

2-VECTOR SPACES AND GROUPOIDS

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ABSTRACT. This paper describes a relationship between essentially finite groupoids and 2-vector spaces. In particular, we show to construct 2-vector spaces of **Vect**-valued *presheaves* on such groupoids. We define 2-linear maps corresponding to functors between groupoids in both a covariant and contravariant way, which are ambidextrous adjoints. This is used to construct a representation—a weak functor—from $\text{Span}(\mathbf{FinGpd})$ (the bicategory of essentially finite groupoids and spans of groupoids) into **2Vect**. In this paper we prove this and give the construction in detail.

1. INTRODUCTION

In this paper, I will describe an extension of the *groupoidification* program of Baez and Dolan [2]. Groupoidification refers to the program of treating parts of linear algebra as arising from spans of groupoids (categories whose morphisms are all invertible) by a process of “degroupoidification”, which produces complex vector spaces associated to groupoids, and linear maps associated to spans. The extension described here shows a connection of the setting of groupoids and spans with 2-vector spaces and 2-linear maps, a categorical analog of linear algebra. (We will assume that all groupoids are essentially finite - that is, equivalent to finite groupoids - although there is work in progress on how to extend these results to infinite groupoids, and in particular Lie groupoids.)

A simple example of the groupoidification program can be seen in terms of spans of finite sets (i.e. finite trivial groupoids). In that program, groupoids give corresponding vector spaces and spans of groupoids give corresponding linear maps. In particular, the special case of trivial groupoids (equivalently, sets) gives a useful illustration. Given a finite set S , there is a finite dimensional vector space $L(S)$ consisting of all complex linear combinations of elements of S . Now, consider a span in **FinSet**: that is, a diagram of the form:

$$(1) \quad \begin{array}{ccc} & X & \\ s \swarrow & & \searrow t \\ Y & & Z \end{array}$$

To the span, there is a corresponding linear map $L(X) : L(Y) \rightarrow L(Z)$, represented by a matrix T whose (i, j) -component is $|(s, t)^{-1}(y_i, z_j)$

So the set in Figure 1 gives rise to the linear transformation:

$$(2) \quad \begin{array}{l} y_1 \mapsto z_1 \\ y_2 \mapsto z_1 + z_2 \\ y_3 \mapsto z_2 \end{array}$$

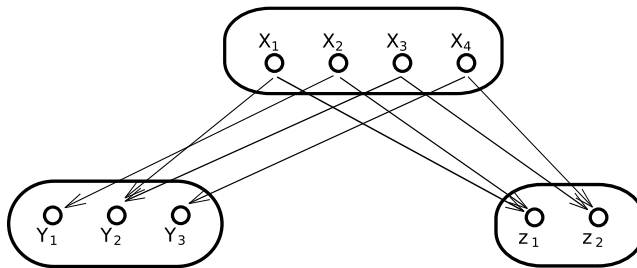


FIGURE 1. A Span of Sets

This makes sense for spans of finite sets. Similarly, we will be considering an analogous construction for spans of *essentially finite* groupoids.

There is a physical motivation here in quantum mechanics. If Y is the (discrete) set of classical (pure) states of a system, then $L(Y) = \mathbb{C}[Y]$, the space of linear combinations of states in Y , is the Hilbert space of the corresponding quantum mechanical system. (More generally, if Y is a measure space, one takes $L^2(Y)$). In the span, we think of X as a set of “processes” x , each with a designated “source” or starting state $s(x) \in Y$ and “target” or ending state $t(x) \in Z$. Then the linear transformation described by the matrix T can be seen in the following way, which we shall generalize later on:

Given a linear combination of elements of Y (that is, a function $f : Y \rightarrow \mathbb{C}$), transport f to X by “pulling back” along s . That is, $s^*f(x) = f(s(x))$. Then “push forward” to Z by taking the sum over all elements of X mapping down to a chosen one in Z :

$$(3) \quad t_*s^*f(z) = \sum_{t(x)=z} s^*f(x)$$

This precisely gives matrix multiplication by the matrix described above, and can clearly also be seen as a “sum over histories”: the value of $t_*s^*f(z)$ is a sum, over all histories x ending at z , of the value of f at the source $s(x)$. This illustrates a contrast between classical and quantum *processes*. Classically, states succeed each other by exactly one process. In the quantum picture, *every* possible process contributes to evolution of a state. In particular, there is an interpretation of quantum processes in terms of “matrix mechanics”, which takes a sum (in the form of matrix multiplication) over all histories joining fixed start and end states. This is exactly what is shown in our example.

It is not too difficult to check that the linearization of spans of sets gets along with composition, so that the composite of spans (by pullback, giving a set of *composite paths*) agrees with composition of linear maps. That is, that the process is *functorial*. This fact makes it possible to think of categorifying this process, in order to explicitly include symmetries of both states and histories as fundamental concepts. A categorified version of this process should be a 2-functor.

One way to generalize spans of sets, which is seen in [2], uses groupoids (categories whose morphisms are all invertible) instead of sets. One reason to consider this is that it often happens that the configuration space can naturally be thought of not as a set but as a groupoid. This happens particularly when there are symmetry operations acting on the set of configurations, and we explicitly represent

such symmetries as morphisms of the groupoid. The existence of a group action on the set would be one example. In such a categorified picture, X has objects which represent states of a system, and morphisms denoting *symmetries* of states. Then L gives vector spaces which are linear combinations of *isomorphism classes of objects* of the groupoids. The components of the linear maps uses groupoid cardinality instead of set cardinality:

$$(4) \quad L(X)_{[y_i], [z_k]} = \sum_{x \in \widehat{(s,t)^{-1}(y_i, z_k)}} \frac{\#(\text{Aut}(y_i))}{\#(\text{Aut}(x))} \\ = |\widehat{(y_i, z_k)}| \cdot \#(\text{Aut}(y_i))$$

where $\widehat{(y_i, z_k)}$ is the *essential preimage* of y_i and z_k , and its cardinality is the groupoid cardinality described by Baez and Dolan [4] (the other cardinality is the order of the group). This uses a weighting of contributions from intermediate elements depending on the size of their symmetry group. The groupoid cardinality of a finite groupoid X is:

$$(5) \quad |X| = \sum_{[x] \in X} \frac{1}{\# \text{Aut}(x)}$$

where the cardinality in the sum denotes the order of the group.

Here, however, we want to do something a little different: this process is still a functor, and we wanted a *2-functor*. Since we want to think of X as a category, rather than look at functions from the objects of X into \mathbb{C} , we should look at functors from X into some category which plays the role of \mathbb{C} . In particular, this category will be **Vect**, whose objects are vector spaces over \mathbb{C} , and whose morphisms are linear maps. When categorifying, therefore, we will want to find an analogous 2-functor, which requires specifying more data.

Then there will be a “free 2-vector space” $\Lambda(X)$ of all functors from X into **Vect**. We think of the objects as “2-linear combinations” of classical states, each with an *internal state space* which carries a representation of the symmetry group of that state. For most physically realistic systems, X would be an infinite set with a measure, and in fact a symplectic manifold. In general, to deal with L^2 spaces involves some issues in analysis, such as the measure on X . Then instead of $L(X)$ we consider $L^2(X)$. A similar caveat should apply in the categorified setting. Restricting to the situation of a finite groupoid helps to more clearly illustrate some of the purely category-theoretic aspects of the “free 2-vector space” construction. We do expect that for well-behaved smooth groupoids, for example, similar results to those considered in this paper will hold, involving infinite dimensional 2-vector spaces one could denote $2L^2(X)$. But this will be addressed in a companion paper.

Finally, we remark that this construction is used in the construction of an Extended Topological Quantum Field Theory (ETQFT) in the author’s Ph.D. thesis [15], where the groupoids in question are topological invariants of manifolds. By analogy, it could be used to give “extended quantum theories” in other settings where spans of groupoids appear.

Another view of a related process involves the 2-functor into additive categories (which, in the \mathbb{C} -linear case, are the KV 2-vector spaces) from a 2-category **Bim**, whose objects are rings, morphisms in $\text{hom}_{\mathbf{Bim}}(R, S)$ are (R, S) -bimodules, and whose 2-morphisms are bimodule homomorphisms. This is a dual picture to that of

spans. Indeed, the type of “pull-push” construction given here is ubiquitous (as its appearance in linear algebra suggests), due to the universal properties of categories of spans (see [7]). Similar notions also appear in the theory of Mackey functors (see [16]), and in the study of “correspondences” in noncommutative geometry, algebraic geometry, and elsewhere.

In this paper, we will begin by describing the source and target categories, $\text{Span}(\mathbf{FinGpd})$ in section 2, and $\mathbf{2Vect}$ in section 3. In particular, to categorify the functor L , we need a 2-category to correspond to \mathbf{Vect} , and this will be the 2-category of all Kapranov-Voevodsky 2-vector spaces. A KV 2-vector space is an abelian category with some extra structure, just as a vector space is a special type of abelian group. In section 3 we give some background and collect some fundamental results about them which are widely known, but whose proofs are seldom given. For example, we show that 2-vector spaces, understood as a semisimple \mathbb{C} -linear additive category, are all equivalent to \mathbf{Vect}^k for some nonnegative integer k .

In section 4, we give the object level of our construction for a 2-functor Λ , which, to (essentially finite) groupoids assigns KV 2-vector spaces. Analogously with sets, we obtain 2-linear maps for spans of groupoids. In fact, just as with sets, this is a consequence of an even simpler correspondence. Namely, there is the “pullback” and “push-forward” of a function mentioned in the description of the linear map from a span of sets as a *sum over histories* (the sum occurs in the “push-forward” operation, and corresponds to the sum in matrix multiplication). The groupoid situation is more complicated than that for sets, however, because of the existence of automorphisms of the objects, and the condition that maps between groupoids are functors. This means, in particular, that for each object x in a groupoid, the functor determines a homomorphism from the automorphism group of x to that of its image. The push-forward operation can be interpreted as a Kan extension and has both an object and a morphism level.

Section 5 describes how these results define 2-linear maps associated to spans of groupoids. It begins with a brief discussion of the bicategory whose objects are groupoids, whose morphisms are spans of groupoids, and whose 2-morphisms are span maps. This is followed by an explicit construction of the morphism level of the 2-functor $\Lambda : \text{Span}(\mathbf{FinGpd}) \rightarrow \mathbf{2Vect}$ and shows that it preserves composition of spans in the weak sense: that is, up to a specified isomorphism. (Technical details of this proof are reserved for appendix A). Finally, using Frobenius reciprocity, it describes a simple explicit matrix representation for the 2-functor constructed.

Section 6 then continues by describing how this representation works at the level of 2-morphisms. This is analogous to the 1-morphism level, in that it consists of a “pullback and pushforward” process. This is most easily described in terms of the linear maps between corresponding vector spaces which appear in the matrix representation of the 2-linear maps associated to a pair of spans from A to B . We give this construction and show it preserves both vertical and horizontal composition of 2-morphisms in the appropriate ways.

The results shown in sections 5 and 6 do much of the work involved in showing that our representation is really a 2-functor. The remainder of this proof is given in section 7.

Now we begin to describe the 2-linearization process by collecting some key facts about the bicategory of 2-vector spaces, including a canonical construction of one for each (essentially finite) groupoid.

2. THE BICATEGORY $\text{Span}(\mathbf{FinGpd})$

The main purpose of this paper is to describe a weak 2-functor

$$(6) \quad \Lambda : \text{Span}(\mathbf{FinGpd}) \rightarrow \mathbf{2Vect}$$

In this section, we will describe the source bicategory, $\text{Span}(\mathbf{FinGpd})$.

First, the objects of $\text{Span}(\mathbf{FinGpd})$.

Definition 2.0.1. *An **essentially finite** groupoid is one which is equivalent to a finite groupoid. A **finitely generated** groupoid is one with a finite set of objects, and all of whose morphisms are generated under composition by a finite set of morphisms. An **essentially finitely generated** groupoid is one which is equivalent to a finitely generated one.*

Note that, in particular, essentially finitely generated groupoids must be essentially finite, since every object has an identity morphism. We will use the term “essentially finite” to mean both of these conditions. Next we describe the morphisms of $\text{Span}(\mathbf{FinGpd})$.

In any category \mathbf{C} , a *span* is a diagram of the form:

$$(7) \quad \begin{array}{ccc} & X & \\ s \swarrow & & \searrow t \\ A_1 & & A_2 \end{array}$$

In particular, we want to reproduce the “linearization” associated to spans of sets which we discussed in the introduction. The idea is that given a span of groupoids, as in Figure 2 (which suppresses the homomorphisms labelling the strands in the span, but should be compared with Figure 1), there will be a “transfer” 2-linear map from the KV 2-vector space associated to the source of the span, to that associated to the target.

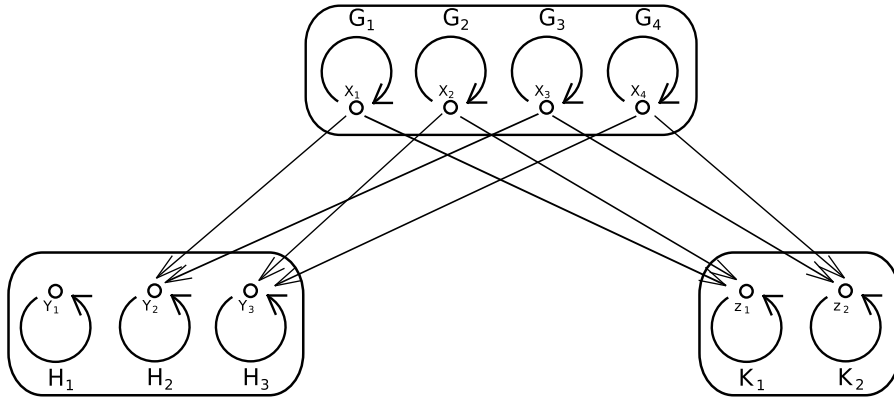


FIGURE 2. A Span of Groupoids

If \mathbf{C} has pullbacks, we can define composition of spans using them:

$$(8) \quad \begin{array}{ccccc} & & X' \circ X & & \\ & \swarrow S & & \searrow T & \\ & X & & X' & \\ \swarrow s & & & & \searrow t' \\ A_1 & & A_2 & & A_3 \\ \nwarrow t & & \nwarrow s' & & \end{array}$$

where we define $X' \circ X$ to be the object, unique up to isomorphism, which makes the central square a pullback square. That is, it is a terminal occupant of this niche. If \mathbf{C} is, in addition, a concrete category, the pullback is a subobject of the product $X \times X'$.

$$(9) \quad X' \circ X = X \times_{A_2} X' = \bigcup_{a \in A_2} t^{-1}(a) \times (s')^{-1}(a)$$

the fibred product of X' and X over A_2 . (Indeed, if \mathbf{C} is Cartesian, any span can be factored through a product.)

Now, for any category \mathbf{C} with pullbacks, there is a category $\text{Span}(\mathbf{C})$ whose objects are the objects of \mathbf{C} , and whose morphisms are isomorphism classes spans in \mathbf{C} composed by pullback. Here we are taking spans up to isomorphisms $\alpha : X_1 \rightarrow X_2$ which are commuting diagrams of the form:

$$(10) \quad \begin{array}{ccccc} & & X_1 & & \\ & \swarrow s_1 & \downarrow \alpha & \searrow t_1 & \\ A_1 & \xleftarrow{s_2} & X_2 & \xrightarrow{t_2} & A_2 \end{array}$$

However, we will do something slightly different. We will be interested in spans of groupoids. Since groupoids naturally form a 2-category, we should weaken the notion of composition, and give an appropriate notion of 2-morphism, for a bicategory $\text{Span}(\mathbf{FinGpd})$.

Given any 2-category \mathbf{C} with weak pullbacks, one can again form a bicategory $\text{Span}(\mathbf{C})$ with the same objects as \mathbf{C} and with spans for morphisms. The composite of spans $A_1 \xleftarrow{s} X \xrightarrow{t} A_2$ and $A_2 \xleftarrow{s'} X' \xrightarrow{t'} A_3$ is an object $X' \circ X$ together with a 2-morphism α making this diagram commute, and terminal for such choices:

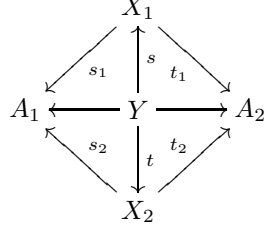
$$(11) \quad \begin{array}{ccccc} & & X' \circ X & & \\ & \swarrow S & & \searrow T & \\ & X & \xrightarrow[\sim]{\alpha} & X' & \\ \swarrow s & & & & \searrow t' \\ A_1 & & A_2 & & A_3 \\ \nwarrow t & & \nwarrow s' & & \end{array}$$

(Note: In the original image, curved arrows labeled $s \circ S$ and $t' \circ T$ connect A_1 to X' and A_3 to X respectively, passing above the main diagram.)

A 2-morphism in $\text{Span}(\mathbf{C})$ will be an isomorphism class of *spans of span maps*. That is, consider a span of *2-morphisms* in the usual $\text{Span}(\mathbf{C})$. This is a diagram

of the form:

(12)



In principle we need only require this diagram commute weakly: that is, there are isomorphisms $\zeta_s : s_1 \circ s \rightarrow s_2 \circ t$ and $\zeta_t : t_1 \circ s \rightarrow t_2 \circ t$. For the most part, since the construction we mean to give is invariant under equivalence of groupoids, and taking the A_i to be skeletal makes this strict, we will assume strict commutativity, though we shall indicate where the argument must be changed to accommodate the weak case.

We are considering such diagrams only *up to isomorphism*: that is, the inner span $X_1 \leftarrow Y \rightarrow X_2$ in the 2-morphism 12 is only considered up to an isomorphism of spans in the sense of 10.

The reason for considering these is as follows. First, taking a category \mathbf{C} and passing to $\text{Span}(\mathbf{C})$ amounts to formally adjoining duals for morphisms in \mathbf{C} . The dual of any span is the span which has the same maps, considered in the reverse orientation, exchanging the role of source and target object. When we apply Λ , these duals will in fact become adjoints, as we shall see.

Now, we are interested in the case $\mathbf{C} = \mathbf{FinGpd}$, so the diagram for the composite of spans between groupoids contains a weak pullback square: composition is only preserved up to isomorphism. In particular, the objects are now groupoids, which are themselves categories with objects and morphisms. Since it makes sense to speak of two objects of a groupoid being isomorphic, the weakest meaningful condition is that objects of groupoids X and X' should need only project to isomorphic objects on A_2 . But there are potentially different isomorphisms between those objects. So the weak pullback is a larger groupoid than a strict pullback, since its objects come with a specified isomorphism between the two restrictions.

That this is a weak pullback square of functors between groupoids means that this diagram commutes up to the natural isomorphism $\alpha : t \circ S \rightarrow s' \circ T$. (The fact that α is iso is what makes this a *weak* pullback rather than a *lax* pullback, where α is only a natural transformation.) This is an example of a *comma category* (the concept, though not the name, was introduced by Lawvere in his doctoral thesis [12]). We recall some background about this construction in Appendix A.1.

Now, the process of finding higher morphisms by taking spans of span maps could obviously be continued: each new level of span naturally gives maps of spans as morphisms. We could repeat the process of adjoining duals by passing to spans of such span maps, and so on recursively as far as we wish. For our purposes here, however, we will stop at 2-morphisms for two reasons. First, we want to describe a representation into $\mathbf{2Vect}$, which is a 2-category. This in turn is since the objects of \mathbf{FinGpd} are themselves categories, and our 2-functor Λ will represent them as categories - so a 2-category is the natural setting for them.

Collecting the definition together, we then have the following.

Definition 2.0.2. *The bicategory $\text{Span}(\mathbf{FinGpd})$ has:*

- **Objects:** *Essentially finite groupoids*
- **Morphisms:** *Spans of groupoids, composed by weak pullback*
- **2-Morphisms:** *Isomorphism classes of spans of span maps, composed by weak pullback both horizontally and vertically*

Having defined $\text{Span}(\mathbf{FinGpd})$, the source bicategory of the 2-functor Λ we aim to describe here, we next describe its target, $\mathbf{2Vect}$.

3. KAPRANOV-VOEVODSKY 2-VECTOR SPACES

There are two major philosophies regarding how to categorify the concept “vector space”. A Baez-Crans (BC) 2-vector space is a category object in \mathbf{Vect} —that is, a category having a vector space of objects and of morphisms, where source, target, composition, etc. are linear maps. This is a useful concept for some purposes—it was developed to give a categorification of Lie algebras. The reader may refer to the paper of Baez and Crans [3] for more details. However, a BC 2-vector space turns out to be equivalent to a 2-term chain complex and for many purposes this is too strict. This is not the concept of 2-vector space which concerns us here.

The other, earlier, approach is to define a 2-vector space as a category having operations such as a monoidal structure analogous to the addition on a vector space. In particular, we will restrict our attention to *complex* 2-vector spaces.

This ambiguity about the correct notion of “2-vector space” is typical of the problem of categorification. Since the categorified setting has more layers of structure, there is a choice of level to which the structure in the concept of a vector space should be lifted. Thus in the BC 2-vector spaces, we have literal vector addition and scalar multiplication within the objects and morphisms. In KV 2-vector spaces and their cousins, we only have this for morphisms, and for objects there is a categorified analog of addition, in the sense that they are additive categories. The key difference between the two notions of 2-vector space lies in which category plays the role of the “base field”: in the BC definition, this is the ring category $\mathbb{C}_{[0]}$ whose objects are complex numbers, whereas for the KV definition it is \mathbf{Vect} , whose objects are complex vector spaces. This is discussed by Josep Elgueta [8].

Indeed, Elgueta [8] shows several different types of “generalized” 2-vector spaces, and relationships among them. In particular, while KV 2-vector spaces can be thought of as having a *set* of basis elements, a generalized 2-vector space may have a general *category* of basis elements. The free generalized 2-vector space on a category is denoted $\mathbf{Vect}[\mathcal{C}]$. Then KV 2-vector spaces arise when \mathcal{C} is a discrete category with only identity morphisms. This is essentially a set S of objects. Thus it should not be surprising that KV 2-vector spaces have a structure analogous to free vector spaces generated by some finite set - which are isomorphic to \mathbb{C}^k .

3.1. Definition. The standard example of this approach is the Kapranov-Voevodsky (KV) definition of a 2-vector space [10], which is the form we shall use (at least when the situation is finite-dimensional). To motivate the KV definition, consider the idea that, in categorifying, one should replace the base field \mathbb{C} with a monoidal category. Specifically, it turns out, with \mathbf{Vect} , the category of finite dimensional complex vector spaces. This leads to the following replacements for concepts in elementary linear algebra:

- Vectors = k -tuples of scalars \mapsto 2-vectors = k -tuples of vector spaces
- Addition \mapsto Direct Sum

- Multiplication \mapsto Tensor Product

So just as \mathbb{C}^k is the standard example of a complex vector space, \mathbf{Vect}^k will be the standard example of a 2-vector space. But we should define these precisely.

To begin with, a KV 2-vector space is a \mathbb{C} -linear additive category with some properties, so we begin by explaining this. The property of *additivity* for categories, is here seen as the analog of the group structure of a vector space, though additivity in a category is somewhat different. The motivating example for us is the *direct sum* operation in \mathbf{Vect} . Such an operation plays the role in a 2-vector space which vector addition plays in a vector space.

Definition 3.1.1. *If a category \mathbf{C} is enriched in abelian groups, a **biproduct** is an operation giving, for any objects x and y in \mathbf{C} an object $x \oplus y$ equipped with morphisms ι_x, ι_y from x and y respectively into $x \oplus y$; and morphisms π_x, π_y from $x \oplus y$ into x and y respectively, which satisfy the biproduct relations:*

$$(13) \quad \pi_x \circ \iota_x = \text{id}_x \text{ and } \pi_y \circ \iota_y = \text{id}_y$$

and

$$(14) \quad \iota_x \circ \pi_x + \iota_y \circ \pi_y = \text{id}_{x \oplus y}$$

Whenever biproducts exist, they are always both products and coproducts.

Definition 3.1.2. *A **\mathbb{C} -linear additive category** is a category \mathbf{V} enriched in \mathbf{Vect} (i.e. $\forall x, y \in \mathbf{V}$, $\text{hom}(x, y)$ is a vector space over \mathbb{C}), such that composition is a bilinear map, and such that \mathbf{V} has a zero object (i.e. 0 which is both initial and terminal). A **\mathbb{C} -linear functor** between \mathbb{C} -linear categories is one where morphism maps are \mathbb{C} -linear. A **simple object** in \mathbf{V} is $x \in \mathbf{V}$ such that $\text{hom}(x, x) \cong \mathbb{C}$.*

As important fact about KV 2-vector spaces is that they have (finite) bases: they are generated by finitely many simple objects.

Definition 3.1.3. *A **Kapranov–Voevodsky 2-vector space** is a \mathbb{C} -linear additive category which is semisimple (every object can be written as a finite biproduct of simple objects). A **2-linear map** between 2-vector spaces is a \mathbb{C} -linear functor.*

Remark 3.1.4. It is a consequence of \mathbb{C} -linearity that a 2-linear map also preserves biproducts, since the images of the π and ι maps still satisfy the definition of a biproduct (and the universal properties for product and coproduct follow automatically). The above definition of a 2-linear map is sometimes given in the equivalent form requiring that the functor preserve exact sequences. Indeed, since every object is a finite biproduct of simple objects, a 2-vector space is an *abelian* category. (See e.g. Freyd [9].)

Example 3.1.5. The standard example [10] of a KV 2-vector space highlights the analogy with the familiar vector space \mathbb{C}^k . The 2-vector space \mathbf{Vect}^k is a category whose objects are k -tuples of vector spaces, maps are k -tuples of linear maps. The *additive* structure of the 2-vector space \mathbf{Vect}^k comes from applying the direct sum in \mathbf{Vect} component-wise.

Note that there is an equivalent of *scalar multiplication*, using the tensor product:

$$(15) \quad V \otimes \begin{pmatrix} V_1 \\ \vdots \\ V_k \end{pmatrix} = \begin{pmatrix} V \otimes V_1 \\ \vdots \\ V \otimes V_k \end{pmatrix}$$

and

$$(16) \quad \begin{pmatrix} V_1 \\ \vdots \\ V_k \end{pmatrix} \oplus \begin{pmatrix} W_1 \\ \vdots \\ W_k \end{pmatrix} = \begin{pmatrix} V_1 \oplus W_1 \\ \vdots \\ V_k \oplus W_k \end{pmatrix}$$

As the correspondence with linear algebra would suggest, 2-linear maps $T : \mathbf{Vect}^k \rightarrow \mathbf{Vect}^l$ amount to $k \times l$ matrices of vector spaces, acting by matrix multiplication using the direct sum and tensor product instead of operations in \mathbb{C} :

$$(17) \quad \begin{pmatrix} T_{1,1} & \cdots & T_{1,k} \\ \vdots & & \vdots \\ T_{l,1} & \cdots & T_{l,k} \end{pmatrix} \begin{pmatrix} V_1 \\ \vdots \\ V_k \end{pmatrix} = \begin{pmatrix} \bigoplus_{i=1}^k T_{1,i} \otimes V_i \\ \vdots \\ \bigoplus_{i=1}^k T_{l,i} \otimes V_i \end{pmatrix}$$

The natural transformations between these are matrices of linear transformations:

$$(18) \quad \alpha = \begin{pmatrix} \alpha_{1,1} & \cdots & \alpha_{1,k} \\ \vdots & & \vdots \\ \alpha_{l,1} & \cdots & \alpha_{l,k} \end{pmatrix} : \begin{pmatrix} T_{1,1} & \cdots & T_{1,k} \\ \vdots & & \vdots \\ T_{l,1} & \cdots & T_{l,k} \end{pmatrix} \longrightarrow \begin{pmatrix} T'_{1,1} & \cdots & T'_{1,k} \\ \vdots & & \vdots \\ T'_{l,1} & \cdots & T'_{l,k} \end{pmatrix}$$

where each $\alpha_{i,j} : T_{i,j} \rightarrow T'_{i,j}$ is a linear map in the usual sense.

These natural transformations give 2-morphisms between 2-linear maps, so that \mathbf{Vect}^k is a bicategory with these as 2-cells:

$$(19) \quad \begin{array}{ccc} & \xrightarrow{F} & \\ \mathbf{Vect}^k & \Downarrow \alpha & \mathbf{Vect}^l \\ & \xrightarrow{G} & \end{array}$$

In our example above, the finite set of simple objects of which every object is a sum is the set of 2-vectors of the form

$$(20) \quad \begin{pmatrix} 0 \\ \vdots \\ \mathbb{C} \\ \vdots \\ 0 \end{pmatrix}$$

which have the zero vector space in all components except one (which can be arbitrary). We can call these *standard basis 2-vectors*. Clearly every object of \mathbf{Vect}^k is a finite biproduct of these objects, and each is simple (its vector space of endomorphisms is 1-dimensional).

3.2. Classification Theorems. The most immediately useful fact about KV 2-vector spaces is the following well known characterization:

Theorem 3.2.1. *Every KV 2-vector space is equivalent as a category to \mathbf{Vect}^k for some $k \in \mathbb{N}$.*

Proof. Suppose \mathbf{K} is a KV 2-vector space with a basis of simple objects $X_1 \dots X_k$. Then we construct an equivalence $E : \mathbf{K} \rightarrow \mathbf{Vect}^k$ as follows:

E should be an additive functor with $E(X_i) = V_i$, where V_i is the k -tuple of vector spaces having the zero vector space in every position except the i^{th} , which has a copy of \mathbb{C} . But any object X , is a sum $\bigoplus_i X_i^{n_i}$, so by linearity (i.e. the fact that E preserves biproducts) X will be sent to the sum of the same number of copies of the V_i , which is just a k -tuple of vector spaces whose i^{th} component is \mathbb{C}^{n_i} . So every object in K is sent to an k -tuple of vector spaces. By \mathbb{C} -linearity, and the fact that hom-vector spaces of simple objects are one-dimensional, this determines the images of all morphisms.

But then the weak inverse of E is easy to construct, since sending V_i to X_i gives an inverse at the level of objects, by the same linearity argument as above. At the level of morphisms, the same argument holds again. \square

This is a higher analog of the fact that every finite dimensional complex vector space is isomorphic to \mathbb{C}^k for some $k \in \mathbb{N}$. So, indeed, the characterization of 2-vector spaces in our example above is generic: every KV 2-vector space is equivalent to one of the form given. Moreover, our picture of 2-linear maps is also generic, as shown by this argument, analogous to the linear algebra argument for representation of linear maps by matrices:

Lemma 3.2.2. *Any 2-linear map $T : \mathbf{Vect}^n \rightarrow \mathbf{Vect}^m$ is naturally isomorphic to a map of the form (17).*

Proof. Any 2-linear map T is a \mathbb{C} -linear additive functor between 2-vector spaces. Since any object in a 2-vector space can be represented as a biproduct of simple objects—and morphisms likewise—such a functor is completely determined by its effect on the basis of simple objects and morphisms between them.

But then note that since the automorphism group of a simple object is by definition just all (complex) multiples of the identity morphism, there is no choice about where to send any such morphism. So a functor is completely determined by the images of the basis objects, namely the 2-vectors $V_i = (0, \dots, \mathbb{C}, \dots, 0) \in \mathbf{Vect}^n$, where V_i has only the i^{th} entry non-zero.

On the other hand, for any i , $T(V_i)$ is a direct sum of some simple objects in \mathbf{Vect}^m , which is just some 2-vector, namely a k -tuple of vector spaces. Then the fact that the functor is additive means that it has exactly the form given. \square

And finally, the analogous fact holds for natural transformations between 2-linear maps:

Lemma 3.2.3. *Any natural transformation $\alpha : T \rightarrow T'$ from a 2-linear map $T : \mathbf{Vect}^n \rightarrow \mathbf{Vect}^m$ to a 2-linear map $T' : \mathbf{Vect}^n \rightarrow \mathbf{Vect}^m$, both in the form (17) is of the form (18).*

Proof. By Lemma 3.2.2, the 2-linear maps T and T' can be represented as matrices of vector spaces, which act on an object in \mathbf{Vect}^n as in (17). A natural transformation α between these should assign, to every object $X \in \mathbf{Vect}^n$, a morphism $\alpha_X : T(X) \rightarrow T'(X)$ in \mathbf{Vect}^m , such that the usual naturality square commutes for every morphism $f : X \rightarrow Y$ in \mathbf{Vect}^n .

Suppose X is the n -tuple (X_1, \dots, X_n) , where the X_i are finite dimensional vector spaces. Then

$$(21) \quad T(X) = (\bigoplus_{k=1}^n V_{1,k} \otimes X_k, \dots, \bigoplus_{k=1}^n V_{m,k} \otimes X_k)$$

where the $V_{i,j}$ are the components of T , and similarly

$$(22) \quad T'(X) = (\oplus_{k=1}^n V'_{1,k} \otimes X_k, \dots, \oplus_{k=1}^n V'_{m,k} \otimes X_k)$$

where the $V'_{i,j}$ are the components of T' .

Then a morphism $\alpha_X : T(X) \rightarrow T'(X)$ consists of an m -tuple of linear maps:

$$(23) \quad \alpha_j : \oplus_{k=1}^n V_{j,k} \otimes X_k \rightarrow \oplus_{k=1}^n V'_{j,k} \otimes X_k$$

but by the universal property of the biproduct, this is the same as having an $(n \times m)$ -indexed set of maps

$$(24) \quad \alpha_{jk} : V_{j,k} \otimes X_k \rightarrow \oplus_{r=1}^n V'_{j,r} \otimes X_r$$

and by the dual universal property, this is the same as having $(n \times n \times m)$ -indexed maps

$$(25) \quad \alpha_{jkr} : V_{j,k} \otimes X_k \rightarrow V'_{j,r} \otimes X_r$$

However, we must have the naturality condition for every morphism $f : X \rightarrow X'$:

$$(26) \quad \begin{array}{ccc} T(X) & \xrightarrow{T(f)} & T(X') \\ \alpha_X \downarrow & & \downarrow \alpha_{X'} \\ T'(X) & \xrightarrow{T'(f)} & T'(X') \end{array}$$

Note that each of the arrows in this diagram is a morphism in \mathbf{Vect}^m , which are linear maps in each component—so in fact we have a separate naturality square for each component.

Also, since T and T' act on X and X' by tensoring with fixed vector spaces as in (21), one has $T(f)_i = \oplus_j 1_{V_{ij}} \otimes f_j$, having no effect on the V_{ij} . We want to show that the components of α affect *only* the V_{ij} .

Additivity of all the functors involved implies that the assignment α of maps to objects in \mathbf{Vect}^n is additive. So consider the case when X is one of the standard basis 2-vectors, having \mathbb{C} in one position (say, the k^{th}), and the zero vector space in every other position. Then, restricting to the naturality square in the k^{th} position, the above condition amounts to having m maps (indexed by j):

$$(27) \quad \alpha_{j,k} : V_{j,k} \rightarrow V'_{j,k}$$

So by linearity, a natural transformation is determined by an $n \times m$ matrix of maps as in (18). \square

The fact that 2-linear maps between 2-vector spaces are functors between categories recalls the analogy between linear algebra and category theory in the concept of an *adjoint*. If V and W are inner product spaces, the adjoint of a linear map $F : V \rightarrow W$ is a map F^\dagger for which $\langle Fx, y \rangle = \langle x, F^\dagger y \rangle$ for all $x \in V_1$ and $y \in V_2$. A (right) adjoint of a functor $F : \mathbf{C} \rightarrow \mathbf{D}$ is a functor $G : \mathbf{D} \rightarrow \mathbf{C}$ for which $\text{hom}_{\mathbf{D}}(Fx, y) \cong \text{hom}_{\mathbf{C}}(x, Gy)$ (and then F is a left adjoint of G).

In the situation of a KV 2-vector space, the categorified analog of the adjoint of a linear map is indeed an adjoint functor. (Note that since a KV 2-vector space has a specified basis of simple objects, it makes sense to compare it to an inner product space.) Moreover, the adjoint of a functor has a matrix representation which is much like the matrix representation of the adjoint of a linear map. We summarize

this in the following (a variant of proposition 25 in [1], shown there for 2-Hilbert spaces):

Theorem 3.2.4. *Given any 2-linear map $F : V \rightarrow W$, there is a 2-linear map $F^\dagger : W \rightarrow V$ which is both a left and right adjoint to F .*

Proof. By Theorem 3.2.1, we have $V \simeq \mathbf{Vect}^n$ and $W \simeq \mathbf{Vect}^m$ for some n and m . By composition with these equivalences, we can restrict to this case. But then we have by Lemma 3.2.2 that F is naturally isomorphic to some 2-linear map given by matrix multiplication by some matrix of vector spaces $[F_{i,j}]$:

$$(28) \quad \begin{pmatrix} F_{1,1} & \cdots & F_{1,n} \\ \vdots & & \vdots \\ F_{m,1} & \cdots & F_{m,n} \end{pmatrix}$$

We claim that a (two-sided) adjoint functor F^\dagger is given by the “dual transpose matrix” of vector spaces $[F_{i,j}]^\dagger$:

$$(29) \quad \begin{pmatrix} F_{1,1}^\dagger & \cdots & F_{1,n}^\dagger \\ \vdots & & \vdots \\ F_{m,1}^\dagger & \cdots & F_{m,n}^\dagger \end{pmatrix}$$

where $F_{i,j}^\dagger$ is the vector space dual $(F_{i,j})^*$ (note the transposition of the matrix).

We note that this prescription is symmetric, since $[T]^\dagger = [T]$, so if F^\dagger is always a left adjoint of F , then F is also a left-adjoint of F^\dagger , hence F^\dagger a right adjoint of F . So if this prescription gives a left adjoint, it gives a two-sided adjoint. Next we check that it does.

Suppose $x = (X_i) \in \mathbf{Vect}^n$ is the 2-vector with vector space X_i in the i^{th} component, and $y = (Y_j) \in \mathbf{Vect}^m$ has vector space Y_j in the j^{th} component. Then $F(x) \in \mathbf{Vect}^m$ has j^{th} component $\bigoplus_{i=1}^n F_{i,j} \otimes X_i$. Now, a map in \mathbf{Vect}^m from $F(x)$ to y consists of a linear map in each component, so it is an m -tuple of maps:

$$(30) \quad f_j : \bigoplus_{i=1}^n F_{i,j} \otimes X_i \rightarrow Y_j$$

for $j = 1 \dots m$. But since the direct sum (biproduct) is a categorical coproduct, this is the same as an $m \times n$ matrix of maps:

$$(31) \quad f_{ij} : V_{i,j} \otimes X_i \rightarrow Y_j$$

for $k = 1 \dots n$ and $j = 1 \dots m$, and $\text{hom}(F(x), y)$ is the vector space of all such maps.

By the same argument, a map in \mathbf{Vect}^n from x to $F^\dagger(y)$ consists of an $n \times m$ matrix of maps:

$$(32) \quad g_{ji} : X_i \rightarrow V_{j,i}^* \otimes Y_j \cong \text{hom}(V_{j,i}, Y_j)$$

for $i = 1 \dots n$ and $j = 1 \dots m$, and $\text{hom}(x, F^\dagger(y))$ is the vector space of all such maps.

But then we have a natural isomorphism $\text{hom}(F(x), y) \cong \text{hom}(x, F^\dagger(y))$ by the duality of hom and \otimes , so in fact F^\dagger is a right adjoint for F , and by the above argument, also a left adjoint.

Moreover, no other non-isomorphic matrix defines a 2-linear map with these properties, and since any functor is naturally isomorphic to some matrix, this is the sole F^\dagger which works. \square

3.3. Example: Group 2-Algebra. We conclude this section by giving an example of a 2-vector space:

Example 3.3.1. As an example of a KV 2-vector space, consider the *group 2-algebra* on a finite group G , defined by analogy with the group algebra:

The group algebra $\mathbb{C}[G]$ consists of the set of elements formed as formal linear combinations elements of G :

$$(33) \quad b = \sum_{g \in G} b_g \cdot g$$

where all but finitely many b_g are zero. We can think of these as complex functions on G . The algebra multiplication on $\mathbb{C}[G]$ is given by the multiplication in G :

$$(34) \quad b \star b' = \sum_{g, g' \in G} (b_g b'_{g'}) \cdot gg'$$

This does not correspond to the multiplication of functions on G , but to *convolution*:

$$(35) \quad (b \star b')_g = \sum_{h \cdot h' = g} b_h b'_{h'}$$

Similarly, the *group 2-algebra* $A = \mathbf{Vect}[G]$ is the *category* of G -graded vector spaces. That is, direct sums of vector spaces associated to elements of G :

$$(36) \quad V = \bigoplus_{g \in G} V_g$$

where $V_g \in \mathbf{Vect}$ is a vector space. This is a G -graded vector space. We can take direct sums of these pointwise, so that $(V \oplus V')_g = V_g \oplus V'_g$, and there is a “scalar” product with elements of \mathbf{Vect} given by $(W \otimes V)_g = W \otimes V_g$. There is also a *group 2-algebra* product of G -graded vector spaces, involving a convolution on G :

$$(37) \quad (V \star V')_h = \bigoplus_{g \cdot g' = h} V_g \otimes V'_{g'}$$

The category of G -graded vector spaces is clearly a KV 2-vector space, since it is equivalent to \mathbf{Vect}^k where $k = |G|$. However, it has the additional structure of a 2-algebra because of the group operation on the finite set G .

Example 3.3.2. Given a finite group G , the category $\mathbf{Rep}(G)$ has:

- **Objects:** Complex representations of G (i.e. functors $\rho : G \rightarrow \mathbf{Vect}$, where G is seen as a one-object groupoid)
- **Morphisms:** Intertwining operators between reps (i.e. natural transformations)

This is clearly a 2-vector space generated by the irreducible representations of G .

In the next section, we will see that a similar construction shows that the representation categories of finite groupoids are KV 2-vector spaces. This will be the beginning of our definition of Λ .

This highlights one motivation for thinking of 2-vector spaces: the fact that, in quantum mechanics, one often “quantizes” a classical system by taking the Hilbert

space of (square integrable) \mathbb{C} -valued functions on its phase space. Similarly, one approach to finding a higher-categorical version of a quantum theory is to take **Vect**-valued functors, as we discuss in more detail in Section 4.

By restricting our attention to the (essentially) finite case, we avoid here the analytical issues involved in finding an analog for $L^2(X)$.

4. KV 2-VECTOR SPACES AND FINITE GROUPOIDS

We have now seen that we can get a 2-vector space as a category of functions from some finite set S into **Vect**, and this may have extra structure if S does. However, this is somewhat unnatural, since **Vect** is a category and S a mere set. It seems more natural to consider functor categories into **Vect** from some category **C**. These are examples of the generalized 2-vector spaces described by Elgueta [8]. Then the above way of looking at a KV 2-vector space can be reduced to the situation when **C** is a discrete category with a finite set of objects. However, there are interesting cases where **C** is not of this form, and the result is still a KV vector space. A relevant class of examples, as we shall show, come from special kinds of groupoids.

4.1. Free 2-Vector Space on a Finite Groupoid. Since we want our 2-vector spaces to have finitely many generators, we need a condition on the sorts of groupoids we are talking about here. Of course, since often one works with topological groupoids which may be uncountable, the kind of finiteness condition we will have to apply seems restrictive. A full treatment of, for example, Lie groupoids, would require much more consideration of infinite dimensional 2-vector spaces (and indeed 2-Hilbert spaces). In the meantime, we can only consider groupoids which are essentially finite.

We first show that essentially finite groupoids are among the special categories **C** we want to consider:

Lemma 4.1.1. *If X is an essentially finite groupoid, $\mathbf{Rep}(X) = [X, \mathbf{Vect}]$ is a 2-vector space*

Proof. The groupoid X is equivalent to its skeleton, \underline{X} , which contains a single object in each isomorphism class. Since X is essentially finite, this is a finite set of objects, and each object has a finite group of endomorphisms. So

$$(38) \quad X \simeq \coprod_{x \in \underline{X}} \text{Aut}(x)$$

where the groups $\text{Aut}(x)$ are seen as one-object groupoids.

Then

$$(39) \quad [X, \mathbf{Vect}] \simeq \prod_{x \in \underline{X}} [\text{Aut}(x), \mathbf{Vect}]$$

$$(40) \quad = \prod_{x \in \underline{X}} \text{Rep}(\text{Aut}(x))$$

This inherits the biproducts from the categories $\text{Rep}(\text{Aut}(x))$. An irreducible representation of an essentially finite groupoid amounts to a choice of isomorphism class of objects $[x]$, and an irreducible representation of the group $\text{Aut}(x)$. By Schur's Lemma, these are indeed simple objects, since irreducible representations of a group are simple.

□

We notice that we are speaking here of groupoids, and any groupoid \mathbf{X} is equivalent to its opposite category \mathbf{X}^{op} , by an equivalence that leaves objects intact and replaces each morphism by its inverse. So there is no real difference between $[\mathbf{X}, \mathbf{Vect}]$, the category of \mathbf{Vect} -valued functors from \mathbf{X} , and $[\mathbf{X}^{op}, \mathbf{Vect}]$, the category of \mathbf{Vect} -valued *presheaves* (or just “ \mathbf{Vect} -presheaves”) on \mathbf{X} . (We also should note that, since our groupoids are discrete, there is no distinction here between sheaves and presheaves).

Figure 3 is an illustration of an object in $[\mathbf{X}, \mathbf{Vect}]$.

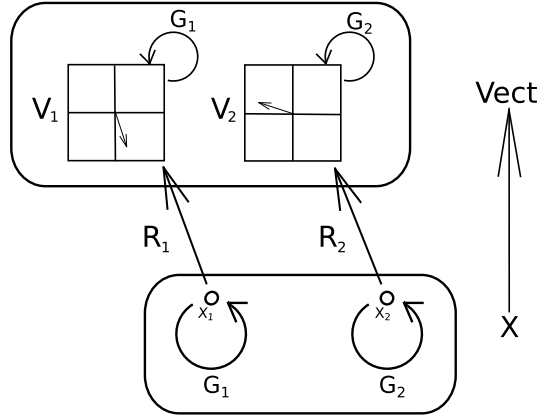


FIGURE 3. A \mathbf{Vect} -valued Presheaf on X

We will use the terminology of “presheaves” for objects of $[\mathbf{X}, \mathbf{Vect}]$ for the sake of highlighting the connection between these results and the usual facts about presheaves of sets in topos theory - which again raises questions about topologically interesting groupoids. This will be addressed in later work, but for now we consider the algebraic aspect of the 2-linearization construction by itself.

4.2. The Ambidextrous Adjunction. Now we want to highlight a result analogous to a standard result for set-valued presheaves (see, e.g. MacLane and Moerdijk [14], Theorem 1.9.2). This is that functors between groupoids induce 2-linear maps between the 2-vector spaces of \mathbf{Vect} -presheaves on them. For \mathbf{Set} -presheaves, there will be a left and a right adjoint to this functor. For \mathbf{Vect} -presheaves, these coincide, as we have seen in Theorem 3.2.4 (an inspection of the proof shows that this is essentially because a finite dimensional vector space V is naturally isomorphic to its double dual $V^{\dagger\dagger}$, while the analogous statement is false for sets). Thus, one says that the “pushforward” map is an *ambidextrous adjoint* for the pullback. For much more on ambidextrous adjunctions and their relation to TQFTs, see Lauda [11]). This is one important motivation for the present work. We summarize the statement as follows.

Proposition 4.2.1. *If \mathbf{X} and \mathbf{Y} are essentially finite groupoids, a functor $f : \mathbf{Y} \rightarrow \mathbf{X}$ gives two 2-linear maps between KV 2-vector spaces:*

$$(41) \quad f^* : [\mathbf{X}, \mathbf{Vect}] \rightarrow [\mathbf{Y}, \mathbf{Vect}]$$

called “pullback along f ” and

$$(42) \quad f_* : [\mathbf{Y}, \mathbf{Vect}] \rightarrow [\mathbf{X}, \mathbf{Vect}]$$

the (two-sided) adjoint to f^* , called “pushforward along f ”

Proof. For any functor $F : \mathbf{X} \rightarrow \mathbf{Vect}$,

$$(43) \quad f^*(F) = F \circ f$$

which is a functor from \mathbf{Y} to \mathbf{Vect} , the pullback of F along f .

To see that this is a 2-linear map, we recall that it is enough to show it is \mathbb{C} -linear, since then biproducts will automatically be preserved. But a linear combination of maps in some hom-category in $[\mathbf{X}, \mathbf{Vect}]$ is taken by f^* to the corresponding linear combination in the hom-category in $[\mathbf{Y}, \mathbf{Vect}]$, where maps are now between vector spaces thought of over $y \in \mathbf{Y}$.

So indeed there is a 2-linear map f^* . But then by Theorem 3.2.4, there is a two-sided adjoint of f^* , denoted f_* . \square

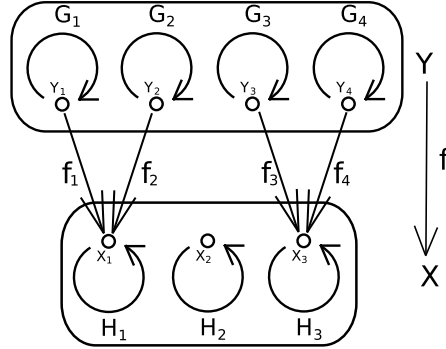


FIGURE 4. A Functor $F : \mathbf{Y} \rightarrow \mathbf{X}$ Between Groupoids

In Figure 4, we see the essential information contained in a functor of groupoids. Any groupoid is equivalent to a skeletal one (that is, one with just one object in each isomorphism class), so we illustrate this case. A skeletal groupoid can be seen as a set of objects, each labelled by a group. A functor between groupoids is a set map, where each “strand” of the set map (i.e. each pair (y_i, x_j) of source and image under the map) is labelled by a homomorphism f_i . This takes the group G_i of automorphisms of the source y_i to the group H_j of automorphisms of the target x_j .

It will be useful to have another, more explicit, way to describe the “pushforward” map than the matrix-dependent view of Theorem 3.2.4. Fortunately, there is a more intrinsic way to describe the 2-linear map f_* , the adjoint of f^* .

Definition 4.2.2. For a given $x \in \mathbf{X}$, the comma category $(f \downarrow x)$ has objects which are objects $y \in \mathbf{Y}$ equipped with maps $f(y) \rightarrow x$ in \mathbf{X} , and morphisms which

are morphisms $a : y \rightarrow y'$ whose images make the triangles

$$(44) \quad \begin{array}{ccc} f(y) & \xrightarrow{f(a)} & f(y') \\ \downarrow & \swarrow & \\ x & & \end{array}$$

in \mathbf{X} commute. Given a **Vect**-presheaf G on \mathbf{Y} , define $f_*(G)(x) = \text{colim } G(f \downarrow x)$ —a colimit in **Vect**.

The pushforward of a morphism $b : x \rightarrow x'$ in \mathbf{X} , $f_*(G)(b) : f_*(G)(x) \rightarrow f_*(G)(x')$ is the induced morphism.

The comma category is the appropriate categorical equivalent of a *preimage*—rather than requiring $f(y) = x$, one accepts that they may be isomorphic, in different ways. So this colimit is a categorified equivalent of taking a sum over a preimage. The result is the *Kan extension* of G along f .

Consider the effect of f_* on a 2-vector $G : \mathbf{Y} \rightarrow \mathbf{Vect}$ by describing $f_*G : \mathbf{X} \rightarrow \mathbf{Vect}$. If $F : \mathbf{X} \rightarrow \mathbf{Vect}$ is as above, there should be a canonical isomorphism between $[G, f^*(F)]$ (a hom-set in $[\mathbf{Y}, \mathbf{Vect}]$) and $[p_*(G), F]$ (a hom-set in $[\mathbf{X}, \mathbf{Vect}]$).

The hom-set $[G, f^*(F)]$ is found by first taking the pullback of F along f . This gives a presheaf on \mathbf{Y} , namely $F(f(-))$. The hom-set is then the set of natural transformations $\alpha : G \rightarrow f^*F$. Given an object y in \mathbf{Y} , α picks a linear map $\alpha_y : F(f(y)) \rightarrow G(y)$ subject to the naturality condition.

Now, we have seen that, given $f : \mathbf{Y} \rightarrow \mathbf{X}$, this $f_* : [\mathbf{Y}, \mathbf{Vect}] \rightarrow [\mathbf{X}, \mathbf{Vect}]$ is a 2-linear map, and an ambidextrous adjoint for f^* . We would like to describe f_* more explicitly. We shall want to make use of the units and counits from both the adjunction in which f_* is a left adjoint, and that in which it is a right adjoint. These are described in the next section.

To describe f_* in more detail, we use the fact that both \mathbf{Y} and \mathbf{X} are equivalent to unions of finite groups, and so a **Vect**-presheaf on \mathbf{Y} is a functor which assigns a representation of $\text{Aut}(y)$ to each object $y \in \mathbf{Y}$. Furthermore, if \mathbf{Y} and \mathbf{X} are skeletal, then $f : \mathbf{Y} \rightarrow \mathbf{X}$ on objects can be any set map, taking objects in \mathbf{Y} to objects in \mathbf{X} . For morphisms, f gives, for each object $y \in \mathbf{Y}$, a homomorphism from the group $\text{Aut}(y) = \text{hom}(y, y)$ to the group $\text{Aut}(f(y))$.

So the pullback f^* is fairly straightforward: given $F : \mathbf{X} \rightarrow \mathbf{Vect}$, the pullback $f^*F = F \circ f : \mathbf{Y} \rightarrow \mathbf{Vect}$ assigns to each $y \in \mathbf{Y}$ the vector space $F(f(y))$, and gives a representation of $\text{Aut}(y)$ on this vector space where $g : y \rightarrow y$ acts by $f(g)$. This is the *pullback representation*. If f is an inclusion, this is usually called the *restricted representation*. The *pushforward*, or adjoint of pullback, for an inclusion is generally called finding the *induced representation*. We remark that for the case where f is an inclusion, Sternberg [17] gives some classical discussion of this for complex representations, as does Benson [6] for more modules over the group ring with a more general base ring R . Here we use the same term for the more general case when f is any homomorphism.

For any presheaf F , the pushforward f_*F is determined by the colimit for each component of that essential preimage. Then for each $x \in \mathbf{X}$, we first get:

$$(45) \quad \bigoplus_{g: f(y) \rightarrow x} F(y)$$

Which is just the direct sum (i.e. biproduct) over the isomorphism classes in the essential preimage of the corresponding vector spaces. However, this is not the colimit: an object in the essential preimage is a pair (y, g) , but we note that if y and y' are isomorphic in \mathbf{Y} , such isomorphisms induce isomorphisms of the spaces $F(y)$, and the colimit will be a quotient which identifies these spaces. In general, the colimit will be a direct sum over isomorphism classes $[y]$ in the essential preimage. Each term of the sum is isomorphic to the induced representation of $F(y)$ under the homomorphism determined by f .

Now, consider what the induced representations are for each isomorphism class. Any isomorphism class $[y]$ of objects in \mathbf{Y} determines a group $G = \text{Aut}(y)$, and similarly $[x] \in \mathbf{X}$ determines $H = \text{Aut}(x)$. So this reduces to the case where \mathbf{Y} and \mathbf{X} are just groups (seen as one-object categories), so we have a group homomorphism $f : G \rightarrow H$. Using the induced algebra homomorphism $f : \mathbb{C}[G] \rightarrow \mathbb{C}[H]$, one can directly construct the induced homomorphism as a quotient: $f_* V = \mathbb{C}[H] \otimes_{\mathbb{C}[G]} V$.

So for general groupoids, with $V = F(y)$, we have the direct sum:

$$(46) \quad (f_* F)(x) = \bigoplus_{f(y) \cong x} \mathbb{C}[\text{Aut}(x)] \otimes_{\mathbb{C}[\text{Aut}(y)]} F(y)$$

Figure 5 illustrates the induced representation schematically, for a single object.

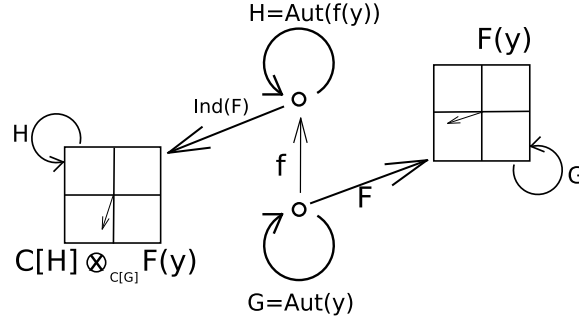


FIGURE 5. Induced Representation from Homomorphism

4.3. Units and Counits. We have observed that the pullback and pushforward maps f^* and f_* are both left and right adjoints. Thus there are two adjunctions to consider: $f^* \dashv f_*$, where pushforward is right adjoint to pullback; and $f_* \dashv f^*$ where pushforward is left adjoint to pullback. For convenience, we refer to these as the “right adjunction” and “left adjunction” respectively, after the position of the pushforward. Each adjunction has unit and counit, so there are four natural transformations to describe. We will identify them as “right” and “left” unit and counit following the convention above. Thus, we have:

$$(47) \quad \eta_L : \text{Id}_{[\mathbf{Y}, \mathbf{Vect}]} \Longrightarrow f^* f_*$$

$$(48) \quad \epsilon_L : f_* f^* \Longrightarrow \text{Id}_{[\mathbf{X}, \mathbf{Vect}]}$$

$$(49) \quad \eta_R : \text{Id}_{[\mathbf{X}, \mathbf{Vect}]} \Longrightarrow f_* f^*$$

$$(50) \quad \epsilon_R : f^* f_* \Longrightarrow \text{Id}_{[\mathbf{Y}, \mathbf{Vect}]}$$

Once again, it is useful for practical calculations to have a coordinate-dependent form for these maps, but there is a convenient intrinsic definition which we shall describe first. Here again, we note that Benson [6] describes the case where f is an inclusion, in a more general setting than the complex representations we consider here.

To begin with, we should describe the functors f^*f_* (“push-pull”), which is an endofunctor on $[\mathbf{Y}, \mathbf{Vect}]$, and f_*f^* (“pull-push”), which is an endofunctor on $[\mathbf{X}, \mathbf{Vect}]$.

For the “push-pull”, f^*f_* , we first push a \mathbf{Vect} -presheaf F on \mathbf{Y} to one on \mathbf{X} , then pull back to \mathbf{Y} . On each object $y \in \mathbf{Y}$, this gives a new presheaf where the vector space $F(y)$ is replaced by the pullback (i.e. induced representation of $Aut(y)$) of $f_*F(f(y))$. But f_*F is a presheaf on \mathbf{X} , which, at each $x \in \mathbf{X}$, gives a colimit over the essential preimage of x in \mathbf{Y} , namely $\bigoplus_{[y']|f(y') \cong x} \mathbb{C}[Aut(x)] \otimes_{\mathbb{C}[Aut(y')]} F(y')$. In the case where $x = f(y)$, this means we get:

$$(51) \quad f^*f_*F(x) = \bigoplus_{[y']|f(y') \cong f(y)} \mathbb{C}[Aut(f(y))] \otimes_{\mathbb{C}[Aut(y')]} F(y')$$

thought of as a (left) representation of $Aut(y)$ in the obvious way (i.e. $g \in Aut(y)$ acts on this space as $f(g)$).

For the “pull-push”, f_*f^* , we first pull a \mathbf{Vect} -presheaf G on \mathbf{X} back to f^*G on \mathbf{Y} . At each $y \in \mathbf{Y}$, this assigns the vector space $f^*G(y) = G(f(y))$ as a representation of $Aut(y)$. We then push forward to \mathbf{X} to get, at each $x \in \mathbf{X}$, that:

$$(52) \quad f_*f^*G(x) = \bigoplus_{[y]|f(y) \cong x} \mathbb{C}[Aut(x)] \otimes_{\mathbb{C}[Aut(y)]} G(x)$$

Note that *a priori* the last space would be $G(f(y))$, but since $f(y) \cong x$, we have also that $G(f(y)) \cong G(x)$ as representations of $Aut(y)$. Here we are implicitly taking a colimit over the essential preimage of x , whose objects are not just y such that $f(y) \cong x$, but rather such y equipped with a specific isomorphism. These therefore induce specific isomorphisms of $G(f(y))$ and $G(x)$, and the quotient implied by the colimit identifies these spaces.

Now, the description above accords with the usual description of these functors in the left adjunction. Since the adjunction is ambidextrous, it applies in both cases, but to describe the unit and counit properly, we should note that in general the canonical description of the left and right adjunctions are different. (Here again we note that Benson [6] shows this for modules over general rings, which in our case are the group algebras $\mathbb{C}[Aut(y)]$ etc., in the case of inclusion) We need to take account of the specific isomorphism between the form we have presented (natural for the left adjunction), and the form which is natural for the right adjunction.

The right adjoint is given as:

$$(53) \quad f_*F(y) = \bigoplus_{[y]|f(y) \cong x} \text{hom}_{\mathbb{C}[Aut(x)]}(\mathbb{C}[Aut(y)], F(y))$$

(Note that the case for groups, namely when \mathbf{Y} and \mathbf{X} have only one object, appears in each term of this direct sum). The *Nakayama isomorphism* gives the duality between the two descriptions of f_* , in terms of $\text{hom}_{\mathbb{C}[Aut(x)]}$ and $\otimes_{\mathbb{C}[Aut(x)]}$, by means of the *exterior trace map*. The groupoid case is just the direct sum of group

cases, which looks like:

$$(54) \quad N : \bigoplus_{[y] \mid f(y) \cong x} \text{hom}_{\mathbb{C}[Aut(y)]}(\mathbb{C}[Aut(x)], F(y)) \rightarrow \bigoplus_{[y] \mid f(y) \cong x} \mathbb{C}[Aut(x)] \otimes_{\mathbb{C}[Aut(y)]} F(y)$$

given by the *exterior trace map* in each factor of the sum:

$$(55) \quad N : \bigoplus_{[y] \mid f(y) \cong x} \phi_y \mapsto \bigoplus_{[y] \mid f(y) \cong x} \frac{1}{\#Aut(y)} \sum_{g \in Aut(x)} g^{-1} \otimes \phi_y(g)$$

Note that the exterior trace map gives an $Aut(x)$ -invariant vector, but the normalization is by the size of $Aut(y)$. In the case where the homomorphism is an inclusion, this is interpreted as trace given by a sum over cosets of $Aut(y)$ in $Aut(x)$, (which is the situation usually presented in the group case). We remark here that this factor will be important in interpreting our 2-functor Λ as a form of groupoidification.

We can now write down the units and counits explicitly for both adjunctions in our preferred notation.

The left and right units are natural transformations which, for each **Vect**-presheaf on \mathbf{Y} or \mathbf{X} respectively, gives a morphism which is itself a natural transformation. So, in particular the left unit

$$(56) \quad \eta_L(F)(y) : F(y) \rightarrow \bigoplus_{[y'] \mid f(y') \cong f(y)} \mathbb{C}[Aut(f(y))] \otimes_{\mathbb{C}[Aut(y')]} F(y')$$

is given by the natural map into the counit:

$$(57) \quad v \mapsto \bigoplus_{[y]} (1 \otimes v)$$

Notice the unit map has no contribution in the image from any y' which is not in the isomorphism class $[y]$. (It is a canonical map out of the limit which gives the usual form for f_{*} .)

The right unit map

$$(58) \quad \eta_R(G)(x) : G(x) \rightarrow \bigoplus_{[y] \mid f(y) \cong x} \mathbb{C}[Aut(x)] \otimes_{\mathbb{C}[Aut(y)]} f^*G(x)$$

is found by composing the Nakayama isomorphism (55) with the groupoid form of the canonical map for the right adjoint. This is a direct sum, in which each factor is given by the multiplication map:

$$(59) \quad v \mapsto (g \mapsto g(v))$$

Thus, the composite is:

$$(60) \quad \eta_R(G)(x) : v \mapsto \bigoplus_{[y] \mid f(y) \cong x} \frac{1}{\#Aut(y)} \sum_{g \in Aut(x)} g^{-1} \otimes g(v)$$

The left and right counits are natural transformations which, for each **Vect**-presheaf on \mathbf{X} or \mathbf{Y} respectively, gives a morphism which is itself a natural transformation. So in particular, the left counit

$$(61) \quad \epsilon_L(G)(x) : \bigoplus_{[y] \mid f(y) \cong x} \mathbb{C}[Aut(x)] \otimes_{\mathbb{C}[Aut(y)]} f^*G(x) \rightarrow G(x)$$

is given by summing multiplication maps:

$$(62) \quad \bigoplus_{[y] \mid f(y) \cong x} g_y \otimes v \mapsto \sum_{[y] \mid f(y) \cong x} f(g_y)v$$

The right counit map

$$(63) \quad \epsilon_R(F)(y) : \bigoplus_{[y'] \mid f(y') \cong f(y)} \mathbb{C}[Aut(f(y))] \otimes_{\mathbb{C}[Aut(y')]} F(y') \rightarrow F(y)$$

is given by composing the inverse of the Nakayama isomorphism (55) with the evaluation map from the canonical form of the right adjoint. Again, the only factor which contributes is $y' \cong y$, and so we have:

$$(64) \quad \bigoplus_{[y'] \mid f(y') \cong f(y)} \phi_{y'} \mapsto \phi_y(1)$$

So finally, (by using that $\mathbb{C}[Aut(f(y))]$ is canonically isomorphic to its dual using the canonical inner product on the group algebra) the composite is:

$$(65) \quad \epsilon_R(F)(y) : \bigoplus_{[y'] \mid f(y') \cong f(y)} g_{y'} \otimes v_{y'} \mapsto \frac{\#Aut(y)}{\#Aut(f(y))} g_y(v_y)$$

Here we are implicitly using the fact that the objects y' in the essential preimage come equipped with isomorphisms $f(y') \rightarrow f(y)$ which induce specified isomorphisms $Aut(f(y')) \cong Aut(f(y))$. In the colimit which gave the direct sum over isomorphism classes, these are all naturally identified.

A straightforward check (cancelling the Nakayama isomorphisms) verifies the unit and counit identities:

$$(66) \quad (\epsilon_L \cdot Id_{f_*}) \circ (Id_{f_*} \cdot \eta_L) = Id_{f_*}$$

$$(67) \quad (Id_{f_*} \cdot \epsilon_L) \circ (\eta_L \cdot Id_{f_*}) = Id_{f_*}$$

$$(68) \quad (\epsilon_R \cdot Id_{f^*}) \circ (Id_{f^*} \cdot \eta_R) = Id_{f^*}$$

$$(69) \quad (Id_{f^*} \cdot \epsilon_R) \circ (\eta_R \cdot Id_{f^*}) = Id_{f^*}$$

5. SPANS OF GROUPOIDS

We have already seen how essentially finite groupoids give rise to 2-vector spaces. In this section, we will show the weak functoriality of these assignments. In particular, we first must describe how our 2-functor Λ will produce 2-linear maps from spans of groupoids.

5.1. 2-Linear Maps from Spans of Groupoids. Given a span of groupoids as in Figure 2, we can apply the functor $[-, \mathbf{Vect}]$ to the span diagram (7). This functor is contravariant, so we get a cospan:

$$(70) \quad \begin{array}{ccc} & [X, \mathbf{Vect}] & \\ s^* \nearrow & & \nwarrow t^* \\ [A_1, \mathbf{Vect}] & & [A_2, \mathbf{Vect}] \end{array}$$

We now recall that the pullbacks s^* and t^* have adjoints: this is a direct consequence of Theorem 4.2.1. This reveals how to transport a \mathbf{Vect} -presheaf on A_1 along this cospan. In fact, it gives two 2-linear maps, which are adjoint. Thinking

of the span as a morphism in $\text{Span}(\mathbf{FinGpd})$ from A_1 to A_2 , we find a corresponding 2-linear map (though the adjoint is equally well defined). We first do a pullback along s , giving a \mathbf{Vect} -presheaf on X . Then we use the adjoint map t_* . So we have the following:

Definition 5.1.1. *For a span of groupoids $X : A_1 \rightarrow A_2$ in $\text{Span}(\mathbf{FinGpd})$ define the 2-linear map:*

$$(71) \quad t_* \circ s^* : [A_1, \mathbf{Vect}] \longrightarrow [A_2, \mathbf{Vect}]$$

Now, by Theorem 4.2.1, both s^* and t_* are 2-linear maps, so the composite $t_* \circ s^*$ is also a 2-linear map.

Remark 5.1.2. We can think of the pullback-pushforward construction as giving—in the language of quantum field theory—a “sum over histories” for evolving a 2-vector. Each 2-vector in $[A_1, \mathbf{Vect}]$ picks out a vector space for each object of A_1 . The 2-linear map we have described tells us how to evolve this 2-vector along a span. First we consider the pullback to $[X, \mathbf{Vect}]$, which gives us a 2-vector consisting of all assignments of vector spaces to objects of X which project to the chosen one in A_1 . Each of these objects could be considered a “history” of the 2-vector along the span. We then “push forward” this assignment to A_2 , which involves a colimit. This is more general than a sum, though so one could describe this as a “colimit of histories”. It takes into account the symmetries between individual “histories” (i.e. morphisms in X).

So, given a span $X : A_1 \rightarrow A_2$, we can write $\Lambda(X)$ in terms of its effect on a \mathbf{Vect} -presheaf G on A_1 , which, at any object $a_2 \in A_2$ gives:

$$(72) \quad \Lambda(X)(G)(a_2) = \bigoplus_{[x] | t(x) \cong a_2} \mathbb{C}[\text{Aut}(a_2)] \otimes_{\mathbb{C}[\text{Aut}(x)]} G(s(x))$$

by exactly the same reasoning as in Section 4.3.

It is sometimes useful—particularly when we look at composition of spans—to break up the direct sum into the contributions from different objects of A_1 , like this:

$$(73) \quad \Lambda(X)(G)(a_2) = \bigoplus_{[a_1] \in \underline{A_1}} \left(\bigoplus_{\substack{[x] \\ s(x) \cong a_1 \\ t(x) \cong a_2}} \mathbb{C}[\text{Aut}(a_2)] \otimes_{\mathbb{C}[\text{Aut}(x)]} G(a_1) \right)$$

Moreover, there is a convenient way to write down the components of the 2-linear map associated to a span, which is given by Frobenius reciprocity.

Proposition 5.1.3. *Given basis elements $(a_1, W_1) \in \Lambda(A_1)$ and $(a_2, W_2) \in \Lambda(A_2)$, the matrix elements are:*

$$(74) \quad \Lambda(X)_{(a_1, W_1), (a_2, W_2)} \simeq \bigoplus_{[x]} \text{hom}_{\text{Rep}(\text{Aut}(x))}[s^*(W_1), t^*(W_2)]$$

Here, the direct sum is taken over equivalence classes $[x]$ in the essential preimage of (a_1, a_2) : that is, objects of X mapping to a_1 and a_2 . For each $[x]$, the functors s and t define homomorphisms

$$(75) \quad s_x : \text{Aut}(x) \rightarrow \text{Aut}(s(x))$$

and

$$(76) \quad t_x : \text{Aut}(x) \rightarrow \text{Aut}(t(x))$$

, which define the induced representations. We think of the terms of the direct sum as “lying over” the objects x .

So using the adjoint 2-linear map

$$(77) \quad t_* : \Lambda(X) \rightarrow \Lambda(A_2)$$

to *push forward* a 2-vector $s^*F : X \rightarrow \mathbf{Vect}$ to one on A_2 , the above is also, by Frobenius reciprocity:

$$(78) \quad \bigoplus_{[x]} \mathrm{hom}_{\mathrm{Rep}(\mathrm{Aut}(a_2))}[(t_x)_* \circ (s_x)^* W_1, W_2]$$

By Schur’s lemma, this says:

$$(79) \quad \Lambda(X)(a_1, W_1) = \bigoplus_{[x]} (t_x)_* \circ (s_x)^*(a_1, W_1)$$

since the components of $\Lambda(X)(a_1, W_1)$ count the number of copies of W_2 in the pushforward of W_1 . (In the remainder of this paper, we will suppress the subscripts and denote s_x by s and t_x by t when the context makes clear that we mean the induced group homomorphism.)

So in fact, $\Lambda(X)(a_1, W_1)$ is a direct sum of irreducible 2-vectors in $\Lambda(A_2)$, given as a sum over $x \in X$ restricting to a_1, a_2 of the induced representations along each restriction map.

5.2. Λ and Composition. Next we show that Λ preserves horizontal composition of functors *weakly*—that is, up to a natural isomorphism. That is, the composition of the 2-linear maps must be compatible, in a weak sense, with composition of spans of groupoids.

To construct the isomorphism explicitly, we look at the weak pullback square in the middle of (11), since the two 2-linear maps being compared differ only by arrows in this square. The square as given is a weak pullback, with the natural isomorphism α “horizontally” across the square. When considering a corresponding square of categories of \mathbf{Vect} -presheaves, the arrows are reversed. So, including the adjoints of t^* and T^* , namely t_* and T_* , we have the square:

$$(80) \quad \begin{array}{ccc} & [X' \circ X, \mathbf{Vect}] & \\ S^* \nearrow & & \nwarrow T_* \\ [X, \mathbf{Vect}] & & [X', \mathbf{Vect}] \\ t_* \searrow & & \nearrow T^* \\ & [A_2, \mathbf{Vect}] & \\ & (s')^* \nearrow & \end{array}$$

Note that there are two squares here—one by taking only the “pull” morphisms $(-)^*$ from the indicated adjunctions, and the other by taking only the “push” morphisms $(-)_*$. The first is just the square of pullbacks along morphisms from the weak pullback square of groupoids. Comparing these is the core of the following theorem, which gives one of the necessary properties for Λ to be a weak 2-functor.

We give a more explicit description of the functors $\Lambda(X' \circ X)$ and $\Lambda(X') \circ \Lambda(X)$ below, but remark that this general result is discussed by Panchadcharam [16] (Proposition 0.0.1), and the general theory behind this elaborated on by Street [18].

Theorem 5.2.1. *The process Λ weakly preserves composition. In particular, there is a natural isomorphism*

$$(81) \quad \beta_{X',X} : \Lambda(X' \circ X) \rightarrow \Lambda(X') \circ \Lambda(X)$$

Proof. Recall that, given the composite of two spans of groupoids in (11), we have 2-linear maps:

$$(82) \quad \Lambda(X' \circ X) = (t' \circ T)_* \circ (s \circ S)^*$$

and

$$(83) \quad \Lambda(X') \circ \Lambda(X) = (t')_* \circ (s')^* \circ t_* \circ (s)^*$$

So we want to show there is a natural isomorphism:

$$(84) \quad \beta_{X',X} : (t' \circ T)_* \circ (s \circ S)^* \rightarrow (t')_* \circ (s')^* \circ t_* \circ (s)^*$$

It suffices to show that there is an isomorphism:

$$(85) \quad \gamma : T_* \circ S^* \rightarrow (s')^* \circ t_*$$

between the upper and lower halves of the square in the middle of (11) since then $\beta_{X',X}$ is obtained by tensoring with identities.

So first taking a **Vect**-presheaf F on X , we get that S^*F is a **Vect**-presheaf on $X' \circ X$. Now over any fixed object $x \in X$, we have a set of objects in $X' \circ X$ which restrict to it: there is one for each choice (g, x') which is compatible with x in the sense that (x, g, x') is an object in the weak pullback - that is, $g : t(x) \rightarrow s'(x')$. Each object of this form is assigned

$$S^*F(x, g, x') = F(x)$$

Further, there are isomorphisms between such objects, namely pairs (h, k) as above. There are thus no isomorphisms except between objects (x, g_1, x') and (x, g_2, x') for some fixed x and x' . For any such fixed x and x' , objects corresponding to g_1 and g_2 are isomorphic if

$$(86) \quad g_2 t(h) = p'_1(k) g_1$$

. Denote the isomorphism class of any g by $[g]$.

Then we get:

$$(87) \quad T_* \circ S^*F(x') = \bigoplus_{x \in X} \left(\bigoplus_{[x'] : t(x) \cong s'(x')} \mathbb{C}[Aut(x')] \otimes_{\mathbb{C}[Aut(x) \times_{Aut(t(x))} Aut(x')]} F(x) \right)$$

since $Aut(x) \times_{Aut(t(x))} Aut(x')$ is the automorphism group of the object in $X' \circ X$ which restricts to x and x' . Notice that although outside direct sum here is written over all objects x on S , the only ones which contribute any factor are those for which $g : t(x) \rightarrow s'(x')$ for some g . The inside direct sum is over all isomorphism classes of elements g for which this occurs: in the colimit, vector spaces over objects with isomorphisms between them are identified.

Note that in the direct sum over $[g]$, there is a tensor product term for each class $[g] : t(x) \rightarrow s'(x')$. By the definition of the tensor product over an algebra, we can pass elements of $\mathbb{C}[Aut(x) \times_{Aut(t(x))} Aut(x')]$ through the tensor product. These are generated by pairs $(h, k) \in Aut(x) \times Aut(x')$ where the images of h and k are conjugate by g so that $t(h)g = gs'(k)$. These are just automorphisms of g : so this says we are considering objects only up to these isomorphisms.

This is the result of the “pull-push” side of the square applied to F . Now consider the “push-pull” side: $(s')^* \circ t_*$.

First, pushing down to A_2 , we get, on any object $a \in A_2$

$$(88) \quad t_* F(a) = \bigoplus_{[x]|t(x) \cong a} \mathbb{C}[Aut(a)] \otimes_{\mathbb{C}[Aut(x)]} F(x)$$

Then, pulling this back up to X' , we find:

$$(89) \quad (s')^* \circ t_* F(x') = \bigoplus_{[x]|t(x) \cong s'(x')} \left(\mathbb{C}[Aut(t(x))] \otimes_{\mathbb{C}[Aut(x)]} F(x) \right)$$

Now we define a natural isomorphism

$$(90) \quad \gamma_{X, X'} : T_* \circ S^* \rightarrow (s')^* \circ t_*$$

as follows. For each x' , this must be an isomorphism between the above vector spaces. The first step is to observe that there is a 1-1 correspondence *between* the terms of the first direct sums, and then secondly to note that the corresponding terms are isomorphic.

Since the outside direct sums are over all objects $x \in X$ for which $t(x) \cong s'(x')$, it suffices to get an isomorphism between each term. That is, between

$$(91) \quad \mathbb{C}[Aut(x')] \otimes_{\mathbb{C}[Aut(x) \times_{Aut(t(x))} Aut(x')]} F(x)$$

and

$$(92) \quad \mathbb{C}[Aut(t(x))] \otimes_{\mathbb{C}[Aut(x)]} F(x)$$

In order to define this isomorphism, first note that both of these vector spaces are in fact $\mathbb{C}[Aut(x')]$ -modules. An element of $Aut(x')$ acts on (91) in each component by the standard group algebra multiplication, giving an action of $\mathbb{C}[Aut(x')]$ by extending linearly. An element $l \in Aut(x')$ acts on (92) by the action of $s'(l)$ on $\mathbb{C}[Aut(t(x))]$. Two elements $l_1, l_2 \in [l]$ in the same equivalence class have the same action on this tensor product, since they differ precisely by $(h, k) \in Aut(x) \times Aut(x')$, so that $l_2 t(h) = s'(k) l_1$.

Also, we notice that, in (91), for each $g \in Aut(t(x))$, the corresponding term of the form $\mathbb{C}[Aut(x')] \otimes_{\mathbb{C}[Aut(x) \times_{Aut(t(x))} Aut(x')]} F(x)$ is generated by elements of the form $k \otimes v$, for $k \in \mathbb{C}[Aut(x')]$ and $v \in F(x)$. These are subject to the relations that, for any $(h, k_1) \in \mathbb{C}[Aut(x)] \times \mathbb{C}[Aut(x')]$ such that $t(h) = g^{-1} s'(k_1) g$:

$$(93) \quad k k_1 \otimes v = k(h, k_1) \otimes v = k \otimes (h, k_1) v = k \otimes h v$$

since elements of $\mathbb{C}[Aut(x)] \times \mathbb{C}[Aut(x')]$ act on $F(x)$ and $\mathbb{C}[Aut(x')]$ by their projections into the first and second components respectively.

Now, we define the map $\gamma_{x, x'}$. First, for any element of the form $k \otimes v \in \mathbb{C}[Aut(x')] \otimes_{\mathbb{C}[Aut(x) \times_{Aut(t(x))} Aut(x')]} F(x)$ in the g component of the direct sum (91):

$$(94) \quad \gamma_{x, x'}(k \otimes v) = s'(k) g^{-1} \otimes v$$

which we claim is in $\mathbb{C}[Aut(t(x))] \otimes_{\mathbb{C}[Aut(x)]} F(x)$. This map extends linearly to the whole space.

To check this is well-defined, suppose we have two representatives $k_1 \otimes v_1$ and $k_2 \otimes v_2$ of the class $k \otimes v$. So these differ by an element of $\mathbb{C}[Aut(x) \times_{Aut(t(x))} Aut(x')]$, say (h, k) , so that

$$(95) \quad k_1 = k_2 k$$

, and

$$(96) \quad hv_1 = v_2$$

where

$$(97) \quad t(h) = gs'(k)g^{-1}$$

But then

$$(98) \quad \begin{aligned} \gamma_{x,x'}(k_1 \otimes v_1) &= s'(k_1)g^{-1} \otimes v_1 \\ &= s'(k_2k)g^{-1} \otimes v_1 \\ &= s'(k_2)g^{-1}gs'(k)g^{-1} \otimes v_1 \\ &= s'(k_2)g^{-1}t(h) \otimes v_1 \\ &= s'(k_2)g^{-1} \otimes hv_1 \end{aligned}$$

while on the other hand,

$$(99) \quad \begin{aligned} \gamma_{x,x'}(k_2 \otimes v_2) &= s'(k_2)g^{-1} \otimes v_2 \\ &= s'(k_2)g^{-1} \otimes hv_1 \end{aligned}$$

But these are representatives of the same class in $\mathbb{C}[Aut(t(x))] \otimes_{\mathbb{C}[Aut(x)]} F(x)$, so γ is well defined on generators, and thus extends linearly to give a well-defined function on the whole space.

Now, to see that γ is invertible, note that given an element $m \otimes v \in \mathbb{C}[Aut(t(x))] \otimes_{\mathbb{C}[Aut(x)]} F(x)$, we can define

$$(100) \quad \gamma^{-1}(m \otimes v) = 1 \otimes v \in \bigoplus_{t(x) \rightarrow s'(x')} \mathbb{C}[Aut(x')] \otimes_{\mathbb{C}[Aut(x) \times_{Aut(t(x))} Aut(x')]} F(x)$$

in the component coming from the isomorphism class of $g = m^{-1}$ (we will denote this by $(1 \otimes v)_{m^{-1}}$ to make this explicit, and in general an element in the class of g will be denoted with subscript g whenever we need to refer to g).

Now we check that this is well-defined. Given $m_1 \otimes v_1$ and $m_2 \otimes v_2$ representing the same element $m \otimes v$ of $\mathbb{C}[Aut(t(x))] \otimes_{\mathbb{C}[Aut(x)]} F(x)$, we must have $h_1 \in Aut(x)$ with

$$(101) \quad m_1 t(h_1) = m_2$$

and

$$(102) \quad h_1 v_2 = v_1$$

But then applying γ^{-1} , we get:

$$(103) \quad \gamma^{-1}(m_1 \otimes v_1) = (1 \otimes v_1)_{m_1^{-1}} = (1 \otimes h_1 v_2)_{m_1^{-1}}$$

and

$$(104) \quad \gamma^{-1}(m_2 \otimes v_2) = (1 \otimes v_2)_{m_2^{-1}} = (1 \otimes v_2)_{t(h_1)^{-1} m_1^{-1}}$$

but these are in the same component, since $g \sim g'$ when $g's'(k) = t(h)g$ for some $h \in Aut(x)$ and $k \in Aut(x')$. But then, taking $k = 1$ and $h = h_1^{-1}$, we get that $m_1^{-1} \sim m_2^{-1}$, and hence the component of $\gamma(m \otimes v)$ is well defined.

But then, consider $m \otimes v = \gamma((k \otimes v)_g) = s'(k)g^{-1} \otimes v$. Applying γ^{-1} we get:

$$(105) \quad \gamma^{-1} \circ \gamma(k \otimes v)_g = (1 \otimes v)_{gs'(k)^{-1}}$$

so we hope that these determine the same element. But in fact, notice that the morphism in the weak pullback which gives that g^{-1} and $s'(k)g^{-1}$ are isomorphic is just labelled by $(h, k) = (1, k)$, which indeed takes k to 1 and leaves v intact. So these are the corresponding elements under this isomorphism.

So γ is invertible, hence an isomorphism. Thus we define

$$(106) \quad \beta_{X, X'} = 1 \otimes \gamma \otimes 1$$

This is the isomorphism we wanted. \square

So the $\beta_{X, X'}$ can now be seen as natural transformations explicitly. First consider $\Lambda(X') \circ \Lambda(X)$, which acts on a presheaf G on A_1 as follows.

$$(107) \quad \begin{aligned} \Lambda(X)(G)(a_2) &= \bigoplus_{[x] | t(x) \cong a_2} \mathbb{C}[Aut(a_2)] \otimes_{\mathbb{C}[Aut(x)]} G(s(x)) \\ &= \bigoplus_{[a_1] \in \underline{A_1}} \left(\bigoplus_{\substack{[x] \\ s(x) \cong a_1 \\ t(x) \cong a_2}} \mathbb{C}[Aut(a_2)] \otimes_{\mathbb{C}[Aut(x)]} (\Lambda(X)(G)(a_1)) \right) \end{aligned}$$

and then applying $\Lambda(X')$ to this, we get, rearranging direct sums suitably:

$$(108) \quad \begin{aligned} &(\Lambda(X') \circ \Lambda(X))(G)(a_3) \\ &= \bigoplus_{[a_2] \in \underline{A_2}} \left(\bigoplus_{\substack{[x'] \\ s'(x') \cong a_2 \\ t'(x') \cong a_3}} \mathbb{C}[Aut(a_3)] \otimes_{\mathbb{C}[Aut(x')] } (\Lambda(X)(G)(a_2)) \right) \\ &= \bigoplus_{[a_1] \in \underline{A_1}} \left(\bigoplus_{[a_2] \in \underline{A_2}} \bigoplus_{\substack{[x] \\ s(x) \cong a_1 \\ t(x) \cong a_2}} \bigoplus_{\substack{[x'] \\ s'(x') \cong a_2 \\ t'(x') \cong a_3}} \mathbb{C}[Aut(a_3)] \otimes_{\mathbb{C}[Aut(x')] } \mathbb{C}[Aut(a_2)] \otimes_{\mathbb{C}[Aut(x)]} G(a_1) \right) \end{aligned}$$

We similarly have:

$$(109) \quad (\Lambda(X') \circ \Lambda(X))(G)(a_3) = \bigoplus_{[a_1] \in \underline{A_1}} \left(\bigoplus_{\substack{[(x, f, x')] \\ ((s \circ S)(x, f, x') \cong a_1 \\ (t' \circ T)(x, f, x') \cong a_3}} \mathbb{C}[Aut(a_3)] \otimes_{\mathbb{C}[Aut(x, f, x')]} G(a_1) \right)$$

The isomorphisms $\beta_{X, X'}$ allow us to identify (108) and (109).

Remark 5.2.2. We can also describe the effect of β in coordinates - that is, in the matrix form for a natural transformation of a 2-linear map. This illustrates the fact that $\mathbb{C}[Aut(a_2)] \cong \bigoplus_W W \otimes W^*$, where the sum is over irreducible representations of $Aut(a_2)$. For suppose we have a composite of spans, $X' \circ X$. By Lemma 3.2.2, we have that the functors $T_* \circ S^*$ and $(s')^* \circ t_*$ can be written in the form of a matrix of vector spaces as in (17).

First, $\Lambda(X' \circ X)$ is given by a matrix indexed by classes of objects and representations $([a_1], W_1)$ from A_1 and $([a_3], W_3)$ from A_3 . In the form (74), we see

that

$$(110) \quad \begin{aligned} & \Lambda(X' \circ X)_{([a_1], W_1), ([a_3], W_3)} \\ & \simeq \bigoplus_{[(x, f, x')]} \text{hom}_{\text{Rep}(\text{Aut}(x, f, x'))}[(s \circ S)^*(W_1), (t' \circ T)^*(W_3)] \end{aligned}$$

where $[(x, f, x')]$ represents an equivalence class of objects in the weak pullback.

The isomorphisms $\beta_{X, X'}$ take this to the matrix product of $\Lambda(X')$ with $\Lambda(X)$, which has components given by a direct sum over classes and representations $([a_2], W_2)$ from A_2 :

$$(111) \quad \begin{aligned} & [\Lambda(X' \circ X)]_{([a_1], W_1), ([a_3], W_3)} \\ & \xrightarrow{\beta_{X, X'}} \bigoplus_{([a_2], W_2)} [\Lambda(X)]_{([a_1], W_1), ([a_2], W_2)} \otimes [\Lambda(X')]_{([a_2], W_2), ([a_3], W_3)} \end{aligned}$$

Recall that $[\Lambda(X)]_{([a_1], W_1), ([a_2], W_2)}$, is a direct sum over isomorphism classes of objects of X which restrict to $[a_1]$ and $[a_2]$, with each component being

$$(112) \quad \text{hom}_{\text{Rep}(\text{Aut}(x))}(s^*W_1, t^*W_2)$$

The component $[\Lambda(X')]_{([a_2], W_2), ([a_3], W_3)}$, is a similar sum over classes of objects of X' which restrict to $[a_2]$ and $[a_3]$.

The isomorphism $\beta_{X', X}$ identifies the composite, whose components are sums over objects of $X' \circ X$, with this product. This β consists of isomorphisms in each component. So in fact, the β are described by their components:

$$(113) \quad \begin{aligned} & \bigoplus_{[(x, f, x')]} [\text{hom}((s \circ S)^*(W_1), (t' \circ T)^*(W_3))] \\ & \xrightarrow{\beta_{X, X'}} \bigoplus_{([a_2], W_2)} [\text{hom}(s^*(W_1), t^*(W_2)) \otimes \text{hom}((s')^*(W_2), (t')^*(W_3))] \end{aligned}$$

Where the second sum is over equivalence classes of (x, f, x') such that $f : t(x) \rightarrow s'(x')$, and for which $s(x) = a_1$ and $t'(x') = a_3$.

Since the choice of $[x]$ and $[x']$ amounts to the same thing as the choice of $[a_2]$, this isomorphism turns a sum over representations W_2 of tensor products of space (of intertwiners), into a sum over isomorphism (conjugacy) classes of $f \in \text{Aut}([a_2])$. This isomorphism is describing how the representations in the big pullback decompose.

6. SPANS OF SPANS

The situation we are interested in can be represented as an equivalence class of spans of spans of the following sort:

$$(114) \quad \begin{array}{ccccc} & & X_1 & & \\ & s_1 \swarrow & \uparrow s & \searrow t_1 & \\ A_1 & & Y & & A_2 \\ & s_2 \swarrow & \downarrow t & \searrow t_2 & \\ & & X_2 & & \end{array}$$

Recall that we assume weak commutativity here - that is, that there are isomorphisms $\zeta_s : s_1 \circ s \rightarrow s_2 \circ t$ and $\zeta_t : t_1 \circ s \rightarrow t_2 \circ t$.

Given this situation, which is a 2-morphism for the associated bicategory of spans, we want to get a 2-morphism in the bicategory $\mathbf{2Vect}$. That is to say, a natural transformation $\alpha = \Lambda(Y)$ between a pair of 2-linear maps. In this section, we show how to construct $\Lambda(Y)$.

6.1. 2-Morphisms from Spans of Spans. We begin by noting that the diagram (114), which weakly commutes up to isomorphisms ζ_s and ζ_t , gives rise to a diagram of pullback functors:

$$(115) \quad \begin{array}{ccccc} & & [X_1, \mathbf{Vect}] & & \\ & \nearrow s_1^* & \downarrow s^* & \nwarrow t_1^* & \\ [A_1, \mathbf{Vect}] & & [Y, \mathbf{Vect}] & & [A_2, \mathbf{Vect}] \\ & \searrow s_2^* & \uparrow t^* & \swarrow t_2^* & \\ & & [X_2, \mathbf{Vect}] & & \end{array}$$

which commutes up to isomorphisms:

$$(116) \quad \zeta_s^* : s^* \circ s_1^*(V) \xrightarrow{\sim} t^* \circ s_2^*(V)$$

and similarly

$$(117) \quad \zeta_t^* : s^* \circ t_1^*(V) \xrightarrow{\sim} t^* \circ t_2^*(V)$$

We want to get a natural transformation from $\Lambda(X_1)$ and $\Lambda(X_2)$ from this diagram. In section 6.2 we show how this can be described as a “pull-push” process, similar to the one used to define the 2-linear maps, but first it can be defined in terms of the unit and counit maps we have already defined.

Definition 6.1.1. *Given a span between spans, $Y : X_1 \rightarrow X_2$, for $X_1, X_2 : A_1 \rightarrow A_2$, then*

$$(118) \quad \Lambda(Y) : \Lambda(X_1) \rightarrow \Lambda(X_2)$$

is the natural transformation given as

$$(119) \quad \Lambda(Y) = \epsilon_{L,t} \circ ((\zeta_t)_* \otimes ((\zeta_s)^*)^{-1}) \circ \eta_{R,s} : (t_1)_* s_1^* \Longrightarrow (t_2)_* s_2^*$$

where $\epsilon_{L,t}$ is the counit (61) for the left adjunction associated to t , and $\eta_{R,s}$ is the unit (58) for the right adjunction associated to s .

We comment here that this composition of left unit followed by right counit can be interpreted as a “pull-push” in a sense that can be seen more precisely when we consider this natural transformation in coordinates. In the special case where $A_i = 1$, this recovers groupoidification in the sense of Baez and Dolan, as shown in Theorem 6.2.1.

Remark 6.1.2. Henceforth, we will assume that the diagram (114) commutes strictly - that is, ζ_s and ζ_t are identity 2-morphisms. This is a mild assumption, since the diagram can always be “strictified” by taking equivalent groupoids for source and target which are skeletal. Similar arguments will follow through if not, but in this simpler case, we simply have:

$$(120) \quad \Lambda(Y) = \epsilon_{L,t} \circ \eta_{R,s} : (t_1)_* s_1^* \Longrightarrow (t_2)_* s_2^*$$

For our construction to give a 2-functor, this must agree with composition in two ways. The first is strict preservation of vertical composition; the second is preservation of horizontal composition as strictly as possible (i.e. up to the isomorphisms β which make comparison possible - as we will see). We will show in

Lemma 6.1.3. *The assignment $\Lambda(Y)$ to spans of spans given in Definition 6.1.1 preserves vertical composition strictly: $\Lambda(Y' \circ Y) = \Lambda(Y') \circ \Lambda(Y)$.*

Proof. Suppose we have a vertical composite of two spans between spans, here written as 2-cells:

$$(121) \quad \begin{array}{ccc} & X_1 & \\ \swarrow & \Downarrow Y & \searrow \\ A_1 & \xrightarrow{X_2} & A_2 \\ \nwarrow & \Downarrow Y' & \nearrow \\ & X_3 & \end{array}$$

The composition is given by a weak pullback (taken up to isomorphism) - that is, a diagram of the form (11), with the Y and Y' in place of X and X' , and the X_i in place of the A_i . We use the same notation for the maps in all these spans. Of course, each object in (121) comes equipped with (commuting) maps into A_1 and A_2 , but we can ignore these here.

So the source and target maps for Y are s and t , and those for Y' are s' and t' , and we can easily write:

$$(122) \quad \Lambda(Y') \circ \Lambda(Y) = \epsilon_{L,t'} \circ \eta_{R,s'} \circ \epsilon_{L,t} \circ \eta_{R,s}$$

Now, to write $\Lambda(Y' \circ Y)$, we recall that the vertical composite is formed by weak pullback of spans, with the resulting source and target maps $s \circ S$ and $t' \circ T$, where the groupoid in the span $Y' \circ Y$ is the comma category whose objects are of the form (y', g_2, y) , with $g_2 : t(y) \rightarrow s'(y')$ in X_2 , and S and T are the natural projections onto Y and Y' . Then of course

$$(123) \quad \Lambda(Y' \circ Y) = \epsilon_{L,(t' \circ T)} \circ \eta_{R,(s \circ S)}$$

Now, a composite of adjunctions is an adjunction (see for instance MacLane [13] IV.8), and the unit and counit of the composite is given in a standard way, so we have:

$$(124) \quad \epsilon_{L,(t' \circ T)} = \epsilon_{L,t'} \circ (\text{Id}_{(t')*} \otimes \epsilon_{L,T} \otimes \text{Id}_{(t')*})$$

and

$$(125) \quad \eta_{R,(s \circ S)} = (\text{Id}_{s*} \otimes \eta_{R,S} \otimes \text{Id}_{s*}) \circ \eta_{R,s}$$

So we get

$$(126) \quad \Lambda(Y' \circ Y) = \epsilon_{L,t'} \circ (\text{Id}_{(t')*} \otimes \epsilon_{L,T} \otimes \text{Id}_{(t')*}) \circ (\text{Id}_{s*} \otimes \eta_{R,S} \otimes \text{Id}_{s*}) \circ \eta_{R,s}$$

So to get strict composition, we just need that

$$(127) \quad (\text{Id}_{(t')*} \otimes \epsilon_{L,T} \otimes \text{Id}_{(t')*}) \circ (\text{Id}_{s*} \otimes \eta_{R,S} \otimes \text{Id}_{s*}) = \eta_{R,s'} \circ \epsilon_{L,t}$$

This follows the same pattern as the proof for the fact that Λ weakly preserves composition of morphisms. Note that the two sides of this expression are the top and bottom of a (weak) pullback square. So in particular, the argument for weak

preservation of composition of spans shows that we also have a pullback square for the induced functors.

In particular (ignoring the identity maps), we first get the right unit for $S : Y' \circ Y \rightarrow Y$, which at $y \in G(y) \cong G(a_1)$ gives:

$$(128) \quad \eta_{R,S} : v \mapsto \bigoplus_{[(y,f,y')]} \frac{1}{\#Aut(y,f,y')} \sum_{g \in Aut(y)} (g^{-1}) \otimes g(v)$$

and the left counit for $T : Y' \circ Y \rightarrow Y'$ takes this to. So this gives

$$(129) \quad \epsilon_{L,T} \circ \eta_{R,S} : v \mapsto \sum_{[(y,f,y')]} \frac{1}{\#Aut(y,f,y')} \sum_{g \in Aut(y)} (g^{-1}) \otimes g(v)$$

since the target space is now already a tensor product over $\mathbb{C}[Aut(x_2)]$. Similarly, on the other side we have first the left counit for $t : Y \rightarrow X_2$, then the right unit for $s' : Y' \rightarrow X_2$, giving:

$$(130) \quad \eta_{R,s'} \circ \epsilon_{L,t} : v \mapsto \sum_{[y'] | s'(y') \cong t(y)} \frac{1}{\#Aut(t(y))} \sum_{g \in Aut(x_2)} g^{-1} \otimes g(v)$$

Since the sources and targets of the maps (61) and (58) are in a pullback square, the coefficients arising from the Nakayama isomorphisms will yield the same group averages, and the terms of the implied direct sum correspond pairwise. So these maps are indeed equal. \square

We also must show that Λ respects horizontal composition weakly. To make this clear, it will be convenient to write source and target 2-linear maps in the form (108) and (109).

Lemma 6.1.4. *The assignment $\Lambda(Y)$ to spans of spans given by Definition 6.1.1 preserves horizontal composition strictly, up to the isomorphism weakly preserving composition of the source and target morphisms:*

$$(131) \quad \begin{array}{c} \Lambda(X'_1 \circ X_1) \\ \swarrow \Lambda(X_1) \quad \downarrow \beta_{X_1, X'_1} \Lambda(X'_1) \quad \searrow \Lambda(X'_1) \\ \Lambda(A_1) \quad \downarrow \Lambda(Y) \quad \Lambda(A'_2) \quad \downarrow \Lambda(Y') \quad \Lambda(A_3) \\ \swarrow \Lambda(X_2) \quad \downarrow \beta_{X_2, X'_2}^{-1} \Lambda(X'_2) \quad \searrow \Lambda(X'_2) \\ \Lambda(X'_2 \circ X_2) \end{array} = \begin{array}{c} \Lambda(X'_1 \circ X_1) \\ \swarrow \Lambda(Y' \circ Y) \quad \downarrow \Lambda(Y' \circ Y) \quad \searrow \Lambda(Y' \circ Y) \\ \Lambda(A_1) \quad \downarrow \Lambda(Y' \circ Y) \quad \Lambda(A_3) \\ \swarrow \Lambda(X'_2 \circ X_2) \quad \downarrow \Lambda(X'_2 \circ X_2) \quad \searrow \Lambda(X'_2 \circ X_2) \\ \Lambda(X'_2 \circ X_2) \end{array}$$

Proof. (Elsewhere, we have used the same notation for horizontal and vertical composition of all kinds, to simplify notation and because context made this unambiguous. In this proof it will be helpful to distinguish the two, so we write vertical composition with no symbol, concatenating natural transformations between 2-linear maps.)

Begin by writing the spans explicitly. The situation for a horizontal composite of 2-morphisms in $\text{Span}(\mathbf{FinGpd})$ looks like:

$$(132) \quad \begin{array}{ccccc} & & X'_1 \circ X_1 & & \\ & \swarrow S_1 & & \searrow T_1 & \\ & X_1 & \xrightleftharpoons[\sim]{\alpha_1} & X'_1 & \\ \swarrow s_1 & \uparrow s & \searrow t_1 & \swarrow s'_1 & \searrow t'_1 \\ A_1 \xleftarrow{\sigma} Y \xrightarrow{\tau} A_2 & & A_2 \xleftarrow{\sigma'} Y' \xrightarrow{\tau'} A_3 & & \\ \swarrow s_2 & \downarrow t & \swarrow t_2 & \downarrow t'_2 & \swarrow t'_2 \\ & X_2 & \xrightleftharpoons[\sim]{\alpha_2} & X'_2 & \\ & \swarrow S_2 & & \searrow T_2 & \\ & & X'_2 \circ X_2 & & \end{array}$$

(Note that here again we are assuming the 2-morphisms Y and Y' are strict, so that $s_1 \circ s = s_2 \circ t$ and similarly for the other composites. We represent these by the dotted arrows σ , τ , σ' and τ' . As before, a similar argument would go through if these spans of span maps were only weakly commuting, but we would need the ζ natural transformations as discussed for (12)).

Now, the functor Λ assigns 2-linear maps to the spans X_1 , X_2 , X'_1 , and X'_2 , and their composites, and natural transformations to Y and Y' . Then the horizontal composite is a natural transformation between 2-linear maps:

$$(133) \quad \Lambda(Y') \circ \Lambda(Y) : \Lambda(X'_1) \circ \Lambda(X_1) \rightarrow \Lambda(X'_2) \circ \Lambda(X_2)$$

And we can calculate as in the proof of Theorem 6.1.3 that:

$$(134) \quad \Lambda(Y')(a_1) = \epsilon_{L,t'} \eta_{R,s'} : v \mapsto \bigoplus_{[y] | s'(y') \cong x_1} \frac{\# \text{Aut}(y)}{\# \text{Aut}(y, f, y')} v$$

and

$$(135) \quad \Lambda(Y) = \epsilon_{L,t} \eta_{R,s}$$

So the composite is just

$$(136) \quad \Lambda(Y') \circ \Lambda(Y) = (\epsilon_{L,t'} \eta_{R,s'}) \circ (\epsilon_{L,t} \eta_{R,s})$$

We recall that $\Lambda(X') \circ \Lambda(X)$ is described explicitly in (108) and $\Lambda(X' \circ X)$ in (109). Finding these for the X_i gives a total of four functors here. We next describe natural transformations between these.

As shown in Theorem 5.2.1, there are comparison isomorphisms

$$(137) \quad \beta_{X_i, X'_i} : \Lambda(X'_i) \circ \Lambda(X_i) \rightarrow \Lambda(X'_i \circ X_i)$$

which will necessarily be involved in the isomorphism we are looking for. These derive from the α isomorphisms in the weak pullback in $X'_i \circ X_i$.

Composing with these comparison isomorphisms as in (131) gives:

$$(138) \quad (\text{Id}_{s_1^*} \circ \beta_{X_1, X'_1} \circ \text{Id}_{(t'_1)^*}) (\epsilon_{L,t'} \eta_{R,s'}) (\epsilon_{L,t} \eta_{R,s}) (\text{Id}_{s_2^*} \circ (\beta_{X_2, X'_2})^{-1} \circ \text{Id}_{(t'_2)^*})$$

Now, the β isomorphisms simply allow us to identify the spaces here, so it suffices to describe the maps, and in particular the coefficients which arise. At any presheaf G on A_1 , in the summand for $[a_1] \in \underline{A_1}$ and $[x'_1] \in X'_1$:

$$(139) \quad (\epsilon_{L,t}\eta_{R,s}) : v \mapsto \sum_{[y] \mid s(y) \cong x_1} \frac{1}{\#Aut(y)} \sum_{g \in Aut(x_1)} g^{-1} \otimes g(v)$$

which is a map between spaces of the form (107) associated to X_1 and X_2 . Now this becomes the v' when we take the full map between spaces like (108), where we have:

$$(140) \quad (\epsilon_{L,t'}\eta_{R,s'}) : v' \mapsto \sum_{[y'] \mid s'(y') \cong x'_1} \frac{1}{\#Aut(y')} \sum_{h \in Aut(x'_1)} h^{-1} \otimes h(v')$$

so finally we get:

$$(141) \quad v \mapsto \sum_{[y'] \mid s'(y') \cong x'_1} \frac{1}{\#Aut(y')} \sum_{h \in Aut(x'_1)} h^{-1} \otimes h \left(\sum_{[y] \mid s(y) \cong x_1} \frac{1}{\#Aut(y)} \sum_{g \in Aut(x_1)} g^{-1} \otimes g(v) \right)$$

which we want to show is the same as the natural transformation associated to $\Lambda(Y' \circ Y)$:

$$(142) \quad \Lambda(Y' \circ Y) = \epsilon_{L,(t,t')} \circ \eta_{R,(s,s')} : \Lambda(X'_1 \circ X_1) \rightarrow \Lambda(X'_2 \circ X_2)$$

which is a map between two spaces of the form (109).

This $Y' \circ Y : X'_1 \circ X_1 \rightarrow X'_2 \circ X_2$ is a span of span maps which is given as follows. We take the horizontal composite of the spans $A_1 \xleftarrow{\sigma} Y \xrightarrow{\tau} A_2$ and $A_2 \xleftarrow{\sigma'} Y' \xrightarrow{\tau'} A_3$. This is a weak pullback taken up to isomorphism. The pullback square commutes weakly, say up to ξ . Then the groupoid $Y' \circ Y$ has maps into Y and Y' , and therefore by composition with s and s' , it has maps into X_1 and X'_1 . By the universal property of the weak pullback $X'_1 \circ X_1$, there is a map $S : Y' \circ Y \rightarrow X'_1 \circ X_1$. Similarly, there is $T : Y' \circ Y \rightarrow X'_2 \circ X_2$.

We can see what this is by taking the weak pullback giving $Y' \circ Y$, which we take to be the comma category whose objects are of the form (y, f, y') where $f : \tau(y) \rightarrow \sigma'(y')$ in A_2 . Then the S and T given by the universal property are just $S = (s, s')$ giving objects like (x_1, f, x'_1) and $T = (t, t')$ giving objects like (x_2, f, x'_2) . In particular, the morphism $f \in A_2$ is left intact. (Different isomorphic choices for weak pullback could of course change f).

Given this, we have, in the summand for a given $[a_1] \in \underline{A_1}$: and a particular $[(x_1, f, x'_1)] \in X'_1 \circ X_1$, we have:

$$(143) \quad \eta_{R,(s,s')} : v \mapsto \bigoplus_{\substack{[y, f, y'] \\ s(y) \cong x_1 \\ s'(y') \cong x'_1}} \frac{1}{\#Aut(y, f, y')} \sum_{(g, h) \in Aut(x, f, x')} (g, h)^{-1} \otimes (g, h)(v)$$

and then

$$(144) \quad \epsilon_{L,(t,t')}\eta_{R,(s,s')} : v \mapsto \sum_{\substack{[y, f, y'] \\ s(y) \cong x_1 \\ s'(y') \cong x'_1}} \frac{1}{\#Aut(y, f, y')} \sum_{(g, g') \in Aut(x, f, x')} (g, h)^{-1} \otimes (g, h)(v)$$

So in fact, since we are in a weak pullback square, the size of the automorphism groups in the two expressions we have found will be in the same ratios, and so it becomes clear that, using the β isomorphisms as seen in the proof of Theorem 5.2.1:

$$(145) \quad \Lambda(Y' \circ Y) = \beta_{X_1, X'_1}(\Lambda(Y')\Lambda(Y))(\beta_{X_2, X'_2})^{-1}$$

as required. \square

6.2. Coordinate Description of 2-Morphisms. In this section, we discuss the behaviour of Λ on 2-morphisms, namely the assignment of a natural transformation to a span of span maps. As discussed in Section 4, any natural transformation between a pair of 2-linear maps between KV 2-vector spaces can be represented as a matrix of linear operators, as in (18). We would like to describe explicitly the linear maps composing $\Lambda(Y)$ and some consequences.

To motivate the rest, we can begin with the special case of $\text{hom}_{\text{Span}(\mathbf{FinGpd})}(\mathbf{1}, \mathbf{1})$, where $\mathbf{1}$ is the trivial groupoid with one object (which we denote \star) and its identity morphism. We can summarize the effect of Λ on this hom-category by the following theorem:

Theorem 6.2.1. *On $\text{hom}_{\text{Span}(\mathbf{FinGpd})}(\mathbf{1}, \mathbf{1})$, the 2-functor Λ , expressed in coordinates, reproduces groupoidification in the sense of (4).*

Proof. First, we note that $\Lambda(\mathbf{1}) \cong \mathbf{Vect}$, since the only irreducible representation of the trivial group (basis object) is \mathbb{C} itself.

Since $\mathbf{1}$ is terminal in \mathbf{FinGpd} , any groupoid has a unique map into it. Thus, any span from $\mathbf{1}$ to $\mathbf{1}$ is of the form:

$$(146) \quad \mathbf{1} \xleftarrow{!} X \xrightarrow{!} \mathbf{1}$$

which just amounts to a choice of X . Then $\Lambda(X) = !_* !^*$ can be described as a 1×1 matrix of vector spaces,

$$(147) \quad \Lambda(X)_{(\star, \mathbb{C}), (\star, \mathbb{C})} = \text{hom}(!^* \mathbb{C}, !_* \mathbb{C})$$

$$(148) \quad \cong \bigoplus_{[x] \in \underline{X}} \mathbb{C}$$

since $!^* \mathbb{C}$ is the representation of X assigning a copy of \mathbb{C} to each object. In particular, inducing up the representation \mathbb{C} gives, at each $x \in X$, the representation

$$(149) \quad \mathbb{C}[Aut(\star)] \otimes_{\mathbb{C}[Aut(x)]} \mathbb{C} = \mathbb{C}$$

since $!^* \mathbb{C}$, is the trivial representation of $Aut(x)$.

For each isomorphism class in \underline{X} , we thus get a copy of $\text{hom } \mathbb{C}, \mathbb{C} \cong \mathbb{C}$. This is the vector space associated to X by groupoidification.

Similarly, a 2-morphism $Y : X_1 \rightarrow X_2$ just amounts to an isomorphism class of spans of groupoids (since Y and the X_i have unique maps to $\mathbf{1}$). Then the linear map

$$(150) \quad \Lambda(Y)_{(\star, \mathbb{C}), (\star, \mathbb{C})} : \Lambda(X_1)_{(\star, \mathbb{C}), (\star, \mathbb{C})} \rightarrow \Lambda(X_2)_{(\star, \mathbb{C}), (\star, \mathbb{C})}$$

just becomes a map

$$(151) \quad T(Y) : \mathbb{C}[\underline{X}_1] \rightarrow \mathbb{C}[\underline{X}_2]$$

given by $T(Y) = \epsilon_{L,t} \circ \eta_{R,s}$.

By the above, (58), using $F(x_1) \cong \mathbb{C}$, can be written:

$$(152) \quad \eta_{R,s}(F)(y) : \mathbb{C} \rightarrow \bigoplus_{[y]|s(y) \cong x_1} \mathbb{C}[Aut(x_1)] \otimes_{\mathbb{C}[Aut(y)]} \mathbb{C}$$

So now for each $[x_1] \in \underline{X}_1$, every y in the essential preimage of x_1 under s gets a copy of the trivial representation \mathbb{C} for each coset of $Im(Aut(y))$ in $Aut((x_1))$. This describes a decomposition of a representation of $Aut(y)$ in terms of irreps (all of which are necessarily trivial in this case). Call this representation G . In particular, a vector in $\Lambda X_{1(\star, \mathbb{C}), (\star, \mathbb{C})}$ gives a complex number at each $[x_1]$. The unit $\eta_{R,s}$ takes such a vector v to, at each y with $s(y) \cong x_1$,

$$(153) \quad \frac{1}{\#Aut(y)} \sum_{g \in Aut(x_1)} g^{-1} \otimes 1$$

By commutativity for the span of span maps (which is necessarily strict here!), we also must have that

$$(154) \quad \bigoplus_{[y']|t(y') \cong t(y)} \mathbb{C}[Aut(t(y))] \otimes_{\mathbb{C}[Aut(y')]} G(y') \cong \bigoplus_{[y']|s(y') \cong s(y)} \mathbb{C}[Aut(s(y))] \otimes_{\mathbb{C}[Aut(y)]} \mathbb{C}$$

Similarly, then, using this (63) can be written:

$$(155) \quad \epsilon_{L,t}(G)(y) : \bigoplus_{[y']|t(y') \cong t(y)} \mathbb{C}[Aut(t(y))] \otimes_{\mathbb{C}[Aut(y')]} G(y') \rightarrow G(y)$$

So now consider the vector $v \in \Lambda(X_1)_{(\star, \mathbb{C}), (\star, \mathbb{C})}$ which gives 1 at $[x_1]$ and 0 elsewhere. (That is, it gives the identity intertwining map between the copies of the representation $!^*\mathbb{C}$ at objects in $[x_1]$ and the zero intertwiner elsewhere). Then the natural transformation induces a map on the coefficient:

$$(156) \quad \eta_{R,s} : v \mapsto \bigoplus_{[y]|s(y) \cong x_1} \frac{1}{\#Aut(y)} \sum_{g \in Aut(x_1)} g^{-1} \otimes 1$$

but then suppose we look for the coefficient of the result at $[x_2] \in \underline{X}_2$. Only those y over $[x_2]$ will contribute, but then, since the g^{-1} have no effect on vectors in \mathbb{C} , we get:

$$(157) \quad \epsilon_{L,t} : \bigoplus_{[y]|s(y) \cong x_1} \frac{1}{\#Aut(x_1)} \sum_{g \in Aut(x_1)} g^{-1} \otimes 1 \mapsto \sum_{y|(s,t)(y) \cong (x_1, x_2)} \frac{\#Aut(x_1)}{\#Aut(y)}$$

But this is just

$$(158) \quad \#Aut(x_1) \sum_{y|(s,t)(y) \cong (x_1, x_2)} \frac{1}{\#Aut(y)} = \#Aut(x_1) |\widehat{(x_1, x_2)}|$$

where the second term is the groupoid cardinality of the essential preimage of (x_1, x_2) . This is just the coefficient we find in groupoidification in the sense of Baez and Dolan. \square

Similar calculations apply for less trivial situations as well, although for these we will require a little more of the representation theory of the groupoids A_i .

Lemma 6.2.2. *Given a (strict) span between spans, $Y : X_1 \rightarrow X_2$, for $X_1, X_2 : A_1 \rightarrow A_2$, then the natural transformation*

$$(159) \quad \Lambda(Y) : \Lambda(X_1) \rightarrow \Lambda(X_2)$$

is a natural transformation given by a matrix of linear operators:

$$(160) \quad \Lambda(Y)_{([a_1], W_1), ([a_2], W_2)} : \Lambda(X_2)_{([a_1], W_1), ([a_2], W_2)} \rightarrow \Lambda(X_2)_{([a_1], W_1), ([a_2], W_2)}$$

or equivalently

$$(161) \quad \Lambda(Y)_{([a_1], W_1), ([a_2], W_2)} : \bigoplus_{[x_1]} \text{hom}_{\text{Rep}(\text{Aut}(x_1))} [s_1^*(W_1), t_1^*(W_2)] \\ \rightarrow \bigoplus_{[x_2]} \text{hom}_{\text{Rep}(\text{Aut}(x_2))} [s_2^*(W_1), t_2^*(W_2)]$$

Such that for each block $([x_1], [x_2])$, the corresponding linear operator behaves as follows: for $f \in \text{hom}[s_1^*(W_1), t_1^*(W_2)]$ we get:

$$(162) \quad \Lambda(Y)_{([a_1], W_1), ([a_2], W_2)}|_{(x_1, x_2)}(f) = |\widehat{(x_1, x_2)}| \sum_{g \in \text{Aut}(x_1)} g^{-1} f g$$

where $\widehat{(x_1, x_2)}$ is the essential preimage of (x_1, x_2) under (s, t) , namely the comma category $((s, t) \downarrow (x_1, x_2))$.

Proof. The argument here is similar to that in Theorem 6.2.1, except that we must deal with nontrivial representations of the $\text{Aut}(x_i)$. That is, when we apply the Nakayama isomorphism, and the evaluation maps, we cannot use triviality. The “group-averaging” acting on intertwiners in the expression we have given is exactly the exterior trace used in the Nakayama isomorphism. Here its function is to project a linear map (a “pulled-back” intertwiner) onto a space of intertwiners as we push it along the functor t .

In particular, the effect of η_R, s on coordinates (i.e. choosing particular representations W_i) is to take an intertwiner $f \in \text{hom}[s_1^*(W_1), t_1^*(W_2)]$ and produce an intertwiner at the representations pulled back to Y . The counit η_L, t “pushes” this down to an intertwiner in $\text{hom}[s_2^*(W_1), t_2^*(W_2)]$. The group averaging ensures this will be an intertwiner itself. \square

Remark 6.2.3. Using the formula for composition of 2-linear maps and natural transformations in a general 2-vector space, we can readily see how horizontal and vertical composition work.

Vertical composition is given by composition of linear maps component-wise, so we have:

$$(163) \quad \Lambda(Y' \circ Y)_{([a_1], W_1), ([a_2], W_2)}$$

with components given by:

$$(164) \quad \bigoplus_{([x_1], [x_3])} |\widehat{(x_1, x_3)}| \sum_{g \in \text{Aut}(x_3)} g^{-1} f g \\ = \bigoplus_{([x_1], [x_3])} \sum_{[x_2]} |\widehat{(x_2, x_3)}| \cdot |\widehat{(x_1, x_2)}| \cdot \# \text{Aut}(x_2) \left(\sum_{g \in \text{Aut}(x_3)} g^{-1} f g \right)$$

This uses the fact that the two group averages each give projections into spaces of intertwiners, which is redundant, so we omit one, taking only the order of the group. We also use that $\widehat{(x_1, x_3)}$ is a subgroupoid of $Y' \circ Y$. In fact, it is a union over all equivalence classes $[x_2]$ in X_2 of the objects in the weak pullback $Y' \circ Y$ based over $[x_2]$, which gives the sum over $[x_2]$ (which performs the matrix multiplication in each component).

The horizontal composite, $\Lambda(Y' \circ Y) \cong \Lambda(Y') \circ \Lambda(Y)$, on the other hand, involves “matrix multiplication” at the level of composition of 2-linear maps. The $(([a_1], W_1), ([a_3], W_3))$ component of the product is a linear map given as a block matrix with one block for each basis 2-vector. The blocks consist of the tensor products of the matrices from the components of $\Lambda(Y)$ and $\Lambda(Y')$. In particular, the β isomorphisms from the horizontal composition of source and target induce an isomorphism which acts on intertwiners $\iota \otimes \iota'$ by:

$$\begin{aligned}
 (165) \quad & \left(\bigoplus_{\substack{([x_1, f_1, x'_1]), \\ ([x_2, f_2, x'_2])}} |(\widehat{(x_1, x'_1)}, \widehat{(x_2, x'_2)})| \sum_{(g, g') \in \text{Aut}([x_1], [x'_1])} (g, g')^{-1} \iota \otimes \iota'(g, g') \right) \\
 & \cong \bigoplus_{([a_2], W_2)} \left(\bigoplus_{([x_1], [x_2])} |(\widehat{x_1}, \widehat{x_2})| \sum_{g \in \text{Aut}(x_1)} g^{-1} \iota g \right) \otimes \left(\bigoplus_{([x'_1], [x'_2])} |(\widehat{x'_1}, \widehat{x'_2})| \sum_{g' \in \text{Aut}(x'_1)} (g')^{-1} \iota' g' \right) \\
 & = \bigoplus_{([a_2], W_2)} \left(\bigoplus_{\substack{([x_1], [x_2]), \\ ([x'_1], [x'_2])}} |(\widehat{x_1}, \widehat{x_2})| \cdot |(\widehat{x'_1}, \widehat{x'_2})| \sum_{(g, g') \in \text{Aut}(x_1) \times \text{Aut}(x'_1)} (g, g')^{-1} \iota \otimes \iota'(g, g') \right)
 \end{aligned}$$

Here, we note that since $Y' \circ Y$ is a weak pullback over A_2 , its objects consist of triples (y, h, y') , we implicitly have a sum over $[y, h, y']$ in the groupoid cardinality, which is $|(\widehat{x_1, x_2})| \cdot |(\widehat{x'_1, x'_2})| \cdot |\text{Aut}(a_2)|$.

7. MAIN THEOREM

Having now described the effect of the functor Λ at each level - groupoids, spans, and spans of spans—it remains to check that these really define a 2-functor of the right kind. We begin by explicitly laying out what this 2-functor is, then verify the remaining properties.

7.1. The 2-Linearization Functor. We have been defining the maps involved in Λ throughout the last few sections, so here we collect the full definition in one place.

Definition 7.1.1. *The 2-linearization process $\Lambda : \text{Span}(\mathbf{FinGpd}) \rightarrow \mathbf{2Vect}$ is defined as follows:*

- For an essentially finite groupoid A it assigns:

$$(166) \quad \Lambda(A) = [A, \mathbf{Vect}]$$

- For a span of groupoids:

$$(167) \quad A \xleftarrow{s} X \xrightarrow{t} B$$

it assigns:

$$(168) \quad \Lambda(S) = t_* \circ s^*$$

- For a (strictly commuting) span of maps between two spans with the same source and target:

$$(169) \quad \begin{array}{ccccc} & & X_1 & & \\ & s_1 \swarrow & \uparrow s & \searrow t_1 & \\ A & & Y & & B \\ & s_2 \swarrow & \downarrow t & \searrow t_2 & \\ & & X_2 & & \end{array}$$

Λ assigns a natural transformation:

$$(170) \quad \Lambda(Y) = \epsilon_{L,t} \circ \eta_{R,s} : (t_1)_* s_1^* \Longrightarrow (t_2)_* s_2^*$$

(and analogously for weakly commuting spans of maps as in Definition 6.1.1)

Λ also associates the following:

- For each composable pair $X : A_1 \rightarrow A_2$ and $X' : A_2 \rightarrow A_3$, a natural isomorphism

$$(171) \quad \beta : \Lambda(X' \circ X) \rightarrow \Lambda(X') \circ \Lambda(X)$$

, as described in Theorem 5.2.1.

- For each object $X \in \mathbf{FinGpd}$, the natural transformation

$$(172) \quad U_B : 1_{\Lambda(B)} \xrightarrow{\sim} \Lambda(1_B)$$

is the natural transformation induced by the equivalence between B and 1_B .

Then we have the following:

Theorem 7.1.2. *The construction given in Definition 7.1.1 defines a weak 2-functor $\Lambda : \text{Span}(\mathbf{FinGpd}) \rightarrow \mathbf{2Vect}$.*

Proof. First, we note that by the result of Lemma 4.1.1, we know that Λ assigns a 2-vector space to each object of $\text{Span}(\mathbf{FinGpd})$.

If $S : B \rightarrow B'$ span of essentially finite groupoids—i.e. a morphism in $\text{Span}(\mathbf{FinGpd})$, the map $\Lambda(S)$ defined in Definition 5.1.1 is a linear functor by the result of Theorem 4.2.1, since it is a composite of two linear maps. This respects composition of morphisms, as shown in Theorem 5.2.1, and of 2-morphisms in both horizontal and vertical directions, as shown in Theorems 6.1.3 and 6.1.4.

Next we need to check that our Λ satisfies the remaining properties of a weak 2-functor: that the isomorphisms from the weak preservation of composition and units satisfy the requisite coherence conditions; and that Λ strictly preserves horizontal and vertical composition of natural transformations.

The coherence conditions for the compositor morphisms

$$(173) \quad \beta_{S,T} : \Lambda(T \circ S) \rightarrow \Lambda(T) \circ \Lambda(S)$$

and the associator say that these must make the following diagram commute for all composable triples (X, X', X'') :

$$(174) \quad \begin{array}{ccccc} & & \Lambda(X'') \circ \Lambda(X') \circ \Lambda(X) & & \\ & \nearrow^{1 \otimes \beta_{2,1}} & & \nwarrow_{\beta_{3,2} \otimes 1} & \\ \Lambda(X'') \circ \Lambda(X' \circ X) & & & & \Lambda(X'' \circ X') \circ \Lambda(X) \\ & \nwarrow_{\beta_{3,21}} & & \nearrow_{\beta_{32,1}} & \\ \Lambda(X'' \circ (X' \circ X)) & \xrightarrow{\Lambda(\alpha_{X'', X', X})} & \Lambda((X'' \circ X') \circ X) & & \end{array}$$

We implicitly assume here a trivial associator for the 2-linear maps in the expression $\Lambda(X'') \circ \Lambda(X') \circ \Lambda(X)$. This is because each 2-linear map is just a composite of functors, so this composition is associative. But note that we can similarly assume, without loss of generality, that the associator α for composition of spans is trivial. The composite $X' \circ X$ is a weak pullback. This is only defined up to isomorphism, but one candidate is the comma category for any $x \in A_2$. Any other candidate is isomorphic to this one. But then, the associator

$$(175) \quad \alpha_{X'', X', X} : \Lambda(X'' \circ (X' \circ X)) \rightarrow \Lambda((X'' \circ X') \circ X)$$

is just given by the obvious canonical map between the comma categories. In particular, both composites give comma categories whose objects are determined by choices (x, f, x', g, x'') where $f : t_1(x) \rightarrow s_2(x')$ and $g : t_2(x') \rightarrow s_3(x'')$, and whose morphisms are triples of morphisms in $X \times X' \times X''$ making the appropriate diagrams commute. However, these comma categories are defined in terms of pairs, with different parenthesizations. So $\alpha_{X'', X', X}$ is the evident isomorphism between these composites.

So it suffices to show that, up to this identification:

$$(176) \quad (1 \otimes \beta_{X', X}) \circ \beta_{X'', X' \circ X} = (\beta_{X'', X'} \otimes 1) \circ \beta_{X'' \circ X', X}$$

The β isomorphisms are given by the α up to which the weak pullbacks commute, and so are given by choices of the functions $f \in A_2$ and $g \in A_3$ in the comma categories. The associator isomorphism induces an corresponding isomorphism between these composite β maps by the correspondence between the choices of f and g in the pullback squares on each side of this equation. So indeed, this is true.

In general, the coherence conditions for the “unit” isomorphism

$$(177) \quad U_A : 1_{\Lambda(A)} \xrightarrow{\sim} \Lambda(1_A)$$

which accomplishes weak preservation of identities, say that it must make the following commute for any span $X : A_1 \rightarrow A_2$:

$$(178) \quad \begin{array}{ccc} & \Lambda(X) & \\ \uparrow 1 \otimes U_{A_1} & \nwarrow \Lambda(r_X) & \\ \Lambda(X) \circ \Lambda(1_{A_1}) & \xleftarrow{\beta_{X, 1_{A_1}}} & \Lambda(X \circ 1_{A_1}) \end{array}$$

where r_{A_1} is the right unitor for A_1 . There is also the symmetric condition for the left unitor.

We notice that, as with $\Lambda(1_{A_1})$, $\Lambda(r_{A_1})$ is equivalent to the identity. The map $r_X : X \circ 1_{A_1} \rightarrow X$ is the canonical isomorphism taking composition of X with an identity span to X which is just a projection from a comma category. Since $X \circ 1_{A_1}$ and X are thus isomorphic, .

So the condition amounts to the fact that $\beta_{X, 1_{A_1}} : \Lambda(X \circ 1_{A_1}) \rightarrow \Lambda(X) \circ \Lambda(1_{A_1}) = \Lambda(X)$ is equivalent to the identity in such a way that (178) commutes. But this is immediate since this β map is being applied to an identity span. \square

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APPENDIX A. WEAK PRESERVATION OF COMPOSITION

In this appendix, we give some background to the definition of composition of spans of groupoids, namely comma categories. We also give a note on a key element of the proof of Theorem 5.2.1, which states that the putative 2-functor Λ weakly preserves this composition. We rely on the fact that a pullback square of groupoids gives rise to a square of 2-linear maps, which satisfies the Beck-Chevalley condition. We discuss this here as well.

A.1. Background on Comma Categories. We now recall some facts about comma categories, which play a role in our construction of our 2-functor Λ in the composition of spans of groupoids, via weak pullback.

Definition A.1.1. *Given a diagram of categories $\mathbf{A} \xrightarrow{F} \mathbf{C} \xleftarrow{G} \mathbf{B}$. Then an object in the **comma category** $(F \downarrow G)$ consists of a triple (a, f, b) , where $a \in \mathbf{A}$ and $b \in \mathbf{B}$ are objects, and $f : F(a) \rightarrow G(b)$ is a morphism in \mathbf{C} . A morphism in $(F \downarrow G)$ from (a_1, f_1, b_1) to (a_2, f_2, b_2) consists of a pair of morphisms $(h, k) \in \mathbf{A} \times \mathbf{B}$ making the*

square

$$(179) \quad \begin{array}{ccc} F(a_1) & \xrightarrow{f_1} & G(b_2) \\ F(h) \downarrow & & \downarrow G(k) \\ F(a_2) & \xrightarrow{f_2} & G(b_2) \end{array}$$

commute.

Remark A.1.2. Note that in a weak pullback, the morphisms f would be required to be an *isomorphism*, but when we are talking about a weak pullback of groupoids, these conditions are the same.

The comma category has projection functors which complete the (weak) pullback square for the two projections:

$$(180) \quad \begin{array}{ccc} & (F \downarrow G) & \\ P_A \swarrow & & \searrow P_B \\ \mathbf{A} & \xrightleftharpoons[\sim]{\alpha} & \mathbf{B} \\ F \searrow & & \swarrow G \\ & \mathbf{C} & \end{array}$$

such that $(F \downarrow G)$ is a universal object (in \mathbf{Cat}) with maps into \mathbf{A} and \mathbf{B} making the resulting square commute up to a natural isomorphism α . This satisfies the universal condition that, given any other category \mathbf{D} with maps to \mathbf{A} and \mathbf{B} , there is an equivalence between $[\mathbf{D}, \mathbf{C}]$ and the comma category (P_A^*, P_B^*) (where P_S^* and P_T^* are the functors from \mathbf{D} to \mathbf{B} which factor through P_S and P_T respectively). This is the weak form of the universal property of a pullback.

So suppose we restrict to the case of a weak pullback of groupoids. This is equivalent to the situation where \mathbf{A} , \mathbf{B} and \mathbf{C} are skeletal - that is, each is just a disjoint union of groups. Then the set of objects of $(F \downarrow G)$ is a disjoint union over all the morphisms of \mathbf{C} (which are all of the form $g : x \rightarrow x$ for some object x) of all the pairs of objects $a \in \mathbf{A}$ and $b \in \mathbf{B}$ with $g : F(a) \rightarrow G(b)$. In particular, since we assume \mathbf{C} is skeletal, this means $F(a) = G(b)$, though there will be an instance of this pair in $(F \downarrow G)$ for each g in the group of morphisms on this object $F(a) = G(b)$.

So as the set of objects in $(F \downarrow G)$ we have a disjoint union of products of sets—for each $c \in \mathbf{C}$, we get $|\text{Aut}(c)|$ copies of $F^{-1}(c) \times G^{-1}(c)$. The set of morphisms is just the collection of commuting squares as in (179) above.

Note that if we choose a particular c and $g : c \rightarrow c$, and choose objects a, b with $F(a) = c$, $G(b) = c$, and if $H = \text{Aut}(a)$, $K = \text{Aut}(b)$ and $M = \text{Aut}(c)$, then the group of automorphisms of $(a, g, b) \in (F \downarrow G)$ is isomorphic to the fibred product $H \times_M K$. In particular, it is a subgroup of the product group $H \times K$ consisting of only those pairs (h, k) with $F(h)g = gG(k)$, or just $F(h) = gG(k)g^{-1}$. We can call it $H \times_M K$, keeping in mind that this fibred product depends on g . Clearly, the group of automorphisms of two isomorphic objects in $(F \downarrow G)$ are isomorphic groups.

Now, as we saw when discussing comma squares, the objects of the weak pullback $X' \circ X$ consist of pairs of objects, $x \in X$, and $x' \in X'$, together with a morphism in

A_2 , $g : t(x) \rightarrow s'(x')$. The morphisms from (x_1, g_1, x'_1) to (x_2, g_2, x'_2) in the weak pullback are pairs of morphisms, $(h, k) \in X \times X'$, making the square

$$(181) \quad \begin{array}{ccc} t(x_1) & \xrightarrow{g_1} & s'(x'_1) \\ t(h) \downarrow & & \downarrow s'(k) \\ t(x_2) & \xrightarrow{g_2} & s'(x'_2) \end{array}$$

commute.

We may assume that the groupoids we begin with are skeletal—if not, we replace the groupoid with its skeleton, so the objects are just isomorphism classes of the original objects. Then recall from Section 5.1 that in this weak pullback the set of objects in $X' \circ X$ is a disjoint union of products of sets - for each $a \in A_2$, we get $|Aut(a)|$ copies of $t^{-1}(a) \times (s')^{-1}(a)$.

A.2. The Beck Condition.

Remark A.2.1. The isomorphism α in the weak pullback square (11) gave rise to a natural isomorphism:

$$(182) \quad \alpha^* : T^* \circ (s')^* \rightarrow S^* \circ t^*$$

Given an object in the composite $X' \circ X$, α gives an isomorphism of the two restrictions to A_2 , through X and X' .

What we proved is that the other square—the “mate” under the adjunctions, also has a natural isomorphism (“vertically” across the square), namely that there exists:

$$(183) \quad \beta_{X, X'} : T_* \circ S^* \rightarrow (s')^* \circ t_*$$

In fact, these are related by the units for both pairs of adjoint functors:

$$(184) \quad \eta_{R, T} : 1_{\Lambda(X' \circ X)} \rightarrow T_* \circ T^*$$

and

$$(185) \quad \eta_{R, t} : 1_{\Lambda(X)} \rightarrow t_* \circ t^*$$

So the desired “vertical” natural transformation across the square (80) is determined by the condition that it complete the following square of natural transformations to make it commute:

$$(186) \quad \begin{array}{ccc} T_* \circ S^* & \xRightarrow{1 \otimes \eta_{R, t}} & T_* \circ S^* \circ t^* \circ t_* \\ \parallel \beta_{X, X'} \downarrow & & \downarrow 1 \otimes (\alpha^*)^{-1} \otimes 1 \\ (s')^* \circ t_* & \xRightarrow{1 \otimes \eta_{R, T}} & T_* \circ T^* \circ (s')^* \circ t_* \end{array}$$

The crucial element of this is the fact that the (weak) pullback square for the groupoids in the middle of the composition diagram gives rise to a (weak) pullback square of **Vect**-presheaf categories. This is shown by Ross Street [18]. This is the *Beck-Chevalley* (BC) condition, which is discussed by Bénabou and Streicher [5], MacLane and Moerdijk [14], and by Dawson, Paré and Pronk [7].

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