ASSOCIATIVE ALGEBRAS RELATED TO CONFORMAL ALGEBRAS

PAVEL KOLESNIKOV

ABSTRACT. In this note, we introduce a class of algebras that are in some sense related to conformal algebras. This class (called TC-algebras) includes Weyl algebras and some of their (associative and Lie) subalgebras. By a conformal algebra we generally mean what is known as *H*-pseudo-algebra over the polynomial Hopf algebra $H = \Bbbk[T_1, \ldots, T_n]$. Some recent results in structure theory of conformal algebras are applied to get a description of TC-algebras.

1. INTRODUCTION

Conformal algebras, initially introduced as an algebraic tool for studying operator product expansion (OPE) in conformal field theory, become objects of pure algebraic study. By the axiomatic definition appeared in [9], a conformal algebra is a linear space C over a field \Bbbk (char $\Bbbk = 0$) endowed with a linear map T and with a family of operations $(\cdot_{(n)} \cdot) : C \otimes C \to C$, where n ranges over the set of non-negative integers, such that:

- (C1) for every $a, b \in C$ only a finite number of $a_{(n)} b$ is nonzero;
- (C2) $Ta_{(n)} b = -na_{(n-1)} b;$

(C3) $a_{(n)} Tb = T(a_{(n)} b) + na_{(n-1)} b.$

These relations provide a formalization of the following structure. Suppose A is an algebra (not necessarily associative), and let $A[[z, z^{-1}]]$ stands for the space of all formal power series over A in one variable z. Consider

$$T = \frac{d}{dz}, \quad (a_{(n)} b)(z) = \operatorname{Res}_{w} a(w) b(z) (w - z)^{n}, \quad n \ge 0,$$

where $\operatorname{Res}_w x(w, z) \in A[[z, z^{-1}]]$ denotes the coefficient of w^{-1} in a formal power series $x \in A[[z, z^{-1}, w, w^{-1}]]$. A pair of series $a, b \in A[[z, z^{-1}]]$ is said to be local, if $a(w)b(z)(w-z)^N = 0$ for sufficiently large N. A space of pairwise mutually local series which is closed with respect to T and $(\cdot_{(n)}, \cdot), n \ge 0$, is a conformal algebra.

Structure theory of conformal algebras started with [5, 9, 10], where the classification of simple and semisimple associative and Lie conformal algebras was obtained in "finite" case, i.e., if C is a finitely generated $\Bbbk[T]$ -module. In [17], the same result was obtained for Jordan conformal algebras. In [12, 13], the next step was done: the structure of some classes of infinite (i.e., infinitely generated over $\Bbbk[T]$) associative conformal algebras was clarified.

Various features of ordinary algebras can be translated to conformal algebras. Such a translation is not a "word-to-word" one, but there is a general approach

Partially supported by RFBR 05–01–00230, Complex Integration Program SB RAS (2006–1.9) and SB RAS grant for young researchers N. 29. The author gratefully acknowledges the support of the Pierre Deligne fund based on his 2004 Balzan prize in mathematics.

how to get an analogue of some notion (construction) for conformal algebras. This comes from the fact that ordinary and conformal algebras are just algebras in certain multicategories corresponding to Hopf algebras \Bbbk and $\Bbbk[T]$, respectively.

The notion of a multicategory goes back to Lambek [14], but we will mainly follow [3] in the exposition. An algebra in a multicategory is a functor from an operad C to the category. Given a Hopf algebra H, one may construct a multicategory $\mathcal{M}^*(H)$ associated with H. An algebra in $\mathcal{M}^*(H)$ is called an H-pseudo-algebra [2].

In Section 2 we state the general definition of a multicategory. Section 3 is devoted to the construction of the category $\mathcal{M}^*(H)$, where H is a cocommutative Hopf algebra. In Section 4 we introduce what is a TC-algebra, i.e., an ordinary algebra with conformal structure, and state structure theorems for such algebras.

Throughout the paper, k is an algebraically closed field of characteristic zero.

2. Multicategories

Let us first fix some notation.

Suppose $m \ge n \ge 1$ are two integers. An *n*-tuple of integers $\pi = (m_1, \ldots, m_n)$, $m_i \ge 1$, such that $m_1 + \cdots + m_n = m$ we call an *n*-partition of *m* (but we do not assume $m_i \le m_{i+1}$). Every partition π can be considered as a surjection (also denoted by π) from $\{1, \ldots, m\}$ onto $\{1, \ldots, n\}$ in the natural way:

$$\pi(1) = \dots = \pi(m_1) = 1,$$

 $\pi(m_1 + 1) = \dots = \pi(m_1 + m_2) = 2,$
 \dots
 $\pi(m - m_n) = \dots = \pi(m) = n.$

Therefore, π determines one-to-one correspondence between $\{1, \ldots, m\}$ and the set of pairs $\{(i, j) \mid i = 1, \ldots, n, j = 1, \ldots, m_i\}$. This correspondence we will denote by $\stackrel{\pi}{\leftrightarrow}$.

By S_n , $n \ge 1$, we denote the symmetric group on $\{1, \ldots, n\}$. Suppose $\pi = (m_1, \ldots, m_n)$ is a partition, $\sigma \in S_n$. Then $\sigma \pi = (m_{\sigma^{-1}(1)}, \ldots, m_{\sigma^{-1}(n)})$ is again a partition. Thus, S_n acts on the set of all *n*-partitions of *m*.

Given an *n*-partition π of $m, \sigma \in S_n$, and a family $\tau_i \in S_{m_i}$ (i = 1, ..., n), one may define $\sigma^{\pi}(\tau_1, \ldots, \tau_n) \in S_m$ in the following way: if $k \stackrel{\pi}{\leftrightarrow} (i, j)$ then

$$\sigma^{\pi}(\tau_1,\ldots,\tau_n)(k) \stackrel{\sigma_{\pi}}{\leftrightarrow} (\sigma(i),\tau_i(j)).$$

This composition rule turns the collection of S_n , $n \ge 1$, into an operad, known as the operad of symmetries.

Also, define what is to be a composition of two partitions: if $p \ge m \ge n \ge 1$, $\tau = (p_1, \ldots, p_m)$ is an *m*-partition of p, $\pi = (m_1, \ldots, m_n)$ is an *n*-partition of m, then there exists an *n*-partition of p, denoted $\pi \tau = (q_1, \ldots, q_n)$, where $q_1 = p_1 + \ldots p_{m_1}$, $q_2 = p_{m_1+1} + \cdots + p_{m_1+m_2}$, etc.

2.1. **Definition of a multicategory.** The following definition is due to [14]. A similar notion was defined in [3] as *pseudo-tensor category* (see also [2]). We will mainly follow [15] in the exposition. Another approach to (essentially) same structures was developed in a recent book [7].

Definition 1. Let \mathcal{A} be a class of objects such that

- (M1) for any integer $n \ge 1$ and for any family of objects $A_1, \ldots, A_n, A \in \mathcal{A}$ there exists a set $P_n^{\mathcal{A}}(A_1, \ldots, A_n; A)$ (simply, $P_n(\{A_i\}; A)$) of *n*-morphisms;
- (M2) for any $A_1, \ldots, A_m \in \mathcal{A}, B_1, \ldots, B_n \in \mathcal{A}, C \in \mathcal{A}$, and for any partition $\pi = (m_1, \ldots, m_n)$ of m we are given a map

$$\operatorname{Comp}^{\pi} : P_n(\{B_i\}; C) \times \prod_{i=1}^n P_{m_i}(\{A_{ij}\}; B_i) \to P_m(A_1, \dots, A_m; C),$$
(1)

We will denote

$$\operatorname{Comp}^{\pi}(\varphi,\psi_1,\ldots,\psi_n)=\operatorname{Comp}^{\pi}\left(\varphi,(\psi_i)_{i=1}^n\right)=\varphi(\psi_1,\ldots,\psi_n).$$

Such a map Comp is called a *composition map*;

(M3) there is a symmetric group action on *n*-morphisms, i.e.,

$$\sigma: \quad P_n^{\mathcal{A}}(A_1, \dots, A_n; A) \to P_n^{\mathcal{A}}(A_{\sigma^{-1}(1)}, \dots, A_{\sigma^{-1}(n)}; A), \qquad (2)$$
$$f \mapsto \sigma f,$$

such that $(\sigma \tau)f = \tau(\sigma f), \ \sigma, \tau \in S_n$.

If these structures satisfy the following axioms, then \mathcal{A} is said to be a *multicategory*.

(A1) The composition map is associative. Namely, suppose we have three families of objects $A_h \in \mathcal{A}$ $(h = 1, ..., p), B_j \in \mathcal{A}$ $(j = 1, ..., m), C_i \in \mathcal{A}$ (i = 1, ..., n), a partition $\tau = (p_1, ..., p_m)$ of p, and a partition $\pi = (m_1, ..., m_n)$ of m. Also, let D be an object of \mathcal{A} , and

$$\psi_j \in P_{p_i}(\{A_{jt}\}; B_j), \quad \chi_i \in P_{m_i}(\{B_{it}\}; C_i), \quad \varphi \in P_n(\{C_i\}; D)$$

be multimorphisms of \mathcal{A} . Then

 $\operatorname{Comp}^{\tau}\left(\operatorname{Comp}^{\pi}\left(\varphi,(\chi_{i})_{i=1}^{n}\right),(\psi_{j})_{j=1}^{m}\right)$

$$= \operatorname{Comp}^{\pi\tau} \left(\varphi, \left(\operatorname{Comp}^{\tau_i} \left(\chi_i, (\psi_{it})_{t=1}^{m_i} \right)_{i=1}^n \right) \right), \quad (3)$$

where $\tau_i = (p_{i1}, \ldots, p_{im_i})$ is the "subpartition" of τ .

(A2) For any $A \in \mathcal{A}$ there exists a "unit", i.e., 1-morphism $id_A \in P_1(A; A)$ such that

$$\operatorname{Comp}^{\operatorname{id}(n)}(f, \operatorname{id}_{A_1}, \dots, \operatorname{id}_{A_n}) = \operatorname{Comp}^{\varepsilon}(\operatorname{id}_A, f) = f \tag{4}$$

for any $f \in P_n(\{A_i\}; A)$; here id(n) is the identity partition $(1, \ldots, 1)$, ε is the trivial one: $\varepsilon = (n)$.

(A3) The composition map is equivariant with respect to the symmetric group action, i.e., if $\sigma \in S_n$, $\pi = (m_1, \ldots, m_n)$ is a partition of m, $\tau_i \in S_{m_i}$ $(i = 1, \ldots, n), \psi_i \in P_{m_i}(\{A_{ij}\}, B_i), \varphi \in P_n(\{B_i\}, C)$, then

$$\operatorname{Comp}^{\sigma\pi}\left(\sigma\varphi, (\tau_{\sigma^{-1}(i)}\psi_{\sigma^{-1}(i)})_{i=1}^{n}\right) = \sigma^{\pi}(\tau_{1}, \dots, \tau_{n})\operatorname{Comp}^{\pi}\left(\varphi, (\psi_{i})_{i=1}^{n}\right).$$
(5)

Suppose \mathcal{A} and \mathcal{B} are two multicategories. A functor $F : \mathcal{A} \to \mathcal{B}$ is a rule $A \mapsto F(A), A \in \mathcal{A}, F(A) \in \mathcal{B}$, such that for any $\varphi \in P_n(\{A_i\}; A)$ there is $F(\varphi) \in P_n(\{F(A_i)\}; F(A))$ and

• F preserves composition maps, i.e.,

$$F\left(\operatorname{Comp}^{\pi}\left(\varphi,\left(\psi_{i}\right)_{i=1}^{n}\right)\right) = \operatorname{Comp}^{\pi}\left(F(\varphi),\left(F(\psi_{i})\right)_{i=1}^{n}\right);$$

•
$$F(\operatorname{id}_A) = \operatorname{id}_{F(A)};$$

•
$$F(\sigma\varphi) = \sigma F(\varphi), \ \sigma \in S_n.$$

A multicategory \mathcal{A} can be considered as an ordinary category with respect to $\operatorname{Mor}^{\mathcal{A}}(A,B) = P_1^{\mathcal{A}}(A;B).$

It is suitable to distinguish a "linear version" of a multicategory, assuming

- all $P_n^{\mathcal{A}}(\{A_i\}; A)$ are linear spaces over a ground field \Bbbk ;
- the composition map is polylinear;
- symmetric group action is linear.

In the linear case, functors are supposed to be linear maps of spaces of multimorphisms.

2.2. Algebras in a multicategory. One of the most natural examples of a multicategory is the category \mathcal{V}_{\Bbbk} of linear spaces over a field \Bbbk , where $P_n^{\mathcal{V}_{\Bbbk}}(A_1, \ldots, A_n; A)$ consists of all polylinear maps from $A_1 \times \cdots \times A_n$ to A.

A multicategory with only one object is known as an *operad* (see, e.g., [15]). If C is a (linear) operad, C is the object of C, then the multicategory structure on C is completely determined by the collection of spaces $C(n) = P_n^C(C, \ldots, C; C)$ endowed with the structures (M1)–(M3).

Let us state an example of an operad that will be used later. Suppose $X = \{x_1, x_2, \ldots\}$ is a countable set of variables, and $\Bbbk\{X\}$ is the free (non-associative) algebra generated by X over \Bbbk .

Example 1. Let *I* be a collection of homogeneous polylinear polynomials in *X*. Denote by (*I*) the T-ideal of $\Bbbk\{X\}$ generated by *I*. Then $A = \Bbbk\{X\}/(I)$ is the free algebra in the corresponding variety. Define $C_I(n)$ to be the k-linear span of images in *A* of all non-associative words of length *n* in x_1, \ldots, x_n .

The action of S_n on $C_I(n)$ is just the permutation of variables (it is well-defined), and the composition Comp^{π} is the substitution with relabelling variables (also welldefined):

$$\operatorname{Comp}^{\pi} : (f(x_1, \dots, x_n), g_1(x_1, \dots, x_{m_1}), \dots, g_n(x_1, \dots, x_{m_n}) \mapsto f(g_1(x_{11}, \dots, x_{1m_1}), g_2(x_{21}, \dots, x_{2m_2}), \dots, g_n(x_{n1}, \dots, x_{nm_n})),$$

 $\pi = (m_1, \ldots, m_n)$. Here we identify x_{ij} with x_k via the partition π as above. Note that any multimorphism $f \in \mathcal{C}_I(n)$ can be obtained (up to permutation of variables) as a composition of id $\in \mathcal{C}_I(1)$ and $\mu = \overline{x_1 x_2} \in \mathcal{C}_I(2)$.

Definition 2 ([3]). Let \mathcal{A} be a multicategory, and let \mathcal{C} be an operad. A *C*-algebra in \mathcal{A} is a functor $F : \mathcal{C} \to \mathcal{A}$.

This definition allows to consider the class of C-algebras as a category, but not as a multicategory. We propose slightly different definition in order to make the class of algebras to be a multicategory.

Suppose \mathcal{A} and \mathcal{B} are two multicategories, and let

$$F, G: \mathcal{A} \to \mathcal{B}$$

be two functors.

Definition 3. Consider the class of objects $\mathcal{P}(\mathcal{A}, F, G)$ that consists of pairs (\mathcal{A}, μ) , where $\mathcal{A} \in \mathcal{A}, \mu \in P_1^{\mathcal{B}}(F(\mathcal{A}); G(\mathcal{A}))$. Define multimorphisms on $\mathcal{P}(\mathcal{A}, F, G)$ as those $f \in \mathcal{P}_n^{\mathcal{A}}(\mathcal{A}_1, \ldots, \mathcal{A}_n; \mathcal{A})$ that satisfy the relation

$$G(f)(\mu_1, \dots, \mu_n) = \mu(F(f)) \in P_n^{\mathcal{B}}(\{F(A_i)\}; G(A)),$$
(6)

4

where (A_i, μ_i) , (A, μ) are objects of $\mathcal{P}(\mathcal{A}, F, G)$. The structure obtained is a multicategory of (F, G)-pseudo-algebras in \mathcal{A} .

For example, if $\mathcal{A} = \mathcal{B} = \mathcal{V}_{\Bbbk}$, $F(A) = A \otimes A$, G(A) = A then an (F, G)-pseudoalgebra in \mathcal{A} is just an ordinary algebra over a field. It is easy to see that in this case $\mu \in P_2^{\mathcal{V}_{\Bbbk}}(A, A; A)$, so there exists a functor $\Phi : \mathcal{C}_{\emptyset} \to \mathcal{V}_{\Bbbk}$, defined by $\Phi(\overline{x}_1) = \mathrm{id}_A$, $\Phi(\overline{x}_1 \overline{x}_2) = \mu$. Therefore, this is also an algebra in the sense of Definition 2. Later we will consider another example of a multicategory, that leads to the notion of a conformal algebra.

3. H-PSEUDO-ALGEBRAS AND H-CONFORMAL ALGEBRAS

In this section, we generally follow [2].

Let H be a cocommutative bialgebra with a coproduct Δ . We will use the Sweedler's notation

$$\Delta(f) = f_{(1)} \otimes f_{(2)}, \quad (\Delta \otimes \operatorname{id})\Delta(f) = (\operatorname{id} \otimes \Delta)\Delta(f) = f_{(1)} \otimes f_{(2)} \otimes f_{(3)}, \quad \text{etc.}$$

Define Δ^k , $k \geq 0$, as follows: $\Delta^0 = \mathrm{id}_H$, $\Delta^k = (\mathrm{id} \otimes \Delta^{k-1}) \Delta$, i.e., $\Delta^k(f) = f_{(1)} \otimes \cdots \otimes f_{(k)}$.

Consider the class of objects \mathcal{M} that consists of left modules over H. The following structure turns \mathcal{M} into a multicategory called H-pseudo-tensor category:

$$P_n^{\mathcal{M}}(A_1,\ldots,A_n;A) = \operatorname{Hom}_{H^{\otimes n}}(A_1 \otimes \cdots \otimes A_n, H^{\otimes n} \otimes_H A)$$
(7)

(the tensor product \otimes without any index is assumed to be over the ground field \Bbbk). Here we consider $H^{\otimes k}$ as the outer product of regular right *H*-modules, i.e.,

$$F \cdot f = F\Delta^k(f), \quad F \in H^{\otimes k}, \ f \in H$$

Symmetric groups S_n act on $P_n^{\mathcal{M}}(\{A_i\}; A)$ by permutations of arguments from A_1, \ldots, A_n together with the corresponding permutation of tensor factors in $H^{\otimes n}$:

$$(\sigma f)(a_1,\ldots,a_n) = (\sigma \otimes_H \operatorname{id}_A) f(a_{\sigma(1)},\ldots,a_{\sigma(n)}),$$

 $f \in P_n^{\mathcal{M}}(\{A_i\}; A), \sigma \in S_n, a_i \in A_i, i = 1, \ldots, n$. Note that σ is an endomorphism of *H*-module $H^{\otimes n}$ since *H* is cocommutative.

To define a composition of multimorphisms Comp^{π} , $\pi = (m_1, \ldots, m_n)$ is a partition of m, let us first extend $f \in P_n^{\mathcal{M}}(\{A_i\}; A)$ to an $H^{\otimes m}$ -linear map

$$f: \bigotimes_{i=1}^{n} (H^{\otimes m_{i}} \otimes_{H} A_{i}) \to H^{\otimes m} \otimes_{H} A$$

$$\tag{8}$$

as follows: if $F_i \in H^{\otimes m_i}$, $a_i \in A_i$, and

$$f(a_1,\ldots,a_n) = \sum_j G_j \otimes_H b_j, \quad G_j \in H^{\otimes n},$$

then

$$f(F_1 \otimes_H a_1, \dots, F_n \otimes_H a_n) = (F_1 \otimes \dots \otimes F_n) \sum_j (\Delta^{m_1 - 1} \otimes \dots \otimes \Delta^{m_n - 1}) (G_j) \otimes_H b_j.$$
(9)

Suppose $\pi = (m_1, \ldots, m_n)$ is a partition, and consider $A_{i1}, \ldots, A_{im_i} \in \mathcal{M}$, $B_i \in \mathcal{M}, C \in \mathcal{M}, f_i \in P_{m_i}^{\mathcal{M}}(\{A_{ij}\}; B_i), f \in P_n^{\mathcal{M}}(\{B_i\}; C), i = 1, \ldots, n$. If

$$f_i(a_{i1},\ldots,a_{im_i}) = \sum_j F_{ij} \otimes_H b_{ij}, \quad F_{ij} \in H^{\otimes m_i}, \ b_{ij} \in B_n,$$

then define $\operatorname{Comp}^{\pi}(f, f_1, \dots, f_n) = f(f_1, \dots, f_n)$ by (9):

$$f(f_1, \dots, f_n)(a_{11}, \dots, a_{1m_1}, \dots, a_{n1}, \dots, a_{nm_n}) = f(f_1(a_{11}, \dots, a_{1m_1}), \dots, f_n(a_{n1}, \dots, a_{nm_n})) \quad (10)$$

The class \mathcal{M} with multimorphisms (7) and their compositions (10) is a multicategory denoted by $\mathcal{M}^*(H)$ [2].

If $H = \mathbb{k}$, i.e., dim H = 1, then this category is just the category of linear spaces $\mathcal{V}_{\mathbb{k}}$ with polylinear maps as multimorphisms.

Consider the following two functors F and G from $\mathcal{M}^*(H)$ to $\mathcal{M}^*(H^{\otimes 2})$, where $H \otimes H$ is considered as the tensor product of bialgebras: if A is an H-module then

$$F: A \mapsto A \otimes A, \quad G: A \mapsto H^{\otimes 2} \otimes_H A. \tag{11}$$

Indeed, if $f \in P_n^{\mathcal{M}^*(H)}(A_1, \dots, A_n; A)$,

$$f(a_1,\ldots,a_n) = \sum_j f_{1j} \otimes \cdots \otimes f_{nj} \otimes_H c_j,$$

$$f(b_1,\ldots,b_n) = \sum_j g_{1j} \otimes \cdots \otimes g_{nj} \otimes_H d_j,$$

then

$$F(f)(a_1 \otimes b_1, \ldots, a_n \otimes b_n) = \sum_{j,k} f_{1j} \otimes g_{1k} \otimes \cdots \otimes f_{nj} \otimes g_{nk} \otimes_{H^{\otimes 2}} (c_j \otimes d_k).$$

This expression is well-defined.

In the same way, G acts on multimorphisms as follows: if $f \in P_n^{\mathcal{M}^*(H)}(\{A_i\}; A)$,

$$f(a_1,\ldots,a_n) = \sum_j f_{1j} \otimes \cdots \otimes f_{nj} \otimes_H b_j,$$

then

$$G(f)(G_1 \otimes_H a_1, \dots, G_n \otimes_H a_n) = \sum_j G_1 \Delta(f_{1j}) \otimes \dots \otimes G_n \Delta(f_{nj}) \otimes_{H^{\otimes 2}} (1 \otimes 1 \otimes_H b_j),$$

 $G_i \in H^{\otimes 2}$, $f_{ij} \in H$, $b_j \in A$, i = 1, ..., n. This definition is correct since H is cocommutative. Therefore, $G : A \mapsto H \otimes H \otimes_H A$ is indeed a functor of multicategories.

Definition 4. An *H*-pseudo-algebra is an (F, G)-pseudo-algebra in $\mathcal{M}^*(H)$, where F and G are the functors defined by (11).

This definition is equivalent to the one of [2], where a pseudo-algebra was defined as an *H*-module endowed with an $(H \otimes H)$ -linear map (pseudo-product) $* : A \otimes A \to H \otimes H \otimes_H A$.

Let C_I be the operad from Example 1. Then a C_I -algebra in $\mathcal{M}^*(H)$ in the sense of Definition 2 is also an *H*-pseudo-algebra since μ is the image of $\overline{x_1x_2}$. Conversely, for any *H*-pseudo-algebra (A, μ) one may construct a functor $\Phi_A : C_{\emptyset} \to \mathcal{M}^*(H)$ in such a way that $\Phi_A(C) = A$, $\Phi_A(\overline{x_1}) = \mathrm{id}_A$, $\Phi_A(\overline{x_1x_2}) = \mu$. (Recall that all

 $\mathbf{6}$

multimorphisms of \mathcal{C}_{\emptyset} can be constructed via compositions and permutations of variables from $\overline{x_1}$ and $\overline{x_1x_2}$.)

In order to show that for $H = \Bbbk[T]$ (with the canonical Hopf algebra structure) an H-pseudo-algebra is the same as a conformal algebra, we have to describe the pseudo-product $\mu: A \otimes A \to H \otimes H \otimes_H A$ in terms of "usual" algebraic operations.

Suppose that H is a Hopf algebra with an antipode S. One may consider a linear isomorphism $\Phi: H \otimes H \to H \otimes H$ defined as follows:

$$\Phi: f \otimes g \mapsto fS(g_{(1)}) \otimes g_{(2)}.$$

The inverse is easy to find: $\Phi^{-1}(f \otimes g) = fg_{(1)} \otimes g_{(2)}$. Therefore, one may choose an *H*-basis of the product $H \otimes H$ of regular right modules in the form $\{h_i \otimes 1\}_{i \in I}$, where $\{h_i\}_{i \in I}$ is a basis of H. Then we have a well-defined map

$$\iota: (H \otimes H) \otimes_H A \to H \otimes 1 \otimes A \simeq H \otimes A.$$

Thus, the pseudo-product can be completely described by a family of binary k-linear operations

$$(\cdot_{(x)} \cdot) : A \otimes A \to A,$$

where x ranges over the dual space $X = H^*$. These operations are defined by

$$a_{(x)} b) := (\langle x, S(\cdot) \rangle \otimes \mathrm{id}_A) \iota(a * b), \tag{12}$$

and satisfy the following axioms:

- $\begin{array}{ll} (\text{H0}) & (a_{(\alpha x+\beta y)} \ b) = \alpha (a_{(x)} \ b) + \beta (a_{(y)} \ b); \\ (\text{H1}) & (\text{locality}) \ \text{codim} \{ x \in X \mid (a_{(x)} \ b) = 0 \} < \infty; \end{array}$
- (H2) (sesqui-linearity)

$$(ha_{(x)} b) = a_{(xh)} b, \quad (a_{(x)} hb) = h_{(2)}(a_{(S(h_{(1)})x)} b).$$

An algebraic system obtained on a left *H*-module *A* with operations $(\cdot_{(x)} \cdot)$, $x \in X = H^*$, satisfying the axioms (H0)–(H2) is called an *H*-conformal algebra [2]. In particular, if $H = \Bbbk$ then this is just an ordinary algebra. If $H = \Bbbk[T]$ then

 $X \simeq \mathbb{k}[[t]]$, and it is enough to consider $x = t^n$, $\langle t^n, T^m \rangle = \delta_{n,m} n!$. Then

$$(a_{(n)} b) = (a_{(t^n)} b), \quad n \ge 0,$$

together with the action of T on A determines the structure of a conformal algebra on A in the sense of [9] (see Introduction). Therefore, a conformal algebra is just an *H*-pseudo-algebra in the sense of Definition 4.

The notions of associative (commutative, Lie, etc.) conformal algebras can also be translated to the language of pseudo-algebras in a general way. One approach was proposed in [16], using the notion of a coefficient algebra. Namely, for any conformal algebra A there exists uniquely defined ordinary algebra Coeff A (coefficient algebra, or annihilation algebra [10]) such that (Coeff A)[$[z, z^{-1}]$] "universally contains" the conformal algebra A. If \mathcal{V} is a variety of algebras and Coeff $A \in \mathcal{V}$ then A is said to be a \mathcal{V} -conformal algebra. Such algebras form a category with respect to homomorphisms of conformal algebras.

Another approach is to apply Definition 2. If \mathcal{V} is a variety of ordinary algebras, then there exists a collection $I = I(\mathcal{V})$ of polylinear homogeneous defining identities. If A is an H-pseudo-algebra and the functor $\Phi_A : \mathcal{C}_{\emptyset} \to A$ can be restricted to \mathcal{C}_I then it is natural to say that A is a \mathcal{V} -algebra in $\mathcal{M}^*(H)$. The class of all such pseudo-algebras form a category.

In [11] it was shown that the last approach is equivalent to the one of [16].

Theorem 1 ([11]). Let \mathcal{V} be a variety of algebras. A conformal algebra A is a \mathcal{V} -algebra in $\mathcal{M}^*(\Bbbk[T])$ if and only if Coeff A is a \mathcal{V} -algebra.

4. TC-ALGEBRAS

In this section, by conformal algebras we mean (F, G)-pseudo-algebras in $\mathcal{M}^*(H)$, $H = \Bbbk[T_1, \ldots, T_n]$, corresponding to the functors $F : X \mapsto X \otimes X$, $G : X \mapsto$ $H \otimes H \otimes_H X$. The algebra H is considered as a topological algebra with basic neighborhoods of zero given by powers of the augmentation ideal (T_1, \ldots, T_n) .

Suppose A is a (Hausdorff) topological algebra endowed with continuous derivations $\partial_1, \ldots, \partial_n$. A map $a: H \to A$ is said to be translation-invariant (T-invariant, for short) if

$$a\frac{\partial}{\partial T_i} = \partial_i a, \quad i = 1, \dots, n.$$
 (13)

Denote the set of all continuous T-invariant maps from H to A by $\mathcal{F}(A)$.

Example 2. (i) Consider $A = \mathbb{A}_n$, where

$$\mathbb{A}_{n} = \mathbb{k} \langle p_{1}, \dots, p_{n}, q_{1}, \dots, q_{n} \mid [p_{i}, p_{j}] = [q_{i}, q_{j}] = 0, \ [q_{i}, p_{j}] = \delta_{i,j} \rangle$$

(the *n*th Weyl algebra) with respect to *q*-adic topology, i.e., the system of basic neighborhoods of zero is given by left ideals $Q_1 \supset Q_2 \supset \ldots$,

$$Q_k = \sum_{i_1,\dots,i_k=1}^n \mathbb{A}_n q_{i_1}\dots q_{i_k}, \quad k \ge 1.$$

Derivations $\partial_i = [\cdot, p_i]$ are continuous, and for any $f \in \Bbbk[p_1, \ldots, p_n]$ the map

$$a_f: T_1^{a_1} \dots T_n^{a_n} \mapsto fq_1^{a_1} \dots q_n^{a_n}$$

is continuous and T-invariant.

(ii) Consider $W_n \subset \mathbb{A}_n$, $W_n = \sum_{i=1}^n \Bbbk[p_1, \ldots, p_n]q_i$, i.e., the space of all derivations of H. This is a topological Lie algebra with respect to p-adic topology. Derivations $\partial_i = [q_i, \cdot]$ are continuous, and for any $i = 1, \ldots, n$ the map

$$a_i: T_1^{a_1} \dots T_n^{a_n} \mapsto p_1^{a_1} \dots p_n^{a_n} q_n$$

is continuous and T-invariant.

Note that $\mathcal{F}(A)$ can be considered as an *H*-module with respect to

$$(T_i a)(f) = -a\left(\frac{\partial f}{\partial T_i}\right) = -\partial_i a(f), \quad a \in \mathcal{F}(A), \ f \in H.$$

If B is a subspace of A then by $\mathcal{F}(B)$ we denote the space of all B-valued maps from $\mathcal{F}(A)$. If C is an H-submodule of $\mathcal{F}(A)$ then by $\mathcal{A}(C)$ we denote the subspace

$$\mathcal{A}(C) = \{ a(f) \mid a \in C, f \in H \}.$$

Note that $B \supseteq \mathcal{A}(\mathcal{F}(B)), \mathcal{F}(\mathcal{A}(C)) \supseteq C$.

Definition 5. A topological algebra A with continuous derivations ∂_i , i = 1, ..., n, is said to be a *TC-algebra* if $A = \mathcal{A}(\mathcal{F}(A))$.

For any TC-algebra A the derivations ∂_i necessarily commute and each of them is locally nilpotent.

For example, the polynomial algebra $H = \Bbbk[T_1, \ldots, T_n]$ is an associative TCalgebra, $\mathrm{id}_H \in \mathcal{F}(H)$. If n is even then the same H with respect to the Poisson bracket

$$\{f,g\} = \sum_{i=1}^{k} \frac{\partial f}{\partial T_i} \frac{\partial g}{\partial T_{k+i}} - \frac{\partial f}{\partial T_{k+i}} \frac{\partial g}{\partial T_i}, \quad n = 2k, \ f,g \in H,$$

is a Lie TC-algebra (as usual, denoted by P_n).

The Weyl algebra from Example 2 is an associative TC-algebra. The Lie algebra $W_n \subset \mathbb{A}_n^{(-)}$ is also a TC-algebra, as well as its classical subalgebras $S_n = \{D \in W_n \mid Dv = 0\}$ and $H_n = \{D \in W_n \mid Ds = 0\}, v = dT_1 \wedge \cdots \wedge dT_n, s = \sum_{i=1}^k dT_i \wedge dT_{k+i}, n = 2k.$

Remark 1. Note that W_n is not a TC-subalgebra of $\mathbb{A}_n^{(-)}$. One has to consider p-adic topology on W_n and set $\partial_i = [q_i, \cdot] = \frac{\partial}{\partial p_i}$. The same settings work for S_n and H_n . Let us consider H_n in details. It is well-known that H_n is a homomorphic image of P_n : $f \mapsto \{f, \cdot\} \in H_n, f \in H$. It is obvious that the map $a : H \to H_n$, $a(f) = \{f, \cdot\}$ is continuous and T-invariant.

TC-algebras form a category where morphisms are continuous homomorphisms of algebras commuting with ∂_i , i = 1, ..., n. By an ideal of a TC-algebra A we mean a ∂_i -invariant ideal of A.

Proposition 1. (i) If A is a TC-algebra then the matrix algebra $\mathbb{M}_N(A)$ is also a TC-algebra.

(ii) If A is an associative TC-algebra and $\sigma : A \to A$ is a continuous ∂_i -invariant involution then $\text{Sym}(A, \sigma)$ and $\text{Skew}(A, \sigma)$ are Jordan and Lie TC-algebras, respectively.

(iii) If A is a TC-algebra for $H' = \mathbb{k}[T_1, \ldots, T_r], n \ge r$, then the polynomial algebra $A[T_{r+1}, \ldots, T_n]$ is a TC-algebra for $H = \mathbb{k}[T_1, \ldots, T_n]$.

Proof. (i) Suppose $x = (a_{ij}(f_{ij})) \in \mathbb{M}_N(A)$. Then x can be presented as $x = (b_{ij}(T^{\alpha}))$ for an appropriate $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+$, $T^{\alpha} = T_1^{\alpha_1} \ldots T_n^{\alpha_n}$. Hence, $x = b(T^{\alpha}), b = (b_{ij}) \in \mathbb{M}_N(\mathcal{F}(A))$.

(ii) Just note that for every $a \in \mathcal{F}(A)$ the maps $f \mapsto a(f) \pm \sigma a(f)$, $f \in H$, also belong to $\mathcal{F}(A)$.

(iii) Define additional derivations and topology on $A' = A[T_{r+1}, \ldots, T_n]$ in the usual way. Then for any continuous T-invariant map $a: H' \to A$ the map $a': H \to A'$ given by

$$a'(T_1^{a_1}\dots T_n^{a_n}) = a(T_1^{a_1}\dots T_r^{a_r})T_{r+1}^{a_{r+1}}\dots T_n^{a_n}$$

is also continuous and T-invariant.

The problem of studying a TC-algebra A can be reduced to the space $\mathcal{F}(A)$. The last object is in some sense smaller, but the algebraic structure on $\mathcal{F}(A)$ is more complicated.

One may define a family of bilinear operations

$$(\cdot_{(f)}\cdot): \mathcal{F}(A) \otimes \mathcal{F}(A) \to \mathcal{F}(A), \quad f \in H,$$

in the following way:

$$(a_{(f)}b)(g) = a(f_{(1)})b(S(f_{(2)})g), \quad a, b \in \mathcal{F}(A), \ f, g \in H,$$
(14)

S stands for the standard antipode in H. It is easy to check that $T_i a \in \mathcal{F}(A)$, $(a_{(f)}b) \in \mathcal{F}(A)$ for every $a, b \in \mathcal{F}(A)$.

Identify the space of polynomials H with a subalgebra of H^* (getting product in H^* as dual to the coproduct in H) in the natural way: $T_i \mapsto T_i^*$, where $\langle T_i^*, T_j \rangle = \delta_{i,j}$. Then $\mathcal{F}(A)$ turns into an H-module endowed with bilinear operations indexed by H^* . It is straightforward to show that (H0) and (H2) hold (cf. [2]). However, (H1) does not hold in general. But if C is an H-submodule of $\mathcal{F}(A)$ that consists of elements satisfying (H1), and C is closed under all operations ($\cdot_{(f)} \cdot$), $f \in H$, then C is a conformal algebra.

One may interpret Theorem 1 as follows.

Theorem 2. Suppose a TC-algebra A belongs to a variety \mathcal{V} . Then any conformal algebra $C \subseteq \mathcal{F}(A)$ is a \mathcal{V} -algebra in $\mathcal{M}^*(H)$.

Proof. It is sufficient to perform the same computations as in [11] for multi-dimensional case. \Box

There is a sufficient condition for entire $\mathcal{F}(A)$ to satisfy (H1), i.e., to be a conformal algebra. Now A is supposed to be associative, Lie, or Jordan.

Let M be an H-module, $E = \operatorname{End}_{\Bbbk} M$. One may consider E as a topological algebra with respect to finite topology [8], i.e., such that basis of neighborhoods of zero is given by subspaces

$$U_{u_1,\dots,u_N} = \{ \psi \in E \mid \psi u_j = 0, \, j = 1,\dots,N \}, \quad u_1,\dots,u_N \in M.$$

Then $\partial_i = [\cdot, T_i] \in \text{Der } E, i = 1, \dots, n$, are continuous derivations.

Definition 6. A TC-representation of a TC-algebra A is a continuous ∂_i -invariant representation $\rho: A \to \operatorname{End}_{\Bbbk} M$. In this case, M is said to be a TC-module over A.

For example, the space H itself with respect to the canonical action of the Weyl algebra can be considered as a TC-module over \mathbb{A}_n .

Proposition 2 (cf. [2]). Let A be a TC-algebra. If A has a faithful TC-representation ρ on a finitely generated H-module M then $\mathcal{F}(A)$ is a conformal algebra.

Proof. Suppose M is generated over H by elements $u_1, \ldots, u_N \in M$. It follows from the definition of finite topology that $\dim \rho(a(H))u_i < \infty$ for every $a \in \mathcal{F}(A)$, $i = 1, \ldots, n$. Moreover, there exists $s \ge 1$ such that $\rho(a(I^s))u_i = 0$ for all i, where I is the augmentation ideal of H.

Relation (14) implies that for every $a, b \in \mathcal{F}(A), f \in I^K, K \ge 1$, we have

$$(a_{(f)} b)(g) \in \sum_{k=0}^{K} a(I^k)b(I^{K-k}), \quad g \in H,$$

where $I^0 = \mathbb{k}$.

Let s stands for a number such that $\rho(b(I^s))u_i = 0$ for all i = 1, ..., n. Denote $V = \sum_{i=1}^n \rho(b(H))u_i \subset M$. Since dim $V < \infty$, there exists $m \ge 1$ such that $\rho(a(I^m))V = 0$. If $K_1 > s + m$ then $\rho(a(I^k)b(I^{K_1-k}))u_i = 0$ for all $k = 0, ..., K_1$, i = 1, ..., n. In the same way, we can find K_2 such that $\rho(b(I^{K_2-k})a(I^k))u_i = 0$ for all $k = 0, ..., K_2$, i = 1, ..., n. Since ρ is a representation of A (recall that

A is either associative, or Lie, or Jordan), we obtain $\rho((a_{(f)} b)(g))u_i = 0$ for all $f \in I^K$, $K \ge \max\{K_1, K_2\}$, $g \in H$. It remains to note that ∂_i -invariance of ρ implies $(a_{(f)} b) = 0$.

Corollary 1 (cf. [10]). The following objects are associative conformal algebras:

- $\mathcal{F}(\mathbb{M}_N(H)) = \operatorname{Cur}_N^n$ (current conformal algebra over $\mathbb{M}_N(\Bbbk)$);
- $\mathcal{F}(\mathbb{A}_n) = \operatorname{Cend}_1^n$ (conformal Weyl algebra);
- $\mathcal{F}(\mathbb{M}_N(\mathbb{A}_n)) = \operatorname{Cend}_N^n$ (algebra of conformal endomorphisms of a free N-generated H-module);
- $\mathcal{F}(\mathbb{A}_{n,N,Q}) = \operatorname{Cend}_{N,Q}^{n}$, where $\mathbb{A}_{n,N,Q} = \mathbb{M}_{N}(\mathbb{A}_{n})Q(p_{1},\ldots,p_{n})$, Q is a matrix over $\Bbbk[p_{1},\ldots,p_{n}]$.

The last conformal algebra appears from TC-subalgebra $\mathbb{A}_{n,N,Q}$ of the matrix Weyl algebra. If det $Q \neq 0$ then $\text{Cend}_{N,Q}^n$ is simple [10].

By abuse of terminology, let us call a TC-algebra A satisfying the condition of Proposition 2 by a TC-algebra with finite faithful representation. If, in addition, $\mathcal{F}(A)$ is a finitely generated H-module then A is said to be finite TC-algebra. A conformal algebra which is finitely generated over H is also called finite.

The structure of finite simple and semisimple Lie conformal algebras was completely described in [5] (n = 1) and [2] $(n \ge 1)$. Finite Jordan conformal algebras were considered in [17] (n = 1), where simple and semisimple algebras were described. Our aim is to present a similar result for the general class of not necessarily finite associative TC-algebras with finite faithful representation. The complete solution is known for n = 1 (for n = 0 this is the classical Wedderburn Theorem). Throughout the rest of the paper we consider associative algebras only.

Proposition 3. Let A be a semiprime (i.e., without nonzero nilpotent ideals) TCalgebra with finite faithful representation. Then $\mathcal{F}(A)$ is a semiprime conformal algebra.

Proof. Relation (14) implies that

 $a(f)b(g) = (a_{(f_{(1)})} b)(f_{(2)}g), \quad a, b \in \mathcal{F}(A), \ f, g \in H.$

Therefore, if $\mathcal{F}(A)$ has a nonzero abelian ideal J then $\{a(f) \mid f \in H, a \in J\}$ is a nonzero abelian ideal of A.

Theorem 3. Let A be a TC-subalgebra of $\mathbb{M}_N(\mathbb{A}_n)$. If the subalgebra $A_1 = \mathbb{k}[p_1, \ldots, p_n]A$ acts irreducibly on $M = H \otimes \mathbb{k}^N$ then A_1 is a left ideal in $\mathbb{M}_N(\mathbb{A}_n)$ which is dense with respect to the q-adic topology.

Proof. Let us first note that A_1 is a subalgebra. Indeed, $ap_i = p_i a + \partial_i(a), a \in A$, i = 1, ..., n. Hence, $A_1A_1 \subseteq A_1$.

The centralizer $\mathcal{D} = \operatorname{End}_{A_1} M$ of A_1 in $\operatorname{End} M$ is a division algebra. Consider $0 \neq \varphi \in \mathcal{D}, 0 \neq a \in A, f \in H$. Then $fa = f(p_1, \ldots, p_n)a \in A_1$, and we have

$$[\varphi, f]a = \varphi(fa) - f(\varphi a) = 0.$$

Note that $[\varphi, f] \in \mathcal{D}$ since for every $b \in A_1$ the commutator [f, b] belongs to A_1 . Therefore, for every $f \in H$, $\varphi \in \mathcal{D}$ we have $\varphi f = f\varphi$, so $\mathcal{D} \subseteq \mathbb{M}_N(H)$. Identity endomorphism of End M belongs to \mathcal{D} . But it is clear that $\mathbb{M}_N(H)$ contains no division algebra with the same identity except for $\mathcal{D} = \Bbbk$. Hence, A_1 is a dense subalgebra of $\mathbb{M}_N(\mathbb{A}_n) \subset \text{End } M$ with respect to the finite topology on End M (over \Bbbk). This topology is actually equivalent to the q-adic topology.

Now, consider $\mathcal{F}(A_1) \supseteq \mathcal{F}(A) = C$. Since every TC-algebra *B* acts on $\mathcal{F}(B)$ as on a left module by the rule

$$a(f) \cdot b = (a_{(f)} b), \quad a, b \in \mathcal{F}(B), \ f \in H,$$

we have $A_1 \cdot \mathcal{F}(A_1) \subseteq \mathcal{F}(A_1)$. Since A_1 is dense, $\mathcal{F}(A_1)$ is a left ideal of $\operatorname{Cend}_N^n = \mathcal{F}(\mathbb{M}_N(\mathbb{A}_n))$. It follows from (14) that A_1 is a left ideal of $\mathbb{M}_N(\mathbb{A}_n)$.

For n = 1, the last theorem allows to deduce what is a structure of simple and semisimple conformal algebras with finite faithful representation [13]. In the language of TC-algebras, these results may be stated as follows.

Theorem 4. Let A be a TC-algebra over $H = \Bbbk[T]$ with finite faithful representation.

(i) If A is simple then $A \simeq \mathbb{M}_N(H)$ or $A \simeq \mathbb{A}_{1,N,Q}$, det $Q \neq 0$.

(ii) If A is semiprime then A is isomorphic to a finite direct sum of simple ones from (i).

Proof. By Proposition 3, if A is semiprime then $C = \mathcal{F}(A)$ is semiprime. In [13] it was shown that a semiprime conformal algebra C with finite faithful representation is isomorphic to a direct sum of algebras C_i , $C_i \simeq \operatorname{Cur}_{N_i}$ or $C_i \simeq \operatorname{Cend}_{N_i,Q_i}$,

det $Q_i \neq 0$. Therefore, $A = \mathcal{A}(C) = \mathcal{A}\left(\bigoplus_i C_i\right) = \bigoplus_i \mathcal{A}(C_i)$, that proves (ii). Note that A is even semisimple. Statement (i) is now obvious.

Theorem 5. If A is a TC-algebra with finite faithful representation then its Jacobson radical is nilpotent.

Proof. Consider $C = \mathcal{F}(A)$. It was shown in [13] that an associative conformal algebra with finite faithful representation has a maximal nilpotent ideal (radical).

Suppose R is the radical of C. It is easy to show that C/I is a conformal algebra with finite faithful representation. Then $\mathcal{A}(R) \subseteq A$ is a nilpotent ideal, and $A/\mathcal{A}(I)$ is isomorphic to $\mathcal{A}(C/I)$ which is semisimple. Therefore, $J(A) = \mathcal{A}(R)$.

5. Open problems

1. Describe irreducible conformal subalgebras of Cend_N^n for n > 1, i.e., those C that $\Bbbk[p_1, \ldots, p_n]\mathcal{A}(C)$ acts irreducibly on $M = H \otimes \Bbbk^N$.

This is equivalent to the problem of description all TC-subalgebras $A \subseteq \mathbb{M}_N(\mathbb{A}_n)$ such that $\Bbbk[p_1, \ldots, p_n]A$ acts irreducibly on $M = H \otimes \Bbbk^N$. Theorem 3 provides an important property of such algebras, however, the complete description is obtained only for n = 1 [13] (for n = 0 this is the classical Burnside's theorem). For n = 1, the corresponding TC-algebras are (up to automorphism) $\mathbb{M}_N(H)$ and $W_{N,Q} = \mathbb{M}_N(\mathbb{A}_1)Q(p)$, det $Q \neq 0$.

2. Describe irreducible Lie conformal subalgebras of $gc_N^n = \mathcal{F}(gl(H \otimes \Bbbk^N)), H = \Bbbk[T_1, \ldots, T_n].$

A great advance in this problem was obtained in [6] and in [18], but there is no complete solution even for N = n = 1. In [4], the following conjecture was stated: if C is an infinite irreducible Lie conformal subalgebra of gc_N^1 then the corresponding TC-algebra $L = \mathcal{A}(C)$ is isomorphic either to $W_{N,Q}^{(-)}$ or to $\text{Skew}(W_{N,Q},\sigma)$ for an appropriate TC-involution σ of $W_{N,Q}$, $\det Q \neq 0$.

12

References

- [1] Bakalov, B.: Beilinson-Drinfeld's definition of a chiral algebra, 2002, preprint.
- Bakalov, B., D'Andrea, A., Kac, V.G.: Theory of finite pseudoalgebras. Adv. Math. 162 (1), 1–140 (2001).
- [3] Beilinson, A.A., Drinfeld, V.G.: Chiral algebras. American Mathematical Society Colloquium Publications, vol. 51. AMS, Providence, RI (2004).
- [4] Boyallian, C., Kac, V.G., Liberati, J.I.: On the classification of subalgebras of Cend_N and gc_N. J. Algebra 260 (1), 32–63 (2003).
- [5] D'Andrea, A., Kac, V.G.: Structure theory of finite conformal algebras. Sel. Math. New Ser. 4, 377–418 (1998).
- [6] De Sole, A., Kac, V.G.: Subalgebras of gc_N and Jacobi polynomials. Canad. Math. Bull. 45 (4), 567–605 (2002).
- [7] Fiore, T.M.: Pseudo limits, biadjoints, and pseudo algebras: categorical foundations of conformal field theory. Mem. Amer. Math. Soc. 182 (860) (2006).
- [8] Jacobson, N.: Structure of rings. American Mathematical Society Colloquium Publications, vol. 37. AMS, Providence, RI (1956).
- [9] Kac, V.G.: Vertex algebras for beginners. University Lecture Series, vol. 10. AMS, Providence, RI (1996).
- [10] Kac, V.G.: Formal distribution algebras and conformal algebras. In: XIIth International Congress in Mathematical Physics (ICMP'97), pp. 80–97. Internat. Press, Cambridge, MA (1999).
- [11] Kolesnikov, P.S.: Identities of conformal algebras and pseudoalgebras. Comm. Algebra 34 (6), 1965–1979 (2006).
- [12] Kolesnikov, P.S.: Simple associative conformal algebras of linear growth. J. Algebra 295 (1), 247–268 (2006).
- [13] Kolesnikov, P.S.: Associative conformal algebras with finite faithful representation. Adv. Math. 202 (2), 602–637 (2006).
- [14] Lambek, J.: Deductive systems and categories. II. In: Standard constructions and closed categories, pp. 76–122. Lecture Notes Math., vol. 86. Springer-Verl., Berlin (1969).
- [15] Leinster, T.: Higher operads, higher categories. London Mathematical Society Lecture Note Series, vol. 298. Cambridge University Press, Cambridge (2004).
- [16] Roitman, M.: On free conformal and vertex algebras. J. Algebra 217 (2), 496–527 (1999).
- [17] Zelmanov, E.I.: On the structure of conformal algebras. In: International Conference on Combinatorial and Computational Algebra, Hong Kong, May 24–29, 1999. Contemp. Math. 264, 139–153 (2000).
- [18] Zelmanov, E.I.: Idempotents in conformal algebras. In: Y. Fong, et al (eds.) Proceedings of the Third International Algebra Conference, pp. 257–266 (2003).

Sobolev Institute of Mathematics, Novosibirsk, Russia

NOVOSIBIRSK STATE UNIVERSITY, NOVOSIBIRSK, RUSSIA E-mail address: pavelsk@math.nsc.ru