for the sake of definiteness, the mine has to be *Abandoned* when reaching its maximum lifetime of T^{max} years, thereby preventing the mine from having an infinite lifetime.

Let the node and the inventory of the mine at time t be denoted by N_t and Q_t, respectively, as well as let the copper spot price at time t be denoted by X_t. Then, the resource and information component of the state variable S_t are given by $R_t = (t, N_t, Q_t)$ and $I_t = X_t$, respectively. The state variable is then written as $S_t = (t, N_t, Q_t, X_t)$. Since the mine's commodity inventory can be depleted at an annual output rate q, which is assumed constant, it takes at least Q_0/q years to empty the finite inventory. While the assumption of a finite horizon (T^{max}) may result in an approximated numerical solution, adverse effects can be minimised, even avoided fully, by choosing $T^{max} \gg Q_0/q$.

The binary decision variables $(a_{th})_{h \in b^D(N_t)}$ associated with transitions h ($a_{th} = 1$ if h is made at time t, and 0 otherwise) available at node N_t at time t, $b^D(N_t)$, which are given by:

$$b^D(N_t) = \begin{cases} \{1, 2, 3, 4\}, & \text{if } N_t = 1, \\ \{5, 6, 7\}, & \text{if } N_t = 4, \\ \{\}, & \text{otherwise,} \end{cases} \tag{1}$$

have to satisfy the feasible region $\mathcal{A}_{S_t}$, which is defined by the following set of constraints:

$$a_{t1} + a_{t2} + a_{t3} + a_{t4} = 1 \qquad \text{if } N_t = 1, \tag{2}$$

$$a_{t5} + a_{t6} + a_{t7} = 1 \qquad \text{if } N_t = 4, \tag{3}$$

$$a_{th}q\Delta_h \leq Q_t \qquad \forall h \in \{1, 5\}, \tag{4}$$

$$-Q_t < a_{t2} \leq 1 - \frac{Q_t}{Q_0}, \tag{5}$$

$$t - T^{max} < a_{t2} + a_{t4}, \tag{6}$$

$$t - T^{max} < a_{t7}, \tag{7}$$

where $a_{th} \in \{0, 1\}$, $\forall h \in \mathcal{H}$. These constraints accomplish the following: (2) and (3) ensure that exactly one transition is made at decision node 1 and 4, respectively; (4) make sure the inventory does not become negative[5]; (5) requires the mine to closure if and only if $Q_t = 0$; (6) ensures that if $t = T^{max}$ and $Q_{T^{max}} > 0$ then $a_{T^{max}4} = 1$; and, lastly, (7) makes sure the mine is abandoned if closed at $t = T^{max}$.

After having made a decision subject to these constraints, the resource state R_t evolves deterministically to $R_{t+\Delta_h}$ according to $S^R(\cdot)$, whereas the information state I_t evolves stochastically to $I_{t+\Delta_h}$ under the risk-neutral measure represented by $S^I(\cdot)$. With regard to $S^R(\cdot)$, the evolution of t is straightforward as it simply evolves from t to $t + \Delta_h$, the evolution of N_t is implied by the adjacency matrix of the digraph $(\mathcal{N}, \mathcal{H})$:

$$
\begin{array}{c}
 \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}
\left[\begin{array}{cccc}
1 & 4 & 2 & 3 \\
- & - & - & - \\
- & - & - & - \\
5 & 7 & - & 6
\end{array} \right]
\end{array}
$$

and the evolution of Q_t is specified by the following transition equation for all $h \in \mathcal{H}$:

$$Q_{t+\Delta_h} = Q_t - q\Delta_1(a_{t1} + a_{t5}) \tag{8}$$

On the other hand, the copper spot price at time t, X_t, evolves stochastically to $X_{t+\Delta_h}$ according to the following discrete diffusion process:

$$X_{t+\Delta_h} = X_t \exp\left\{ \left(r - \delta - \frac{\sigma_x^2}{2} \right) \Delta_h + \sigma_x \sqrt{\Delta_h} \epsilon_{t+\Delta_h}^x \right\}, \tag{9}$$

where r is the price trend, δ is the instantaneous convenience yield, σ_x is the standard deviation of price changes, and $\epsilon_{t+\Delta_h}^x$ is the driving zero-mean process—a standard normal random variable whose increments are iid.

The payoff obtained at t when making decision a_t given state S_t is:

$$\Pi_t(S_t, a_t) = \begin{cases} q\Delta_1(X_t - A_t) - f(X_t)\Delta_1, & \text{if } a_{t1} = 1, \\ -M_t\Delta_3 - K_t^c, & \text{if } a_{t3} = 1, \\ q\Delta_5(X_t - A_t) - f(X_t)\Delta_5 - K_t^o, & \text{if } a_{t5} = 1, \\ -M_t\Delta_6, & \text{if } a_{t6} = 1, \\ 0, & \text{otherwise}, \end{cases} \tag{10}$$

where $A_t = A_0 e^{\pi t}$ is the average (per unit) production cost at time t with inflation rate π; $f(X_t) = \tau_1 q X_t + \max\{\tau_2 q(X_t(1 - \tau_1) - A_t), 0\}$ is the sum of royalties and income tax paid at time t with τ_1 the royalty rate and τ_2 the income tax rate; $M_t = M_0 e^{\pi t}$ is the maintenance cost at time t; and $K_t^c = K_0^c e^{\pi t}$ and $K_t^o = K_0^o e^{\pi t}$ are the costs to switch to the *Closed* and *Opened* node at time t, respectively.

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[5] For simplicity, we assume $Q_0 \mod (q\Delta_1) = 0$.

2.3 Portfolio optimisation problem

The problem of determining the optimal value of the copper mine can be formulated as a multi-stage stochastic optimisation problem and then be solved algorithmically, in theory, via stochastic dynamic programming (SDP) recursions, as described by Maier et al. (2020). Let the optimal value of the copper mine at time t given state S_t be denoted by $G_t(S_t)$. The value of the copper mine, $G_0(S_0)$, is then given by the optimal solution of:

$$G_0(S_0) = \max_{(a_t)_{t \in \mathcal{T}}} \mathbb{E}\Big[\sum_{t \in \mathcal{T}} e^{-kt} \Pi_t(S_t, a_t) \big| S_0\Big], \tag{11}$$

where S_0 is the state at time 0, $a_t = (a_{th})_{h \in b^D(N_t)}$, $a_t \in \mathcal{A}_{S_t}$, $a_{th} \in \{0, 1\}$, k is the discount rate, and $S_{t+\Delta_h} = S^M(S_t, a_t, W_{t+\Delta_h})$ with $W_{t+\Delta_h} = \epsilon^x_{t+\Delta_h}$.

To determine an optimal policy, i.e. a decision vector $a_t^* = \big(a_t^*(S_t)\big)_{S_t \in \mathcal{S}_t}$ for all $t \in \mathcal{T}$ that maximises the mine value given the state S_0 at time 0, $G_0(S_0)$, we apply Bellman's "principle of optimality" and hence solve the stochastic optimisation problem (11) recursively using the following value function when in state S_t at t:

$$G_t(S_t) = \max_{a_t} \left(\Pi_t(S_t, a_t) + \mathbb{E}\Big[e^{-k\Delta_h} G_{t+\Delta_h}(S_{t+\Delta_h}) \big| S_t, a_t\Big] \right) \tag{12}$$

$$\text{s.t.} \quad a_t \in \mathcal{A}_{S_t}, \tag{13}$$

$$a_{th} \in \{0, 1\} \qquad \forall h \in b^D(N_t), \tag{14}$$

$$S_{t+\Delta_h} = S^M(S_t, a_t, W_{t+\Delta_h}) \quad \forall h \in b^D(N_t), \tag{15}$$

with the terminal condition $G_t(S_t) = 0$, for all $S_t \in \big\{S_t' \in \mathcal{S}_t : b^D(N_t') = \emptyset\big\}, t \in \mathcal{T}$.

Ultimately, since the SDP recursion (12)-(15) is, in general, computationally intractable, the value of the mine, $G_0(S_0)$, is approximated through a simulation-and-regression-based solution procedure that consists of a forward and backward induction—both are described in "Appendix A", giving the approximation $\bar{G}_0(S_0)$.[6]

2.4 Valuation

With regard to the valuation of this natural resource investment, we used the same parameter values as Brennan and Schwartz (1985), which are shown in Table 1. Furthermore, we considered five decisions to be made per year (i.e. $\Delta_1 = \Delta_3 = \Delta_5 = \Delta_6 = 1/5$, whereas $\Delta_2 = \Delta_4 = \Delta_7 = 0$[7]) and the first six (i.e. $L = 5$) generalized Chebyshev polynomials as basis functions. Also, as in Tsekrekos et al. (2012), we considered 100,000 ($= |\Omega|$) sample paths (half of which antithetic for variance reduction), where Ω is the set of sample realisations. While the inventory of the mine can be depleted as early as 15 years ($=Q_0/q$) after starting operation, a finite time horizon of $T^{max} = 60$ years was chosen for the time by which the right to extract copper from the mine expires. Since there is no payoff associated with transitions 4 and 7 nor a terminal value with the *Abandoned* node, we can use a constant discount rate of 12% ($k = r + \lambda_1$) in our computations.

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[6] It is important to note that $\bar{G}_0(S_0)$ is, in general, a lower bound on the optimal value of the copper mine, $G_0(S_0)$. For a recent application of a duality-based upper bound technique to a selection of standard mining problems see Hinz et al. (2020).

[7] By considering that abandonment and closure takes place instantaneously and does not result in a salvage value and closing costs, respectively, we ensure comparability with Brennan and Schwartz (1985); however, it would be straightforward to account for more realistic assumptions here, e.g., by simply using $\Delta_h > 0$, $h \in \{2, 4, 7\}$, and appropriately adapting (10).

Table 1 Data for hypothetical copper mine of Brennan and Schwartz (1985)

Description	Parameter	Value	Unit
Mine			
Output rate	q	10×10^6	lbs/year
Initial inventory	Q_0	150×10^6	lbs
Initial average production cost	A_0	0.50	US\$/lbs
Initial cost of opening	K_0^o	200×10^3	US\$
Initial cost of closing	K_0^c	200×10^3	US\$
Initial maintenance cost	M_0	500×10^3	US\$/year
Cost inflation rate	π	8%	year^{-1}
Copper			
Convenience yield	δ	1%	year^{-1}
Price variance	σ_x^2	8%	year^{-1}
Nominal interest rate	r	10%	year^{-1}
Taxes			
Royalty	τ_1	0%	–
Income	τ_2	50%	–
Property, *Opened/Closed*	λ_1	2%	year^{-1}
Property, *Abandoned*	λ_3	0%	year^{-1}

Using the initialisation from the forward pass and applying the backward induction procedure of "Appendix A", results are presented and discussed in the following section. Figure 5 provides an illustration of sample paths.

3 Results and discussion: the one-factor benchmark

This section begins with an analysis of the way in which the value of the (initially) opened and closed mine, $\bar{G}_0(S_0)$, characterised by $S_0 = (0, 1, Q_0, X_0)$ and $S_0 = (0, 4, Q_0, X_0)$, respectively, are affected by the initial price of copper, X_0. The results are shown in Table 2 and compared with those of Brennan and Schwartz (1985), who applied PDE-based finite differences. In terms of the mine values (the switching decisions), our IDSR-based numerical results converge very closely (are identical) to the ones obtained by Brennan and Schwartz (1985) and are in line with the conclusions reached by Abdel Sabour and Poulin (2006); Tsekrekos et al. (2012), thus confirming the adequacy of the IDSR approach to correctly value such a natural resource investment. "Appendix C" contains an analysis of the effects of the copper price and its uncertainty on the value of this benchmark natural resource investment.

To evaluate the individual real options available in this natural resource investment, this section now analyses the extent to which the mine value with different configurations of option portfolios depends on the initial copper price, X_0. Table 3 shows the sensitivity of the value of different portfolio configurations when X_0 is in the range from US\$ 0.30 to 1.00 per pound. Column (–) gives the expected value of the fixed-output-rate mine, which assumes it is operated at the rate of 10 million pounds/year until the 15-year inventory is fully

Table 2 Value (in US$ millions) of mine for different copper prices according to alternative numerical methods

Price (US$/lbs)	Brennan-Schwartz Finite Differences		Simulation-and-regression-based		Relative error (in %)	
X_0	Opened	Closed	Opened	Closed	Opened	Closed
0.30	1.25[a]	1.45	1.250[a]	1.450	−0.020	−0.020
0.40	4.15[a]	4.35	4.174[a]	4.374	0.568	0.542
0.50	7.95	8.11	7.975	8.164	0.314	0.667
0.60	12.52	12.49	12.548	12.524	0.224	0.270
0.70	17.56	17.38	17.585	17.399	0.143	0.109
0.80	22.88	22.68[b]	22.914	22.714[b]	0.150	0.151
0.90	28.38	28.18[b]	28.439	28.239[b]	0.208	0.210
1.00	34.01	33.81[b]	34.101	33.901[b]	0.268	0.269

[a]Optimal to close mine
[b]Optimal to open mine

exhausted. As can be seen, this value is negative for copper prices of US$ 0.50 per pound and below, making operation unprofitable.[8] As described in Sect. 2.1, columns (a-i) and (a-ii) display the value of the mine if it can be temporarily mothballed and irreversibly abandoned, respectively. Having the flexibility provided by the former (latter) option enables the copper mine to become economically viable for prices of US$ 0.50 (0.40) per pound and above, thus allowing the mine with such options to become viable in situations where the fixed-output-rate mine is not. By contrast, with the option to switch, whose value is shown in column (a) and which can be interpreted as the portfolio of options to mothball and abandon, the mine is economically viable for all copper prices under consideration. The IDs[9] illustrating the managerial flexibility available in the setting of columns (–), (a-i), (a-ii) and (a) are shown by Figs. 2a, 2c, 2b and 1, respectively.

In addition, Table 3 also displays the value added by the portfolio's individual options—both in isolation and within the portfolio of options. Columns (a-i)-(–), (a-ii)-(–) and (a)-(–) report the value of the options to mothball the mine, to abandon the mine and to switch, respectively. These values were determined by the difference between the mine values with these individual options—shown in columns (a-i), (a-ii) and (a)—and column (–), which gives the value of the fixed-output-rate mine. As can be seen, the real options considered add substantial value to this natural resource investment. For all copper prices under consideration, abandoning the mine was found to be more valuable than mothballing, and switching more valuable than abandoning. Importantly, the option to switch, which in itself is a portfolio of options containing the other two options, will always be at least as valuable than its constituent options. As expected, the values of these options decrease as the operating margin increases since operational flexibility becomes less attractive. Nevertheless, the added value of switching is still almost 13% for the highest copper price considered, which is twice the cost of production.

Columns (a)-(a-ii) and (a)-(a-i) of the table give the value of the option to mothball and to abandon the mine, respectively, within the portfolio of options. In other words, these

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[8] It should be noted that these are not the critical copper prices, i.e. the point at which it becomes optimal to invest, which largely depend on the chosen input data as mentioned by Brennan and Schwartz (1985); however, our approach can easily be used to accurately estimate these critical prices.

[9] Although not shown here for simplicity, the ID in Fig. 2c also contains two transitions to the *Abandoned* node for the case $t = T^{max}$.
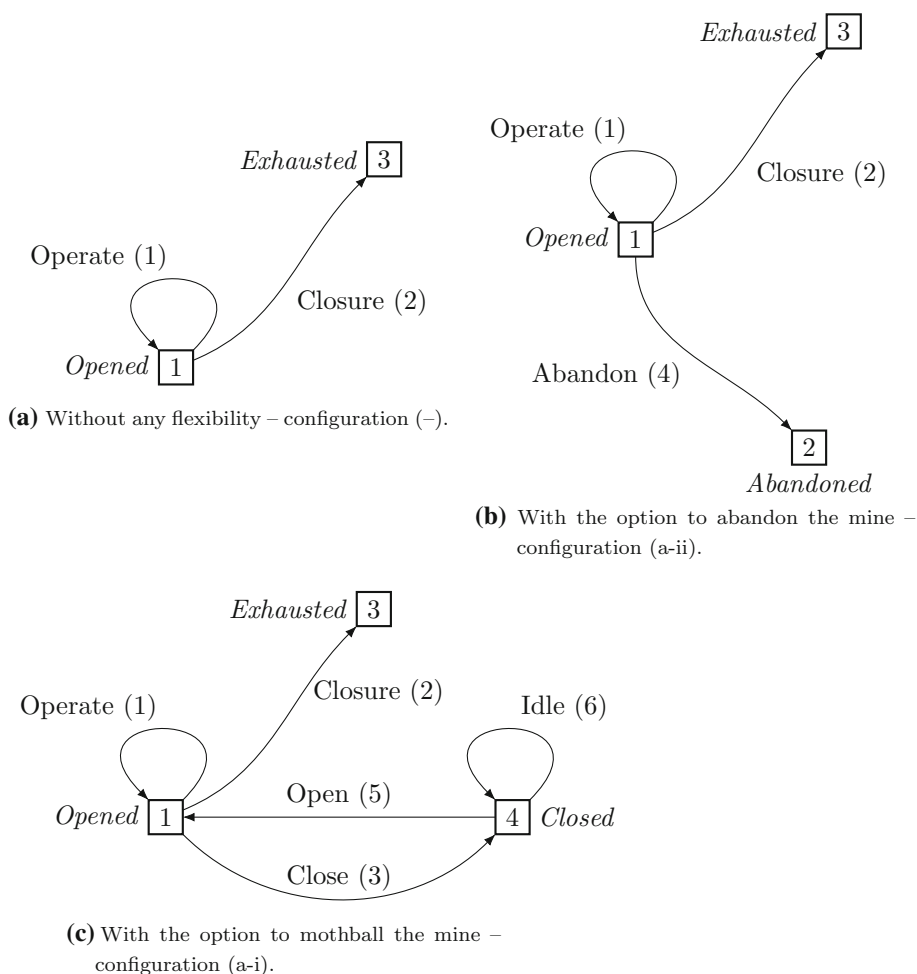
Table 3 Value of opened copper mine, $\bar{G}_0(S_0)$ (in US$ millions), for different copper price levels

Copper Price (US$/lbs) X_0	Value of Fixed-Output-Rate Mine (−)	In Isolation		Value of Option to Switch (a)-(−)	In Portfolio		Value with Option to Mothball (a-i)	Value with Option to Abandon (a-ii)	Value with Portfolio of Options (a)
		Value of Mothball (a-i)-(−)	Option to Abandon (a-ii)-(−)		Value of Mothball (a)-(a-ii)	Option to Abandon (a)-(a-i)			
0.30	-22.831*	18.793 (−)**	22.831 (−)	24.080 (−)	1.250 (5.19)	5.288 (21.96)	-4.038*	0*	1.250
0.40	-13.338*	13.322 (−)	15.515 (−)	17.512 (−)	1.997 (11.40)	4.190 (23.93)	-0.016*	2.177	4.174
0.50	-4.757*	9.205 (−)	11.259 (−)	12.732 (−)	1.473 (11.57)	3.527 (27.70)	4.448	6.502	7.975
0.60	2.950	6.582 (223.12)	8.527 (289.07)	9.598 (325.37)	1.071 (11.16)	3.016 (31.43)	9.532	11.477	12.548
0.70	10.142	4.836 (47.68)	6.716 (66.22)	7.443 (73.39)	0.727 (9.77)	2.607 (35.03)	14.978	16.858	17.585
0.80	17.017	3.618 (21.26)	5.441 (31.97)	5.898 (34.66)	0.457 (7.75)	2.279 (38.65)	20.635	22.457	22.914
0.90	23.685	2.784 (11.76)	4.481 (18.92)	4.754 (20.07)	0.273 (5.74)	1.970 (41.43)	26.470	28.166	28.439
1.00	30.213	2.210 (7.31)	3.736 (12.36)	3.888 (12.87)	0.152 (3.91)	1.678 (43.16)	32.423	33.949	34.101

The sets of transitions available in the different settings are as follows: $\mathcal{H}^- = \{1, 2\}$ in (−); $\mathcal{H}^- \cup \{3, 5, 6\}$ in (a-i); $\mathcal{H}^- \cup \{4\}$ in (a-ii); and $\mathcal{H}$ in (a)

*No investment

**Added value of option(s) in %

(a) Without any flexibility – configuration (–).

(b) With the option to abandon the mine – configuration (a-ii).

(c) With the option to mothball the mine – configuration (a-i).

Fig. 2 Influence diagrams for different copper mine settings

columns report how much an individual real option adds to the portfolio assuming that the other individual option is already contained in the portfolio. To determine the value of one option, the value of the mine with the options portfolio was measured against the value of the mine with the other option. For example, the difference between column (a) and (a-ii) results in the values shown in column (a)-(a-ii). Comparing the values shown in column (a)-(a-ii) with the ones of column (a)-(a-i) shows that, while the value of either option in the portfolio generally decreases as X_0 increases[10], abandoning adds substantially more value to the portfolio than mothballing, especially for high copper prices. This result is very intuitive given the above presented valuation of the options to mothball and abandon in isolation. It is interesting to note, however, that the relative portfolio value of mothballing generally decreases as the initial price of copper increases, whereas the relative value of adding the abandonment option to the option to mothball the mine always increases in X_0. This indicates

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[10] The initial increase displayed in column (a)-(a-ii) is due to the non-negative value of the mine with the option to abandon—see column (a-ii)—, which is bounded below by zero.

Table 4 Value (in US$ millions) of copper mine for different copper price levels according to Brennan and Schwartz (1985)

Copper Price (US$/lbs) X_0	Mine Value		Value of Fixed-Output-Rate Mine (4)	Value of Closure Option (5)
	Open (2)	Closed (3)		
0.30	1.25[a]	1.45	0.38	1.07
0.40	4.15[a]	4.35	3.12	1.23
0.50	7.95	8.11	7.22	0.89
0.60	12.52	12.49	12.01	0.51
0.70	17.56	17.38	17.19	0.37
0.80	22.88	22.68[b]	22.61	0.27
0.90	28.38	28.18[b]	28.18	0.20
1.00	34.01	33.81[b]	33.85	0.16

[a]Optimal to close mine
[b]Optimal to open mine

that adding strategic flexibility (to limit downside risk by abandoning the mine early) to the mine that already has operational flexibility (to exploit upside risk by deviating from the immediate extraction of copper) is more valuable in this portfolio context than the other way around.

Although Brennan and Schwartz (1985) were able to provide several insights into the valuation of this small options portfolio, there are some technical issues in their analysis that impacted on the reliability of some of their results. Table 4 reports the results of their numerical analysis. According to Brennan and Schwartz (1985), the relevant columns of their table are defined as follows:

> Column 4 gives the value of the mine assuming that it cannot be closed down but must be operated at the rate of 10 million pounds per year until the inventory is exhausted in 15 years. The difference between column 4 and the greater of the values shown in columns 2 and 3 represents the value of the option to close down or abandon the mine if the price of copper falls far enough. The value of this closure option is shown in column 5. (Brennan and Schwartz 1985)

However, while the values reported in columns (2) and (3) have been widely confirmed in the literature including this work (see Table 2), both the definitions of the values and the actual values shown in columns (4) and (5) are flawed.

Firstly, the values of the fixed-output-rate mine shown in column (4) are given as positive (and convex in X_0) for all copper prices under consideration. By contrast, and as we would expect, we have found that the expected value of the mine—assuming copper must be extracted immediately until the 15-year inventory is fully exhausted—is highly negative if operating margins are low, and that this value is an increasing yet concave function of X_0. As seen in column (–) of Table 3, the value of the fixed-output-rate mine is negative for copper prices of US$ 0.50 per pound and below. Our results are in line with Cortazar et al. (2008), who performed a comparative static analysis and showed that, considering the three-factor commodity model of Cortazar and Schwartz (2003), the expected NPV of the mine without any flexibility is negative for low spot prices of copper—see also Castillo-Ramirez (2000); Lin and Wang (2012). In addition to noting that the value of the mine with the option to switch is convex in X_0, the figure shown by Cortazar et al. (2008) also appears to illustrate

that the value of the fixed-output-rate mine is concave in X_0, whilst converging to the value of the opened mine for high commodity prices.

The detailed portfolio analysis presented here gives insights into why the fixed-output-rate mine might have been overvalued in the analysis of Brennan and Schwartz (1985). Comparing these values in column (4) of Table 4 with our results in Table 3 shows that there is a similarity between their values and the values we have obtained for the mine with the option to abandon—see column (a-ii). This similarity suggests that their fixed-output-rate mine actually includes the option to abandon. Even though our results tend to be slightly low-biased in relation to theirs, particularly for low copper prices, the overall patterns are very similar. Surprisingly, our hypothesis that the fixed-output-rate mine of Brennan and Schwartz (1985) includes the abandonment option even seems to be confirmed by the authors themselves, who stated in their paper, two paragraphs below the one we cited above, that:

> Ownership of a mine […] involves three distinct types of decision possibilities or options: first, the decision to begin operations; second, the decision to close the mine when it is currently operating (and possibly to reopen it later), which we have referred to as the closure option; and third, the decision to abandon the mine early, before the inventory is exhausted. (Brennan and Schwartz 1985)

This statement seems inconsistent with the authors' earlier definition presented above. In the earlier statement, Brennan and Schwartz (1985) define close down or abandon the mine as representing the closure option. However, the case of their fixed-output-rate mine represents actually a mine with the early abandonment option, and their closure option does not correspond with "the option to close down or abandon the mine" (Brennan and Schwartz 1985), which we have referred to as the option to switch. Instead, their closure option corresponds to an option that looks like our option to temporarily mothball the mine. Using IDs to graphically illustrate this inconsistency, for the fixed-output-rate mine, it appears that Brennan and Schwartz (1985) have considered the case that corresponds with the ID of Fig. 2b instead of the correct one shown by Fig. 2a, and the flexibility of their closure option (which is problematic in itself, as discussed in the following paragraph) corresponds to the ID of Fig. 2c rather than to the ID shown in Fig. 1.

Secondly, the definition of the closure option in Brennan and Schwartz (1985) is inconsistent, and this leads to an incorrect valuation. Given by the "difference between column 4 and the greater of the values shown in columns 2 and 3" (Brennan and Schwartz 1985), the value of their closure option is determined by subtracting a benchmark mine value—i.e. the value of the "fixed-output-rate mine"—from the maximum value of two different mines—an opened and a closed one. However, the minuend of this subtraction is inconsistent with its subtrahend given that the latter assumes the mine is opened at time $t = 0$ ($N_0 = 1$), whereas the former represents a mine that is either opened ($N_0 = 1$) or closed ($N_0 = 4$) at the beginning, so performing this subtraction does not generate meaningful data. The data thus obtained is therefore irrelevant regardless of the benchmark applied, that is whether the (real) value of the fixed-output-rate mine (Fig. 2a) or the value of the mine with the option to abandon (Fig. 2b) is being used since both have the same initial state. Moreover, since their benchmark values in column (4) seemingly correspond with the values of the mine with the option to abandon, the values shown in column (5) do not represent, as implied by the authors' definition, the value of the closure option in isolation but instead within the portfolio that already contains the early abandonment option. Our analysis—the relevant values are given by column (a)-(a-ii) of Table 3—seems to confirm this when taking into account the above mentioned bias and inconsistency.

4 Natural resource investments: two- and three-factor model extensions

In this section we extend the mine example by integrating the two- and three-factor model of Gibson and Schwartz (1990) and Schwartz (1997), respectively.

4.1 Problem setting

While the original copper mine example of Brennan and Schwartz (1985) contained a portfolio of interdependent real options, it only treated the commodity spot price, i.e. the price of copper, to be stochastic. Here we extend their example by additionally considering both the instantaneous convenience yield and the instantaneous interest rate to be stochastic. As such, in terms of options portfolio considered this setting is the same as the one described in Sect. 2. In terms of uncertainties considered, however, we replace the one-factor setting of Brennan and Schwartz (1985) by the three-factor model of Schwartz (1997), which nests the two-factor model of Gibson and Schwartz (1990). Let the copper spot price, the instantaneous convenience yield, and the instantaneous interest rate at time t be denoted by X_t, δ_t, and r_t, respectively. As in Tsekrekos et al. (2012), the evolution of these three stochastic factors is described by discretised versions of the continuous stochastic processes of Schwartz (1997). Note that the two-factor model (copper price and convenience yield are stochastic) of Gibson and Schwartz (1990) is obtained by making the interest rate constant, i.e. by setting $r_t = r_0 \, \forall t \in \mathcal{T}$.

4.2 Modelling

The modelling of this investment problem is to a large extent identical to the modelling presented in Sect. 2.2; however, adaptations are necessary in the following two areas: the information state and its transition function. The information state component is given by $I_t = (X_t, \delta_t, r_t)$. Hence, $S_t = (t, N_t, Q_t, X_t, \delta_t, r_t)$. The information state I_t evolves to $I_{t+\Delta_h}$ according to:

$$X_{t+\Delta_h} = X_t \exp \left\{ \left(r_t - \delta_t - \frac{\sigma_x^2}{2} \right) \Delta_h + \sigma_x \sqrt{\Delta_h} \epsilon_{t+\Delta_h}^x \right\}, \tag{16}$$

$$\delta_{t+\Delta_h} = \left(1 - e^{-\kappa_\delta \Delta_h} \right) \theta_\delta + e^{-\kappa_\delta \Delta_h} \delta_t + \sigma_\delta \sqrt{\frac{1 - e^{-2\kappa_\delta \Delta_h}}{2\kappa_\delta}} \epsilon_{t+\Delta_h}^\delta, \tag{17}$$

$$r_{t+\Delta_h} = \left(1 - e^{-\kappa_r \Delta_h} \right) \theta_r + e^{-\kappa_r \Delta_h} r_t + \sigma_r \sqrt{\frac{1 - e^{-2\kappa_r \Delta_h}}{2\kappa_r}} \epsilon_{t+\Delta_h}^r, \tag{18}$$

where σ_x, σ_δ and σ_r are the standard deviations of changes in X_t, δ_t and r_t, respectively; κ_δ and κ_r are positive mean reversion (speed of adjustment) coefficients; θ_δ and θ_r are the long run mean of convenience yield and interest rate, respectively; and $\epsilon_{t+\Delta_h}^x$, $\epsilon_{t+\Delta_h}^\delta$ and $\epsilon_{t+\Delta_h}^r$ are correlated standard normal random variables (mean 0, variance 1) with correlation matrix (= covariance matrix Σ here, see Glasserman (2003)):

$$\begin{pmatrix} 1 & \rho_{x,\delta} & \rho_{x,r} \\ \rho_{x,\delta} & 1 & \rho_{r,\delta} \\ \rho_{x,r} & \rho_{r,\delta} & 1 \end{pmatrix}$$

4.3 Overall volatility and model equivalence

One of the key insights from (real) options theory is that there is a non-negative relationship between option value and underlying volatility, so higher volatility in the underlying asset generally results in a higher option value as flexibility becomes more valuable (Dixit and Pindyck 1994). In recent years, this relationship between option value and underlying uncertainty has received increasing interest by academics, e.g. see Caballero (1991); Sarkar (2000); Cappuccio and Moretto (2001); Lund (2005); Gryglewicz et al. (2008). Existing works have studied the investment-uncertainty relationship from a range of perspectives, that is considering different interpretations of this relationship such as the impact of uncertainty on the optimal investment trigger, or the probability that investment will take place in a certain time interval. In this work, we interpret the investment-uncertainty relationship as the effect of overall (or total) volatility on the value of the investment with the portfolio of real options.

As a proxy measure of the investment's actual overall volatility, we use the volatility of commodity futures returns. With regard to commodity futures, Schwartz (1997) derived analytical expressions that describe the volatilities implied by the two- and three-factor model, which, in the limiting case[11], converge to:

$$\sigma_{M_2}^2 = \sigma_x^2 + \frac{\sigma_\delta^2}{\kappa_\delta^2} - \frac{2\rho_{x,\delta}\sigma_x\sigma_\delta}{\kappa_\delta}, \tag{19}$$

and

$$\sigma_{M_3}^2 = \sigma_x^2 + \frac{\sigma_\delta^2}{\kappa_\delta^2} + \frac{\sigma_r^2}{\kappa_r^2} - \frac{2\rho_{x,\delta}\sigma_x\sigma_\delta}{\kappa_\delta} + \frac{2\rho_{x,r}\sigma_x\sigma_r}{\kappa_r} - \frac{2\rho_{r,\delta}\sigma_\delta\sigma_r}{\kappa_\delta\kappa_r}. \tag{20}$$

Even though (19) and (20) describe the volatility of futures returns in the two- and three-factor model, respectively, so consider commodity futures contracts rather than a natural resource investment with flexibility, it may be reasonable to assume that the overall volatility of the copper mine investment with the options portfolio could be represented by a similar functional relationship in terms of the two models' parameters involved. In fact, since these expressions were obtained by Schwartz (1997) from the solution to the PDEs that must be satisfied by futures prices in the respective model, the concept of contingent claims analysis (Cortazar and Schwartz 1994) suggests that if the contingent claim is an investment project—in our case the copper mine—instead of a futures contract with linear payoff, then term structure of the volatility may be obtained, in theory, by expanding the valuation model's PDE(s) accordingly. In this sense, it can be expected that, in the two-factor setting, (19) reflects the investment's actual overall volatility more accurately than (20) does in the three-factor setting, because in the former δ_t affects the valuation only indirectly through X_t, whereas when using the more complex three-factor model r_t has both indirect (via X_t) and direct (as a discount factor) effects on the valuation.

To numerically analyse the effects of the multi-factor models' parameters on the mine value, we perform an equivalence analysis of the three stochastic models. In doing so, we investigate the influence of parameters of the convenience yield (σ_δ, κ_δ, $\rho_{x,\delta}$ and $\rho_{r,\delta}$) and

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[11] For simplicity, we consider the case when time to maturity (of the futures contract) is infinity.

Table 5 Parameters of	Spec.#	$\theta_\delta = \delta_0$	σ_δ	κ_δ
convenience yield process for	1	0.01	0.05	0.30
different specifications of	11	0.01	0.10	0.50
Tsekrekos et al. (2012)	21	0.01	0.15	0.80

interest rate process (σ_r, κ_r, and $\rho_{x,r}$)—described by (17) and (18), respectively—on the implied volatilities of (19)-(20). In fact, we can eliminate the contribution of the convenience yield process to $\sigma_{M_2}^2$ of (19) as well as the contributions of both the convenience yield and interest rate process to $\sigma_{M_3}^2$ of (20) by determining, e.g., the values of $\rho_{x,\delta}$ at which both $\sigma_{M_2}^2$ and $\sigma_{M_3}^2$ equal σ_x^2. In other words, we can determine the respective $\rho_{x,\delta}$-values such that the sum of the 2nd and 3rd term of the right side of (19) becomes zero, and such that the sum of the 2nd to 6th term of the right side of (20) becomes zero, thereby having $\sigma_{M_2}^2 = \sigma_x^2$ and $\sigma_{M_3}^2 = \sigma_x^2$. Analytical expressions for these values of the correlation coefficient $\rho_{x,\delta}$, which we refer to as equivalence correlations, are given by:

$$\rho_{x,\delta}^* = \frac{\sigma_\delta}{2\kappa_\delta \sigma_x}, \tag{21}$$

and

$$\rho_{x,\delta}^* = \frac{\sigma_\delta}{2\kappa_\delta \sigma_x} + \frac{\sigma_r}{\kappa_r}\left[\frac{\sigma_r \kappa_\delta}{2\kappa_r \sigma_\delta \sigma_x} + \frac{\rho_{x,r}\kappa_\delta}{\sigma_\delta} - \frac{\rho_{r,\delta}}{\sigma_x}\right]. \tag{22}$$

Hence, when $\rho_{x,\delta}$ equals the respective $\rho_{x,\delta}^*$ then the volatilities in the multi-factor models equal the copper price variance, σ_x^2, of the one-factor model of Brennan and Schwartz (1985), in which only the copper spot price is stochastic. The mine values from Brennan and Schwartz (1985) are therefore used as benchmark in our equivalence analysis.

4.4 Valuation

For the valuation of this extended mine example, we used the parameter values presented in Sect. 2.4 for the copper mine and, to ensure comparability, of Tsekrekos et al. (2012) for the two multi-factor models. For the sake of our numerical analysis, yet without loss of generality, we focus on the three combinations of parameters of the convenience yield process shown in Table 5. These three specifications correspond with the 1st, 11th, and 21st specification of Tables 3, 4, 5 and 6 of Tsekrekos et al. (2012) and are the most relevant specifications used by the authors. This choice is sufficient for our analysis, more specifically, for studying the effects of different parameters of the two models on the investment value. Additional parameters used for the three-factor model are: $\kappa_r = 0.50$, $\theta_r = r_0 = 0.10$, $\sigma_r = 0.015$, $\rho_{r,\delta} = 0.10$ and $\rho_{x,r} = 0.15$. Also, as Tsekrekos et al. (2012), we considered $X_0 = 0.70$, 100,000 paths (half of which antithetic) and the complete set of polynomials in the parametric model, but, unlike the authors, we used generalised Chebyshev polynomials (with $L = 5$, which implies 56 basis functions).

To evaluate this extension, we adapted the backward procedure of "Appendix A" (now $k_t = r_t + \lambda_1$) and changed the second step of the forward pass to: use (18), (17) and (16) to sample $|\Omega|$ paths of r_t, δ_t and X_t, respectively, giving $(X_t(\omega), \delta_t(\omega), r_t(\omega))_{\omega \in \Omega}$, $\forall t \in \mathcal{T}$. Figure 6 gives an illustration of sample paths.

Table 6 Value of opened mine, $\bar{G}_0(S_0)$ (in US\$ millions), under the two- and three-factor model for the three specifications of Table 5 according to different numerical methods

Spec.#	Our results – IDSR-based			Tsekrekos et al. (2012) – LSM-based[a]		
	$\rho_{x,\delta} = 0.40$	$\rho_{x,\delta} = 0.60$	$\rho_{x,\delta} = 0.80$	$\rho_{x,\delta} = 0.40$	$\rho_{x,\delta} = 0.60$	$\rho_{x,\delta} = 0.80$
Two-factor model						
1	16.210	13.960	11.958	22.625	23.221	25.060
11	16.703	13.518	10.768	23.497	25.095	27.271
21	16.328	13.239	10.500	23.532	24.079	25.789
Three-factor model						
1	16.301	14.043	12.042	24.072	24.865	25.225
11	16.759	13.587	10.844	23.747	25.245	28.174
21	16.403	13.300	10.580	24.545	24.576	26.177

[a]Their results under the two- and three-factor model were obtained from Table 3 on page 552 and Table 5 on page 557, respectively

5 Results and discussion: two- and three-factor model extensions

This section begins with an analysis of the way in which the value of the mine is affected by the different dynamics of the two multi-factor models. Table 6 summarises the results under both the two-factor model of Gibson and Schwartz (1990)—described by (16)-(17) with $r_t = r_0 \, \forall t \in \mathcal{T}$—and the three-factor model of Schwartz (1997) given by (16)-(18), whilst considering the three specifications of Table 5 and three different values of the correlation between the copper price and convenience yield process, $\rho_{x,\delta}$. In addition, Table 6 reports the corresponding results of Tsekrekos et al. (2012), who used the LSM approach. Comparing their results with ours shows that results are noticeably different. Not only (i) are our mine values consistently lower than theirs, they also (ii) exhibit the opposite behaviour with respect to changes in $\rho_{x,\delta}$. Indeed, our mine values decrease in $\rho_{x,\delta}$, whereas Tsekrekos et al. (2012) found values to be increasing in $\rho_{x,\delta}$:

> For the given set of parameters for the short-rate process, project values are found to be increasing […] in the correlation between spot price and convenience yield changes, […] much like in Section 3 where interest rates were assumed constant. (Tsekrekos et al. 2012)

With regard to (ii), it should be noted that the value of a real options portfolio can be affected positively or negatively by correlation between the underlying stochastic factors (Brosch 2008). From (19) and (20) we can observe that the correlation coefficient $\rho_{x,\delta}$ negatively affects the volatility in both the two- ($\sigma^2_{M_2}$) and three-factor model ($\sigma^2_{M_3}$). As such, an increase in $\rho_{x,\delta}$ generally results in a decrease of the value of the mine as overall volatility in the underlying decreases. This is, however, in contrast to what has been found by Tsekrekos et al. (2012). Intuitively, we would expect such a negative relationship considering the way in which δ_t of (17) is nested in the dynamics of X_t in (16). For non-negative $\rho_{x,\delta}$ (Schwartz 1997), Fig. 3 plots the value of an opened mine ($\bar{G}_0(S_0)$) and the volatilities implied by both the two- and three-factor model as a function of $\rho_{x,\delta}$. It can be seen from these figures that the implied volatilities $\sigma^2_{M_2}$ and $\sigma^2_{M_3}$ and, hence, $\bar{G}_0(S_0)$ decrease as $\rho_{x,\delta}$ increases, for the three specifications under consideration. While the implied volatilities decrease linearly in

(a) Specification # 1.

(b) Specification # 11.

(c) Specification # 21.
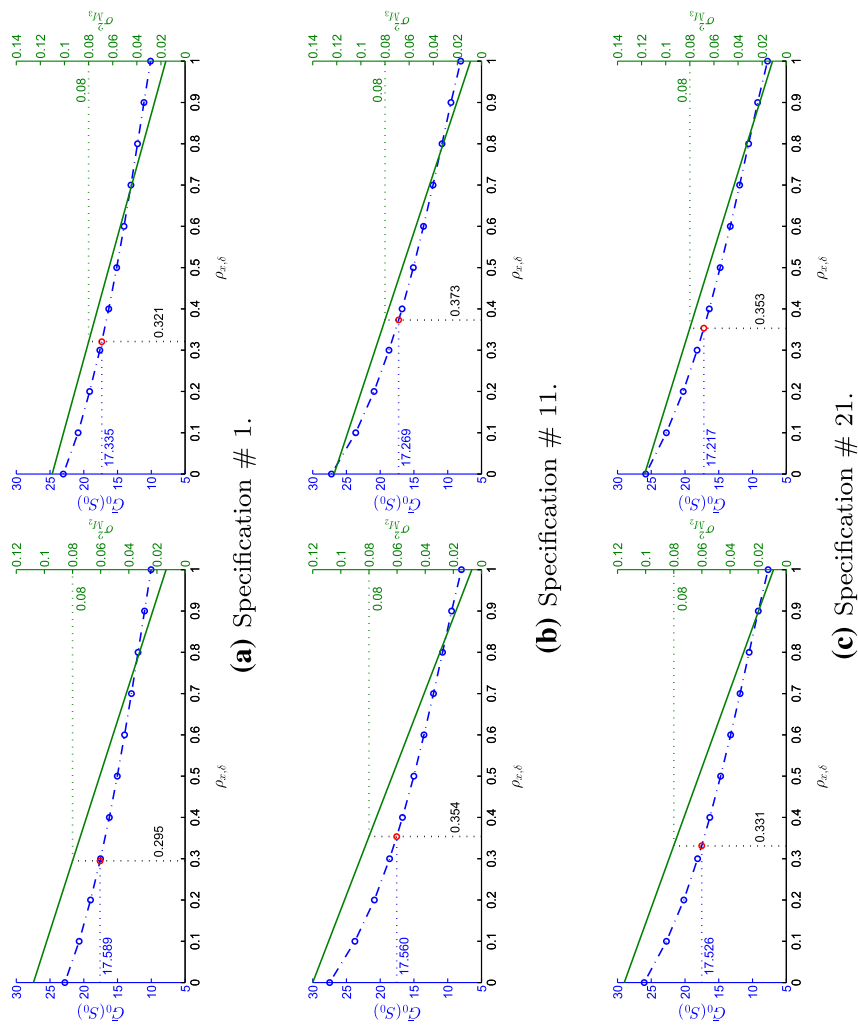
Fig. 3 Value of opened mine, $\bar{G}_0(S_0)$ (in US\$ millions), and volatility in two-factor model ($\sigma_{M_2}^2$, left) and in three-factor model ($\sigma_{M_3}^2$, right) as a function of correlation between copper price and convenience yield process ($\rho_{x,\delta}$)

Table 7 Results of equivalence analysis for both the two- and three-factor model with the three specifications of Table 5

Spec.#	One-factor model[a]		Two-factor model			Three-factor model[b]		
	σ_x^2	$\hat{G}_0(S_0)$	$\rho_{x,\delta}^*$	$\sigma_{M_2^*}^2$	$\bar{G}_0(S_0)$	$\rho_{x,\delta}^*$	$\sigma_{M_3^*}^2$	$\bar{G}_0(S_0)$
1	0.08	17.56	0.295	0.08	17.589	0.321	0.08	17.335
11	0.08	17.56	0.354	0.08	17.560	0.373	0.08	17.269
21	0.08	17.56	0.331	0.08	17.526	0.353	0.08	17.217

[a]Obtained from Table 2, row $X_0 = 0.70$, of Brennan and Schwartz (1985)
[b]With $\kappa_r = 0.50$, $\theta_r = r_0 = 0.10$, $\sigma_r = 0.015$, $\rho_{r,\delta} = 0.10$ and $\rho_{x,r} = 0.15$

$\rho_{x,\delta}$, as evident from (19)-(20), $\bar{G}_0(S_0)$ is a nonlinear function of $\rho_{x,\delta}$ and it is apparent that the decline in mine value, $\frac{\partial \bar{G}_0(S_0)}{\partial \rho_{x,\delta}}$, is larger—i.e. more negative—for lower (higher) values of $\rho_{x,\delta}$ ($\sigma_{M_2}^2$ and $\sigma_{M_3}^2$). This is consistent with the results reported in Fig. 7a and the nonlinear relationship[12] is in line with Schwartz (1997):

> When the option element of the investment is considered, the values obtained under the different models will be nonlinear functions of the spot price (and also of the other factors in the particular model). (Schwartz 1997)

To address (i) and verify the vertical position of the $\bar{G}_0(S_0)$-curves in Fig. 3, we preformed an equivalence analysis. The results are shown by Table 7 and are also included in Fig. 3. It can be observed in Table 7 that the values of the opened mine, $\bar{G}_0(S_0)$, under the two-factor model converge very closely to the benchmark mine value, $\hat{G}_0(S_0)$, for all three specifications under consideration. Even though mine values under the three-factor model are marginally below the one-factor benchmark values, these results are in line with the previously mentioned (and to be expected) differences in quality of the implied volatilities as proxy measures of the mine project's actual overall volatility. This is due to the higher complexity of the three- over the two-factor model as well as other influencing factors related to both the numerical procedure applied here and non-linearities in parameters such as X_0, δ_0 and r_0 (e.g., see Fig. 7a).

According to the above analysis, consistent with option pricing theory, the value of the copper mine decreases in the correlation coefficient $\rho_{x,\delta}$ as a consequence of the decrease in overall volatility. Tsekrekos et al. (2012) also claimed that "values under a stochastic mean-reverting convenience yield will be higher than those under a constant convenience yield assumption". However, it is misleading to suggest that this is always the case. Our equivalence analysis, as indicated in Fig. 3, demonstrates that for $\rho_{x,\delta}$-values below the equivalence correlation ($0 \leq \rho_{x,\delta} < \rho_{x,\delta}^*$), $\bar{G}_0(S_0)$-values under both models are indeed higher than the benchmark mine value under the one-factor model, $\hat{G}_0(S_0)$, which assumes a constant convenience yield. At $\rho_{x,\delta} = \rho_{x,\delta}^*$, we approximately have $\bar{G}_0(S_0) = \hat{G}_0(S_0)$. However, for $\rho_{x,\delta}^* < \rho_{x,\delta} \leq 1$, mine values $\bar{G}_0(S_0)$ are lower than $\hat{G}_0(S_0)$ and, as $\rho_{x,\delta}$ approaches 1, these are even considerably lower than the constant one-factor benchmark value, which was obtained in a constant convenience yield setting and is therefore independent of $\rho_{x,\delta}$.

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[12] It should be noted that it is not entirely clear why Tsekrekos et al. (2012) obtained different results. Intuitively, one might expect results to converge at $\rho_{x,\delta} = 0$. However, comparing their results in Table 6 with ours of Fig. 3 indicates that our mine values would be substantially larger than their values at $\rho_{x,\delta} = 0$, which, although not reported by the authors, can be estimated through extrapolation.

With regard to the three-factor model, Tsekrekos et al. (2012) have also analysed how variations in both the standard deviation of changes in the interest rate (σ_r) and the correlation between the interest rate and convenience yield process ($\rho_{r,\delta}$) affect the value of the opened mine. The authors stated[13]:

> Moreover, […] Figure 3 demonstrates that the value of the investment is increasing in the volatility of the short rate and its correlation with convenience yield changes, since higher variability in expected project cash flows makes the flexibility to alter the operating mode of the project more valuable. (Tsekrekos et al. 2012)

We also performed this analysis and report results for θ_δ equalling 0.15 and 0.12 in Figs. 4 and 8, respectively. The figures on the left- and right-hand sides show $\sigma^2_{M_3}$-values and $\bar{G}_0(S_0)$-values, respectively. Since the authors' choice of value for κ_δ is not given, rather than choosing only one κ_δ-value, as in Tsekrekos et al. (2012), we ran our valuation algorithm for all three possible κ_δ-values of 0.30, 0.50 and 0.80, with results shown in Figures 4a–c, respectively. Of these, it appears that our results for $\kappa_\delta = 0.80$ are qualitatively most similar to those of Tsekrekos et al. (2012). It is evident that the mine value surfaces obtained here are in exceptionally good agreement with the volatility surfaces implied by the three-factor model. For $\sigma_r = 0$, $\bar{G}_0(S_0)$-values are constant because $\sigma^2_{M_3}$ is, as evident from (20), independent of $\rho_{r,\delta}$. For $\sigma_r > 0$, as is apparent from Panel (a) of their figure, we also find mine values to be decreasing in $\rho_{r,\delta}$.

In contrast to Tsekrekos et al. (2012), however, our results demonstrate that the mine value is not always increasing in the volatility of the interest rate process. As we can see from Figs. 4a and 4b, which consider κ_δ=0.30 and κ_δ=0.50, respectively, $\bar{G}_0(S_0)$-values are increasing in σ_r for low values of $\rho_{r,\delta}$, yet decreasing for relatively high $\rho_{r,\delta}$-values. Interestingly, we observe from Fig. 4c (κ_δ=0.80) that while $\bar{G}_0(S_0)$-values increase in σ_r for the four lowest $\rho_{r,\delta}$-values under consideration, there is a twofold effect of the degree of σ_r on the investment value for $0.4 \leq \rho_{r,\delta} \leq 0.8$: $\bar{G}_0(S_0)$-values actually decrease in σ_r for low σ_r-values, but increase in σ_r for high σ_r-values; this change from decrease to increase seems to occur at higher σ_r-values the higher the value of the correlation coefficient $\rho_{r,\delta}$. The evidence provided by our analysis, particularly the volatility surface of Fig. 4c, seems to confirm that the nonlinear dependency of $\sigma^2_{M_3}$ on σ_r is the cause of this non-monotonic effect. It can be inferred therefore that the implied volatilities can be used as a proxy measure to accurately describe how the mine value will be affected by changes in the complex multi-factor models' parameters.

6 Discussion and conclusions

In this work, we have re-evaluated a well-known natural resource investment under three different commodity price models using the IDSR approach. Despite having many advantages as a framework to represent sequential decision problems, IDs have rarely been used in the context of ROA. A reason for this might be, as Wallace (2010) suggests, that real option analysts, like their financial counterparts, are generally interested in determining the value of single well-defined options (possibly compound but still predefined), rather than identifying and defining the portfolio of options. This focus on valuing single options is perhaps derived from financial option theory, which addresses decision making prob-

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[13] However, it should be noted that their statement is not consistent with their illustration because from their Figure it can be seen that the investment value is actually decreasing in the correlation coefficient.
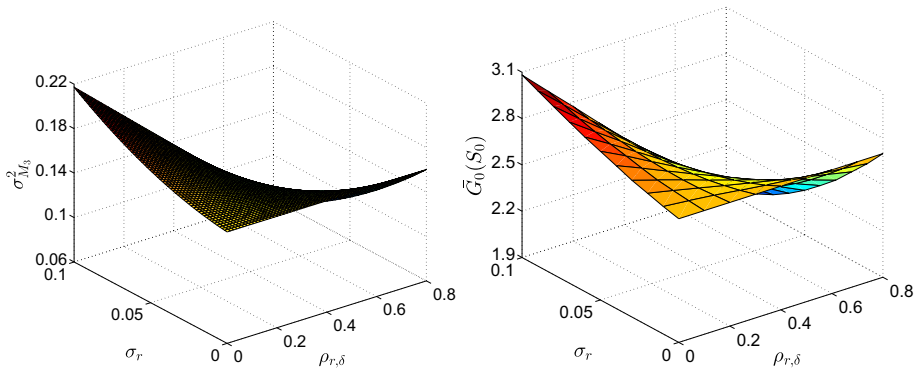
(a) With $\kappa_\delta = 0.30$.
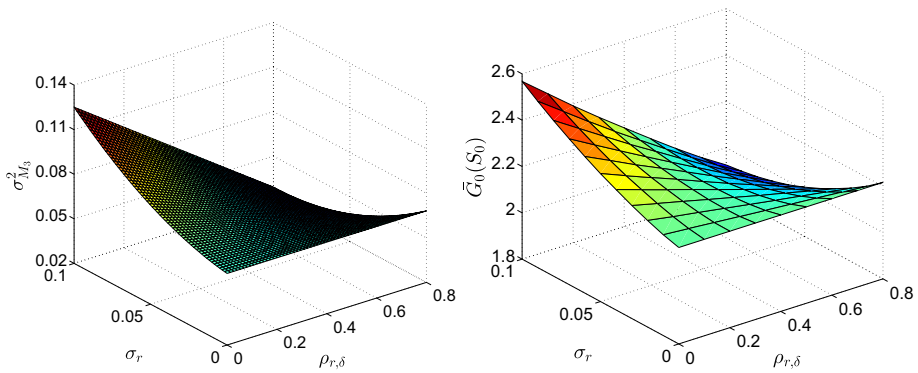
(b) With $\kappa_\delta = 0.50$.
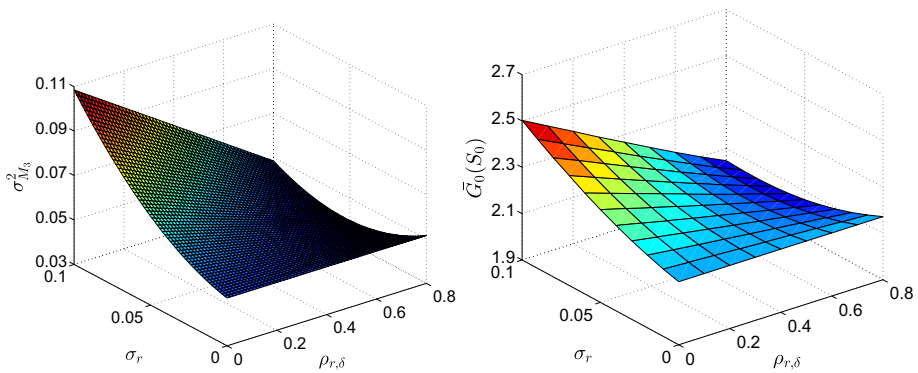
(c) With $\kappa_\delta = 0.80$.

Fig. 4 Volatility in three-factor model ($\sigma_{M_3}^2$) and value of opened mine, $\bar{G}_0(S_0)$ (in US\$ millions), as a function of both the standard deviation of the interest rate (σ_r) and the correlation between the interest rate and convenience yield process ($\rho_{r,\delta}$), with $\theta_\delta = 0.15$

lems in which the representation of the investment proposition requires less sophistication than when considering complex physical assets. However, realistic and practical real option problems are generally more complex, so their analysis benefits from the more sophisticated representation of their underlying decision problems that can be addressed via IDs. Indeed, unlike the intuitively unappealing modelling based on PDEs (Brandão et al. 2005a), the flexibilities available to decision makers in alternative option portfolio configurations can be simply and intuitively represented by an ID, as demonstrated in Figs. 1 and 2.

In order to approximate the value of the portfolio of interdependent real options embedded in the copper mine project, we applied simulation in combination with regression. In contrast, the mathematically more complex valuation approach of Brennan and Schwartz (1985) used PDE-based modelling to describe the value of the mine and then applied a finite difference technique to approximately solve their valuation problem. As shown in Sect. 3, however, there are technical issues at the portfolio decomposition stage in their illustrative example, and given that the inaccuracy was in this relatively simple example, this suggests that the approach is not a good basis for the analysis of real option portfolios. By contrast, the IDSR approach, which can be easily implemented[14] and efficiently applied (e.g. parallel computing), is capable of correctly evaluating both the mine with the options portfolio and its individual real options through simply adapting the optimisation problem's feasible region. Our portfolio analysis showed that the value of the fixed-output-rate mine of Brennan and Schwartz (1985) is incorrect and that this mine appears to include an abandonment option. As a result, the value of their closure option represents the value of mothballing within the portfolio rather than the closure option in isolation.

A controversial issue in the real options community is whether to apply option pricing or decision analysis approaches. Adequately evaluating natural resource investments under uncertainty requires the proper modelling of the embedded sequential stochastic decision problem. Only then is it possible to devise and apply adequate and powerful algorithmic strategies for the valuation of complex and risky investments. We agree with Wallace (2010) in that option pricing theory has traditionally tended to focus on valuation, whilst neglecting the modelling of the underlying sequential decision problem, whereas decision analysis and the related tools generally have the decision context as a starting point. An example of this can be found in Christiansen and Wallace (1998), who compared a decision analysis (decision tree solved via dynamic programming) and an option pricing approach (valuation by arbitrage via a replication argument) using a simple example, and showed that although both approaches deliver the same result, they are methodologically different, with the latter focusing on determining the optimal value whilst delivering the optimal decisions as a consequence, and vice versa.

So are option pricing and decision analysis approaches just two sides of the same coin? This question not only highlights one of the more contentious debates in the field of ROA, but also implies that the approach taken here of integrating PDE-based analytical expressions into our simulation-and-regression-based numerical analysis may well be a more promising approach. While the LSM approach has inherent limitations when it comes to valuing complex real option portfolios (Smith 2005; Brandão et al. 2005a; Maier et al. 2020), it has been successfully extended to such simple switching problems as the one-factor benchmark mine. We therefore believe the issues found in the two multi-factor extensions of Tsekrekos et al. (2012) are related to the simulation steps, most likely to the correlation matrix used for

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[14] Requirements are a standardly available solver (integer programming) and simple least squares.

sampling correlated paths; the fact that the exact source of the error is undetecable is a criticism of "black box" simulation-based approaches (Brandão et al. 2005a; Lambrecht 2017).[15] To validate the values obtained in our re-evaluation, we use a key insight from options theory: higher underlying volatility generally means higher investment value. As a proxy measure of the investment's actual overall volatility, we use PDE-based analytical expressions that describe the volatilities implied by the multi-factor models. This enabled us to validate in a transparent and intuitive way our results, and to provide important insights into the investment-uncertainty relationship.

To conclude this discussion, we believe our re-evaluation of natural resource investments under uncertainty directly addresses a number of open and important research questions in the field of ROA. While there is certainly no "magic bullet" (Smith 2005) for evaluating complex option problems, this study revisits highly influential works in the field and demonstrates an alternative way to solve real-life problems that can provide new insights. Traditional option pricing approaches based on PDE modelling and finite difference approximations—such as the approach introduced in Brennan and Schwartz (1985), which remains as a cornerstone of the real options literature—are known to become impractical in multi-factor situations, but our study also casts doubt on their practicality and reliability in portfolio situations, i.e. when there are multiple, possibly interdependent real options. Indeed, based on our numerical analyses and the above discussions, we believe the limitations of PDE-based approaches owing to (mathematical) complexity and lack of intuition (Glasserman 2003; Brandão et al. 2005b) probably invalidate their use in real option portfolio applications. At the same time, however, analytical expressions obtained by PDE modelling—such as those describing the volatilities implied by multi-factor models—can contribute to validating simulation-based, numerical analyses of complex problems. As demonstrated, the integration of such expressions into our IDSR-based numerical analysis provides important benefits including transparency and intuition, thereby addressing the key challenge (Brandão et al. 2005a; Lambrecht 2017) of simulation-based valuation approaches.

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[15] It should be noted that while the identified errors could probably have been recognised by a similar analysis using lattice-based or other simulation-and-regression-based techniques, the former become impractical in multi-factor situations (see Hinz et al. (2020) for a recent discussion of efficient simulation-and-regression algorithms for pathwise dynamic programming).

Appendix A: Simulation-and-regression-based valuation algorithm

The IDSR algorithm's forward pass[16] consists of the following steps:

1. Determine the set of decision times, $\mathcal{T}_n$, for all decisions nodes $n \in \{1, 4\}$:

$$
\mathcal{T}_n =
\begin{cases}
\{i\Delta_1 : i \in \mathbb{Z}_{\geq 0}, 0 \leq i\Delta_1 \leq T^{max}\}, & \text{if } n = 1, \\
\{i\Delta_1 : i \in \mathbb{Z}_{\geq 0}, \Delta_1 \leq i\Delta_1 \leq T^{max}\}, & \text{if } n = 4,
\end{cases}
\tag{23}
$$

2. Use (9) to sample $|\Omega|$ paths of X_t, giving a set of copper price realisations $\{X_t(\omega) : \omega \in \Omega\}, \forall t \in \mathcal{T}$, where $\mathcal{T} = \bigcup_{n \in \{1,4\}} \mathcal{T}_n$
3. Generate the resource state space $\mathcal{R}_{nt}$ for each decision node n and time t:

$$
\mathcal{R}_{nt} =
\begin{cases}
\{(t, 1, i\Delta_1 q) : i \in \mathbb{Z}_{\geq 0}, Q_0 - \min(Q_0, tq) \leq i\Delta_1 q \leq Q_0 - q\min(\Delta_1, t)\}, & \text{if } n=1, t \in \mathcal{T}_1, \\
\{(t, 4, i\Delta_1 q) : i \in \mathbb{Z}_{\geq 0}, Q_0 - \min(Q_0, tq) + \Delta_1 q \leq i\Delta_1 q \leq Q_0\}, & \text{if } n=4, t \in \mathcal{T}_4.
\end{cases}
\tag{24}
$$

The backward induction procedure is shown by Algorithm 1, with $\check{\Phi}_t(R_t, a_t)$ being an appropriate lower bound (e.g. $\check{\Phi}_t(\cdot) = 0$ if abandoning is possible) and the parametric model and the least-squares regression are shown by (25) and (26), respectively.

Algorithm 1: Approximation of the mine value given by (12)-(15)

Data: From forward induction procedure and Subsections 2.4 and 4.4
Result: $\bar{G}_0(S_0)$

1 **for** $t = \max\{\mathcal{T} \setminus 0\}$ **do**
2   **forall** $R_t \in \{R'_t \in \mathcal{R}_t : b^D(N'_t) \neq \emptyset\}$ **do**
3   **forall** $a_t \in \mathcal{A}_{S_t}$ **do**
4   Use both $\check{\Phi}_t(R_t, a_t)$ and (25)-(26) to determine:

$$F_t(S_t(\omega), a_t) \leftarrow \max\left\{\check{\Phi}_t(R_t, a_t), \hat{\Phi}_t^L(S_t(\omega), a_t)\right\} \forall \omega \in \Omega$$

5   **end**
6   **forall** $\omega \in \Omega$ **do**
7   Compute pathwise optimisers:

$$\hat{a}_t(\omega) \leftarrow \underset{a_t(\omega) \in \mathcal{A}_{S_t(\omega)}}{\arg\max} \left\{\Pi_t(S_t(\omega), a_t(\omega)) + F_t(S_t(\omega), a_t(\omega))\right\}$$

8   Approximate optimal portfolio value along each path ω:

$$\bar{G}_t(S_t(\omega)) \leftarrow \Pi_t(S_t(\omega), \hat{a}_t(\omega)) + e^{-k\Delta_h}\bar{G}_{t+\Delta_h}\left(S^R(R_t, \hat{a}_t(\omega)), I_{t+\Delta_h}(\omega)\right)$$

9   **end**
10   **end**
11     $\mathcal{T} \leftarrow \mathcal{T} \setminus t$
12 **end**
13 At $t = 0$, determine:

$$\bar{G}_0(S_0) \leftarrow \underset{a_0 \in \mathcal{A}_{S_0}}{\max}\left(\Pi_0(S_0, a_0) + \frac{1}{|\Omega|}\sum_{\omega \in \Omega} e^{-k\Delta_h}\bar{G}_{\Delta_h}\left(S^R(R_0, a_0), I_{\Delta_h}(\omega)\right)\right)$$

$$
\hat{\Phi}_t^L(S_t, a_t) = \sum_{l=0}^{L} \hat{\alpha}_l(S^R(R_t, a_t))\phi_l(I_t),
\tag{25}
$$

---

[16] For simplicity, we show only steps for a mine that is opened at time $t = 0$, i.e. $N_0 = 1$.

where L is the model's dimension, the functions $\{\phi_l\}_{l=0}^{L}$ are called basis functions, and the optimal values of the coefficients, $(\alpha_l(S^R(R_t, a_t)))_{l=0}^{L}$, are estimated by:

$$\left(\hat{\alpha}_l(R_{t+\Delta_h})\right)_{l=0}^{L} = \underset{(\alpha_l(\cdot))_{l=0}^{L}}{\arg\min} \left\{ \sum_{\omega \in \Omega} \left[e^{-k\Delta_h} \bar{G}_{t+\Delta_h}(S_{t+\Delta_h}(\omega)) - \sum_{l=0}^{L} \alpha_l(R_{t+\Delta_h})\phi_l(I_t(\omega)) \right]^2 \right\},$$

(26)

where $R_{t+\Delta_h} = S^R(R_t, a_t)$ and $S_{t+\Delta_h}(\omega) = (R_{t+\Delta_h}, I_{t+\Delta_h}(\omega))$.

Appendix B: Illustration of generated sample paths

Considering $X_0 = 0.70$, Fig. 5 shows the evolution of X_t for five generated paths in the one-factor model, whereas Figs. 6a, 6b and 6c show the evolution of X_t, δ_t and r_t, respectively, for five generated paths in the three-factor model.

Fig. 5 Selection of 5 equally likely paths for the evolution of the copper price, X_t

(a) Copper spot price.

(b) Instantaneous convenience yield.


(c) Instantaneous interest rate.
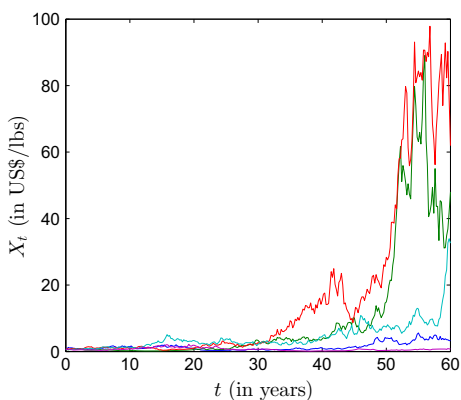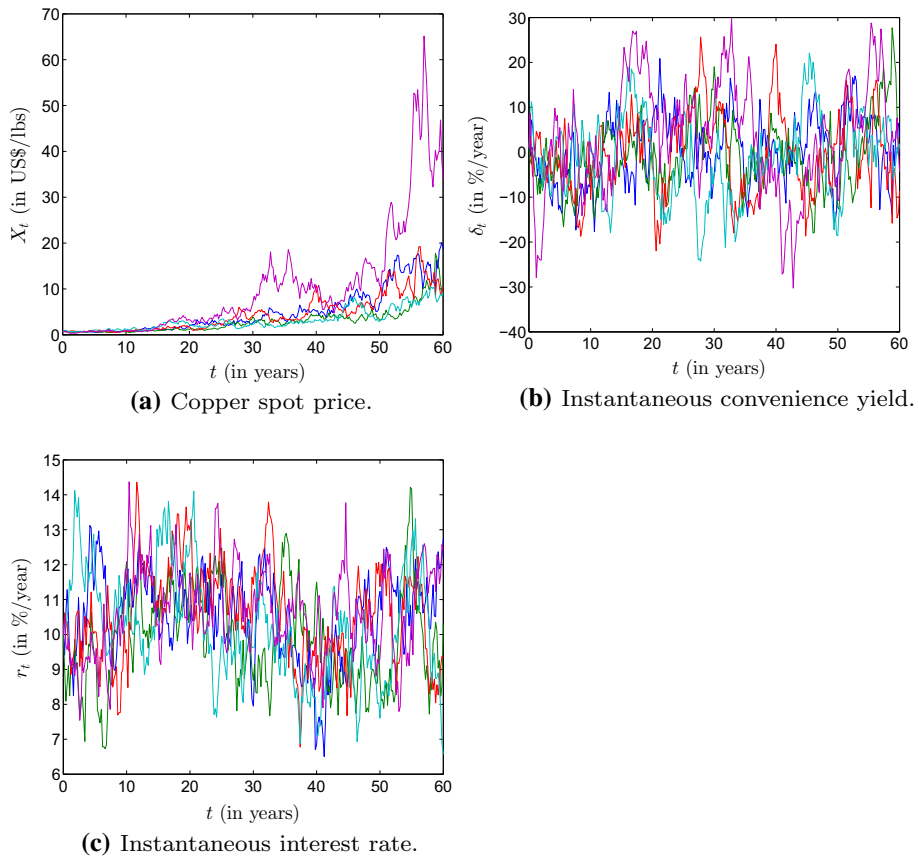
Fig. 6 Selection of 5 equally likely paths for the evolution of the three stochastic factors with specification # 11 and $\rho_{x,\delta} = 0.60$

Appendix C: Effect of copper spot price and its uncertainty on mine value

To illustrate the combined effects of the degrees of the operating margin and copper price uncertainty on the value of the opened mine, Fig. 7 shows the way in which the initial copper price, X_0, and the standard deviation of the copper price, σ_x, affect the investment value. As we would expect, once positive, the value of both the mine with the portfolio of options (i.e. the option to switch , see Fig. 7a) and the mine without options (see Fig. 7b), which applies a static "now-or-never strategy", increases in X_0, and the value of the former also increases as copper price uncertainty increases, whereas the latter decreases in σ_x.

Fig. 7 Value of opened mine (in US$ millions) with portfolio of options and without options as a function of initial copper price (X_0) and price uncertainty (σ_x)
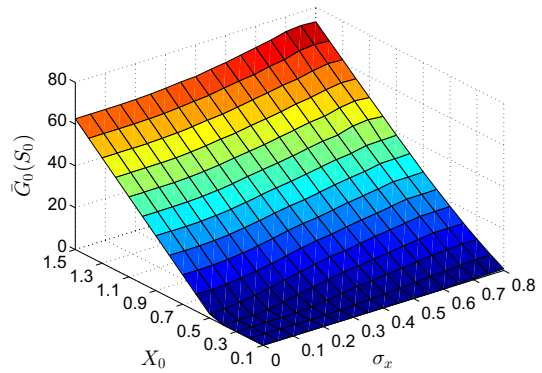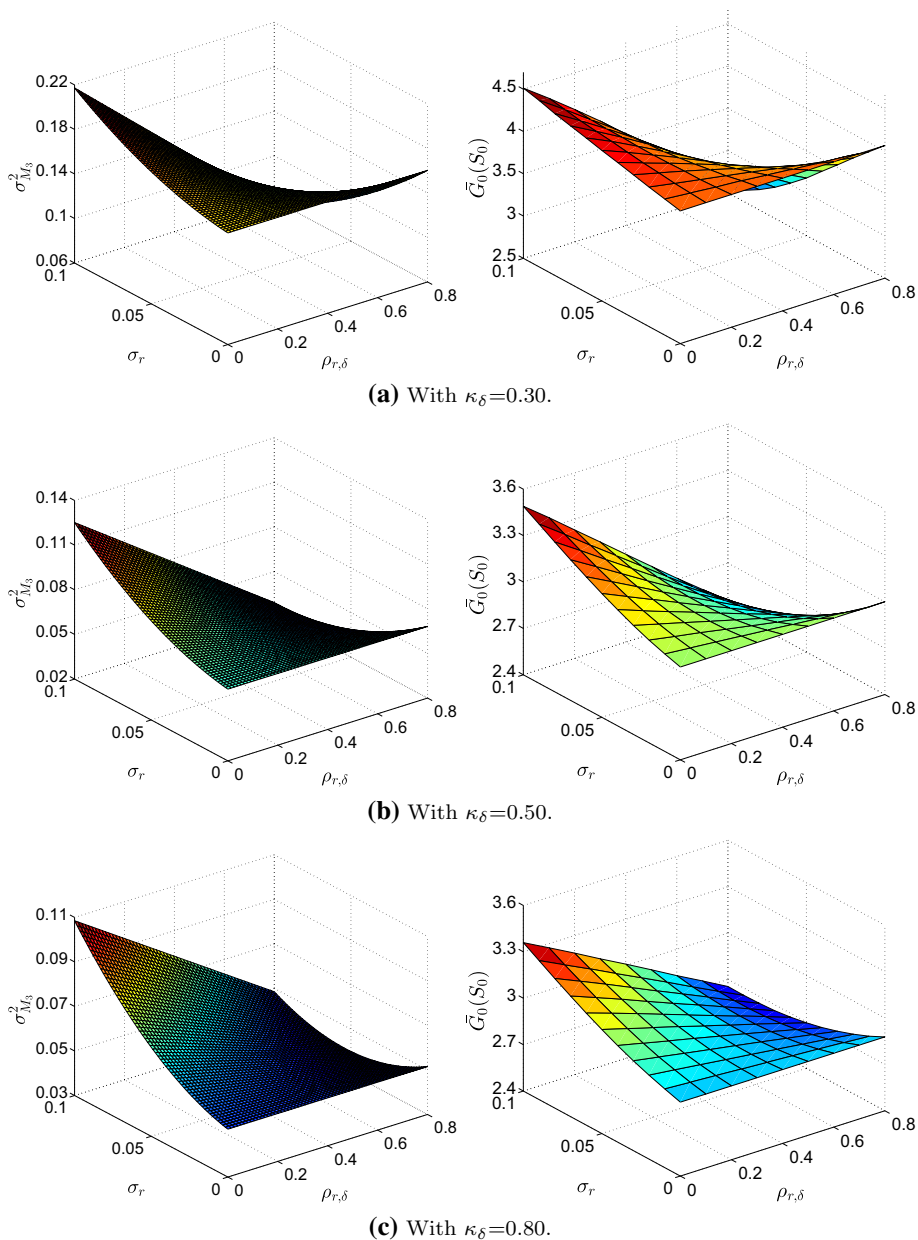

(a) With options portfolio.
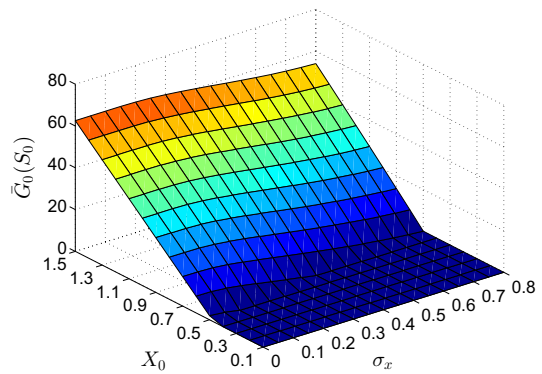
(b) Without options.

Appendix D: Effect of interest rate uncertainty and correlation on valuation

Figure 8 illustrates the same analysis as in Fig. 4 but using $\theta_\delta = 0.12$, with results for κ_δ equalling 0.30, 0.50 and 0.80 shown by Figs. 8a, 8b and 8c, respectively.

Fig. 8 Volatility in three-factor model ($\sigma_{M_3}^2$) and value of opened mine, $\bar{G}_0(S_0)$ (in US\$ millions), as a function of both the standard deviation of the interest rate (σ_r) and the correlation between the interest rate and convenience yield process ($\rho_{r,\delta}$), with $\theta_\delta = 0.12$

References

Abdel Sabour, S. A., & Poulin, R. (2006). Valuing real capital investments using the least-squares monte carlo method. *The Engineering Economist*, *51*(2), 141–160.

Bailey, W. (1991). Valuing agricultural firms: An examination of the contingent-claims approach to pricing real assets. *Journal of Economic Dynamics and Control*, *15*(4), 771–791.

Barraquand, J., & Martineau, D. (1995). Numerical valuation of high dimensional multivariate American securities. *Journal of Financial and Quantitative Analysis*, *30*(3), 383–405.

Bjerksund, P., & Ekern, S. (1990). Managing investment opportunities under price uncertainty: From "last chance" to "wait and see" strategies. *Financial Management*, *19*(3), 65–83.

Boyle, P. P. (1977). Options: A monte carlo approach. *Journal of Financial Economics*, *4*(3), 323–338.

Boyle, P. P. (1988). A lattice framework for option pricing with two state variables. *The Journal of Financial and Quantitative Analysis*, *23*(1), 1–12.

Brandão, L. E., Dyer, J. S., & Hahn, W. J. (2005a). Response to comments on Brandão et al. (2005). *Decision Analysis*, *2*(2), 103–109.

Brandão, L. E., Dyer, J. S., & Hahn, W. J. (2005b). Using binomial decision trees to solve real-option valuation problems. *Decision Analysis*, *2*(2), 69–88.

Brennan, M. J., & Schwartz, E. S. (1985). Evaluating natural resource investments. *The Journal of Business*, *58*(2), 135–157.

Brosch, R. (2008). *Portfolios of real options, Lecture notes in economics and mathematical systems* (Vol. 611). Berlin: Springer.

Caballero, R. J. (1991). On the sign of the investment-uncertainty relationship. *The American Economic Review*, *81*(1), 279–288.

Cappuccio, N., & Moretto, M. (2001). Comments on the investment-uncertainty relationship in a real option model. FEEM Working Paper 28.2001, Fondazione Eni Enrico Mattei.

Carriere, J. F. (1996). Valuation of the early-exercise price for options using simulations and nonparametric regression. *Insurance: Mathematics and Economics*, *19*(1), 19–30.

Casassus, J., & Collin-Dufresne, P. (2005). Stochastic convenience yield implied from commodity futures and interest rates. *The Journal of Finance*, *60*(5), 2283–2331.

Castillo-Ramirez, A. (2000). An application of natural resource evaluation using a simulation-dynamic programming approach. *Journal of Computational Finance*, *3*(2), 91–107.

Christiansen, D. S., & Wallace, S. W. (1998). Option theory and modeling under uncertainty. *Annals of Operations Research*, *82*, 59–82.

Clarke, H. R., & Reed, W. J. (1989). The tree-cutting problem in a stochastic environment: The case of age-dependent growth. *Journal of Economic Dynamics and Control*, *13*(4), 569–595.

Cortazar, G., & Casassus, J. (1998). Optimal timing of a mine expansion: Implementing a real options model. *The Quarterly Review of Economics and Finance*, *38*(3), 755–769.

Cortazar, G., Gravet, M., & Urzua, J. (2008). The valuation of multidimensional american real options using the LSM simulation method. *Computers & Operations Research*, *35*(1), 113–129.

Cortazar, G., & Schwartz, E. S. (1994). The valuation of commodity contingent claims. *The Journal of Derivatives*, *1*(4), 27–39.

Cortazar, G., & Schwartz, E. S. (1997). Implementing a real option model for valuing an undeveloped oil field. *International Transactions in Operational Research*, *4*(2), 125–137.

Cortazar, G., & Schwartz, E. S. (1998). Monte carlo evaluation model of an undeveloped oil field. *Journal of Energy Finance & Development*, *3*(1), 73–84.

Cortazar, G., & Schwartz, E. S. (2003). Implementing a stochastic model for oil futures prices. *Energy Economics*, *25*(3), 215–238.

Cortazar, G., Schwartz, E. S., & Casassus, J. (2001). Optimal exploration investments under price and geological-technical uncertainty: A real options model. *R&D Management*, *31*(2), 181–189.

Cox, J. C., Ross, S. A., & Rubinstein, M. (1979). Option pricing: A simplified approach. *Journal of Financial Economics*, *7*(3), 229–263.

Dixit, A. K., & Pindyck, R. S. (1994). *Investment under uncertainty*. Princeton: Princeton University Press.

Ekern, S. (1988). An option pricing approach to evaluating petroleum projects. *Energy Economics*, *10*(2), 91–99.

Gamba, A. (2003). Real options valuation: A Monte Carlo approach. Working Paper 2002/3, Faculty of Management, University of Calgary

Gibson, R., & Schwartz, E. S. (1990). Stochastic convenience yield and the pricing of oil contingent claims. *The Journal of Finance*, *45*(3), 959–976.

Glasserman, P. (2003). *Monte Carlo methods in financial engineering, applications of mathematics* (1st ed., Vol. 53). New York: Springer.

Gryglewicz, S., Huisman, K. J., & Kort, P. M. (2008). Finite project life and uncertainty effects on investment. *Journal of Economic Dynamics and Control*, *32*(7), 2191–2213.

Hinz, J., Tarnopolskaya, T., & Yee, J. (2020). Efficient algorithms of pathwise dynamic programming for decision optimization in mining operations. *Annals of Operations Research*, *286*, 586–615.

Kelly, S. (1998). A binomial lattice approach for valuing a mining property ipo. *The Quarterly Review of Economics and Finance*, *38*(3), 693–709.

Lambrecht, B. M. (2017). Real options in finance. *Journal of Banking and Finance*, *81*, 166–171.

Lin, C. G., & Wang, Y. S. (2012). Evaluating natural resource projects with embedded options and limited reserves. *Applied Economics*, *44*(12), 1471–1482.

Longstaff, F. A., & Schwartz, E. S. (2001). Valuing american options by simulation: A simple least-squares approach. *Review of Financial Studies*, *14*(1), 113–147.

Lund, D. (2005). How to analyze the investment-uncertainty relationship in real option models? *Review of Financial Economics*, *14*(3), 311–322.

Maier, S., Polak, J. W., & Gann, D. M. (2020). Valuing portfolios of interdependent real options using influence diagrams and simulation-and-regression: A multi-stage stochastic integer programming approach. *Computers & Operations Research*, *115*, 104505.

McCarthy, J., & Monkhouse, P. H. L. (2002). To open or not to open-or what to do with a closed copper mine. *Journal of Applied Corporate Finance*, *15*(2), 63–73.

Morck, R., Schwartz, E., & Stangeland, D. (1989). The valuation of forestry resources under stochastic prices and inventories. *Journal of Financial and Quantitative Analysis*, *24*(4), 473–487.

Paddock, J. L., Siegel, D. R., & Smith, J. L. (1988). Option valuation of claims on real assets: The case of offshore petroleum leases. *Quarterly Journal of Economics*, *103*(3), 479–508.

Pindyck, R. S. (1980). Uncertainty and exhaustible resource markets. *Journal of Political Economy*, *88*(6), 1203–1225.

Quigg, L. (1993). Empirical testing of real option-pricing models. *The Journal of Finance*, *48*(2), 621–640.

Raymar, S. B., & Zwecher, M. J. (1997). Monte carlo estimation of american call options on the maximum of several stocks. *The Journal of Derivatives*, *5*(1), 7–23.

Sarkar, S. (2000). On the investment-uncertainty relationship in a real options model. *Journal of Economic Dynamics and Control*, *24*(2), 219–225.

Schwartz, E. S. (1997). The stochastic behavior of commodity prices: Implications for valuation and hedging. *The Journal of Finance*, *52*(3), 923–973.

Smith, J. E. (2005). Alternative approaches for solving real-options problems (comment on Brandão et al. 2005). *Decision Analysis*, *2*(2), 89–102.

Tilley, J. A. (1993). Valuing american options in a path simulation model. *Transactions of the Society of Actuaries*, *45*, 499–520.

Tourinho, O. A. F. (1979). The option value of reserves of natural resources. Working Paper 94, University of California at Berkeley.

Trigeorgis, L. (1990). A real-options application in natural-resource investments. *Advances in Futures and Options Research*, *4*, 154–164.

Trigeorgis, L., & Tsekrekos, A. E. (2018). Real options in operations research: A review. *European Journal of Operational Research*, *270*(1), 1–24.

Tsekrekos, A. E., Shackleton, M. B., & Wojakowski, R. (2012). Evaluating natural resource investments under different model dynamics: Managerial insights. *European Financial Management*, *18*(4), 543–575.

Tsitsiklis, J. N., & Van Roy, B. (2001). Regression methods for pricing complex american-style options. *IEEE Transactions on Neural Networks*, *12*(4), 694–703.

Wallace, S. W. (2010). Stochastic programming and the option of doing it differently. *Annals of Operations Research*, *177*(1), 3–8.