THE VALUE OF KNOWING THE MARKET PRICE OF RISK

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ABSTRACT. This paper presents an optimal allocation problem in a financial market with one risk-free and one risky asset, when the market is driven by a stochastic market price of risk. We solve the problem in continuous time, for an investor with a Constant Relative Risk Aversion (CRRA) utility, under two scenarios: when the market price of risk is observable (the *full information case*), and when it is not (the *partial information case*). The corresponding market models are complete in the partial information case and incomplete in the other case, hence the two scenarios exhibit rather different features. We study how the access to more accurate information on the market price of risk affects the optimal strategies and we determine the maximal price that the investor would be willing to pay to get such information. In particular, we examine two cases of additional information, when an exact observation of the market price of risk is available either at time 0 only (the *initial information case*), or during the whole investment period (the *dynamic information case*).

Keywords: Portfolio optimization, Power utility, Martingale Method, Partial Information.

1. INTRODUCTION

Ours is a classical expected utility maximization problem in continuous time, first studied by Merton (1969) [28]. We solve it via the martingale method, proposed by Karatzas el al. (1987) [22] and by Cox and Huang (1989) [9]. The martingale method relies on duality theory and transforms the original dynamic problem, usually solved through the Hamilton-Jacobi-Bellman (HJB) partial differential equation, into an equivalent static optimization problem. It has two main advantages over the more direct HJB approach: it leads to a quasi-linear partial differential equation that is usually easier to solve and it provides an expression of the optimal wealth as a function of the state price density, that can be used to relate the optimal strategy to the current state of the market.

A necessary assumption to apply the standard martingale method is that the state price density is unique, that is the market is complete. However for our problem, in the

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full information case, the trading strategies may also depend on the market price of risk, that is a not traded asset, and this makes the corresponding market model incomplete. Therefore we must rely on a modification of the standard approach, the so called minimax martingale method, introduced by He and Pearson (1991) [20]. The minimax method exploits the fact that, in an incomplete market, there are infinitely many state price densities but they all assign the same values to the marketable claims, i.e. those claims attainable by admissible trading strategies involving the market securities. Hence, the optimal final wealth is determined by selecting the state price density which minimizes the maximal expected utility of the final wealth, the *minimax state price density*.

We model the stock as a geometric brownian motion with a market price of risk given by an Ornstein-Uhlenbeck process, which is a Gaussian, mean reverting process. This is a convenient assumption, adopted by several studies which will be mentioned below, and that may be justified by empirical evidence. To solve the investment problem under partial information it is necessary to identify the *filter*, that is the conditional distribution of the unobservable process given the available information. The assumptions on the model of the market allows us to apply the linear finite dimensional Kalman filter to identify the dynamics of the filter and characterize its conditional distribution, see, e.g. Lipster and Shiryaev (2001) [27]. Then, following Fleming and Pardoux (1982) [14], we transform the original optimization problem into an equivalent one where all the state variables are adapted to the same filtration. Under this transformation the market model is complete and the classical martingale method can be used, see e.g. Björk, Davis and Landen (2010) [6].

Another important consequence of the assumptions on the dynamics of the assets and on the utility of the investor is that the state variables of the market model, represented by the (logarithm of) minimax state price density and the market price of risk in the full information case, and by the unique state price density and the filter in the partial information case, are jointly affine. This fact allows us to compute the corresponding optimal wealths (and strategies) in closed form, after solving a system of Riccati equations that is homogeneous under full information and non-homogeneous under partial information.

We apply the results for the full and partial information problem to compute the value of initial and dynamic information. Of course, by increasing the information set, the investor gets a higher expected optimal utility. To measure the subjective value of such enlargements we compute the corresponding reservation prices. Again, because of the structure of the model, their expressions are simple. The last part of the paper is devoted to numerical examples that illustrate a few applications of our results. Some of the results are rather unexpected: to mention at least one of them, we will see that to maximize the Sharpe ratio of an investment, one should follow the strategy of a partially informed portfolio manager with CRRA utility rather than that of a fully informed one!

Our study relies on a long list of previous contributions, which we will mention below, but its closest references are Kim and Omberg (1996) [23] and Brendle (2006) [7], who

study optimal investment problems similar to ours by using the HJB approach, showing that the HJB equations can be reduced to a system of Riccati equations. In particular, while Kim and Omberg (1996) [23] are interested in the full information case for HARA utility functions, Brendle (2006) [7] also focuses on a partially informed investor endowed with bounded CRRA preferences, extending his analysis to a multi-dimensional market model. We rely on many of their results, especially those related to the solutions to the Riccati equations. However, both these papers fail to provide a verification theorem for their results, that is, they only show necessary, but not sufficient, conditions for the optimality of their proposed solutions. In particular, Kim and Omberg (1996) [23] write $\hat{a}AIJ(...)$ we are not acquainted with any verification theorem that fits the model above. despite its relative simplicity (...) àÅİ. They mention the fact that the classical verification theorems cannot be applied because the indirect utility that solves the Bellman equation is a function of the investor wealth that is not restricted to a closed set. Hence, they can only solve numerically their HJB equation to suggest that, for a given choice of parameters, there should be no signs of multiple solutions. Instead, by using the martingale approach, we prove verification theorems for both the cases of our interest (Theorem 3.2 and 4.1), we apply them to our solutions to show that they are effectively optimal (Theorem 3.3) and 4.3), and we provide sufficient conditions that are easy to verify directly on any set of parameters (Proposition 3.4 and 4.4).

Having mentioned what we believe are the most important theoretical contributions of this paper, we summarize the other ones. We derive the distribution, conditional and unconditional, of the optimal wealth under full and under partial information at any time within the investment horizon (Proposition 3.5 and 4.5). To the best of our knowledge such results are new and have never been addressed before, despite the fact that the knowledge of the distribution of the optimal wealth may be useful for applications in portfolio and risk management. Another novelty inspired the title of our paper, that is we assign a price to the information that an investor may buy from an expert who is able to provide the value of the market price of risk either at the beginning of the investment period or continuously in time. We hope that this result may provide a new tool to measure the level of uncertainty on the returns of an asset, to be used along with the standard ones based on volatility or implied volatility. To support our theoretical findings we provide a rich numerical analysis that also shows interesting and sometimes unexpected results. Last, but not least, by solving the optimization problem under full information, we show a new application of the powerful minimax martingale approach where it is possible to explicitly identify the minimax state price density and the associated penalization process. This is a nice example that may be useful for didactical purposes.

Before completing this introduction with a literature review, we provide a short description of the rest of the paper. In Section 2 we introduce the modeling framework. Section 3 solves the optimization problem under full information, while Section 4 under partial information. In both sections we characterize the optimal investment strategy and provide a closed form representation for the optimal wealth as a function of the relevant state variables as well as for its distribution. In Section 5 we define and compute the value of initial and dynamic information. A numerical study to illustrate the effects of the parameters on the distribution of the wealth and on the value of information is presented in Section 6. Section 7 concludes.

1.1. Literature review. Optimal investment problem in continuous time started from the work of Merton (1969) [28], and extended since then in many directions with the scope of including more realistic situations. The extension considered by us, when the drift of asset prices is not directly observable by the investor, leads to problems of partial information. Problems of this type have been addressed considering several modelizations of the unobservable risk factors. Contributions in the case where prices are modeled as diffusions can be found for instance in Lackner (1995) [24], Lackner (1998) [25] and Brendle (2006) [7] under different approaches. In Brennan (1998) [8] and Xia (2001) [34] the authors discuss the effect of learning on the portfolio choices. The setting where prices depend on an unobservable Markov chain is considered, for instance in Bäuerle and Rieder (2005) [5] and Haussmann and Sass (2004) [19] and Barucci and Marazzina (2015) [4]. Considering investors endowed with different levels of information motivates the work of Fouque, Papanicolaou, and Sircar (2015) [15] who analyze the loss of utility due to partial information. In Frey, Gabih and Wunderlich (2012) [17] and Gabih et al. (2014) [18] expert opinions in the form of signals at random discrete time points are investigated. This idea is extended to the case where expert opinions arrive continuously in time by Fouque, Papanicolaou and Sircar (2017) [16] and by Davis and Lleo (2013) [11] who implement the BlackâASLitterman model in a continuous time setting and use separability of the filtering problem and the stochastic control problem to incorporate analyst views and non-tradeable assets as additional source of observation to estimate the filter. A similar setting has been studied by Danilova, Monoyios and Ng (2010) [10] who assumed that the investor has partial information about the drift of the stock price but also some privileged information about the future of stock price. In Putschögl and Sass (2008) [32] an optimal investment and consumption problem under partial observation is analysed using Malliavin calculus. Investment problems in a market with two cointegrated assets under partial information are studied in some recent works as for instance Lee and Papanicolaou (2016) [26] and Altay et al. (2018, 2019) [1], [2].

The issue of assessing the value of information is a classical one in economics and finance. Pikovsky and Karatzas (1996) [30] presented the problem of computing the value of initial information for an investor, defined as the informational gain in terms of incremental utility provided by the access to an enlarged filtration. Amendinger, Becherer and Schweizer (2003) [3] addressed the problem of computing the value of information for a trader who has the opportunity to buy some extra information. The problem is formulated for a complete market in the mathematical framework of an initially enlarged filtration, and the value of information is derived via a comparison of the expected utility from terminal wealth. Chau, Cosso and Fontana (2018) [13] extended their approach to estimate the value of an insider information that may allow for an arbitrage opportunity, assuming that unbounded profits cannot be reached with bounded risk.

We model the market price of risk as an Ornstein-Uhlenbeck mean reverting process. We refer to Wachter (2002) [33] for a review of the most important empirical contributions justifying such assumption. Wachter (2002) [33] solved the optimal investment and consumption problem in a complete model where the market price of risk and the stock return are perfectly negatively correlated. A setting close to ours under full information where returns of the risky asset are driven by an Ornstein-Uhlenbeck process was proposed by Kim and Omberg (1996) [23], who studied the portfolio optimization problem for a HARA investor and discussed the existence of the so called *nirvana solutions*, which happen when an infinite expected utility is reached in finite time. An application of this setting to the problem of a fund manager whose compensation depends on the relative performance with respect to a benchmark can be found in Nicolosi, Angelini and Herzel (2018) [29], and Herzel and Nicolosi (2019) [21].

2. The general setting

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a fixed probability space endowed with a complete and right continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{\{0 \le t \le T\}}$ representing the global information flow where T is a fixed time horizon. All processes defined below are assumed to be \mathbb{F} -adapted. We consider a market model with one risky asset S, the stock, and one risk-free asset B with dynamics

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma dZ_t^S \quad S_0 = s_0 \in \mathbb{R}^+,$$
$$\frac{dB_t}{B_t} = rdt, \quad B_0 = 1$$

where $\sigma > 0$ and $r \ge 0$ are constant, Z^S is a standard, one dimensional, \mathbb{F} -Brownian motion and the drift process μ is of the form

$$\mu_t = r + \sigma X_t.$$

The process X represents the *market price of risk* X and is assumed to follow an Ornstein-Uhlenbeck process with dynamics

$$dX_t = -\lambda (X_t - \bar{X})dt + \sigma_X dZ_t^X,$$

where X_0 is a normally distributed random variable with mean π_0 and variance R_0 , $\lambda > 0$ is a constant representing the strength of attraction toward the long term expected mean $\bar{X} \ge 0$, $\sigma_X > 0$ is the volatility of the market price of risk and Z^X is a standard onedimensional F-Brownian motion correlated with Z^S with

$$d\langle Z^X, Z^S \rangle_t = \rho \ dt,$$

for a constant correlation coefficient $\rho \in [-1, 1]$. Let Z^{\perp} be a Brownian motion independent of Z^S such that $Z^X = \rho Z^S + \sqrt{1 - \rho^2} Z^{\perp}$. Then without loss of generality we can assume that \mathbb{F} is the complete and right continuous filtration generated by (Z^S, Z^{\perp}) .

An investor trades the risky asset and the risk-free asset continuously in time, starting from an initial capital w, to maximize the expected utility of her final wealth at time T. Her trading strategy is self-financing and given by the process $\theta = \{\theta_t, t \in [0, T]\}$ representing the proportion of portfolio value invested in the risky asset at time $t \in [0, T]$. We assume the standard integrability condition on θ

$$\mathbf{E}\left[\int_{0}^{T} \left(\left|\theta_{t} X_{t}\right| + \theta_{t}^{2}\right) dt\right] < \infty.$$
(2.1)

to ensure that the associated wealth process

$$\frac{dW_t}{W_t} = (r + \theta_t \sigma X_t)dt + \theta_t \sigma dZ_t^S, \quad W_0 = w > 0.$$
(2.2)

is well defined (note that we also assume that no dividends are paid by the stock before time T) and to exclude arbitrage opportunities. Further restrictions on the measurability of the process θ depending on the information set of the investor will be given in the next sections.

The investor has a power utility function

$$u(x) = \frac{1}{1 - \gamma} x^{1 - \gamma},$$
(2.3)

for every x > 0 and for a positive risk aversion parameter $\gamma \neq 1$. By setting $\gamma = 1$ we get the logarithmic utility $u(x) = \log x$. Note that for $\gamma > 1$ the function u(x) is bounded above while it becomes unbounded when $\gamma < 1$.

We will solve optimization problems corresponding to two different assumptions on the information flow available to the investor. In the first case, we assume that the investor observes both the stock price and the market price of risk; therefore her information flow is given by the global filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$. In particular the initial information \mathcal{F}_0 is given by the sigma algebra generated by X_0 enlarged with the collection of **P**-null sets. In the second case, we assume that the investor directly observes stock prices but not the market price of risk. At any time $t \in [0,T]$, the value of X has to be inferred from the available information, represented by the natural filtration generated by the stock price process completed by the **P**-null sets and denoted by \mathbb{F}^S , where the initial information is $\mathcal{F}_0^S = \{\emptyset, \Omega\}$.

3. Optimal investment under full information

In this section we consider a fully informed investor who observes the path of the stock and of the market price of risk. The investor wants to maximize the expected utility of her wealth at a time T > 0, hence her problem is

$$\max_{\theta \in \mathcal{A}(w)} \mathbf{E}\left[u(W_T)\right] \tag{3.1}$$

where $\mathcal{A}(w)$ is the set of \mathbb{F} -predictable self-financing strategies satisfying the integrability condition (2.1) starting from an initial wealth w. Problem (3.1) is equivalent to

$$\max_{\theta \in \mathcal{A}(w)} \mathbf{E} \left[\frac{1}{1 - \gamma} W_T^{1 - \gamma} | \mathcal{F}_0 \right].$$
(3.2)

In fact, a strategy $\theta^* \in \mathcal{A}(w)$ is optimal for (3.1) if and only if it is optimal for (3.2) almost surely, because θ_0^* is \mathcal{F}_0 -measurable. In the sequel we will denote by \mathbf{E}_t the conditional expectation given \mathcal{F}_t .

In this setting there are two risk factors and only one asset that can be used as a hedging instrument, therefore the market is not complete. The state price densities ξ^{ν} satisfy the equation

$$\frac{d\xi_t^{\nu}}{\xi_t^{\nu}} = -rdt - X_t dZ_t^S - \nu_t \sqrt{1 - \rho^2} dZ_t^{\perp}, \quad \xi_0^{\nu} = 1$$
(3.3)

where $\nu = \{\nu_t, t \in [0, T]\}$ is such that $\mathbf{E}\left[\int_0^T \nu_t^2 dt\right] < \infty$ and $\mathbf{E}\left[\int_0^T |\xi_t^\nu|^2 dt\right] < \infty$.

The process Z_t^{\perp} is orthogonal to the space of attainable payoffs (i.e. payoffs that can be reached by feasible self-financing strategies). If the stock price process and the market price of risk are perfectly correlated (positively or negatively), $\sqrt{1-\rho^2}$ vanishes, the market becomes complete and the state price density is unique.

To solve the optimal investment problem in an incomplete market we follow He and Pearson (1991) [20] and apply the martingale approach to transform the dynamic problem (3.2) into the equivalent static one

$$\min_{\nu} \max_{W_T} \mathbf{E}_0[u(W_T)], \tag{3.4}$$

subject to the constraint

$$w = \mathbf{E}_0[\xi_T^{\nu} W_T]. \tag{3.5}$$

The optimal ν^* for the problem (3.4)-(3.5) determines the minimax state price density process ξ^* . The role of the process ν^* is to penalize contingent claims that cannot be replicated by feasible portfolio strategies. For example, $\nu_t^* = 0$ **P**-a.s. for $t \in [0, T]$ implies that no penalization is necessary and a feasible optimal strategy is naturally chosen by the investor. For power utility functions of the form (2.3), a sufficient condition for the existence of ξ^* is that $\gamma > 1$, see He and Pearson (1991) [20, Theorem 4 and Theorem 6].

The Lagrangian function associated to problem (3.4)-(3.5) is

$$\mathcal{L}(W_T, \lambda_0) = \mathbf{E}_0[u(W_T)] - \lambda_0(\mathbf{E}_0[\xi_T^* W_T] - w),$$

where λ_0 is the multiplier from the constraint (3.5). From standard results (e.g. Karatzas et al. (1987) [22]), the optimal final wealth satisfies

$$W_T^* = g(\lambda_0 \xi_T^*) \tag{3.6}$$

where function $g(\cdot)$ is the inverse of the marginal utility u'. For the power utility (2.3) $g(y) = y^{-\frac{1}{\gamma}}$. Since ξ^* is a state-price density process, the optimal wealth at time t is

$$W_t^* = \xi_t^{*-1} \mathbf{E}_t \left[\xi_T^* g(\lambda_0 \xi_T^*) \right]$$
$$= \xi_t^{*-1} \lambda_0^{-\frac{1}{\gamma}} \mathbf{E}_t \left[\xi_T^{*1-\frac{1}{\gamma}} \right].$$

Remark 3.1. This approach can also be applied to logarithmic utility functions. In this case $g(y) = \frac{1}{y}$, the optimal wealth at time t is

$$W_t^* = \xi_t^{*-1} \mathbf{E}_t \left[\xi_T^* g(\lambda_0 \xi_T^*) \right] = (\lambda_0 \xi_t^*)^{-1}$$

By applying Itô formula to (3.8) we get

$$\frac{dW_t^*}{W_t^*} = (r + X_t^2 + (1 - \rho^2)(\nu_t^*)^2)dt + X_t dZ_t^S + \sqrt{1 - \rho^2}\nu_t^* dZ_t^\perp.$$
(3.9)

By equating the predictable quadratic covariations of W^* and Z^S computed from (2.2) and (3.9) we get the optimal strategy $\theta_t^* = \frac{X_t}{\sigma}$. This strategy is called myopic because it does not depend on the investment horizon. Note that in this case the minimax state price density is associated to the penalty process $\nu_t^* = 0$.

The following verification theorem states sufficient conditions to solve the optimization problem.

Theorem 3.2 (Verification Theorem under full information). Let the function F(Y, X, t) be the solution to the partial differential equation (where subscripts denote partial derivatives)

$$\frac{1}{2} \quad F_{YY}Y^2 \left(X^2 + \nu^* (Y, X, t)^2 (1 - \rho^2) \right) + F_{XY}Y\sigma_X \left(\rho X + \nu^* (Y, X, t) (1 - \rho^2) \right) + \quad \frac{1}{2}F_{XX}\sigma_X^2 - F_X \left(\sigma_X \left(\rho X + \nu^* (Y, X, t) (1 - \rho^2) \right) + \lambda (X - \bar{X}) \right) + \quad F_t - rF + rF_YY = 0$$

where

$$\nu^*(Y, X, t) = -\frac{\sigma_X F_X(Y, X, t)}{Y F_Y(Y, X, t)}$$

with the boundary conditions

$$F(Y, X, T) = Y^{\frac{1}{\gamma}}$$
$$F(Y_0, X_0, 0) = w$$

for some constant $Y_0 > 0$, $F_Y(Y, X, t) \neq 0$, and $F(Y, X, t) \rightarrow F(Y, X, T)$ as $t \rightarrow T$. Assume that the following conditions hold:

(i) the function $\nu^*(Y, X, t)$ is sublinear and locally Lipschitz for $(Y, X, t) \in \mathbb{R}^+ \times \mathbb{R} \times [0, T]$ (this condition implies that the process ξ_t^* satisfying (3.3) with $\nu_t^* := \nu^*(Y_0(\xi_t^*)^{-1}, X_t, t)$ is a well defined local martingale),

$$\mathbf{E}\left[\int_{0}^{T} ((Y_{t}^{-1}F(Y_{t},X_{t},t))^{2} + (F_{Y}(Y_{t},X_{t},t))^{2})(X_{t}^{2} + (\nu^{*}(Y_{t},X_{t},t))^{2})dt\right] < \infty, \qquad (3.14)$$

where $Y_t := Y_0(\xi_t^*)^{-1}$, for $t \in [0, T]$.

Then the process ξ_t^* is the minimax state price density, the optimal wealth is $W_t^* = F(Y_t, X_t, t)$ and the optimal investment strategy is

$$\theta_t^* = \frac{F_Y(Y_t, X_t, t) Y_t X_t + \rho \sigma_X F_X(Y_t, X_t, t)}{\sigma F(Y_t, X_t, t)},$$
(3.15)

for every $t \in [0, T]$.

Proof. To show that $F(Y_t, X_t, t)$ is the optimal wealth process, we need to verify that the initial wealth satisfies the budget constraint and that the final wealth satisfies the first order condition (3.6) and is attainable by a self-financing strategy.

The budget constraint (3.5) follows from (3.13), and the first order condition from (3.12), since $g(y) = y^{-\frac{1}{\gamma}}$. Since ξ_t^* is well defined by condition (i) and Y_0 is given by (3.13), we can define the process $Y_t := Y_0 (\xi_t^*)^{-1}$. From Itô formula the dynamics of Y_t is

$$\frac{dY_t}{Y_t} = (r + X_t^2 + (1 - \rho^2)(\nu^*(Y_t, X_t, t))^2)dt + X_t dZ_t^S + \sqrt{1 - \rho^2}\nu^*(Y_t, X_t, t)dZ_t^{\perp},$$

and, still applying Itô,

$$F(Y_t, X_t, t) = F(Y_0, X_0, 0) + \int_0^t \mathcal{L}F(Y_s, X_s, s)ds + \int_0^t \left(\rho\sigma_X F_X(Y_s, X_s, s) + Y_s X_s F_Y(Y_s, X_s, s)\right) dZ_s^S + \int_0^t \sqrt{1 - \rho^2} \left(\sigma_X F_X(Y_s, X_s, s) + Y_s \nu^*(Y_s, X_s, s)F_Y(Y_s, X_s, s)\right) dZ_s^\perp, \quad (3.16)$$

where \mathcal{L} is the differential operator

$$\mathcal{L}F = F_t + \frac{1}{2} F_{YY} Y^2 \left(X^2 + \nu^* (Y, X, t)^2 (1 - \rho^2) \right) + \sigma_X F_{XY} Y \left(\rho X + \nu^* (Y, X, t) (1 - \rho^2) \right) + \frac{1}{2} F_{XX} \sigma_X^2 - F_X \lambda (X - \bar{X}) + F_Y Y (r + X^2 + (\nu^* (Y, X, t))^2 (1 - \rho^2)).$$

By (3.11), the integral with respect to Z^{\perp} in (3.16) vanishes, therefore the final wealth $F(Y_T, X_T, T)$ belongs to the space of attainable payoffs. To show that it can be obtained by a self-financing strategy starting from w it remains to show that the process $\xi_t^* F(Y_t, X_t, t)$

is a true martingale. From Itô formula and assumption (3.10), we get

$$\begin{split} \xi_t^* F(Y_t, X_t, t) &= F(Y_0, X_0, 0) \\ &+ \int_0^t \xi_s^* \left(\rho \sigma_X F_X(Y_s, X_s, s) + Y_s X_s F_Y(Y_s, X_s, s) - F(Y_s, X_s, s) X_s\right) dZ_s^S \\ &- \sqrt{1 - \rho^2} \int_0^t \xi_s^* F(Y_s, X_s, s) \nu^*(Y_s, X_s, s) dZ_s^\perp, \end{split}$$

which is a true martingale because

$$\begin{split} \mathbf{E} \left[\int_{0}^{T} \left((\xi_{s}^{*})^{2} \left(\rho \sigma_{X} F_{X}(Y_{s}, X_{s}, s) + Y_{s} X_{s} F_{Y}(Y_{s}, X_{s}, s) - F(Y_{s}, X_{s}, s) X_{s} \right)^{2} \\ &+ (1 - \rho^{2}) (\xi_{s}^{*})^{2} (F(Y_{s}, X_{s}, s))^{2} (\nu^{*}(Y_{s}, X_{s}, s))^{2} \right) dt \right] \\ = \mathbf{E} \left[\int_{0}^{T} ((\xi_{s}^{*})^{2} (Y_{s} F_{Y}(Y_{s}, X_{s}, s) (X_{s} - \rho \nu^{*}(Y_{s}, X_{s}, s)) - F(Y_{s}, X_{s}, s) X_{s})^{2} \\ &+ (1 - \rho^{2}) (\xi_{s}^{*})^{2} (F(Y_{s}, X_{s}, s))^{2} (\nu^{*}(Y_{s}, X_{s}, s))^{2} \right) dt \right] \\ &\leq c_{1} \mathbf{E} \left[\int_{0}^{T} \left((X_{s}^{2} + (\nu^{*}(Y_{s}, X_{s}, s))^{2}) (F_{Y}(Y_{s}, X_{s}, s))^{2} \\ &+ (\xi_{s}^{*})^{2} (F(Y_{s}, X_{s}, s))^{2} (X_{s}^{2} + (\nu^{*}(Y_{s}, X_{s}, s))^{2}) \right) dt \right], \end{split}$$

that is bounded by (3.14) (c_1 is a positive constant). Note that the first equality comes from (3.11), and in the inequality we have used $(a + b)^2 \leq 2(a^2 + b^2)$, $\rho^2 < 1$, $1 - \rho^2 < 1$, and the definition of Y_t .

Therefore, $W_t^* = F(Y_t, X_t, t)$ is the optimal wealth process and ξ_t^* is the minimax state price density (see He and Pearson (1991) [20, Theorem 8]). Finally, by equating the predictable quadratic covariations of W^* and Z^S computed from (2.2) and (3.16) we get the optimal strategy (3.15).

To determine a closed form expression for W_t^* we guess that the joint process $(\log(\xi^*), X, X^2)$ is affine. From this guess it follows that the conditional expectation in (3.7) is

$$\mathbf{E}_{t}\left[\xi_{T}^{*\,1-\frac{1}{\gamma}}\right] = \xi_{t}^{*\,1-\frac{1}{\gamma}} e^{A(t)+B(t)X_{t}+\frac{1}{2}C(t)X_{t}^{2}}$$

where the functions A(t), B(t) and C(t) satisfy the system of Riccati equations

$$\begin{cases} \frac{dC}{dt} = -a - bC(t) - cC(t)^2, \\ \frac{dB}{dt} = -C(t)\lambda\bar{X} - \left(\frac{b}{2} + cC(t)\right)B(t), \\ \frac{dA}{dt} = \frac{\gamma - 1}{\gamma}r - B(t)\lambda\bar{X} - \frac{1}{2}C(t)\sigma_X^2 - \frac{1}{2}cB(t)^2 \end{cases}$$
(3.17)

with boundary conditions

$$A(T) = B(T) = C(T) = 0, (3.18)$$

for constants

$$a = \frac{1-\gamma}{\gamma^2}, \quad b = 2\left(-\lambda + \frac{1-\gamma}{\gamma}\rho\sigma_X\right), \quad c = \sigma_X^2\left(\rho^2 + \gamma(1-\rho^2)\right).$$

To prove that our guess is correct we must show that it satisfies Theorem 3.2. Before that, we discuss the behavior of the solutions to the problem (3.17)-(3.18) without reporting them explicitly, as they can be found, for instance, in Kim and Omberg (1996) [23].

Let us define $\Delta := b^2 - 4ac = 4\left(p - \frac{q}{\gamma}\right)$, where $p := \lambda^2 + 2\lambda\rho\sigma_X + \sigma_X^2$, and $q := 2\lambda\rho\sigma_X + \sigma_X^2$. It is easily seen that p > q and $p \ge 0$. In particular, if $\rho \ne -1$ and $\sigma_X \ne \lambda$, then $p \ne 0$. We define the critical correlation value

$$\rho^* = -\frac{\sigma_X}{2\lambda} \lor -1, \tag{3.19}$$

and, for $\rho \ge \rho^*$, the critical risk aversion parameter

$$\gamma^* = \frac{q}{p}.\tag{3.20}$$

Note that $0 \le \gamma^* < 1$, where the lower bound follows from the assumption on ρ . According to the classification by Kim and Omberg (1996) [23], there are four possible cases:

- i. If $\rho \ge \rho^*$ and $\gamma^* < \gamma < 1 \cup \gamma > 1$, then $\Delta > 0$ and the solution, called "well-behaved normal", exists for every t in [0,T].
- ii. If $\rho > \rho^*$ and $0 < \gamma < \gamma^*$, then $\Delta < 0$. The solution is called "tangent" and is defined on $[0, T^*)$, where

$$T^* = \frac{\pi}{\eta} - \frac{2}{\eta} \arctan \frac{b}{\eta}$$
(3.21)

with $\eta = \sqrt{-\Delta}$. In such a case the investment horizon T has to be lower than T^* in order that solution exists over the entire interval [0, T].

- iii. If $\rho > \rho^*$ and $\gamma = \gamma^*$, we get the "well-behaved hyperbolic" solution which exists for every t in [0,T]. This case corresponds to $\Delta = 0$ and b < 0.
- iv. If $\rho < \rho^*$ and $\gamma > 0, \gamma \neq 1$, then q < 0 and hence $\Delta > 0$. The solution is "well-behaved normal" and exists for every t in [0, T].

Wachter (2002) [33] noted that, for real market data, the correlation between returns and the market price of risk is usually negative and close to -1, hence case (iv) should be the most relevant for financial applications.

After providing the conditions under which the system of Riccati equations has a solution, we can state some sufficient conditions for our guess to provide the optimal wealth and the optimal policy for the full information case. **Theorem 3.3.** Let the functions A(t), B(t) and C(t) satisfy (3.17)- (3.18) on [0,T], let

$$\nu_t^* := -\gamma \left(B(t) + C(t) X_t \right) \sigma_X,$$

$$\lambda_0 := \left[\frac{e^{A(0) + B(0) X_0 + \frac{1}{2}C(0) X_0^2}}{w} \right]^{\gamma},$$

and let ξ_t^* be the state price density process associated to ν_t^* . If

$$\mathbf{E}\left[\int_{0}^{T} \left((\xi_{t}^{*})^{1-\frac{1}{\gamma}} e^{A(t)+B(t)X_{t}+\frac{1}{2}C(t)X_{t}^{2}} \right)^{2} (1+X_{t}^{2}) dt \right] < \infty,$$
(3.24)

then ξ_t^* is the minimax state price density, λ_0 is the Lagrange multiplier for problem (3.4)-(3.5), and

$$W_{t}^{*} = (\lambda_{0}\xi_{t}^{*})^{-\frac{1}{\gamma}} e^{A(t) + B(t)X_{t} + \frac{1}{2}C(t)X_{t}^{2}},$$

$$\theta_{t}^{*} = \frac{1}{\gamma}\frac{X_{t}}{\sigma} + \rho\frac{\sigma_{X}}{\sigma}(B(t) + C(t)X_{t})$$
(3.25)

are the optimal wealth and the optimal strategy.

Proof. Let us define

$$F(Y, X, t) := Y^{\frac{1}{\gamma}} e^{A(t) + B(t)X + \frac{1}{2}C(t)X^2},$$
(3.26)

for $(Y, X, t) \in \mathbb{R}^+ \times \mathbb{R} \times [0, T]$. We need to check that the function F satisfies the assumptions of Theorem 3.2, and hence the optimal wealth process is equal to $F(Y_t, X_t, t)$, where $Y_t := Y_0(\xi_t^*)^{-1}$.

From assumption (3.22) we see that ν^* verifies Equation (3.11); moreover, it is sublinear and locally Lipschitz for $(Y, X, t) \in \mathbb{R}^+ \times \mathbb{R} \times [0, T]$, hence condition (i) of Theorem 3.2 is satisfied. Since the functions A(t), B(t) and C(t) satisfy (3.17), function F(Y, X, t) solves (3.10). Moreover $F_Y(Y, X, t) \neq 0$. The boundary conditions (3.18) imply that

$$F(Y, X, T) = Y^{\frac{1}{\gamma}},$$

and therefore condition (3.12) is also true. Moreover, imposing the budget constraint

$$Y_0^{\frac{1}{\gamma}} e^{A(0) + B(0)X_0 + \frac{1}{2}C(0)X_0^2} = w$$

we get that the value of $Y_0 > 0$ that satisfies condition (3.13) is $Y_0 = \lambda_0^{-1}$, where λ_0 is given by (3.23). From (3.6) it follows that λ_0 is the Lagrange multiplier for problem (3.4)-(3.5).

To check that condition (3.14) in Theorem 3.2 is satisfied we use the fact that $Y_t = (\lambda_0 \xi_t^*)^{-1}$, and the definitions of F given in (3.26) and of ν^* in (3.22), to get

$$\mathbf{E} \left[\int_{0}^{T} ((Y_{t}^{-1}F(Y_{t},X_{t},t))^{2} + (F_{Y}(Y_{t},X_{t},t))^{2}) (X_{t}^{2} + (\nu^{*}(Y_{t},X_{t},t))^{2}) dt \right] \\
= \mathbf{E} \left[\int_{0}^{T} ((\lambda_{0}\xi_{t}^{*})^{1-1/\gamma} e^{A(t)+B(t)X+\frac{1}{2}C(t)X^{2}})^{2}) (X_{t}^{2} + \gamma^{2}\sigma_{X}^{2}(B(t) + C(t)X_{t})^{2}) dt \right] \\
\leq c_{2}\mathbf{E} \left[\int_{0}^{T} ((\xi_{t}^{*})^{1-1/\gamma} e^{A(t)+B(t)X+\frac{1}{2}C(t)X^{2}})^{2}) (1+X_{t}^{2}) dt \right]$$

for some constant $c_2 > 0$, because B(t) and C(t) are continuous functions on [0, T]. The last term is bounded by assumption (3.24). This completes the proof.

The following result provides some conditions for Theorem 3.3 that are easier to check than (3.24).

Proposition 3.4. If at least one of the following two holds:

(i) $\gamma > 1$ (ii) the functions A(t), B(t) and C(t) satisfy (3.17)- (3.18) on [0,T] and

$$1 - 4C(0) \max\left(R_0, R_0 e^{-2\lambda T} + \frac{\sigma_X^2}{2\lambda} (1 - e^{-2\lambda T})\right) > 0.$$
 (3.27)

Then all assumptions of Theorem 3.3 are verified.

Proof. Recall that for $\gamma > 1$ the functions A, B, C are well defined on [0, T].

Then, we only need to show that condition (3.24) is satisfied. By Cauchy Schwartz inequality, using Fubini and $Y_t = (\lambda_0 \xi_t^*)^{-1}$,

$$\begin{split} \mathbf{E} \left[\int_{0}^{T} e^{2A(t)} \left(\xi_{t}^{*1-\frac{1}{\gamma}} \right)^{2} e^{2B(t)X_{t}+C(t)X_{t}^{2}} (1+X_{t}^{2}) dt \right] \\ & \leq \kappa \int_{0}^{T} e^{2A(t)} \mathbf{E} \left[\xi_{t}^{*8(1-\frac{1}{\gamma})} \right]^{\frac{1}{4}} \mathbf{E} \left[e^{4B(t)X_{t}+2C(t)X_{t}^{2}} \right]^{\frac{1}{2}} \mathbf{E} [(1+X_{t}^{2})^{4}]^{\frac{1}{4}} dt, \end{split}$$

where k is a positive constant. Considering each expectation separately, first we have

$$\mathbf{E}\left[\xi_t^{*8(1-\frac{1}{\gamma})}\right] < \infty,$$

for every $t \in [0, T]$, since X is an Ornstein-Uhlenbeck process (see, e.g. Revuz and Yor (2013) [31, Chapter 8, Ex. 3.14]). Second,

$$\mathbf{E}[(1+X_t^2)^4] < \infty$$

for every $t \in [0, T]$, since X_t is a Gaussian random variable and hence has moments of all orders. Finally

$$\mathbf{E}\left[e^{4B(t)X_t+2C(t)X_t^2}\right] < \infty$$

for every $t \in [0,T]$ if and only if $1 - 4C(t)v_t > 0$, where $v_t = R_0 e^{-2\lambda t} + \frac{\sigma_X^2}{2\lambda}(1 - e^{-2\lambda t})$ is the variance of X_t .¹

To show that $1 - 4C(t)v_t > 0$ we use that C(t) is strictly negative and increasing on [0,T] if $\gamma > 1$, and is strictly positive and decreasing if $\gamma < 1$ (see Kim and Omberg (1996) [23, Equation (23)]). Then, for $\gamma > 1$, C(t) < 0, therefore $1 - 4C(t)v_t > 0$. When $\gamma < 1$, C(t) is positive and decreasing, hence C(t) < C(0). Moreover, let $v_{\infty} := \frac{\sigma_X^2}{2\lambda}$, then v_t is increasing and $R_0 \leq v_t \leq v_T$ if $R_0 < v_{\infty}$ and decreasing with $v_T \leq v_t \leq R_0$ otherwise. This means that $1 - 4C(t)v_t > 1 - 4C(0) \max(R_0, v_T)$. The result then follows from (3.27).

Condition (3.27) can be easily verified on any set of parameters, but it is more restrictive than Condition (3.24), that is more difficult to check. In the section devoted to the applications we show graphically, in Figure 2, how much restrictive Condition (3.27) is with respect to the domain of existence of the corresponding system of Riccati equations.

Kim and Omberg (1996) [23] and Brendle (2006) [7] solved a problem similar to ours by using the Hamilton-Jacobi-Bellman (HJB) approach. To recover their results, we compute the expected optimal utility at time t

$$\begin{aligned} \mathbf{E}_{t} \left[u(W_{T}^{*}) \right] &= \frac{1}{1 - \gamma} \lambda_{0}^{1 - \frac{1}{\gamma}} \mathbf{E}_{t} \left[(\xi_{T}^{*})^{1 - \frac{1}{\gamma}} \right] \\ &= \frac{1}{1 - \gamma} \lambda_{0} \xi_{t}^{*} W_{t}^{*} \\ &= \frac{1}{1 - \gamma} W_{t}^{*1 - \gamma} e^{\gamma (A(t) + B(t)X_{t} + \frac{1}{2}C(t)X_{t}^{2})}, \end{aligned}$$

where (3.28) follows from (3.6), (3.29) from (3.7), and (3.30) from (3.25). Equation (3.30) corresponds to the formulas [23, Equation (16)] and [7, Equation (14)].

By plugging (3.25) into (3.30), we can also compute the conditional expected optimal utility as

$$\mathbf{E}_{t}\left[u(W_{T}^{*})\right] = \frac{1}{1-\gamma} \left(\lambda_{0}\xi_{t}^{*}\right)^{1-1/\gamma} e^{A(t)+B(t)X_{t}+\frac{1}{2}C(t)X_{t}^{2}}.$$
(3.31)

An advantage of the martingale approach over HJB is that it allows us to compute both the optimal wealth (3.25) and the expected optimal utility (3.31) as functions of the minimax state price density ξ^* and of the market price of risk X. This may be useful to study the dependence on the current state of the market.

From (3.30), we can also derive the (unconditional) expected optimal utility, that exists when

$$Q(0) := 1 - \gamma C(0)R_0$$

 $[\]frac{1}{\sqrt{1-c\sigma^2}} \text{ For a random variable } \varepsilon \sim \mathcal{N}(\mu, \sigma^2), \text{ if } 1 - c\sigma^2 > 0, \mathbf{E}[e^{a+b\varepsilon+\frac{1}{2}c\varepsilon^2}] = \frac{1}{\sqrt{1-c\sigma^2}} \exp\left(a + \frac{b^2\sigma^2}{2(1-c\sigma^2)} + \frac{b\mu}{1-c\sigma^2} + \frac{c\mu^2}{2(1-c\sigma^2)}\right).$

is strictly positive under the hypotheses of Proposition 3.4. ² Indeed, since $X_0 \sim N(\pi_0, R_0)$, and using the formula provided in Footnote 1, we get

$$\mathbf{E} \left[u(W_T^*) \right] = \mathbf{E} \mathbf{E}_0 \left[u(W_T^*) \right] = \frac{w^{1-\gamma}}{1-\gamma} \mathbf{E} \left[e^{\gamma (A(0) + B(0)X_0 + \frac{1}{2}C(0)X_0^2)} \right]$$

$$= \frac{w^{1-\gamma}}{(1-\gamma)\sqrt{Q(0)}} e^{\gamma A(0) + \frac{\gamma}{2Q(0)} \left(\gamma B(0)^2 R_0 + 2\pi_0 B(0) + C(0)\pi_0^2 \right)}$$

To study the conditional distribution of the optimal wealth we compute the conditional moment generating function of $\ln W_t^*$,

$$\phi_s(t,z) := \mathbf{E}_s\left[(W_t^*)^z \right], \tag{3.33}$$

on its domain of existence.

Proposition 3.5. Let $\phi_s(t, z) < \infty$ for $0 \le s \le t \le T$ and z > 0. Then

$$\phi_s(t,z) = (\lambda_0 \xi_s^*)^{-\frac{z}{\gamma}} e^{D(s;t,z) + E(s;t,z)X_s + \frac{1}{2}H(s;t,z)X_s^2}$$
(3.34)

where the functions $D: [0,t] \to \mathbb{R}$, $E: [0,t] \to \mathbb{R}$ and $H: [0,t] \to \mathbb{R}$ satisfy the system of differential equations³

$$\begin{cases} \frac{dH}{ds} = d(s) + 2e(s)H(s) - \sigma_X^2 H(s)^2 \\ \frac{dE}{ds} = f(s) + (e(s) - \sigma_X^2 H(s)) E(s) + g(s)H(s) \\ \frac{dD}{ds} = h(s) + g(s)E(s) - \frac{\sigma_X^2}{2}(H(s) + E(s)^2) \end{cases}$$
(3.35)

with boundary conditions

$$D(t) = zA(t),$$
 $E(t) = zB(t),$ $H(t) = zC(t),$ (3.36)

and

$$d(s) = -\left(\frac{z^2}{\gamma^2} + \frac{z}{\gamma}\right) \left(1 + \gamma^2 \sigma_X^2 (1 - \rho^2) C(s)^2\right) e(s) = \lambda - \frac{z}{\gamma} \sigma_X \rho + z \sigma_X^2 (1 - \rho^2) C(s) f(s) = -(z^2 + z\gamma) \sigma_X^2 (1 - \rho^2) B(s) C(s) g(s) = z \sigma_X^2 (1 - \rho^2) B(s) - \lambda \bar{X} h(s) = -\frac{1}{2} (z^2 + z\gamma) \sigma_X^2 (1 - \rho^2) B(s)^2 - \frac{zr}{\gamma}$$

where the functions $A(\cdot), B(\cdot)$ and $C(\cdot)$ solve (3.17)–(3.18).

²In the next section we will provide further conditions for the positiveness of Q(0) (see Proposition 4.2).

³The dependence on t and z for the functions D, E, H in system (3.35) is omitted for ease of notation.

Proof. From (3.25) and the fact that process $(\ln \xi^*, X, X^2)$ is affine, it follows that

$$\phi_s(t,z) = \lambda_0^{-z/\gamma} \mathbf{E}_s \left[\xi_t^{*-z/\gamma} e^{zA(t) + zB(t)X_t + \frac{1}{2}zC(t)X_t^2} \right] = \lambda_0^{-z/\gamma} G(s,\xi_s^*,X_s;t,z)$$

where

$$G(s,\xi_s^*,X_s;t,z) = \xi_s^{*-z/\gamma} e^{D(s;t,z) + E(s;t,z)X_s + \frac{1}{2}H(s;t,z)X_s^2}.$$
(3.37)

The boundary conditions (3.36) follow from $\phi_t(t, z) = W_t^{*z}$. The function $G(s, \xi, x; t, z)$ is differentiable with respect to s, and twice differentiable with respect to ξ and x. Moreover, by definition, the process $(G(s, \xi_s, X_s; t, z))_{\{s \in [0,t]\}}$ is a martingale. Hence, by applying Itô's formula we get

$$0 = \frac{\partial G}{\partial s} - r\xi \frac{\partial G}{\partial \xi} - \lambda (x - \bar{X}) \frac{\partial G}{\partial x} + \frac{1}{2} \left(\frac{\partial^2 G}{\partial x^2} \sigma_X^2 + \xi^2 \frac{\partial^2 G}{\partial \xi^2} (\nu^2 (1 - \rho^2) + x^2) + 2\xi \sigma_X \frac{\partial^2 G}{\partial \xi \partial x} (-\nu (1 - \rho^2) - x\rho) \right) (3.38)$$

By plugging (3.37) into Equation (3.38) and collecting the constant term and the factors of X and X^2 , we get that D, E, H are the unique solution to problem (3.35)-(3.36) (see Filipović (2009) [12, Lemma 10.1]).

We note that the solution to problem (3.35)-(3.36) assumes a simple form in the special case corresponding to the computation of the conditional expectation of $W_T^{*1-\gamma}$. In fact, from (3.33) and (3.34), we get

$$\mathbf{E}_{t}\left[u(W_{T}^{*})\right] = \frac{1}{1-\gamma}\phi_{t}(T, 1-\gamma) = \frac{1}{1-\gamma}\left(\lambda_{0}\xi_{t}^{*}\right)^{1-1/\gamma}e^{D(t;T,1-\gamma)+E(t;T,1-\gamma)X_{t}+\frac{1}{2}H(t;T,1-\gamma)X_{t}^{2}}.$$

Hence, from (3.31),

Such relations can also be directly verified by substituting $z = 1 - \gamma$ and t = T in (3.35)-(3.36).

4. Optimal investment under partial information

In this section we assume that the investor observes only the stock prices and not the market price of risk. Hence, the available information is carried by the filtration \mathbb{F}^S generated by the process S and the investor can only adopt \mathbb{F}^S -adapted portfolio strategies. Here the standard procedure is to apply separability and transform the optimization problem under partial information into an equivalent one by means of filtering, see, e.g. Fleming and Pardoux (1982) [14]. The first step of this procedure consists of replacing the unobservable quantities by their filtered estimates. In this way, the dynamics of stock price and of the "filtered" market price of risk turn out to be driven by a single, one-dimensional Brownian motion, the so called *Innovation process*. Hence, after this transformation, we are in a complete market model and, in the second step of the procedure, we can solve the optimization problem by following the standard martingale approach.

Let us consider the information filtration $\mathbb{F}^{S} := (\mathcal{F}_{t}^{S})_{t \in [0,T]}$, where, at any time $t \in [0,T]$, $\mathcal{F}_{t}^{S} := \sigma \{S_{u}, 0 \leq u \leq t\} \vee \mathcal{N}$ and \mathcal{N} is the collection of **P**-null sets. We recall that \mathcal{F}_{0}^{S} is the trivial σ -algebra. We denote by π the conditional expectation of X, given the information flow, that is $\pi_{t} = \mathbf{E} \left[X_{t} | \mathcal{F}_{t}^{S} \right]$, for every $t \in [0,T]$ and by R the conditional variance, $R_{t} := \mathbf{E} \left[(X_{t} - \mathbf{E}[X_{t} | \mathcal{F}_{t}^{S}])^{2} | \mathcal{F}_{t}^{S} \right]$ for every $t \in [0,T]$. It is well known that the conditional distribution of X is Gaussian and hence completely identified by the dynamics of expectation and variance.

To characterize these dynamics we introduce the innovation process $I = \{I_t, t \in [0, T]\},\$

$$I_t := Z_t^S + \int_0^t (X_u - \pi_u) du,$$

for every $t \in [0, T]$. Following Lipster and Shiryaev (2001) [27, Chapter 10], it can be proved that I is an $(\mathbb{F}^S, \mathbf{P})$ -Brownian motion and that the processes π and R are the unique solutions to the system

$$d\pi_t = -\lambda(\pi_t - X)dt + (R_t + \rho\sigma_X)dI_t, \quad \pi_0 \in \mathbb{R},$$

$$dR_t = \left[\sigma_X^2 - 2\lambda R_t - (R_t + \rho\sigma_X)^2\right]dt, \quad R_0 \in \mathbb{R}^+.$$
 (4.1)

From equation (4.1), we see that R_t is a deterministic function of time. Therefore to emphasise this fact, from now on we will write R(t) instead of R_t .

The semimartingale representations with respect to the information filtration \mathbb{F}^S of the stock price process and of the wealth produced by a strategy θ are

$$\frac{dS_t}{S_t} = (r + \sigma \pi_t)dt + \sigma dI_t,$$

$$\frac{dW_t}{W_t} = (r + \theta_t \sigma \pi_t)dt + \theta_t \sigma dI_t.$$
 (4.2)

The investor wants to solve the problem

$$\max_{\theta \in \mathcal{A}^{S}(w)} \mathbf{E}\left[\frac{1}{1-\gamma} W_{T}^{1-\gamma}\right]$$

where $\mathcal{A}^{S}(w)$ is the set of \mathbb{F}^{S} -predictable self-financing strategies satisfying the integrability condition (2.1) with initial wealth w. The state price density process in this case is unique and is given by

$$\frac{d\xi_t}{\tilde{\xi_t}} = -rdt - \pi_t dI_t, \quad \tilde{\xi_0} = 1.$$

By the martingale method we formulate the equivalent static problem

$$\max_{W_T} \mathbf{E}[u(W_T)],\tag{4.3}$$

subject to the constraint

$$w = \mathbf{E}[\tilde{\xi}_T W_T]. \tag{4.4}$$

We note that, since \mathbb{F}^{S} -predictable strategies are also \mathbb{F} -predictable, the optimal utility under partial information is always lower than that under full information, and hence if problem (3.4)-(3.5) is bounded, problem (4.3)-(4.4) is also bounded.

By the usual Lagrangian approach, since $\tilde{\xi}$ is the state price density process, the optimal wealth satisfies

$$\tilde{W}_t^* = \tilde{\lambda}_0^{-\frac{1}{\gamma}} \tilde{\xi}_t^{-1} \mathbf{E}[\tilde{\xi}_T^{1-\frac{1}{\gamma}} | \mathcal{F}_t^S]$$

where $\tilde{\lambda}_0$ is the Lagrangian multiplier from the budget constraint (4.4).

We can now state a verification theorem for the partial information setting.

Theorem 4.1 (Verification Theorem under partial information). Let the function $F(Y, \pi, t)$ solve the equation

$$\frac{1}{2} \quad F_{YY}Y^2\pi^2 + F_{\pi Y}Y\pi(R + \rho\sigma_X) + \frac{1}{2}F_{\pi\pi}(R + \rho\sigma_X)^2 + F_t \\ = \ rF - rF_YY + F_{\pi}\left((R + \rho\sigma_X)\pi + \lambda(\pi - \bar{X})\right)$$

with boundary conditions

$$F(Y, \pi, T) = Y^{\frac{1}{\gamma}}, \quad and \quad F(Y_0, \pi_0, 0) = w$$

for some constant $Y_0 > 0$ and $F(Y, \pi, t) \to F(Y, \pi, T)$ as $t \to T$. Let $Y_t := Y_0 \left(\tilde{\xi}_t\right)^{-1}$ and assume that $\mathbf{E}\left[\int_{0}^{T} \left((F_{Y}(Y_{t},\pi_{t},t)\pi_{t})^{2} + (Y_{t}^{-1}F_{\pi}(Y_{t},\pi_{t},t))^{2} + (Y_{t}^{-1}\pi_{t}F(Y_{t},\pi_{t},t))^{2} \right) dt \right] < \infty.$ (4.7)

Then the optimal wealth is $\tilde{W}_t^* = F(Y_t, \pi_t, t)$ and the optimal investment strategy is

$$\tilde{\theta}_t^* = \frac{F_Y(Y_t, \pi_t, t)Y_t\pi_t + (R(t) + \rho\sigma_X)F_\pi(Y_t, \pi_t, t)}{\sigma F(Y_t, \pi_t, t)},$$
(4.8)

for all $t \in [0, T]$.

Proof. Similarly to the proof of Theorem 3.2, we need to show that the initial wealth satisfies the budget constraint and that the final wealth satisfies the first order condition and is attainable by a self-financing strategy.

The budget constraint and the first order condition follow from (4.6). By Itô formula we get

$$\frac{dY_t}{Y_t} = (r + \pi_t^2)dt + \pi_t dI_t.$$

Hence,

$$F(Y_t, \pi_t, t) = F(Y_0, \pi_0, 0) + \int_0^t \widetilde{\mathcal{L}} F(Y_s, \pi_s, s) ds + \int_0^t \left((R_t + \rho \sigma_X) F_\pi(Y_s, \pi_s, s) + Y_s \pi_s F_Y(Y_s, \pi_s, s) \right) dI_s$$
(4.9)

where $\widetilde{\mathcal{L}}$ is the differential operator

$$\begin{aligned} \widetilde{\mathcal{L}}F = & F_t + \frac{1}{2}F_{YY}Y^2\pi^2 + F_{\pi Y}Y\pi(R_t + \rho\sigma_X) \\ & + \frac{1}{2}F_{\pi\pi}(R_t + \rho\sigma_X)^2 - F_{\pi\lambda}(\pi - \bar{X}) + F_YY(r + \pi^2). \end{aligned}$$

To show that the optimal wealth can be obtained by a self-financing strategy starting from w it remains to prove that the process $\tilde{\xi}_t F(Y_t, \pi_t, t)$ is a true martingale. By applying the product rule and using (4.5), we get

$$\tilde{\xi}_t F(Y_t, \pi_t, t) = F(Y_0, \pi_0, 0) + \int_0^t \tilde{\xi}_s \left((R_t + \rho \sigma_X) F_\pi(Y_s, \pi_s, s) + Y_s \pi_s F_Y(Y_s, \pi_s, s) - F(Y_s, \pi_s, s) \pi_s \right) dI_s.$$

By (4.7) and the fact that R_t is the solution to the Riccati equation (4.1) on [0, T], we get that the integral with respect to I is a true martingale. Then $\tilde{W}_t^* = F(Y_t, \pi_t, t)$ is the optimal wealth process and the optimal investment strategy in (4.8) is obtained by equating the predictable covariation processes with respect to I from (4.2) and (4.9).

To obtain a closed form representation for the optimal wealth we guess that

$$\mathbf{E}[\tilde{\xi}_{T}^{1-\frac{1}{\gamma}}|\mathcal{F}_{t}^{S}] = \tilde{\xi}_{t}^{1-\frac{1}{\gamma}} e^{\tilde{A}(t)+\tilde{B}(t)\pi_{t}+\frac{1}{2}\tilde{C}(t)\pi_{t}^{2}}$$

where the functions $\tilde{A}(t), \tilde{B}(t)$ and $\tilde{C}(t)$ satisfy the system of Riccati Equations

$$\begin{cases} \frac{d\tilde{C}}{dt} = \tilde{a} + \tilde{b}(t)\tilde{C}(t) + \tilde{c}(t)\tilde{C}(t)^2, \\ \frac{d\tilde{B}}{dt} = -\tilde{C}(t)\lambda\bar{X} + \left(\frac{\tilde{b}(t)}{2} + \tilde{c}(t)\tilde{C}(t)\right)\tilde{B}(t), \\ \frac{d\tilde{A}}{dt} = \frac{\gamma - 1}{\gamma}r - \tilde{B}(t)\lambda\bar{X} + \frac{1}{2}\tilde{c}(t)\left(\tilde{B}(t)^2 + \tilde{C}(t)\right) \end{cases}$$
(4.10)

with boundary conditions

$$\tilde{A}(T) = \tilde{B}(T) = \tilde{C}(T) = 0, \qquad (4.11)$$

where

$$\tilde{a} = \frac{\gamma - 1}{\gamma^2}, \quad \tilde{b}(t) = 2\left(\lambda + \frac{\gamma - 1}{\gamma}\left(R(t) + \rho\sigma_X\right)\right), \quad \tilde{c}(t) = -\left(R(t) + \rho\sigma_X\right)^2.$$

The solutions to the non-homogeneous system of Riccati equations (4.10)-(4.11) are related to the solutions of the homogeneous system (3.17)-(3.18) arising in the full information case. This fact, shown in the next proposition, will be exploited to get simpler expressions for many quantities of interest.

Proposition 4.2. Let the pairs of functions B(t), C(t) and $\tilde{B}(t), \tilde{C}(t)$ satisfy the problems (3.17)-(3.18) and (4.10)-(4.11) on [0,T], respectively and let

$$Q(t) := 1 - \gamma C(t) R(t).$$

Then, for all t in [0,T], Q(t) is strictly positive and

$$\hat{C}(t) = Q(t)^{-1}C(t),$$

 $\tilde{B}(t) = Q(t)^{-1}B(t).$

Moreover, the functions C(t) and $\tilde{C}(t)$ are strictly positive and decreasing on [0,T] if $\gamma < 1$ and are strictly negative and increasing if $\gamma > 1$.

Proof. The fact that the function C(t) is strictly positive and decreasing on [0,T] if $\gamma < 1$ and it is negative and increasing for $\gamma > 1$ has been proven by Kim and Omberg (1996) [23, Equation (23)].

The function Q(t) is continuous, hence the set $\mathcal{T} := \{t \in [0, T] | Q(t) = 0\}$ is closed; we want to show that it is empty. By contradiction, let us assume that it is not empty and let \bar{t} be its maximum. From the boundary condition (3.18) we see that Q(T) = 1, hence $\bar{t} < T$. Relations (4.12) and (4.13) hold in the set $\mathcal{T}^C \cap [0, T]$, where \mathcal{T}^C is the complement of \mathcal{T} . In fact they follow from the fact that $Q(t)^{-1}C(t)$ and $Q(t)^{-1}B(t)$ satisfy (4.10)-(4.11) when C(t), B(t) satisfy (3.17) -(3.18), as it can be shown by following Brendle (2006) [7, Equations (28)-(29)]. Therefore, for any $\epsilon > 0$ such that $\bar{t} + \epsilon < T$, $Q(\bar{t} + \epsilon)\tilde{C}(\bar{t} + \epsilon) = C(\bar{t} + \epsilon)$ and, by continuity of all the functions involved in the equality, $Q(\bar{t})\tilde{C}(\bar{t}) = C(\bar{t})$. Since C(t) is a monotone function (either increasing or decreasing, depending on the parameter γ) and C(T) = 0, then $C(\bar{t}) \neq 0$, hence $\bar{t} \notin \mathcal{T}$, that is a contradiction and \mathcal{T} is the empty set.

Since \mathcal{T} is empty, Q(t) is continuous on [0,T] and C(T) = 1, it follows that Q(t) is strictly positive on [0,T], hence the functions C(t) and $\tilde{C}(t)$ must have the same sign (positive for $\gamma < 1$ and negative for $\gamma > 1$).

Finally, we prove that for $\gamma < 1$, $\tilde{C}(t)$ is strictly decreasing on [0, T]. Consider the equation

$$\frac{d\tilde{C}(t)}{dt} = f(\tilde{C}(t)),$$

where $f(\tilde{C}(t))$ is the right hand side of the first equation in (4.10)-(4.11). The boundary condition implies that $\tilde{C}(T) = 0$ and that $f(0) = \frac{\gamma - 1}{\gamma^2} < 0$. Then the function f(t) must be negative on [0, T] for the boundary condition to be satisfied and hence $\tilde{C}(t)$ is strictly decreasing. The same argument applies to the case $\gamma > 1$ where the derivative of $\tilde{C}(t)$ is positive and hence $\tilde{C}(t)$ is strictly increasing. \Box

We remark that from (4.12)-(4.13), we can get an explicit expression for $\tilde{B}(t)$ and $\tilde{C}(t)$ from those of B(t) and C(t). Then $\tilde{A}(t)$ can be obtained explicitly by integrating the right hand side of the third equation in system (4.10)-(4.11).

We are now ready to determine the optimal wealth and the optimal investment strategy for the partial information problem.

Theorem 4.3. Let the functions $\tilde{A}(t)$, $\tilde{B}(t)$ and $\tilde{C}(t)$ satisfy (4.10)–(4.11) on [0,T] and let

$$\tilde{\lambda}_0 = \left[\frac{e^{\tilde{A}(0) + \tilde{B}(0)\pi_0 + \frac{1}{2}\tilde{C}(0)\pi_0^2}}{w}\right]^{\gamma}.$$

Assume that

$$\mathbf{E}\left[\int_{0}^{T} \left(\tilde{\xi}_{t}^{1-\frac{1}{\gamma}} e^{\tilde{A}(t)+\tilde{B}(t)\pi_{t}+\frac{1}{2}\tilde{C}(t)\pi_{t}^{2}}\right)^{2} (1+\pi_{t}^{2})\right] < \infty.$$
(4.14)

Then $\tilde{\lambda}_0$ is the Lagrangian multiplier from the budget constraint (4.4) and the optimal wealth and the optimal investment strategy are given by

$$\tilde{W}_t^* = (\tilde{\lambda}_0 \tilde{\xi}_t)^{-\frac{1}{\gamma}} e^{\tilde{A}(t) + \tilde{B}(t)\pi_t + \frac{1}{2}\tilde{C}(t)\pi_t^2}, \qquad (4.15)$$
$$\tilde{\theta}_t^* = \frac{1}{\gamma} \frac{\pi_t}{\sigma} + \frac{(R(t) + \rho\sigma_X)}{\sigma} (\tilde{B}(t) + \tilde{C}(t)\pi_t),$$

for every $t \in [0, T]$.

Proof. The proof follows from the same argument of the analogous result under full information, Theorem 3.3, and hence is omitted. \Box

In the next proposition we provide sufficient conditions to apply Theorem 4.3 that are easier to check for a given set of parameters.

Proposition 4.4. Let the functions $\tilde{A}(t)$, $\tilde{B}(t)$ and $\tilde{C}(t)$ satisfy (4.10)–(4.11) on [0,T] and assume that at least one of the following two holds

- (i) $\gamma > 1$
- (ii) The functions A(t), B(t) and C(t) satisfy (3.17)-(3.18) on [0,T] and

$$1 - 4\frac{C(0)}{Q(0)} \max\left(R_0, R_0 e^{-2\lambda T} + \frac{\sigma_X^2}{2\lambda} (1 - e^{-2\lambda T})\right) > 0.$$
(4.16)

Then all assumptions of Theorem 4.3 are satisfied.

Proof. We only need to show that the integrability condition (4.14) in Theorem 4.3 is satisfied.

Using Fubini and the Cauchy Schwartz inequality we get

$$\mathbf{E}\left[\int_{0}^{T} e^{2\tilde{A}(t)} \left(\tilde{\xi}_{t}^{1-\frac{1}{\gamma}}\right)^{2} e^{2\tilde{B}(t)\pi_{t}+\tilde{C}(t)\pi_{t}^{2}} (1+\pi_{t}^{2}) dt\right] \\
\leq \kappa_{1} \int_{0}^{T} e^{2\tilde{A}(t)} \mathbf{E}\left[\tilde{\xi}_{t}^{8(1-\frac{1}{\gamma})}\right]^{\frac{1}{4}} \mathbf{E}\left[e^{4\tilde{B}(t)\pi_{t}+2\tilde{C}(t)\pi_{t}^{2}}\right]^{\frac{1}{2}} \mathbf{E}[(1+\pi_{t}^{2})^{4}]^{\frac{1}{4}} dt$$

Since π_t is Gaussian, $\mathbf{E}[(1 + \pi_t^2)^4] < \infty$. The expectation $\mathbf{E}\left[\tilde{\xi}_t^{8(1-\frac{1}{\gamma})}\right]$ is finite since π is Ornstein-Uhlenbeck (see, Revuz and Yor (2013)[31, Chaper 8, Ex. 3.14]). Finally, $\mathbf{E}\left[e^{4\tilde{B}(t)\pi_t+2\tilde{C}(t)\pi_t^2}\right]$ is finite for all $t \in [0,T]$ if and only if $1-4\tilde{C}(t)\tilde{v}_t > 0$ where $\tilde{v}_t = v_t - R_t$ is the variance of π_t (and v_t is the variance of X_t).

If $\gamma > 1$, from Proposition 4.2, $\tilde{C}(t) < 0$. Hence $1 - 4\tilde{C}(t)\tilde{v}_t > 0$ and (4.14) is satisfied. If $\gamma < 1$, still from Proposition 4.2 $\tilde{C}(t)$ is strictly positive and decreasing in [0, T]. Therefore

$$1 - 4\tilde{C}(t)\tilde{v}_t > 1 - 4\tilde{C}(0)v_t \ge 1 - 4\frac{C(0)}{Q(0)}\max\left(R_0, R_0e^{-2\lambda T} + \frac{\sigma_X^2}{2\lambda}(1 - e^{-2\lambda T})\right),$$

where the first inequality follows from the monotonicity of \tilde{C} and from the fact that $\tilde{v}_t < v_t$. The second inequality follows from (4.12) and from the fact that v_t is always lower than its maximum value on [0, T] that is equal to R_0 or to v_T depending on R(t) being decreasing or increasing. Then the result follows immediately from (4.16).

Now we can compute the conditional moment generating function of the optimal wealth under the partial information,

$$\tilde{\phi}_s(t,z) := \mathbf{E}\left[(\tilde{W}_t^*)^z | \mathcal{F}_s^S \right].$$

Proposition 4.5. Let $\tilde{\phi}_s(t, z) < \infty$ for $0 \le s \le t \le T$ and z > 0. Then

$$\tilde{\phi}_s(t,z) = (\tilde{\lambda}_0 \tilde{\xi}_s)^{-\frac{z}{\gamma}} e^{\tilde{D}(s;t,z) + \tilde{E}(s;t,z)\pi_s + \frac{1}{2}\tilde{H}(s;t,z)\pi_s^2}$$

where $\tilde{D}: [0,t] \to \mathbb{R}, \ \tilde{E}: [0,t] \to \mathbb{R}$ and $\tilde{H}: [0,t] \to \mathbb{R}$ satisfy the system of differential equations⁴

$$\begin{cases} \frac{d\tilde{H}}{ds} = \tilde{d}(s) + 2\tilde{e}(s)\tilde{F}(s) + \tilde{f}(s)\tilde{H}(s)^2\\ \frac{d\tilde{E}}{ds} = \left(\tilde{e}(s) + \tilde{f}(s)\tilde{H}(s)\right)\tilde{E}(s) - \lambda\bar{X}\tilde{H}(s)\\ \frac{d\tilde{D}}{ds} = -\frac{zr}{\gamma} - \lambda\bar{X}\tilde{E}(s) + \frac{1}{2}\tilde{f}(s)(\tilde{H}(s) + \tilde{E}(s)^2) \end{cases}$$

with boundary conditions

$$\tilde{D}(t) = z\tilde{A}(t), \qquad \tilde{E}(t) = z\tilde{B}(t), \qquad \tilde{H}(t) = z\tilde{C}(t), \qquad (4.17)$$

⁴ Note that the functions $\tilde{D}(s)$, $\tilde{E}(s)$ and $\tilde{H}(s)$ depend on t and z. We do not report such dependence into the formulas for a simpler notation.

and

$$\begin{split} \tilde{d}(s) &= -\left(\frac{z^2}{\gamma^2} + \frac{z}{\gamma}\right), \\ \tilde{e}(s) &= \lambda - \frac{z}{\gamma}(R(s) + \rho\sigma_X), \\ \tilde{f}(s) &= -(R(s) + \rho\sigma_X)^2, \end{split}$$

for every $s \leq t$, and where the functions \tilde{A} , \tilde{B} and \tilde{C} satisfy (4.10)-(4.11).

Proof. The proof replicates the steps of the proof of Proposition 3.5. Using that the process $(\ln \tilde{\xi}, \pi, \pi^2)$ is affine we have

$$\begin{split} \tilde{\phi}_s(t,z) &= \tilde{\lambda}_0^{-z/\gamma} \mathbf{E} \left[\tilde{\xi}_t^{-z/\gamma} e^{z\tilde{A}(t) + z\tilde{B}(t)\pi_t + \frac{1}{2}z\tilde{C}(t)\pi_t^2} | \mathcal{F}_s^S \right] \\ &= \tilde{\lambda}_0^{-z/\gamma} \tilde{G}(s, \tilde{\xi}_s, \pi_s; t, z) \end{split}$$

where

$$\tilde{G}(s,\tilde{\xi}_s,\pi_s;t,z) = \tilde{\xi}_s^{-z/\gamma} e^{\tilde{D}(s;t,z) + \tilde{E}(s;t,z)\pi_s + \frac{1}{2}\tilde{H}(s;t,z)\pi_s^2}$$

The boundary conditions (4.17) follow from $\tilde{\phi}_t(t,z) = (\tilde{W}_t^*)^z$. The function $\tilde{G}(s,\tilde{\xi},\pi;t,z)$ is differentiable with respect to s, and twice differentiable with respect to $\tilde{\xi}$ and π . Moreover, by definition, the process $(\tilde{G}(s,\tilde{\xi}_s,\pi_s;t,z))_{\{s\in[0,t]\}}$ is a martingale with respect to filtration \mathbb{F}^S . Hence, by applying Itô's formula we get that the function \tilde{G} satisfies the equation

$$0 = \frac{\partial \tilde{G}}{\partial s} - r \tilde{\xi}_s \frac{\partial \tilde{G}}{\partial \tilde{\xi}} - \lambda (\pi_s - \bar{X}) \frac{\partial \tilde{G}}{\partial \pi} + \frac{1}{2} \left(\frac{\partial^2 \tilde{G}}{\partial \pi^2} (R_s + \rho \sigma_X)^2 + \tilde{\xi}_s^2 \pi_s^2 \frac{\partial^2 \tilde{G}}{\partial \tilde{\xi}^2} - 2 \tilde{\xi}_s \pi_s \frac{\partial^2 \tilde{G}}{\partial \tilde{\xi} \partial \pi} (R_s + \rho \sigma_X) \right).$$

This completes the proof.

Note that, also under partial information, formulas simplify when t = T and $z = 1 - \gamma$, in fact:

$$\begin{split} \tilde{D}(s;T,1-\gamma) &=& \tilde{A}(s),\\ \tilde{E}(s;T,1-\gamma) &=& \tilde{B}(s),\\ \tilde{H}(s;T,1-\gamma) &=& \tilde{C}(s). \end{split}$$

This allows to compute the optimal expected utility in closed form, since

$$\mathbf{E} \left[u(\tilde{W}_{T}^{*}) | \mathcal{F}_{s}^{S} \right] = \frac{1}{1 - \gamma} \tilde{\phi}_{s}(T, 1 - \gamma)$$

$$= \frac{1}{1 - \gamma} (\tilde{\lambda}_{0} \tilde{\xi}_{s})^{1 - 1/\gamma} e^{\tilde{A}(s) + Q(s)^{-1}B(s)\pi_{s} + \frac{1}{2}Q(s)^{-1}C(s)\pi_{s}^{2}}$$

$$= \frac{1}{1 - \gamma} (\tilde{W}_{s}^{*})^{1 - \gamma} e^{\gamma(\tilde{A}(s) + Q(s)^{-1}B(s)\pi_{s} + \frac{1}{2}Q(s)^{-1}C(s)\pi_{s}^{2})}$$

where we have used the explicit expression of \tilde{C} and \tilde{B} in terms of C and B, given in (4.12)-(4.13) and where the last equality is obtained from (4.15).

5. The value of information

We are now ready to define the value of information, that is to assign a monetary value to the possibility of improving the knowledge of the market price of risk. We start by computing the reservation price, that is the maximal amount of money that a partially informed investor would be willing to pay to get extra information. We will focus on two kinds of information, which we call *initial* and *dynamic*. While the initial information gives the exact knowledge of X_0 that is the value of the market price of risk at time 0, the dynamic information provides the run-time values X_t at all times $t \in [0, T]$.

A partially informed investor, endowed with a starting wealth w, with a prior $X_0 \sim N(\pi_0, R_0)$ obtains, at time T, the final (optimal) wealth $\tilde{W}_T^*(w)$. Let us now assume that the value assumed by X_0 is revealed to the investor at time 0. Then she will be able to implement the optimal strategy, still under partial information because the following path of X will remain unknown to her, but this time starting from the exact value X_0 . Let us denote by $\tilde{W}_T^I(w)$ the optimal wealth obtained at time T, where the index I highlights the *Initial* information case. When dynamic information is provided to the investor, she will reach the wealth produced at time T by the optimal strategy under full information, that is $W_T^*(w)$. Since the sets of feasible strategies for the three scenarios are strictly increasing, the following inequalities hold

$$\mathbf{E}\left[u(\tilde{W}_{T}^{*}(w))\right] \leq \mathbf{E}\left[u(\tilde{W}_{T}^{I}(w))\right]$$
(5.1)

$$\leq \mathbf{E}\left[u(W_T^*(w))\right].\tag{5.2}$$

The maximum amount that the investor is willing to pay to receive the initial information is the quantity $\Delta w < w$ that satisfies

$$\mathbf{E}[u(\tilde{W}_T^*(w))] = \mathbf{E}\left[u(\tilde{W}_T^I(w - \Delta w))\right].$$
(5.3)

Notice that $\Delta w > 0$ because of (5.1) and the fact that the expected utility is increasing with respect to the initial wealth. From (4.18) computed for s = 0 we get

$$\mathbf{E}\left[u(\tilde{W}_{T}^{*}(w))\right] = \frac{w^{1-\gamma}}{1-\gamma} e^{\gamma\left(\tilde{A}_{0}(R_{0}) + Q_{0}^{-1}B_{0}\pi_{0} + \frac{1}{2}Q_{0}^{-1}C_{0}\pi_{0}^{2}\right)}$$
(5.4)

where we use the notation $Q_0 := Q(0)$, $B_0 := B(0)$, $C_0 := C(0)$ and $\tilde{A}_0(R_0) := \tilde{A}(0)$ to highlight the dependence of $\tilde{A}(0)$ on R_0 . With an analogous computation, setting $\pi_0 = X_0$ and $R_0 = 0$, we get

$$\mathbf{E}\left[u(\tilde{W}_{T}^{I}(w-\Delta w))|X_{0}\right] = \frac{(w-\Delta w)^{1-\gamma}}{1-\gamma}e^{\gamma\left(\tilde{A}_{0}(0)+B_{0}X_{0}+\frac{1}{2}C_{0}X_{0}^{2}\right)}.$$

Hence, the right hand side of (5.3) is

$$\mathbf{E}\left[\mathbf{E}\left[u(\tilde{W}_{T}^{I}(w-\Delta w))|X_{0}\right]\right] = \frac{(w-\Delta w)^{1-\gamma}}{(1-\gamma)\sqrt{Q_{0}}}e^{\gamma\tilde{A}_{0}(0)+\frac{\gamma}{2Q_{0}}\left(\gamma B_{0}^{2}R_{0}+2B_{0}\pi_{0}+C_{0}\pi_{0}^{2}\right)}$$
(5.5)

which holds when $Q_0 = 1 - \gamma C_0 R_0 > 0$. Solving equation (5.3) using the explicit expressions in (5.4) and (5.5) we get Δw and we can define the Value of Initial Information \mathcal{V}^I as the ratio $\Delta w/w$, that is

$$\mathcal{V}^{I} = 1 - \left(\sqrt{Q_{0}} e^{\gamma(\tilde{A}_{0}(R_{0}) - \tilde{A}_{0}(0)) - \frac{\gamma^{2} B_{0}^{2} R_{0}}{2Q_{0}}}\right)^{\frac{1}{1-\gamma}}.$$
(5.6)

We remark that \mathcal{V}^I does not depend on the expected value of the market price of risk π_0 but only on the variance of the initial estimate R_0 .

Let us now compute the reservation price for the dynamic information. Let the quantity Δw be the solution to the equation

$$\mathbf{E}\left[u(\tilde{W}_{T}^{*}(w))\right] = \mathbf{E}\left[u(W_{T}^{*}(w-\Delta w))\right].$$
(5.7)

Inequality (5.2) implies that $0 < \Delta w < w$. To compute the right hand side of (5.7) we use equation(3.32). The left hand side of (5.7) is given in (4.18) for s = 0. Again we shorten the notation by using $A_0 = A(0), B_0 = B(0), C_0 = C(0), Q_0 = Q(0)$ and $\tilde{A}_0(R_0) = \tilde{A}(0)$. Hence, we can extract the reservation price Δw from (5.7) and define the Value of Dynamic Information \mathcal{V}^D as the ratio $\Delta w/w$, that is

$$\mathcal{V}^{D} = 1 - \left(\sqrt{Q_0} e^{\gamma(\tilde{A}_0(R_0) - A_0) - \frac{\gamma^2 B_0^2 R_0}{2Q_0}}\right)^{\frac{1}{1 - \gamma}} .$$
(5.8)

From inequalities (5.1)- (5.2) we get

$$0 < \mathcal{V}^I \le \mathcal{V}^D < 1. \tag{5.9}$$

We remark that the expression for \mathcal{V}^D can be obtained from that of \mathcal{V}^I (5.6) by replacing $\tilde{A}_0(0)$ with A_0 . We also note that \mathcal{V}^D does not depend on the expected value of the market price of risk.

6. Applications

In this section we discuss some applications of our results with the parameters of Table 1 obtained from the estimates provided by Xia (2001) [34, Table I] on the U.S. stock market, from 1950 to 1997.

Figure 1 shows the optimal penalization factor ν_0^* for the incomplete market under full information derived in (3.22), as a function of the risk aversion parameter γ , for three different values of the correlation between the stock price process and the market price of risk and assuming $X_0 = \pi_0$. We see that ν_0^* grows in absolute value as γ tends to zero. This is explained by the fact that investors with smaller risk-aversion need a higher penalization factor (i.e. greater in absolute value) to be diverted from unattainable claims. The case $\gamma = 1$ corresponds to logarithmic utility. Here no penalty is necessary: investor

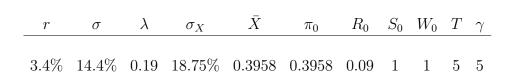


TABLE 1. Parameter set adopted in this section (expressed on a yearly basis and derived from [34, Table I])

is myopic and selects only attainable claims (see Remark 1). For γ larger than 1 the size of ν_0^* is first increasing and then decreasing towards zero. In fact, for values of γ slightly larger than 1, investors are less myopic and attracted by not marketed claims. When investors are more risk averse they put a larger part of their wealth in the risk-free asset, and hence the penalization becomes again less necessary.

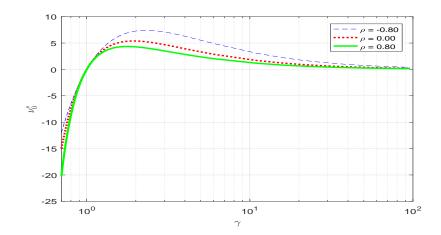


FIGURE 1. The optimal penalization factor ν_0^* , Equation (3.22), as a function of the risk aversion parameter γ , when $X_0 = \pi_0$, for correlations: $\rho = 0.8$, continuous line; $\rho = 0$ dotted line; $\rho = -0.8$ dashed line.

Figure 2 represents the critical time T^* , given by (3.21), that is the maximal horizon of existence for the solution to (3.17)-(3.18), as a function of γ for two values of the correlation ρ . The analysis of existence of the system of Riccati equations states that T^* is finite when ρ is larger than $\rho^* \simeq -0.4934$ given by (3.19), and for values of γ smaller than the value γ^* defined by (3.20). In this case γ^* is equal to 0.4933 when $\rho = 0$, and to 0.7185 when $\rho = 0.8$. When $\rho = 0$ Figure 2 (left panel) shows that the critical time corresponding to $\gamma = 0.4$ is about 20 years and it becomes larger than 20 years for $\gamma > 0.4$. In other words, for $\gamma > 0.4$ the solution to the system (3.17)-(3.18) is well defined up to an investment horizon of at least 20 years, and it is well defined for any horizon when $\gamma > \gamma^*$. In the same plot we also report T^{**} which is the maximal time such that (3.27) is satisfied. Remind that (3.27) is a sufficient condition for the optimal wealth under full information W_t^* to be expressed as in (3.25). When $\rho > \rho^*$ and $\gamma < \gamma^*$ there is a large region in the plane γ, T where the solution to the Riccati system if well defined but (3.27) does not hold, hence to state that formula (3.25) provides the optimal wealth, one should prove that the more general condition (3.24) of Theorem 3.3 holds.

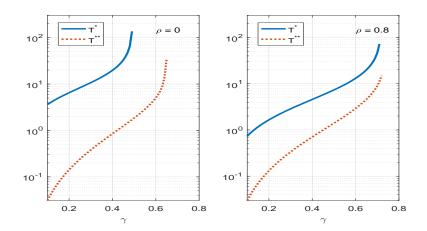


FIGURE 2. Critical times T^* (continuous line), Equation (3.21), for the system of Riccati equations (3.17)-(3.18), as a function of γ , for correlations $\rho = 0$ (left panel) and $\rho = 0.8$ (right panel). Superimposed the maximal time T^{**} (dotted line) such that (3.27) is satisfied.

Propositions 3.5 and 4.5 characterize the moment generating functions of the optimal wealth under full or partial information. Applying those results and Fast Fourier Transform we can compute the corresponding probability distributions very efficiently. Figure 3 represents the probability density functions of the optimal wealth in T for a fully informed investor with $\gamma = 4.03$ and for a partially informed one with $\gamma = 2.08$. We also plot the empirical distributions, obtained by simulations, for a visual check of the precision of our code. The values of γ have been chosen so that the expected returns of the two strategies are equal to 15%. Albeit with the same mean, the two distributions have very different shapes, with the full information density being more skewed and with a heavier right tail. This has interesting consequence on the mean-variance curves corresponding to the full and the partial information investment strategies for different level of risk aversions, represented in Figure 4. To connect Figure 4 and Figure 3 we also indicate the points corresponding to the two values of γ for which we computed the densities. We see that the curve of expected returns under partial information dominates the full information one. Hence, if an investor following a mean-variance criterion (as, for instance, maximizing the Sharpe ratio of her investment) had to choose between optimal strategies under full or under partial information, she would always select the partial information one. This is a consequence of the heavier right tail of the wealth distribution under full information (clearly shown in Figure 3), a feature not much appreciated by a mean-variance kind of investor.

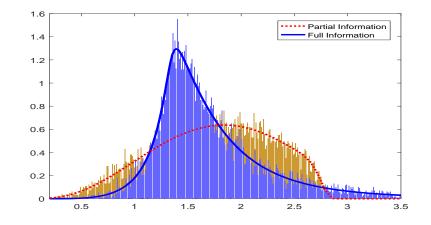


FIGURE 3. Probability distribution for the optimal final wealth under full (continuous line) and partial (dotted line) information starting from w = 1. The two distributions have the same mean, and are obtained by setting $\gamma = 2.08$ for the partially informed investor and $\gamma = 4.03$ for the fully informed one.

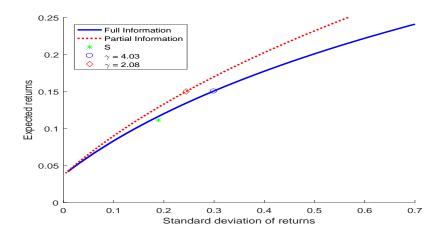


FIGURE 4. Expected returns of optimal strategies under full (continuous line) or partial (dotted line) information as functions of their standard deviations. The points obtained for $\gamma = 2.08$ with partial information and $\gamma = 4.03$ with full information are reported, for reference with Figure 3. The point S represents the risky asset.

The cumulative probability distributions for the optimal final wealth under full and partial information are represented in Figure 5. The plot shows that the optimal wealth under full information (continuous line) stochastically dominates the optimal wealth in partial information (dotted line). However such a dominance is lost if the partially informed investor adds to the initial budget w the reservation price for Dynamic Information Δw . In this case, by definition, the investor attains the same expected utility as the fully informed investor and hence her optimal wealth is sometimes lower sometimes higher than the one obtained by the fully informed investor.

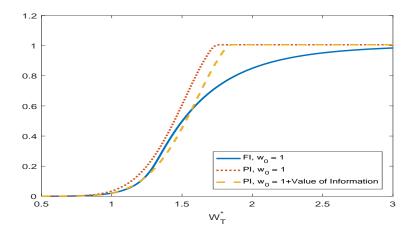


FIGURE 5. Cumulative probability distribution for the optimal final wealth under full (continuous line) and partial (dotted line) information starting from w = 1 and for the optimal wealth under partial information starting from $w = 1 + \Delta w$ (dashed line), where Δw is the reservation price of Dynamic Information.

The certainty equivalent of the optimal utility under partial information with respect to the initial conditional variance R_0 , computed from (5.4), is represented in Figure 6. The expected utility does not always grow as the precision of the initial estimates increases. In particular, for different values of ρ , the certainty equivalent is either increasing or it takes on the minimum value within the interval (0.1, 1). The intuitive explanation for this fact is that, when the expected value of the market price of risk π_0 is fixed, a greater uncertainty on its estimate may increase the likelihood of a better or a more favorable outcome, consequently raising the expected utility of the optimal wealth.

Figure 7 shows the value of the Initial Information \mathcal{V}^{I} (see Equation (5.6)) as a function of R_{0} , for three values of the correlation ρ . As expected, the higher the uncertainty on the initial estimate, the higher \mathcal{V}^{I} . It is perhaps less expected that the value is greater for $\rho = -0.9$ than for the other two cases. Why is the investor willing to pay a larger share of her initial wealth when the correlation of the changes in the market price of risk with the stock returns is more negative? In our opinion, this is a combination of two effects: the first effect is related to the precision of the estimate of the market price of risk, the second effect to the expected return of the optimal strategy. To explain the first effect, we note that, when $\rho = 0.9$, the variance of the estimate, R(t), decreases faster to the steady state $R_{\infty} = 0.0092$, while for $\rho = -0.9$ and $\rho = 0$, it decreases, at a slower rate, towards $R_{\infty} = 0.0632$ and $R_{\infty} = 0.0769$, respectively. Hence a more accurate information on X_{0} must be worth less when $\rho = 0.9$. As for the second effect, the certainty equivalent of the optimal strategy under partial information when $R_{0} = 0$ obtained from (4.18), is 32.11% of the initial wealth for $\rho = -0.9$, 26.11% for $\rho = 0$ and 26.58% for $\rho = 0.9$. Therefore, when $\rho = -0.9$, the investor is expecting a higher return, and hence she is willing to invest a larger share of her initial wealth to know the exact value of X_0 .

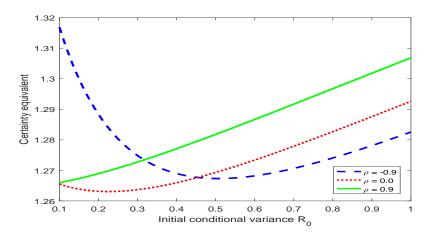


FIGURE 6. The certainty equivalent under partial information computed from (5.4) as a function of the initial variance of the estimate R_0 .

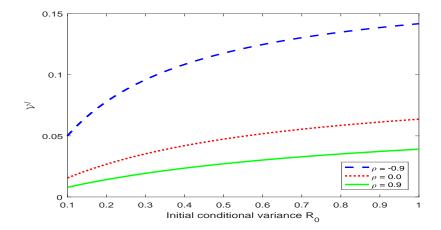


FIGURE 7. The value of Initial Information (5.6) as a function of the initial variance of the estimate R_0 , for different correlation ρ .

Figure 8 presents the ratio of the value of Dynamic Information \mathcal{V}^D , (5.8), over the value of Initial Information \mathcal{V}^I (5.6), as a function of the initial uncertainty R_0 , and for different values of ρ . The ratio is always positive and greater than 1 because of (5.9). It is decreasing with R_0 and converges to a constant as R_0 increases. When R_0 goes to zero, \mathcal{V}^I also goes to zero while \mathcal{V}^D converges to a positive value, hence the ratio grows unbounded. The ratio is larger for $\rho = 0$ and the difference between $\rho = 0.9$ and $\rho = -0.9$ is small. Intuitively, when the correlation is close to 1 or -1, the knowledge of the starting

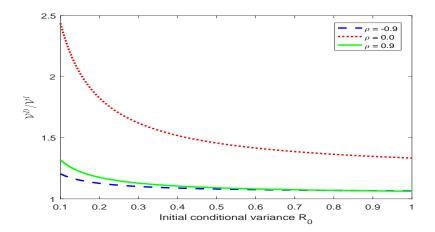


FIGURE 8. The ratio of the values of information: Dynamic Information, Equation (5.8), over Initial Information, Equation (5.6), as a function of the initial uncertainty R_0 .

value for the process X is sufficient to estimate with good precision also its next values, and hence the value added by the full knowledge of X is low (for our set of parameters it is around 5% of the value of knowing only X_0). Instead, when there is no correlation $(\rho = 0)$, knowing X_0 alone is not sufficient to get a good future estimate for X, and hence the value added by the dynamic information is more appreciated by the investor.

Figure 9 provides the ratio $\mathcal{V}^D/\mathcal{V}^I$ as a function of the investment horizon T, for a fixed value $R_0 = 0.09$. The ratio increases almost linearly with T, but more steeply for $\rho = 0$, that is when having access to a dynamic information on the market price of risk adds a significant improvement to the investment policy.

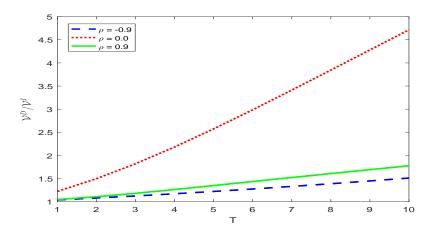


FIGURE 9. The ratio of the values of information: dynamic information (5.8) over initial information (5.6), as a function of the length of the investment period T, for fixed $\gamma = 5$.

7. Conclusions

We studied a portfolio optimization problem for an investor who aims to maximize her expected utility from terminal wealth under two different hypotheses on the information flows when the market price of risk is stochastic and mean-reverting. We solved the problem via the martingale approach and found an explicit representation for the optimal wealth and its associated utility as function of the current state-price density process and of the market price of risk X in the full information case, or of its best estimate π under partial information. We also provided verification theorems for our results.

We introduced the notion of value of information as the maximum percentage of the initial wealth that an investor would be willing to pay to access to more accurate information on the market price of risk X. In particular we considered the value of knowing the whole path of X on-the-run and the value of knowing only its initial value X_0 . Using the structure of the solutions of the Riccati equations that characterize the optimal wealth, we determined closed form representations of such values. We provided applications to illustrate some consequences of our results. The empirical analysis of the distribution of the optimal wealth under full and partial information showed several features that could not be guessed a priori, like for instance the fact that, under our parameter setting, an investor who cares for the Sharpe ratio of her investment would better allocate her wealth to a partially informed portfolio manager rather than to a fully informed one. Our measure for the value of information can be applied to real market data, for example to determine in which periods of time the access to a better knowledge on the market price of risk is more valuable. Our approach may also be used to assess the value of an improvement of the initial prior on the market parameters, and consequently to address issues related to the evaluation of model error.

Acknowledgements

The authors would like to thank the Referees for their useful suggestions. The work on this paper initiated when Katia Colaneri was visiting the Department of Economics, University of Perugia, as a part of the ACRI Young Investigator Training Program (YITP): the Association of Italian Banking Foundations and Savings Banks (ACRI) partially supported this work. Katia Colaneri also received partial financial support from INdAM-GNAMPA under grant UFMBAZ- 2018/000349 and UFMBAZ-2019/000436. The research of Stefano Herzel and Marco Nicolosi was partially funded by the Swedish Research Council grant 2015-01713. Part of this article was written while Katia Colaneri was affiliated with the School of Mathematics of the University of Leeds (UK).

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