

Symmetric bimonoidal intermuting categories and $\omega \times \omega$ reduced bar constructions

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Abstract

A new, self-contained, proof of a coherence result for categories equipped with two symmetric monoidal structures bridged by a natural transformation is given. It is shown that this coherence result is sufficient for $\omega \times \omega$ -indexed family of iterated reduced bar constructions based on such a category.

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1 Introduction

This paper presents a reduced bar construction which is usually the initial part of the results connecting various monoidal categories with 1-fold, 2-fold, n -fold and infinite loop spaces (see [11], [13], [1] and references therein). By the reduced bar construction I mean a construction of a simplicial object based on a monoid in a category whose monoidal structure is given by finite products (exactly the same as the notion used in [13]), which in particular, for a special monoid in the category *Cat*, may be iterated in order to obtain a lax functor from an arbitrary power of $(\Delta^+)^{op}$ to *Cat*. This construction is based here on a category equipped with two symmetric monoidal structures, given by the tensors \vee and \wedge , and the units \perp and \top . These two structures are bridged by a natural transformation, called *intermutation* in [3], given by the family of arrows

$$(A \wedge B) \vee (C \wedge D) \rightarrow (A \vee C) \wedge (B \vee D).$$

Such categories appeared under the name *symmetric bimonoidal intermuting categories* in [3]. As a concrete example of a symmetric bimonoidal intermuting category one can take any category with all finite coproducts and all finite products in which product of initial objects is initial and coproduct of terminal objects is terminal (see [3], Sections 13 and 15).

Since I am not a topologist, I will not go further in the procedure of delooping, which is very well traced by the work of Thomason in [13]. This procedure leads to an $\omega \times \omega$ -indexed family of deloopings of the classifying space of a symmetric bimonoidal intermuting category. (According to this, one can make the following hierarchy of infinite loop spaces; simply symmetric monoidal structure corresponds to an infinite loop space with ω -indexed family of deloopings, double symmetric monoidal structure without intermutation corresponds to an infinite loop space with $\omega + \omega$ -indexed family of deloopings, double symmetric monoidal structure with intermutation corresponds to an infinite loop space with $\omega \times \omega$ -indexed family of deloopings, etc.)

This is not a paper in categorial proof theory. However, it gives, as a by-product, a complete formulation of a fragment of linear derivations in classical and intuitionistic propositional logic. Logic also helped us in [3] to find that something is inappropriate in the unbalanced treatment of units versus tensors in [1]. (Derivations of sequents of the form $A \vee B \vdash A \wedge B$ and $A \vee B \vdash B \wedge A$ are undesirable in logic.) I keep to the notation $\vee, \wedge, \perp, \top$ for tensors and units which is inspired by logic. This is partly because at one point (see Section 4, Lemma 4.1) there is a reference to a coherence result from [2] where this notation is primary. Also, some easy lemmata in Section 5 are taken over from [3]. Otherwise, this paper is self-contained.

I am aware that a result analogous to the coherence obtained in [1], but with relaxed treatment of units, would be stronger to a great extent than the coherence result given here. I will not give a refinement of the notion of n -fold monoidal category from [1]—one can see from the paper how the units should be treated in a relaxed notion and how to use a restricted coherence result to show that the iterated reduced bar construction delivers a lax functor (see Lemma 7.2).

The first part of the paper is devoted to a coherence result for symmetric bimonoidal intermuting categories. At one point, for technical reasons, a strictification with respect to both associativity and symmetry is used, and since the latter is not so standard, although it is explained in details in [2], a sketch of a proof why it actually works is given in Section 4. In the second part of the paper, this coherence result is used to establish that for every pair (n, m) of natural numbers one can iterate the reduced bar construction using first n times the monoidal structure given by \vee, \perp and then m times the monoidal structure given by \wedge, \top of a symmetric bimonoidal intermuting category \mathcal{C} in order to obtain a lax functor mapping an $n + m$ -tuple (k_1, \dots, k_{n+m}) of natural numbers, regarded as objects of the simplicial category, to $\mathcal{C}^{k_1 \dots k_{n+m}}$.

The coherence result for symmetric bimonoidal intermuting categories is already present in [3]. Although that paper is not easy to read, this result, as well as the other coherence results given there, is correct. The proof presented here is just more self-contained and because of that, by my opinion, easier for reading. However, the mathematical content remains the same. So, the correct referring to this coherence result should go through [3].

Some parts of the paper may be skipped (Sections 4 and 6 are optional) and for experts it is, perhaps, sufficient to see the definition of symmetric bimonoidal

intermuting categories (Section 2), then the statement of a coherence result for these categories (Section 3, Theorem 3.1) and eventually Section 7, especially Lemma 7.2, which makes this coherence result sufficient for the construction of a lax functor with desired properties. This paper is influenced very much by nicely written [1].

2 Symmetric bimonoidal intermuting categories

We say that a category \mathcal{C} is *symmetric bimonoidal intermuting (SMI)* category when it has two symmetric monoidal structures given by $\langle \mathcal{C}, \vee, \perp, \check{b}, \check{c}, \check{\delta}, \check{\sigma} \rangle$ and $\langle \mathcal{C}, \wedge, \top, \hat{b}, \hat{c}, \hat{\delta}, \hat{\sigma} \rangle$ (here b 's, c 's, δ 's and σ 's stay for associativity, symmetry, right and left identity natural isomorphisms; $\check{b}_{A,B,C}^{\rightarrow}: A \vee (B \vee C) \rightarrow (A \vee B) \vee C$ has the inverse $\check{b}_{A,B,C}^{\leftarrow}$, etc.) together with a natural transformation c^k (note that k is not an index here) given by the family of arrows

$$c_{A,B,C,D}^k: (A \wedge B) \vee (C \wedge D) \rightarrow (A \vee C) \wedge (B \vee D),$$

two isomorphisms

$$\hat{w}_{\perp}^{\leftarrow}: \perp \rightarrow \perp \wedge \perp, \quad \check{w}_{\top}^{\rightarrow}: \top \vee \top \rightarrow \top,$$

whose inverses are $\hat{w}_{\perp}^{\rightarrow}$ and $\check{w}_{\top}^{\leftarrow}$ respectively, and an arrow $\kappa: \perp \rightarrow \top$. In addition the following diagrams commute:

$$\begin{array}{ccc} (A \wedge (B \wedge C)) \vee (D \wedge (E \wedge F)) & \xrightarrow{\hat{b}^{\rightarrow} \vee \hat{b}^{\rightarrow}} & ((A \wedge B) \wedge C) \vee ((D \wedge E) \wedge F) \\ \downarrow c^k & & \downarrow c^k \\ (A \vee D) \wedge ((B \wedge C) \vee (E \wedge F)) & \xrightarrow{\quad (1) \quad} & ((A \wedge B) \vee (D \wedge E)) \wedge (C \vee F) \\ \downarrow \mathbf{1} \wedge c^k & & \downarrow c^k \wedge \mathbf{1} \\ (A \vee D) \wedge ((B \vee E) \wedge (C \vee F)) & \xrightarrow{\hat{b}^{\rightarrow}} & ((A \vee D) \wedge (B \vee E)) \wedge (C \vee F) \end{array}$$

$$\begin{array}{ccc} (A \vee (B \vee C)) \wedge (D \vee (E \vee F)) & \xrightarrow{\check{b}^{\rightarrow} \wedge \check{b}^{\rightarrow}} & ((A \vee B) \vee C) \wedge ((D \vee E) \vee F) \\ \uparrow c^k & & \uparrow c^k \\ (A \wedge D) \vee ((B \vee C) \wedge (E \vee F)) & \xrightarrow{\quad (2) \quad} & ((A \vee B) \wedge (D \vee E)) \vee (C \wedge F) \\ \uparrow \mathbf{1} \vee c^k & & \uparrow c^k \vee \mathbf{1} \\ (A \wedge D) \vee ((B \wedge E) \vee (C \wedge F)) & \xrightarrow{\check{b}^{\rightarrow}} & ((A \wedge D) \vee (B \wedge E)) \vee (C \wedge F) \end{array}$$

$$\begin{array}{ccc} (A \wedge B) \vee (C \wedge D) & \xrightarrow{\hat{c} \vee \hat{c}} & (B \wedge A) \vee (D \wedge C) \\ \downarrow c^k & \quad (3) \quad & \downarrow c^k \\ (A \vee C) \wedge (B \vee D) & \xrightarrow{\hat{c}} & (B \vee D) \wedge (A \vee C) \end{array}$$

$$\begin{array}{ccc}
(A \vee B) \wedge (C \vee D) & \xrightarrow{\check{c} \wedge \check{c}} & (B \vee A) \wedge (D \vee C) \\
c^k \uparrow & (4) & \uparrow c^k \\
(A \wedge C) \vee (B \wedge D) & \xrightarrow{\check{c}} & (B \wedge D) \vee (A \wedge C)
\end{array}$$

$$\begin{array}{ccc}
(A \wedge B) \vee (\perp \wedge \perp) & \xrightarrow{c^k} & (A \vee \perp) \wedge (B \vee \perp) \\
\mathbf{1} \vee \hat{w}_{\perp}^{\rightarrow} \downarrow & (5) & \uparrow \check{\delta}^{\leftarrow} \wedge \check{\delta}^{\leftarrow} \\
(A \wedge B) \vee \perp & \xrightarrow{\check{\delta}^{\rightarrow}} & A \wedge B
\end{array}$$

$$\begin{array}{ccc}
(A \wedge \top) \vee (B \wedge \top) & \xrightarrow{c^k} & (A \vee B) \wedge (\top \vee \top) \\
\hat{\delta}^{\rightarrow} \vee \hat{\delta}^{\rightarrow} \downarrow & (6) & \uparrow \mathbf{1} \wedge \check{w}_{\top}^{\leftarrow} \\
A \vee B & \xrightarrow{\hat{\delta}^{\leftarrow}} & (A \vee B) \wedge \top
\end{array}$$

$$\begin{array}{ccc}
\top \vee (\top \vee \top) & \xrightarrow{\check{b}^{\rightarrow}} & (\top \vee \top) \vee \top \\
\mathbf{1} \vee \check{w}_{\top}^{\rightarrow} \searrow & (7) & \swarrow \check{w}_{\top}^{\leftarrow} \vee \mathbf{1} \\
& \top \vee \top &
\end{array}$$

$$\begin{array}{ccc}
\perp \wedge (\perp \wedge \perp) & \xrightarrow{\hat{b}^{\rightarrow}} & (\perp \wedge \perp) \wedge \perp \\
\mathbf{1} \wedge \hat{w}_{\perp}^{\rightarrow} \searrow & (8) & \swarrow \hat{w}_{\perp}^{\leftarrow} \wedge \mathbf{1} \\
& \perp \wedge \perp &
\end{array}$$

$$\begin{array}{ccc}
\top \vee \perp & \xrightarrow{\mathbf{1} \vee \kappa} & \top \vee \top \\
\delta^{\rightarrow} \searrow & (9) & \swarrow \check{w}_{\top}^{\leftarrow} \\
& \top &
\end{array}$$

$$\begin{array}{ccc}
\perp \wedge \perp & \xrightarrow{\mathbf{1} \wedge \kappa} & \perp \wedge \top \\
\hat{w}_{\perp}^{\rightarrow} \searrow & (10) & \swarrow \hat{\delta}^{\leftarrow} \\
& \perp &
\end{array}$$

$$\begin{array}{ccc}
(\top \wedge \perp) \vee (\perp \wedge \top) & \xrightarrow{c^k} & (\top \vee \perp) \wedge (\perp \vee \top) \\
\hat{\sigma}^{\rightarrow} \vee \hat{\delta}^{\rightarrow} \downarrow & (11) & \uparrow \check{\delta}^{\leftarrow} \wedge \check{\sigma}^{\leftarrow} \\
\perp \vee \perp & \xrightarrow{\check{\delta}^{\rightarrow}} \perp & \xrightarrow{\kappa} \top & \xrightarrow{\hat{\delta}^{\leftarrow}} \top \wedge \top
\end{array}$$

$$\begin{array}{ccc}
\top \vee \top & \xrightarrow{\check{c}} & \top \vee \top \\
\check{w}_{\top}^{\rightarrow} \searrow & (12) & \swarrow \check{w}_{\top}^{\leftarrow} \\
& \top &
\end{array}$$

$$\begin{array}{ccc}
\perp \wedge \perp & \xrightarrow{\hat{c}} & \perp \wedge \perp \\
\hat{w}_{\perp}^{\rightarrow} \searrow & (13) & \swarrow \hat{w}_{\perp}^{\leftarrow} \\
& \perp &
\end{array}$$

(Note that the equations (1) and (2) are just unstrictified versions of the internal associativity condition and the external associativity condition of [1].)

Our goal is to prove a coherence result for *SMI* categories which roughly says the following:

*Two canonical arrows $f, g: A \rightarrow B$ of an *SMI* category are equal if either:*

the units \perp and \top do not “essentially” occur in A and B , and f and g have the same graph (defined analogously to the Kelly-Mac Lane graphs in [8]), or

A and B are isomorphic to \perp or to \top .

Since this result has to say something about the canonical structure of an *SMI* category, and this structure is equationally presented, a precise formulation of our coherence result is given in terms of an *SMI* category freely generated by a set of objects.

3 Freely generated *SMI* category

Our category \mathcal{M} (called $\mathbf{SC}^{\mathbf{k}_{\top, \perp}}$ in [3]), which is an *SMI* category freely generated by an infinite set \mathcal{P} of propositional letters, is constructed as follows:

The *objects* of \mathcal{M} are propositional formulae of the language generated from \mathcal{P} , constants \perp and \top , with the binary connectives \vee and \wedge . The *arrows* of \mathcal{M} are equivalence classes of *arrow terms* generated from *primitive arrow terms* $\mathbf{1}_A, \hat{b}_{A,B,C}, \dots, \kappa$, with the help of \circ, \vee and \wedge . These equivalence classes are taken with respect to the smallest equivalence relation on arrow terms which makes out of \mathcal{M} an *SMI* category. So, this equivalence relation captures the equations of both symmetric monoidal structures, naturality of c^k , isomorphism conditions for w 's, the equations brought by the commutative diagrams (1)-(13) of the preceding section, and it is congruent with respect to \circ, \vee and \wedge .

Throughout this section we use the following terminology. We say that an arrow term is a *b-term* if it is built from identities and one occurrence of b with the help of \vee and \wedge . For example, $\mathbf{1}_A \wedge (\hat{b}_{B,C,D} \vee \mathbf{1}_E)$ is a *b-term* and we call $\hat{b}_{B,C,D}$ its *head*. We define analogously *c*, δ , σ , c^k , w and κ -terms and their heads. Note that they are all composition free. We say that an arrow term $f_n \circ \dots \circ f_1 \circ \mathbf{1}_A$ is a *developed* arrow term if each f_i is a *b*, *c*, δ , σ , c^k , w or κ -term. It is easy to see that every arrow term of \mathcal{M} is equal to a developed one.

We say that an arrow term is *defined by b* if it is built from identities and *b*'s (both \check{b} 's and \hat{b} 's) with the help of \circ, \vee and \wedge . We say analogously that an arrow term is *defined by c*, or *b* and *c*, etc.

For every object A of \mathcal{M} let $\nu(A)$ be obtained by iterated replacing in A every subformulae of the form $B \vee \perp$, $\perp \vee B$, $B \wedge \top$ and $\top \wedge B$ by B and every subformula $\perp \wedge \perp$ by \perp and $\top \vee \top$ by \top (c.f. the formal definition of $\nu(A)$ in [3], Section 5). We say that A *reduces by ν* to $\nu(A)$. If no letter

occurs in A , then $\nu(A)$ is either \perp or \top . It is clear that for every A there is an isomorphism $i_A: A \rightarrow \nu(A)$ defined by δ, σ and w with superscripts \rightarrow . We call the arrow terms defined by δ, σ and w , $\mathbf{N}_{\top, \perp}$ -terms as in [3], and when all the superscripts are \rightarrow we call them *directed*. It can be easily shown that for directed arrow terms $f, g: A \rightarrow \nu(A)$ we have $f = g$, hence every definition of the isomorphism $i_A: A \rightarrow \nu(A)$ leads to the same arrow. From this result the following lemma is derived.

LEMMA 3.1. *Every diagram of $\mathbf{N}_{\top, \perp}$ -terms is commutative.*

This is established in the same way as the coherence result for monoidal categories in [9]. Then we can prove the following.

LEMMA 3.2. *If no letter occurs in A and B then for every pair $f, g: A \rightarrow B$ of arrow terms defined by δ, σ, w and κ , we have $f = g$.*

PROOF. We establish first that every arrow term defined by δ, σ, w and κ is either equal to an $\mathbf{N}_{\top, \perp}$ -term or it is equal to a term of the form $h'' \circ \kappa \circ h'$ for $\mathbf{N}_{\top, \perp}$ -terms h' and h'' . To do this, we rely on the equations (9), (10), the following naturality conditions

$$\begin{aligned} \kappa \vee \mathbf{1}_{\perp} &= \check{\delta}_{\top}^{\leftarrow} \circ \kappa \circ \check{\delta}_{\perp}^{\rightarrow}, & \mathbf{1}_{\perp} \vee \kappa &= \check{\sigma}_{\top}^{\leftarrow} \circ \kappa \circ \check{\sigma}_{\perp}^{\rightarrow}, \\ \kappa \wedge \mathbf{1}_{\top} &= \hat{\delta}_{\top}^{\leftarrow} \circ \kappa \circ \hat{\delta}_{\perp}^{\rightarrow}, & \mathbf{1}_{\top} \wedge \kappa &= \hat{\sigma}_{\top}^{\leftarrow} \circ \kappa \circ \hat{\sigma}_{\perp}^{\rightarrow}, \end{aligned}$$

and the fact that there are no arrow terms of the form $\kappa \circ h \circ \kappa$. If f is equal to an $\mathbf{N}_{\top, \perp}$ -term then A is isomorphic to B and so g must be equal to an $\mathbf{N}_{\top, \perp}$ -term too, and vice versa. It only remains to apply Lemma 3.1. \dashv

LEMMA 3.3. *If no letter occurs in A and B then every arrow $f: A \rightarrow B$ may be defined by δ, σ, w and κ .*

PROOF. We rely on the equations of symmetric monoidal categories, the equations (5), (6), (7), (8), (11), (12), (13), the naturality conditions, and the fact that the arrow terms i_A and i_A^{-1} are $\mathbf{N}_{\top, \perp}$ -terms, to eliminate the presence of b 's, c 's and c^k 's. For example, we have

$$\begin{array}{ccc} \perp \wedge ((\top \wedge (\perp \wedge \perp)) \wedge \perp) & \xrightarrow{\hat{b}^{\rightarrow}} & (\perp \wedge (\top \wedge (\perp \wedge \perp))) \wedge \perp \\ \mathbf{1} \wedge (i \wedge \mathbf{1}) \downarrow & (nat) & \uparrow (\mathbf{1} \wedge i^{-1}) \wedge \mathbf{1} \\ \perp \wedge (\perp \wedge \perp) & \xrightarrow{\hat{b}^{\rightarrow}} & (\perp \wedge \perp) \wedge \perp \\ \searrow \mathbf{1} \wedge \hat{w}_{\perp}^{\rightarrow} & (8) & \nearrow \hat{w}_{\perp}^{\leftarrow} \wedge \mathbf{1} \\ & \perp \wedge \perp & \end{array} \quad \dashv$$

As a direct consequence of Lemmata 3.2 and 3.3 we have:

LEMMA 3.4. *If no letter occurs in A and B then for every $f, g: A \rightarrow B$ we have $f = g$.*

Here is the explanation what we meant by not “essential” occurrence of the units in an object. We say that an object A of \mathcal{M} is \perp -pure when there is no occurrence of \perp in $\nu(A)$. It is easy to see that A is not \perp -pure iff either $\nu(A) = \perp$ or there is a conjunction in A (by a conjunction in A we mean a subformula of the form $B \wedge C$) such that one of its conjuncts reduces by ν to \perp and a letter occurs in the other. We define analogously a \top -pure object of \mathcal{M} and derive an analogous characterization. An object of \mathcal{M} is *pure* when it is both \perp -pure and \top -pure.

LEMMA 3.5. *Let $f: A \rightarrow B$ be an arrow of \mathcal{M} . If A is \perp -pure, then B is \perp -pure, and if B is \top -pure, then A is \top -pure.*

PROOF. Since f may be represented by a developed term it is sufficient to verify the lemma for $b, c, \delta, \sigma, c^k, w$ and κ -terms. The only interesting case is when f is a c^k -rem.

Suppose B is not \perp -pure. By using the above-mentioned characterization of such objects of \mathcal{M} , we have two possibilities. If $\nu(B) = \perp$ then we easily conclude that $\nu(A) = \perp$ too. If there is a conjunction in B such that one of its conjuncts is reduced by ν to \perp and a letter occurs in the other conjunct, then we obviously have the same situation in A , except in the case when this conjunction is the target of the head $c_{E,F,G,H}^k: (E \wedge F) \vee (G \wedge H) \rightarrow (E \vee G) \wedge (F \vee H)$ of f . If $\nu(E \vee G)$ is \perp and there is a letter in $F \vee H$, then $\nu(E) = \nu(G) = \perp$ and there is a letter in either F or G . So, A is not \perp -pure. This is sufficient for the first implication and we proceed analogously for the second implication of the lemma. \dashv

COROLLARY. *If $f: A \rightarrow B$ and $g: B \rightarrow C$ are arrows of \mathcal{M} such that A and C are pure, then B is pure.*

LEMMA 3.6. *If $f: A \rightarrow B$ is an arrow term such that A and B are pure, then there is an arrow term $f': \nu(A) \rightarrow \nu(B)$ such that $\delta, \sigma, w, \kappa, \top$ and \perp do not occur in f' and*

$$f = i_B^{-1} \circ f' \circ i_A.$$

PROOF. By the corollary of lemma 3.5, it is sufficient to verify the lemma for $b, c, \delta, \sigma, c^k, w$ and κ -terms. If f is an $\mathbf{N}_{\top, \perp}$ -term then $\nu(A) = \nu(B)$ and by Lemma 3.1 we have $f = i_B^{-1} \circ i_A$.

If f is a b -term whose head is $\hat{b}_{C,D,E}^{\rightarrow}$, then by the following naturality diagram

$$\begin{array}{ccc} X \wedge (Y \wedge Z) & \xrightarrow{\hat{b}^{\rightarrow}} & (X \wedge Y) \wedge Z \\ i \wedge (i \wedge i) \downarrow & (nat) & \uparrow (i^{-1} \wedge i^{-1}) \wedge i^{-1} \\ \nu(X) \wedge (\nu(Y) \wedge \nu(Z)) & \xrightarrow{\hat{b}^{\rightarrow}} & (\nu(X) \wedge \nu(Y)) \wedge \nu(Z) \end{array}$$

we may assume that the indices C , D and E are already reduced by ν . By the assumption that A and B are pure we have the following cases:

- (1) the units do not occur in C , D and E ; hence we are already done,
- (2) one of C , D or E is \top ; we are done by the following commutative diagram delivered by the second monoidal structure (here we assume $C = \top$ and we proceed analogously when $D = \top$ or $E = \top$),

$$\begin{array}{ccc} \top \wedge (D \wedge E) & \xrightarrow{\hat{b}^{\rightarrow}} & (\top \wedge D) \wedge E \\ \searrow \hat{\sigma}^{\rightarrow} & & \nearrow \hat{\sigma}^{\leftarrow} \wedge \mathbf{1} \\ & D \wedge E & \end{array}$$

- (3) $C = D = E = \perp$; we are done by the commutative diagram (8).

The situation is quite similar with the other b and c -terms.

If f is a c^k -term then again by naturality we may assume that all the indices of the head of f are reduced by ν . It is not possible that only one of its indices is reduced to \perp or to \top since then A or B is not pure. If two of its indices are \perp or \top while the units do not occur in the remaining two indices, then by the assumption that A and B are pure, we may apply the equations (5) or (6) to eliminate this c^k . Situation is analogous when three indices of c^k are \perp or \top and the forth is not. If all the indices of c^k are \perp or \top then we have two cases: either we apply the equations (5) or (6) to eliminate c^k , or we apply the equation (11) to reduce c^k to κ and we deal with the new occurrence of κ as in the following last case for f .

If f is a κ -term. Since A and B are pure, f is not just κ , so the head of f is in the immediate scope of \vee or \wedge . If $\mathbf{1}_E \vee \kappa$ is a subterm of f , then since A and B are pure, no letter occurs in E and again we may assume that E is already reduced by ν to \perp or \top . If E is \top then we use the equation (9) to eliminate κ . If E is \perp then we apply the naturality equation $\mathbf{1}_{\perp} \vee \kappa = \hat{\sigma}_{\top}^{\leftarrow} \circ \kappa \circ \hat{\sigma}_{\perp}^{\rightarrow}$, mentioned in the proof of Lemma 3.2. This equation does not eliminate κ but it replaces a κ -term of a greater complexity by a κ -term of lower complexity and by induction κ will be eliminated.

We proceed analogously in all the other possible cases for a κ term f relying on equations (9), (10) or the remaining naturality conditions mentioned in the proof of Lemma 3.2. \dashv

In order to complete the formulation of our coherence result, we have to define graphs corresponding to the arrow terms of \mathcal{M} . However, we avoid the graphs and give an equivalent formulation of the result by introducing the following notion. We say that an object of \mathcal{M} is *diversified* if every letter occurs in it at most once. So, our coherence for *SMI* categories (called Restricted Symmetric Bimonoidal Intermuting Coherence in [3]) is the following:

THEOREM 3.1. *If A and B are either pure and diversified or no letter occurs in them, then there is at most one arrow $f: A \rightarrow B$ in \mathcal{M} .*

One part of the theorem is established by Lemma 3.4. By Lemma 3.6 we have

reduced the rest of the theorem to the case when the units do not occur in A and B and f and g are defined by b , c and c^k . So, to complete the proof of Theorem 3.1 it is sufficient to prove a coherence result for categories like *SMI* categories but without units, which we call as in [3], *symmetric biassociative intermuting (SAI)* categories. The canonical structure of *SAI* categories is given by two biendofunctors \vee and \wedge , natural isomorphisms given by associativities b and symmetries c that satisfy Mac Lane's pentagonal and hexagonal conditions, and a natural transformation c^k satisfying the coherence conditions given by the diagrams (1), (2), (3) and (4).

This coherence result is formulated in terms of the category \mathcal{A} which is freely generated *SAI* category by the same set \mathcal{P} of generators as \mathcal{M} . The construction of \mathcal{A} is analogous to the construction of \mathcal{M} given at the beginning of this section. So, our auxiliary coherence result is the following:

THEOREM 3.2. *If A and B are diversified, then there is at most one arrow $f: A \rightarrow B$ in \mathcal{A} .*

The following two sections contain a proof of this theorem.

4 A note on strictification

In order to provide an easier record of equations of arrow terms in the proof of Theorem 3.2 we will replace our category \mathcal{A} by a symmetric biassociative intermuting category in which associativity and symmetry arrows are identities. Strictification under associativity is a standard procedure in coherence results. For example, this is how Mac Lane reduced his proof of symmetric monoidal coherence in [9] to the standard presentation of symmetric groups by generators and relations. However, strictification under symmetry is not so standard and it may cause a suspicion. (A reference where it is used implicitly is [7].) Although various strictifications, including this with respect to symmetry, are thoroughly investigated in [2], Chapter 3 and §§4.7, 7.6-8, 8.4, we briefly pass through such a strictification of our category \mathcal{A} .

Note first that if we factor the arrow terms of \mathcal{A} by the new equations

$$\check{c}_{A,A} = \mathbf{1}_{A \vee A}, \quad \hat{c}_{A,A} = \mathbf{1}_{A \wedge A}$$

obtaining a new category \mathcal{A}' with the same objects as \mathcal{A} , the full subcategories of \mathcal{A} and \mathcal{A}' on diversified objects are the same. This is because we can easily establish that for every pair of arrow terms $f, g: A \rightarrow B$, if $f = g$ in \mathcal{A}' and $f \neq g$ in \mathcal{A} , then A and B are not diversified. Since the objects A and B are diversified in Theorem 3.2, we can replace the category \mathcal{A} in the formulation of that theorem by the category \mathcal{A}' without losing its strength. We use this fact later on.

Let the arrow terms defined by associativities b and symmetries c (c.f. the beginning of the preceding section) be called *S-terms*. Then we have the following result from [2], §6.5.

LEMMA 4.1. *Every diagram of S -terms commutes in \mathcal{A}' .*

This fact together with the property that every S -term represents an isomorphism of \mathcal{A}' is sufficient for our strictification of \mathcal{A}' with respect to its associative and symmetric structures. Roughly speaking, we can further factor the arrow terms so that associativity and symmetry natural transformations become identity natural transformations. Of, course, this makes some identifications among the objects of \mathcal{A}' too.

We define a relation \equiv on the set of objects of \mathcal{A}' (which are the same as the objects of \mathcal{A}) in the following way. Let $A \equiv B$ iff there is an S -term $f: A \rightarrow B$. Since $\mathbf{1}_A$ is an S -term, every S -term represents an isomorphism whose inverse may be represented by an S -term, and the composition of two S -terms is an S -term, we have that \equiv is an equivalence relation. Let $\llbracket A \rrbracket$ denotes the equivalence class with respect to \equiv of an object A of \mathcal{A}' . We can also denote (not in a unique way) the equivalence class $\llbracket A \rrbracket$ by deleting from the formula A parenthesis tied to \vee in the immediate scope of another \vee and the same for \wedge . For example, the equivalence class $\llbracket (p \wedge q) \wedge ((p \vee r) \vee p) \rrbracket$ is denoted by $p \wedge q \wedge (p \vee r \vee p)$, and the same equivalence class may be denoted by $q \wedge (r \vee p \vee p) \wedge p$ or by $(p \vee p \vee r) \wedge p \wedge q$, etc. We call such equivalence classes of formulas *form multisets* (see [2], §7.7), in particular, when A is diversified we call $\llbracket A \rrbracket$ a *form set*. We use S, T, U, V, W, X, Y and Z , possible with indices, for form multisets and form sets.

Note that if $A_1 \equiv A_2$ and $B_1 \equiv B_2$, then $A_1 \vee B_1 \equiv A_2 \vee B_2$ and $A_1 \wedge B_1 \equiv A_2 \wedge B_2$, hence we may define the operations \vee and \wedge on form multisets as

$$\llbracket A \rrbracket \vee \llbracket B \rrbracket =_{df} \llbracket A \vee B \rrbracket, \quad \llbracket A \rrbracket \wedge \llbracket B \rrbracket =_{df} \llbracket A \wedge B \rrbracket.$$

Let \mathcal{A}^{st} be a category built out of syntactical material, starting from the same set \mathcal{P} of generators as in the case of \mathcal{M} , \mathcal{A} and \mathcal{A}' , whose objects are the form multisets. The only *primitive arrow terms* of \mathcal{A}^{st} are of the form

$$\mathbf{1}_S: S \rightarrow S, \quad \text{or} \quad c_{[S,T,U,V]}^k: (S \wedge T) \vee (U \wedge V) \rightarrow (S \vee U) \wedge (T \vee V),$$

where $[S, T, U, V]$ is an abbreviation for the set $\{\{\{S, T\}, \{U, V\}\}, \{S, U\}, \{T, V\}\}$. (It is straightforward to check that $[S, T, U, V] = [W, X, Y, Z]$ iff $(W, X, Y, Z) \in \{(S, T, U, V), (T, S, V, U), (U, V, S, T), (V, U, T, S)\}$.)

Hence, $c_{[S,T,U,V]}^k, c_{[T,S,V,U]}^k, c_{[U,V,S,T]}^k, c_{[V,U,T,S]}^k$ are the same primitive arrow term which prevents us for having many primitive arrow terms representing the same arrow of \mathcal{A}^{st} . Moreover, the strictified versions of the equations (3) and (4) are now incorporated in our notation, and when we draw the arrow $c^k: (S \wedge T) \vee (U \wedge V) \rightarrow (S \vee U) \wedge (T \vee V)$ in a diagram, one can form the index of c^k in a unique way.

The *arrows* of \mathcal{A}^{st} are equivalence classes of *arrow terms* generated from primitive arrow terms with the help of \circ, \vee and \wedge . These equivalence classes are taken with respect to the smallest equivalence relation on arrow terms which makes out of \mathcal{A}^{st} a strict associative and strict symmetric *SAI* category. So, this equivalence relation is congruent with respect to \circ, \vee and \wedge , and it captures the assumptions that \vee and \wedge are biendofunctors, the following equations

$$\begin{aligned}
s \vee (t \vee u) &= (s \vee t) \vee u, & s \wedge (t \wedge u) &= (s \wedge t) \wedge u, \\
s \vee t &= t \vee s, & s \wedge t &= t \wedge s,
\end{aligned}$$

which are the rudiments of naturality conditions for associativity and symmetry, naturality of c^k , and the equations brought by the following commutative diagrams:

$$\begin{array}{ccc}
& (U \wedge V \wedge W) \vee (X \wedge Y \wedge Z) & \\
c^k \swarrow & & \searrow c^k \\
(U \vee X) \wedge ((V \wedge W) \vee (Y \wedge Z)) & (1s) & ((U \wedge V) \vee (X \wedge Y)) \wedge (W \vee Z) \\
\downarrow c^k & & \downarrow c^k \\
& (U \vee X) \wedge (V \vee Y) \wedge (W \vee Z) &
\end{array}$$

$$\begin{array}{ccc}
& (U \vee V \vee W) \wedge (X \vee Y \vee Z) & \\
c^k \swarrow & & \searrow c^k \\
(U \wedge X) \vee ((V \vee W) \wedge (Y \vee Z)) & (2s) & ((U \vee V) \wedge (X \vee Y)) \vee (W \wedge Z) \\
\downarrow c^k & & \downarrow c^k \\
& (U \wedge X) \vee (V \wedge Y) \vee (W \wedge Z) &
\end{array}$$

This concludes the definition of \mathcal{A}^{st} .

The categories \mathcal{A}' and \mathcal{A}^{st} are equivalent via functors that preserve the *SAI* structure. Here is just a sketch of the proof. We define two functors $H_{\mathcal{G}}: \mathcal{A}' \rightarrow \mathcal{A}^{st}$ and $H: \mathcal{A}^{st} \rightarrow \mathcal{A}'$ in the following way. Let $H_{\mathcal{G}}A =_{df} \llbracket A \rrbracket$, and let $H_{\mathcal{G}}f$ be obtained from the arrow term f by replacing every S -term in it by $\mathbf{1}$ indexed by the equivalence class of the source and the target of this term, and by replacing every $c^k_{A,B,C,D}$ in it by $c^k_{\llbracket A \rrbracket, \llbracket B \rrbracket, \llbracket C \rrbracket, \llbracket D \rrbracket}$. It is not difficult to verify that $H_{\mathcal{G}}$ is indeed a functor, i.e. that if $f = g$ in \mathcal{A}' then $H_{\mathcal{G}}f = H_{\mathcal{G}}g$ in \mathcal{A}^{st} .

On the other hand, to define $H: \mathcal{A}^{st} \rightarrow \mathcal{A}'$ we have first to choose a formula A_H in each equivalence class $\llbracket A \rrbracket$. By Lemma 4.1, there is a unique arrow $\varphi_A: A_H \rightarrow A$ of \mathcal{A}' represented by an S -term. We define

$$H\llbracket A \rrbracket =_{df} A_H,$$

$$H\mathbf{1}_S =_{df} \mathbf{1}_{HS}, \quad H(t \circ s) =_{df} Ht \circ Hs,$$

$$Hc^k_{[S,T,U,V]} =_{df} \varphi_{(HS \vee HU) \wedge (HT \vee HV)}^{-1} \circ c^k_{HS, HT, HU, HV} \circ \varphi_{(HS \wedge HT) \vee (HU \wedge HV)},$$

$$H(s \vee t) = \varphi_{HS_2 \vee HT_2}^{-1} \circ (Hs \vee Ht) \circ \varphi_{HS_1 \vee HT_1}, \quad \text{for } s: S_1 \rightarrow S_2, t: T_1 \rightarrow T_2,$$

and the same for \vee replaced by \wedge .

It can be easily checked that this definition is correct and that so defined H is indeed a functor. It is straightforward that $H_G \circ H$ is the identity functor on \mathcal{A}^{st} and one can verify that φ , defined as above, is a natural isomorphism from $H \circ H_G$ to the identity functor on \mathcal{A}' . (Details of the proof, but in more general context, are given in [2], §3.2.) Hence, \mathcal{A}' and \mathcal{A}^{st} are equivalent via H_G and H . Following the terminology of [2], functor H_G strictly preserves *SAI* structure and H is just strong with respect to this structure.

As a consequence of this equivalence and the fact that \mathcal{A} and \mathcal{A}' have the same full subcategories on diversified objects, we have that the following coherence result is sufficient for Theorem 3.2.

PROPOSITION. *If X and Y are form sets, then there is at most one arrow $t: X \rightarrow Y$ in \mathcal{A}^{st} .*

As we said at the beginning of this section, the strictification of \mathcal{A} enables us to record our derivations in the proof of the Proposition, and there are no other reasons, except these technical, for this step. Note that one can always decorate the arrow terms of \mathcal{A}^{st} (using the functor H) by lengthy compositions of S -terms to get back into a rather natural environment given by the category \mathcal{A} .

5 Proof of the Proposition

In this section we are interested only in form sets (i.e. the equivalence classes of diversified formulae) as objects of \mathcal{A}^{st} . We are going to establish a normalization procedure for arrow terms of \mathcal{A}^{st} that eventually delivers our coherence result. For this we use a sequence of definitions and lemmata. We say that a form set S is a *subformset* of a form set T if there is a formula A in S and a formula B in T (S and T are equivalence classes) such that A is a subformula of B . For example $p \wedge (q \vee r)$ is a subformset of $(r \vee q) \wedge s \wedge p$. We use freely for form sets the terminology which is standard for formulae and say, for example, that $r \vee q$ and $(r \vee q) \wedge p$ are *conjuncts* of the form set $(r \vee q) \wedge s \wedge p$ whose *main connective* is \wedge . We say that a conjunct X of a form set is *prime* if \wedge is not the main connective in X . For example $r \vee q$ is a prime conjunct of $(r \vee q) \wedge s \wedge p$ but $(r \vee q) \wedge p$ is not. Also when \wedge is not the main connective of a form set, we treat this form set as the prime conjunct of itself. We use the same conventions for \vee and, for example, $(r \vee q) \wedge s \wedge p$ is the prime disjunct of itself. We denote by $let(X)$ the set of letters in a form set X .

Every arrow term of \mathcal{A}^{st} is equal to a *developed* arrow term of the form

$$s_n \circ \dots \circ s_1 \circ \mathbf{1}$$

where every s_i (if there is any) is a c^k -term. We tacitly use developed form of arrow terms throughout the proofs of lemmata given below. We take over the following lemma from [3].

LEMMA 5.1 ([3], SECTION 14, LEMMA 1). *If $u: X \rightarrow Y$ is an arrow of \mathcal{A}^{st} , and P is a set of letters such that for every subformset $U \wedge V$ of X*

$$let(U) \subseteq P \quad \text{iff} \quad let(V) \subseteq P,$$

then this equivalence holds for every subformset $U \wedge V$ of Y .

As a corollary (taking $P = let(X_1)$) we have the following:

LEMMA 5.2. *If $u: X_1 \vee X_2 \rightarrow Y$ is an arrow of \mathcal{A}^{st} , then for every subformset $U \wedge V$ of Y we have that*

$$let(U) \subseteq let(X_1) \quad \text{iff} \quad let(V) \subseteq let(X_1).$$

(Since $X_1 \vee X_2$ is the same form set as $X_2 \vee X_1$, it is not necessary to mention that the same holds when we replace X_1 by X_2 in the conclusion of this lemma.) By induction on the complexity of a developed arrow term we can easily show:

LEMMA 5.3. *Every arrow term $t: X' \wedge X'' \rightarrow Y$ of \mathcal{A}^{st} is equal to $t' \wedge t''$ for some arrow terms $t': X' \rightarrow Y'$ and $t'': X'' \rightarrow Y''$.*

Since the primitive equations of \mathcal{A} and \mathcal{A}^{st} are such that the number of occurrences of c^k is the same on the both sides, we have:

LEMMA 5.4. *All the arrow terms representing the same arrow of \mathcal{A} or \mathcal{A}^{st} have the same number of occurrences of c^k .*

We introduce now a procedure of deleting letters from form sets. Roughly speaking, to delete a letter p from a form set (which includes some other letters) means to take a formula in this form set, delete the letter p together with its connective and associated brackets from this formula, and then form its equivalence class. It is not difficult to see that this does not depend on the choice of the formula in a form set. In terms of our notation for form sets we define X^{-p} , for a form set X different from p , in the following way:

if p is not in X , then X^{-p} is X ;

if X is of the form $Y \vee p$ or $Y \wedge p$, then X^{-p} is Y ;

if X is of the form $Y \vee Z$ for Y and Z different from p , then X^{-p} is $Y^{-p} \vee Z^{-p}$, and the same holds when we replace \vee by \wedge .

If $let(X) \setminus \{p, q\} \neq \emptyset$ then it is easy to see that

$$(X^{-p})^{-q} = (X^{-q})^{-p},$$

and we can define, for a finite set $P = \{p_1, \dots, p_n\}$ of letters such that $let(X) \setminus P \neq \emptyset$,

$$X^{-P} =_{df} (\dots (X^{-p_1})^{-p_2} \dots)^{-p_n}.$$

This can be extended to a procedure of letter deletion from the arrow terms of \mathcal{A}^{st} .

Let $u: X \rightarrow Y$ be an arrow term of \mathcal{A}^{st} , and let P be a finite set of letters such that $\text{let}(X) \setminus P \neq \emptyset$ (hence $\text{let}(Y) \setminus P \neq \emptyset$, since $\text{let}(Y) = \text{let}(X)$) and such that, as in Lemma 5.1, for every subformset $U \wedge V$ of X we have $\text{let}(U) \subseteq P$ iff $\text{let}(V) \subseteq P$. We define inductively the arrow term $u^{-P}: X^{-P} \rightarrow Y^{-P}$ in the following way:

if u is $\mathbf{1}_X$, then u^{-P} is $\mathbf{1}_{X^{-P}}$;

if u is $c_{[S,T,U,V]}^k$ then

$$u^{-P} =_{df} \begin{cases} \mathbf{1}_{X^{-P}}, & \text{when } \text{let}(S \wedge T) \subseteq P \text{ or } \text{let}(U \wedge V) \subseteq P \\ c_{[S^{-P}, T^{-P}, U^{-P}, V^{-P}]}^k, & \text{otherwise;} \end{cases}$$

if u is $s \vee t$ for $s: S_1 \rightarrow S_2$ and $t: T_1 \rightarrow T_2$, then

$$u^{-P} =_{df} \begin{cases} s^{-P}, & \text{let}(T_1) \subseteq P \\ t^{-P}, & \text{let}(S_1) \subseteq P \\ s^{-P} \vee t^{-P}, & \text{otherwise;} \end{cases}$$

and we have the same clause when we replace \vee by \wedge ;

if u is $u_2 \circ u_1$, then by Lemma 5.1, both u_1^{-P} and u_2^{-P} are defined and u^{-P} is $u_2^{-P} \circ u_1^{-P}$.

Let X_1 and X_2 be form sets. We say that $c_{[S,T,U,V]}^k$ is (X_1, X_2) -*splitting* when one of $\text{let}(S \wedge T)$, $\text{let}(U \wedge V)$ is a subset of $\text{let}(X_1)$ while the other is a subset of $\text{let}(X_2)$. We say that an arrow term of \mathcal{A}^{st} is (X_1, X_2) -*splitting* when every occurrence of c^k in it is (X_1, X_2) -splitting, and we say that it is (X_1, X_2) -*nonsplitting* when every occurrence of c^k in it is *not* (X_1, X_2) -splitting. For example, $(c_{[p,q,s,t]}^k \wedge \mathbf{1}_{r \vee u}) \circ c_{[p \wedge q, r, s \wedge t, u]}^k$ is a $(p \wedge q \wedge r, s \wedge t \wedge u)$ -splitting arrow term.

One can easily check that if $f = g$ and f is (X_1, X_2) -splitting, then g is (X_1, X_2) -splitting, too. This is not the case when we replace “splitting” by “nonsplitting”. (Take for example the diagram (1s) of the preceding section and let X_1 be $U \wedge X$ and X_2 be $V \wedge Y$, then the left leg of this diagram is (X_1, X_2) -nonsplitting, and the occurrence of c^k in $c^k \vee \mathbf{1}$, in the right leg is (X_1, X_2) -splitting.) It is clear that every (X_1, X_2) -splitting arrow term is equal to a developed (X_1, X_2) -splitting arrow term, and analogously with “splitting” replaced by “nonsplitting”. We take over the following three lemmata from [3].

LEMMA 5.5 ([3], SECTION 14, LEMMA 5). *If $u: X_1 \vee X_2 \rightarrow Y$ is (X_1, X_2) -nonsplitting, then u is equal to $u_1 \vee u_2$ for some arrow terms $u_1: X_1 \rightarrow X'_1$ and $u_2: X_2 \rightarrow X'_2$.*

Note that for X_1, X_2, X'_1 and X'_2 as in Lemma 5.5, an arrow term is (X_1, X_2) -splitting if and only if it is (X'_1, X'_2) -splitting, which we will use later on.

LEMMA 5.6 ([3], SECTION 14, LEMMA 6). *If $u: X_1 \vee X_2 \rightarrow Y$ is (X_1, X_2) -splitting, then Y^{-X_1} is X_2 and Y^{-X_2} is X_1 .*

LEMMA 5.7 ([3], SECTION 14, LEMMA 7). *If $u: X_1 \vee X_2 \rightarrow Y_1 \wedge Y_2$ is (X_1, X_2) -splitting, then the main connective in X_1 and X_2 is \wedge .*

Let $X_1 = S \wedge T$ and $X_2 = U \wedge V$ and let $u \circ c_{[S,T,U,V]}^k : X_1 \vee X_2 \rightarrow Y$ be (X_1, X_2) -splitting. By Lemma 5.3, the main connective in Y is \wedge and by Lemma 5.2, the deletions $-^{X_1}$ and $-^{X_2}$ are defined for every conjunct of Y . By Lemma 5.6, we have $Y^{-X_2} = X_1 = S \wedge T$ and hence Y is of the form $Y_S \wedge Y_T$ for Y_S and Y_T such that $Y_S^{-X_2} = S$ and $Y_T^{-X_2} = T$. Analogously, since $Y^{-X_1} = X_2 = U \wedge V$, we have $Y = Y_U \wedge Y_V$ for $Y_U^{-X_1} = U$ and $Y_V^{-X_1} = V$. We can then prove the following.

LEMMA 5.8. *For $u \circ c_{[S,T,U,V]}^k$ as above, we have $Y_S = Y_U$ and $Y_T = Y_V$.*

PROOF. Suppose $Y_S = Y_U \wedge Z$, and hence, $Y_V = Y_T \wedge Z$. We have

$$c_{[S,T,U,V]}^k : (S \wedge Y_T^{-X_2}) \vee (U \wedge Y_T^{-X_1} \wedge Z^{-X_1}) \rightarrow (S \vee U) \wedge (Y_T^{-X_2} \vee (Y_T^{-X_1} \wedge Z^{-X_1})).$$

By Lemma 5.3, u is of the form $s \wedge t$ for $t : Y_T^{-X_2} \vee (Y_T^{-X_1} \wedge Z^{-X_1}) \rightarrow W$, where W is a conjunct of Y . Since the source and the target of t share the same letters we have that $\text{let}(W) = \text{let}(Y_T) \cup \text{let}(Z^{-X_1})$. Hence W is of the form $Y_T \wedge W'$ for W' such that $\text{let}(W') = \text{let}(Z^{-X_1}) \subseteq X_2$. Since W' is a conjunct of Y , by Lemma 5.2, we have $\text{let}(Y) \subseteq X_2$ which means that $\text{let}(X_1) = \emptyset$, i.e. a contradiction. We proceed in the other cases quite similar. \dashv

In the sequel, for $u \circ c_{[S,T,U,V]}^k$ as above, we denote by Y_{SU} both Y_S and Y_U (which are equal by the preceding lemma), and by the same reasons we denote by Y_{TV} both Y_T and Y_V .

LEMMA 5.9. *If $u: X_1 \vee X_2 \rightarrow Y_1 \wedge Y_2$ is (X_1, X_2) -splitting, then u factors as:*

$$\begin{array}{ccc} (Y_1^{-X_2} \wedge Y_2^{-X_2}) \vee (Y_1^{-X_1} \wedge Y_2^{-X_1}) & \xrightarrow{u} & Y_1 \wedge Y_2 \\ \searrow c^k & & \nearrow u_1 \wedge u_2 \\ & (Y_1^{-X_2} \vee Y_1^{-X_1}) \wedge (Y_2^{-X_2} \vee Y_2^{-X_1}) & \end{array}$$

where $u_1 : Y_1^{-X_2} \vee Y_1^{-X_1} \rightarrow Y_1$ is $(Y_1^{-X_2}, Y_1^{-X_1})$ -splitting and $u_2 : Y_2^{-X_2} \vee Y_2^{-X_1} \rightarrow Y_2$ is $(Y_2^{-X_2}, Y_2^{-X_1})$ -splitting.

PROOF. We proceed by induction on number $n \geq 1$ of occurrences of c^k in u . First we prepare a ground for this induction. By relying on the remark after the definition of (X_1, X_2) -splitting arrow term and on Lemma 5.7, u is equal to an arrow term of the form $v \circ c_{[S,T,U,V]}^k$ for $X_1 = S \wedge T$, $X_2 = U \wedge V$, and for v , which by Lemma 5.4 has $n - 1$ occurrences of c^k , being (X_1, X_2) -splitting.

If we denote $Y_1 \wedge Y_2$ by Y , then by Lemma 5.8 we have $Y = Y_{SU} \wedge Y_{TV}$ such that $Y_{SU}^{-X_2} = S$, $Y_{SU}^{-X_1} = U$, $Y_{TV}^{-X_2} = T$ and $Y_{TV}^{-X_1} = V$. There are several possibilities how the “partition” of Y given by the conjuncts Y_1 and Y_2 may be related to the one given by the conjuncts Y_{SU} and Y_{TV} , among which the following three cases make all the essentially different situations.

- (0) $\{Y_1, Y_2\} = \{Y_{SU}, Y_{TV}\}$, when we are done;
- (1) $Y_1 \wedge Z = Y_{SU}$ and $Y_2 = Y_{TV} \wedge Z$;
- (2) $Y_1 = Z_1 \wedge W_1$, $Y_2 = Z_2 \wedge W_2$, $Y_{SU} = Z_1 \wedge Z_2$ and $Y_{TV} = W_1 \wedge W_2$.

We can start now with the induction.

If $n = 1$, then $v = \mathbf{1}_{(S \vee U) \wedge (T \vee V)}$ and $\{S \vee U, T \vee V\} = \{Y_1, Y_2\}$. Hence we are in case (0) and we are done.

If $n > 1$, then if we are in case (0), then we are done again. If we are in case (1), then by Lemma 5.3 we have $v = s \wedge t$ for (X_1, X_2) -splitting arrow terms $s: S \vee U \rightarrow Y_1 \wedge Z$ and $t: T \vee V \rightarrow Y_{TV}$ with less than n occurrences of c^k in them. Since $\text{let}(S) \subseteq \text{let}(X_1)$ and $\text{let}(U) \subseteq \text{let}(X_2)$, we have that s is (S, U) -splitting and we may apply the induction hypothesis in order to obtain that s is equal to an (S, U) -splitting (and hence (X_1, X_2) -splitting) arrow term of the form

$$(Y_1^{-U} \wedge Z^{-U}) \vee (Y_1^{-S} \wedge Z^{-S}) \xrightarrow{c^k} (Y_1^{-U} \vee Y_1^{-S}) \wedge (Z^{-U} \vee Z^{-S}) \xrightarrow{s_1 \wedge s_2} Y_1 \wedge Z.$$

Consider the following commutative diagram whose upper part is an instance of (1s):

$$\begin{array}{ccc}
& (Y_1^{-U} \wedge Z^{-U} \wedge T) \vee (Y_1^{-S} \wedge Z^{-S} \wedge V) = X_1 \vee X_2 & \\
& \swarrow c^k & \searrow c^k \\
((Y_1^{-U} \wedge Z^{-U}) \vee (Y_1^{-S} \wedge Z^{-S})) \wedge (T \vee V) & & (Y_1^{-U} \vee Y_1^{-S}) \wedge ((Z^{-U} \wedge T) \vee (Z^{-S} \wedge V)) \\
\swarrow c^k \wedge \mathbf{1} & & \searrow \mathbf{1} \wedge c^k \\
(Y_1^{-U} \vee Y_1^{-S}) \wedge (Z^{-U} \vee Z^{-S}) \wedge (T \vee V) & & \\
\downarrow s_1 \wedge s_2 \wedge t & & \\
Y_1 \wedge Z \wedge Y_{TV} = Y_1 \wedge Y_2 & &
\end{array}$$

Since $Y_1^{-U} \wedge Z^{-U} = S$ and $Y_1^{-S} \wedge Z^{-S} = U$, the left leg of this diagram is equal to u . Also, we have $Y_1^{-U} = Y_1^{-X_2}$, $Z^{-U} = Z^{-X_2}$ (hence $Z^{-U} \wedge T = Y_2^{-X_2}$), $Y_1^{-S} = Y_1^{-X_1}$, and $Z^{-S} = Z^{-X_1}$ (hence $Z^{-S} \wedge V = Y_2^{-X_1}$). So, the right leg of this diagram is in the desired form.

If we are in case (2), then we use the induction hypothesis twice and appeal to an instance of the following commutative diagram of \mathcal{A}^{st} obtained by pasting instances of (1s):

$$\begin{array}{ccc}
& (S \wedge T \wedge U \wedge V) \vee (W \wedge X \wedge Y \wedge Z) & \\
& \swarrow c^k & \searrow c^k \\
((S \wedge T) \vee (W \wedge X)) \wedge ((U \wedge V) \vee (Y \wedge Z)) & & ((S \wedge U) \vee (W \wedge Y)) \wedge ((T \wedge V) \vee (X \wedge Z)) \\
\swarrow c^k \wedge c^k & & \searrow c^k \wedge c^k \\
(S \vee W) \wedge (T \vee X) \wedge (U \vee Y) \wedge (V \vee Z) & & \vdash
\end{array}$$

We have also the following three lemmata.

LEMMA 5.10. *If $u: X_1 \vee X_2 \rightarrow Y' \vee Y''$ is (X_1, X_2) -splitting and Y' is a prime disjunct of Y (i.e. \vee is not the main connective in Y'), then $u = u' \vee u''$ for $u': X' \rightarrow Y'$, where either*

for $i = 1$ or $i = 2$, X' is a prime disjunct of X_i and $u' = \mathbf{1}_{X'}$, or

$X' = X'_1 \vee X'_2$ for X'_1 and X'_2 being prime disjuncts of X_1 and X_2 respectively and u' is (X'_1, X'_2) -splitting.

PROOF. By the dual of Lemma 5.3, u is equal to $u' \vee u''$, for u' having Y' as the target. If the source X' of u' is a disjunct of X_1 , by the assumption that u is (X_1, X_2) -splitting there are no occurrences of c^k in u' and hence it must be $\mathbf{1}_{Y'}$, and X' , which is equal to Y' , must be a prime disjunct of X_1 .

If the source X' of u' is of the form $X'_1 \vee X'_2$ for X'_1 a disjunct of X_1 and X'_2 a disjunct of X_2 , then Y' cannot be a letter and hence its main connective is \wedge . Also, u' is (X'_1, X'_2) -splitting and by Lemma 5.7, X'_1 and X'_2 are prime disjuncts of X_1 and X_2 , respectively. \dashv

LEMMA 5.11. *Let $v \circ u: X_1 \vee X_2 \rightarrow Z$ be such that v is a c^k -term that is not (X_1, X_2) -splitting and u is an (X_1, X_2) -splitting arrow term. Then there exist an arrow term w and a c^k -term v' , which is not (X_1, X_2) -splitting, such that $v \circ u = w \circ v'$.*

PROOF. Let $X_1 \vee X_2 \xrightarrow{u} Y \xrightarrow{v} Z$, and let $c^k_{[S,T,U,V]}$ be the head of v . We proceed by induction on “depth” of $(S \wedge T) \vee (U \wedge V)$ in Y .

For the base of this induction we have the case when $S \wedge T$ and $U \wedge V$ are prime disjuncts of Y . If $Y = (S \wedge T) \vee (U \wedge V) \vee Y'''$, then by Lemma 5.10 we have

$$v \circ u = (c^k_{[S,T,U,V]} \circ (u' \vee u'')) \vee u'''$$

for $u': X' \rightarrow S \wedge T$ and $u'': X'' \rightarrow U \wedge V$ satisfying the conditions given by that lemma. (The arrow term $u''': X''' \rightarrow Y'''$ is out of our interest and it does not exist when $Y = (S \wedge T) \vee (U \wedge V)$.)

We have several different situations depending on whether X' or X'' are prime disjuncts of X_1 or of X_2 or they are of the form $X'_1 \vee X'_2$ or $X''_1 \vee X''_2$ for X'_1, X''_1 prime disjuncts of X_1 and X'_2, X''_2 prime disjuncts of X_2 . The following three cases represent essentially different situations:

(0) For $i = 1$ or $i = 2$, X' and X'' are prime disjuncts of X_i . (By the assumption that v is not (X_1, X_2) -splitting, X' and X'' cannot be prime disjuncts one of X_1 and the other of X_2 .) By Lemma 5.10, u' and u'' are identities and we are done.

(1) $X' = X'_1 \vee X'_2$ for X'_1 and X'_2 being prime disjuncts of X_1 and X_2 respectively, and X'' is a prime disjunct of X_1 . By Lemma 5.10, $u'' = \mathbf{1}_{X''} = \mathbf{1}_{U \wedge V}$ and we may apply Lemma 5.9 to $u': X'_1 \vee X'_2 \rightarrow S \wedge T$ which is (X'_1, X'_2) -splitting. So, $c^k_{[S,T,U,V]} \circ (u' \vee u'')$ is equal to the left leg of the following commutative dia-

gram whose upper part is an instance of (2s) and whose lower part is a naturality diagram for c^k .

$$\begin{array}{c}
(S^{-X'_2} \wedge T^{-X'_2}) \vee (S^{-X'_1} \wedge T^{-X'_1}) \vee (U \wedge V) \\
\swarrow c^k \vee \mathbf{1} \quad \searrow \mathbf{1} \vee c^k \\
((S^{-X'_2} \vee U) \wedge (T^{-X'_2} \vee V)) \vee (S^{-X'_1} \wedge T^{-X'_1}) \\
\swarrow c^k \quad \searrow c^k \\
((S^{-X'_2} \vee S^{-X'_1}) \wedge (T^{-X'_2} \vee T^{-X'_1})) \vee (U \wedge V) \quad (S^{-X'_2} \vee S^{-X'_1} \vee U) \wedge (T^{-X'_2} \vee T^{-X'_1} \vee V) \\
\downarrow (u'_1 \wedge u'_2) \vee \mathbf{1} \quad \downarrow (u'_1 \vee \mathbf{1}) \wedge (u'_2 \vee \mathbf{1}) \\
(S \wedge T) \vee (U \wedge V) \quad (S \vee U) \wedge (T \vee V) \\
\searrow c^k \quad \downarrow c^k \\
(S \vee U) \wedge (T \vee V)
\end{array}$$

The right leg of this diagram is of the desired form since it starts with $\mathbf{1} \vee c^k$ which is not (X_1, X_2) -splitting.

(2) $X' = X'_1 \vee X'_2$ and $X'' = X''_1 \vee X''_2$ for X'_1 and X''_1 being prime disjuncts of X_1 , and X'_2 and X''_2 being prime disjuncts of X_2 . Then we apply Lemma 5.9 to $u': X'_1 \vee X'_2 \rightarrow S \wedge T$ and to $u'': X''_1 \vee X''_2 \rightarrow U \wedge V$, and proceed as in case (1) relying on the following commutative diagram of \mathcal{A}^{st} obtained by pasting instances of (2s):

$$\begin{array}{c}
(S \wedge W) \vee (T \wedge X) \vee (U \wedge Y) \vee (V \wedge Z) \\
\swarrow c^k \vee c^k \quad \searrow c^k \vee c^k \\
((S \vee T) \wedge (W \vee X)) \vee ((U \vee V) \wedge (Y \vee Z)) \\
\swarrow c^k \quad \searrow c^k \\
(S \vee T \vee U \vee V) \wedge (W \vee X \vee Y \vee Z)
\end{array}$$

For the induction step, we proceed as follows. If Y is of the form $Y_1 \wedge Y_2$ where Y_1 is a prime conjunct of Y whose subformset is $(S \wedge T) \vee (U \wedge V)$, then, by Lemma 5.9, $v \circ u$ factors as:

$$\begin{array}{ccc}
(Y_1^{-X_2} \wedge Y_2^{-X_2}) \vee (Y_1^{-X_1} \wedge Y_2^{-X_1}) & \xrightarrow{v \circ u} & Z_1 \wedge Y_2 \\
\downarrow c^k & & \uparrow v_1 \wedge \mathbf{1}_{Y_2} \\
(Y_1^{-X_2} \vee Y_1^{-X_1}) \wedge (Y_2^{-X_2} \vee Y_2^{-X_1}) & \xrightarrow{u_1 \wedge u_2} & Y_1 \wedge Y_2
\end{array}$$

where u_1 is $(Y_1^{-X_2}, Y_1^{-X_1})$ -splitting and v_1 is a c^k -term that is not (X_1, X_2) -splitting, and hence it is not $(Y_1^{-X_2}, Y_1^{-X_1})$ -splitting. By the induction hypothesis $v_1 \circ u_1$ is equal to an arrow term of the form $w_1 \circ v'_1$ for v'_1 a c^k -term that is

not $(Y_1^{-X_2}, Y_1^{-X_1})$ -splitting. By Lemma 5.5, we may assume that v'_1 is of the form $v''_1 \vee \mathbf{1}_{Y_1^{-X_1}}$ or $\mathbf{1}_{Y_1^{-X_2}} \vee v''_1$. In both cases we just apply the naturality of c^k and we are done.

If Y is of the form $Y' \vee Y''$ where Y' is the prime disjunct containing $(S \wedge T) \vee (U \wedge V)$ as a subformset, then by Lemma 5.10, $v \circ u$ is of the form

$$X' \vee X'' \xrightarrow{u' \vee u''} Y' \vee Y'' \xrightarrow{v' \vee \mathbf{1}_{Y''}} Z' \vee Y'',$$

and if for $i = 1$ or $i = 2$, X' is a prime disjunct of X_i , then u' is $\mathbf{1}_{X'}$ and we are done. If $X' = X'_1 \vee X'_2$, then we just apply the induction hypothesis to $v' \circ u'$. \dashv

LEMMA 5.12. *For every arrow term $t : X_1 \vee X_2 \rightarrow Y$ there are arrow terms $v_1 : X_1 \rightarrow Y^{-X_2}$, $v_2 : X_2 \rightarrow Y^{-X_1}$ and an (X_1, X_2) -splitting arrow term $u : Y^{-X_2} \vee Y^{-X_1} \rightarrow Y$ such that $t = u \circ (v_1 \vee v_2)$.*

PROOF. We proceed by induction on the number n of occurrences of c^k in t . If $n = 0$, then since identities are at the same time (X_1, X_2) -splitting and (X_1, X_2) -nonsplitting we are done.

For the induction step, take a developed arrow term equal to t . If every c^k -term in it is (X_1, X_2) -splitting, then by Lemma 5.6 we are done. Otherwise, by Lemma 5.11 (applied to this developed arrow term from its right-hand side end up to the rightmost c^k -term in it that is not (X_1, X_2) -splitting) we have $t = t' \circ v'$ where v' is not (X_1, X_2) -splitting c^k -term. By Lemma 5.5, $v' = v'_1 \vee v'_2$ for $v'_1 : X_1 \rightarrow X'_1$ and $v'_2 : X_2 \rightarrow X'_2$. By Lemma 5.4, t' has $n - 1$ occurrences of c^k and since $\text{let}(X_1) = \text{let}(X'_1)$ and $\text{let}(X_2) = \text{let}(X'_2)$, we may apply the induction hypothesis to it. \dashv

We conclude this section with the following proof.

PROOF OF THE PROPOSITION. Let $t : X \rightarrow Y$ be an arrow of \mathcal{A}^{st} . To prove that t is unique, we proceed by induction on the complexity of X and Y . If X is a letter p , then Y must be p too, and $t : p \rightarrow p$ must be $\mathbf{1}_p$.

If $X = X' \wedge X''$, then by Lemma 5.3 and the induction hypothesis, $t = t' \wedge t''$ for unique arrows $t' : X' \rightarrow Y'$ and $t'' : X'' \rightarrow Y''$. We reason analogously when $Y = Y' \vee Y''$.

Suppose $X = X_1 \vee X_2$ and $Y = Y_1 \wedge Y_2$. Then by Lemmata 5.12 and 5.9, and the induction hypothesis, t is equal to the following composition

$$X_1 \vee X_2 \xrightarrow{v_1 \vee v_2} Y^{-X_2} \vee Y^{-X_1} \xrightarrow{c^k} (Y_1^{-X_2} \vee Y_1^{-X_1}) \wedge (Y_2^{-X_2} \vee Y_2^{-X_1}) \xrightarrow{u_1 \wedge u_2} Y_1 \wedge Y_2$$

for unique arrows u_1, u_2, v_1 and v_2 . (Note that all the sources and the targets above are completely determined by X_1, X_2, Y_1 and Y_2 .) So, t is the unique arrow with the source X and the target Y . \dashv

6 A note on reduced bar construction

This section is optional. Its aim is to give an analysis of a reduced bar construction based on a monoid in a category whose monoidal structure is given by finite products. Such a reduced bar construction was used by Thomason in [13]. I believe this analysis is not new, but I couldn't find (or just couldn't recognize) a reference which covers it completely, especially in its graphical approach I intend to use.

Let Δ be algebraists simplicial category defined as in [10], VII.5, for whose arrows we take over the notation used in that book. Let Δ^+ be the topologists simplicial category which is the full subcategory of Δ with objects all nonempty ordinals. In order to use geometric dimension, the objects of Δ^+ are decreased by 1. So, Δ^+ has all finite ordinals as objects, and in this category the source of δ_i^n , for $n \geq 1$ and $0 \leq i \leq n$, is $n-1$ and the target is n , while the source of σ_i^n , for $n \geq 1$ and $0 \leq i \leq n-1$ is n and the target is $n-1$. When we speak of $(\Delta^+)^{op}$, then we denote its arrows $(\delta_i^n)^{op}: n \rightarrow n-1$ by d_i^n and $(\sigma_i^n)^{op}: n-1 \rightarrow n$ by s_i^n .

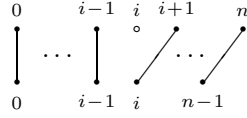
It is known (see, for example, [14], [4], Section 6, [5], Section 6) that the functor $\mathcal{J}: \Delta^{op} \rightarrow \Delta$ defined on objects as $\mathcal{J}(n) = n+1$, and on arrows by the clauses

$$\begin{aligned} \mathcal{J}(\delta_i^n)^{op} = \sigma_i^{n+1} \quad & \begin{array}{c} \begin{array}{ccccccc} 0 & & i-1 & i & i+1 & & n \\ \downarrow & \cdots & \downarrow & \circ & \searrow & \cdots & \searrow \\ 0 & & i-1 & i & n-1 & & n \end{array} & \mapsto & \begin{array}{c} \begin{array}{ccccccc} 0 & & & i & i+1 & & n+1 \\ \downarrow & \cdots & \downarrow & \swarrow & \swarrow & \cdots & \swarrow \\ 0 & & i & i+1 & n & & n+1 \end{array} \end{array} \\ \\ \mathcal{J}(\sigma_i^n)^{op} = \delta_{i+1}^{n+1} \quad & \begin{array}{c} \begin{array}{ccccccc} 0 & & i-1 & i & & n-1 & \\ \downarrow & \cdots & \downarrow & \swarrow & \cdots & \swarrow & \\ 0 & & i-1 & i & i+1 & & n \end{array} & \mapsto & \begin{array}{c} \begin{array}{ccccccc} 0 & & & i & & n & \\ \downarrow & \cdots & \downarrow & \searrow & \cdots & \searrow & \\ 0 & & i & i+1 & & n+1 & \end{array} \end{array} \end{aligned}$$

is faithful and obviously injective on objects. Intuitively, this functor is given by taking complements (pictures on the right-hand sides) of the standard graphical presentations for the arrows of Δ^{op} (pictures on the left-hand sides). So, we may regard of Δ^{op} as a subcategory of Δ . From now on we restrict \mathcal{J} to $(\Delta^+)^{op}$ taking into account that, this time, it is defined on objects by the clause $\mathcal{J}(n) = n+2$.

Let Δ_2 be the subcategory of Δ whose objects are finite ordinals greater or equal to 2 and whose arrows are the arrows of Δ , i.e. order-preserving functions, which preserve, moreover, the first and the last element. The category Δ_2 is the image of $(\Delta^+)^{op}$ under the functor \mathcal{J} . So, Δ_2 is isomorphic to $(\Delta^+)^{op}$ and in the sequel we will represent the arrows of $(\Delta^+)^{op}$ by the standard graphical presentations for the corresponding arrows of Δ_2 .

Let Δ_p be the category whose objects are again finite ordinals and whose arrows are order preserving partial functions. Beside the arrows δ_i^n and σ_i^n , to generate Δ_p we need also the arrows $\rho_i^n: n+1 \rightarrow n$ for $n \geq 0$ and $0 \leq i \leq n$, which are partial functions graphically presented as



The standard list of equations that satisfy δ 's and σ 's should be extended by the following equations:

$$\rho_j \rho_i = \rho_i \rho_{j+1} \quad i \leq j$$

$$\rho_j \delta_i = \begin{cases} \delta_{i-1} \rho_j & i > j \\ \mathbf{1} & i = j \\ \delta_i \rho_{j-1} & i < j \end{cases} \quad \rho_j \sigma_i = \begin{cases} \sigma_{i-1} \rho_j & i > j \\ \rho_i \rho_i & i = j \\ \sigma_i \rho_{j+1} & i < j \end{cases}$$

A counital monad $\langle T, \eta, \mu, \varepsilon \rangle$ in a category X consists of a functor $T: X \rightarrow X$ and three natural transformations

$$\eta: \mathcal{I}_X \rightarrow T, \quad \mu: T^2 \rightarrow T \quad \text{and} \quad \varepsilon: T \rightarrow \mathcal{I}_X$$

such that $\langle T, \eta, \mu \rangle$ is a monad in X , and moreover,

$$\varepsilon \circ \eta = \mathbf{1}_{\mathcal{I}_X}, \quad \varepsilon \circ \mu = \varepsilon \circ \varepsilon_T.$$

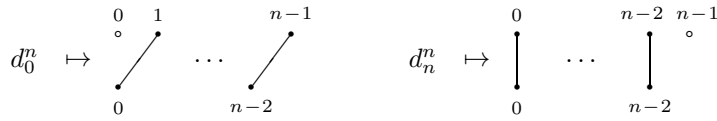
It is not difficult to see that Δ_p is a counital monad freely generated by a single object.

We have that Δ_2 is a subcategory of Δ_p and also we have a functor $\mathcal{H}: \Delta_2 \rightarrow \Delta_p$ defined on objects as $\mathcal{H}(n) = n-2$, and on arrows, for $f: n \rightarrow m$, as

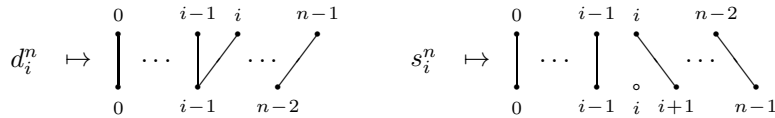
$$\mathcal{H}(f) = \rho_0^{m-2} \circ \rho_{m-1}^{m-1} \circ f \circ \delta_{n-1}^{n-1} \circ \delta_0^{n-2}.$$

(Intuitively, $\mathcal{H}(f)$ is obtained by omitting points 0, $n-1$ from the source, and 0, $m-1$ from the target in the graphical presentation of f together with all edges including them.) By using essentially the property that the arrows of Δ_2 may be built free of δ_0^n and δ_n^n , it is not difficult to check that \mathcal{H} so defined is indeed a functor. (Note that \mathcal{H} is not a functor from Δ to Δ_p .)

The composition $\mathcal{H} \circ \mathcal{J}$ is a functor from $(\Delta^+)^{op}$ to Δ_p which is identity on objects. In this way d_0^n and d_n^n are mapped to the partial functions graphically presented as



while d_i^n , for $0 < i < n$, and s_i^n are mapped to the functions graphically presented as



However, $\mathcal{H} \circ \mathcal{J}$ is not faithful. For example, $d_0^1: 1 \rightarrow 0$ and $d_1^1: 1 \rightarrow 0$ are both mapped to the empty partial function from 1 to 0. So, \mathcal{H} cannot be used for the constructions like, for example, the functor nerve is, where d_0^1 and d_1^1 should be mapped to the source and the target function, respectively.

Let now C be a monoid in a category \mathcal{K} whose monoidal structure is given by finite products. Then the functor $C \times _ : \mathcal{K} \rightarrow \mathcal{K}$ together with η and μ given by the unit and the multiplication of C and ε given by projections, form a counital monad. Since Δ_p is a counital monad freely generated by a single object, we have a functor $\mathcal{F}: \Delta_p \rightarrow \mathcal{K}$ such that $\mathcal{F}(0)$ is a terminal object I of \mathcal{K} and $\mathcal{F}(n+1)$ is $C \times \mathcal{F}(n)$. (If we regard Δ_p as a monoidal category with $+$ as a tensor, then this functor is not strictly monoidal if \mathcal{K} is not strict monoidal category, however, it is always strong monoidal functor.)

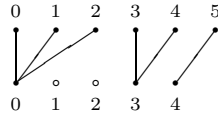
It is not difficult to see that $\overline{WC} = \mathcal{F} \circ \mathcal{H} \circ \mathcal{J}$ underlies the reduced bar construction of [13] when \mathcal{K} is strict monoidal. To avoid permanent decoration with associativity, and right and left identity isomorphisms of the monoidal structure of \mathcal{K} , we will always consider this monoidal structure to be strict, which is supported by the strictification given by [10], XI.3, Theorem 1. So we will write $\mathcal{F}(n) = C^n$ and in this case we have:

$$\overline{WC}(d_0^n) = pr_2: C \times C^{n-1} \rightarrow C^{n-1}, \quad \overline{WC}(d_n^n) = pr_1: C^{n-1} \times C \rightarrow C^{n-1},$$

and for $1 \leq i \leq n-1$ and $0 \leq j \leq n-1$,

$$\overline{WC}(d_i^n) = \mathbf{1}^{i-1} \times \mu \times \mathbf{1}^{n-i-1}, \quad \overline{WC}(s_j^n) = \mathbf{1}^j \times \eta \times \mathbf{1}^{n-j-1}.$$

In particular, we are interested in cases when \mathcal{K} is the category Cat (regarded again as strict monoidal) and when C is a strict monoidal category \mathcal{C} , hence a monoid in Cat . For example, if we take the arrow $f: 4 \rightarrow 3$ of $(\Delta^+)^{op}$ graphically presented as



then for $\overline{WC}: (\Delta^+)^{op} \rightarrow Cat$, we have that $\overline{WC}(f)$, denoted by $f^*: \mathcal{C}^4 \rightarrow \mathcal{C}^3$, is a functor such that

$$f^*(A, B, C, D) = (I, I, C \otimes D),$$

where I is the unit and \otimes is the tensor of the strict monoidal category \mathcal{C} . Note that one needs just a part of the cartesian structure of \mathcal{K} that provides “counits”, for such a reduced bar construction. The construction may work, for example, in any category with a bialgebra object.

7 Iterated reduced bar construction

Let now \mathcal{C} be an *SMI* category which is strict monoidal with respect to both \vee , \perp and \wedge , \top . For n, m such that $n+m \geq 1$ we define, along the lines of [1], a lax functor (c.f. [12]), which is an ordinary functor when $n+m = 1$

$$\overline{WC}_{n,m}: \underbrace{(\Delta^+)^{op} \times \dots \times (\Delta^+)^{op}}_{n+m} \rightarrow Cat.$$

It is defined on objects as $\overline{WC}_{n,m}(k_1, \dots, k_{n+m}) =_{df} \mathcal{C}^{k_1 \dots k_{n+m}}$, and for the arrows we have the following. First, for $1 \leq i \leq n+m$ and $f_i: k_i \rightarrow l_i$ an arrow of $(\Delta^+)^{op}$ let

$$\overline{WC}_{n,m}(\mathbf{1}_{k_1}, \dots, \mathbf{1}_{k_{i-1}}, f_i, \mathbf{1}_{k_{i+1}}, \dots, \mathbf{1}_{k_{n+m}}) =_{df} [\overline{WD}(f_i)]^{k_1 \dots k_{i-1}},$$

where \mathcal{D} is the category $\mathcal{C}^{k_{i+1} \dots k_{n+m}}$ whose monoidal structure is defined componentwise in terms of \vee, \perp when $i \leq n$ and in terms of \wedge, \top when $n < i \leq n+m$. With this in mind, we define $\overline{WC}_{n,m}(f_1, \dots, f_{n+m})$ to be the following composition:

$$\overline{WC}_{n,m}(\mathbf{1}_{l_1}, \dots, f_{n+m}) \circ \dots \circ \overline{WC}_{n,m}(f_1, \mathbf{1}_{k_2}, \dots, \mathbf{1}_{k_{n+m}}).$$

We call this construction of $\overline{WC}_{n,m}$, the (n, m) -reduced bar construction based on \mathcal{C} . (When $n+m = 0$ we may define it to be the functor mapping the object and the arrow of the trivial category $((\Delta^+)^{op})^0$ to \mathcal{C} and to the identity functor on \mathcal{C} , respectively.)

EXAMPLE Let $n = 2$, $m = 1$, and let $f = (f_1, f_2, f_3): (1, 2, 2) \rightarrow (2, 1, 2)$ be the arrow of $(\Delta^+)^{op} \times (\Delta^+)^{op} \times (\Delta^+)^{op}$ graphically presented as

$$\begin{array}{ccc} f_1 & f_2 & f_3 \\ \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & \\ \downarrow & \downarrow & \searrow & \\ 0 & 1 & 2 & 3 \end{array} \end{array} & \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \\ \downarrow & \downarrow & \swarrow & \\ 0 & 1 & 2 & \end{array} \end{array} & \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \\ \downarrow & \downarrow & \swarrow & \downarrow \\ 0 & 1 & 2 & 3 \end{array} \end{array} \end{array}$$

and $g = (g_1, g_2, g_3): (2, 1, 2) \rightarrow (2, 2, 2)$ be the arrow of $(\Delta^+)^{op} \times (\Delta^+)^{op} \times (\Delta^+)^{op}$ graphically presented as

$$\begin{array}{ccc} g_1 & g_2 & g_3 \\ \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \\ \downarrow & \downarrow & \swarrow & \downarrow \\ 0 & 1 & 2 & 3 \end{array} \end{array} & \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & \\ \downarrow & \downarrow & \searrow & \\ 0 & 1 & 2 & 3 \end{array} \end{array} & \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 1 & 2 & 3 \end{array} \end{array} \end{array}$$

With abbreviation h^* for $\overline{WC}_{2,1}(h)$, we have that f^* is the functor from $\mathcal{C}^{1 \cdot 2 \cdot 2}$ to $\mathcal{C}^{2 \cdot 1 \cdot 2}$ defined by

$$(A, B, C, D) \mapsto (A, B, C, D, \perp, \perp, \perp, \perp) \mapsto (A, B, \perp, \perp) \mapsto (A \wedge B, \top, \perp \wedge \perp, \top),$$

and g^* is the functor from $\mathcal{C}^{2 \cdot 1 \cdot 2}$ to $\mathcal{C}^{2 \cdot 2 \cdot 2}$ defined by

$$(A, B, C, D) \mapsto (A \vee C, B \vee D, \perp, \perp) \mapsto (A \vee C, B \vee D, \perp, \perp, \perp, \perp, \perp, \perp).$$

That $\overline{WC}_{2,1}$ is not a functor could be seen from the fact that $g^* \circ f^*$, which is defined by

$$(A, B, C, D) \mapsto ((A \wedge B) \vee (\perp \wedge \perp), \top \vee \top, \perp, \perp, \perp, \perp, \perp, \perp),$$

is different from $(g \circ f)^*$. Here $g \circ f = (g_1 \circ f_1, g_2 \circ f_2, g_3 \circ f_3): (1, 2, 2) \rightarrow (2, 2, 2)$ is graphically presented as

$$\begin{array}{ccc}
\begin{array}{c} 0 \quad 1 \quad 2 \\ \downarrow \quad \downarrow \quad \searrow \\ 0 \quad 1 \quad 2 \quad 3 \end{array} &
\begin{array}{c} 0 \quad 1 \quad 2 \quad 3 \\ \downarrow \quad \downarrow \quad \searrow \quad \downarrow \\ 0 \quad 1 \quad 2 \quad 3 \end{array} &
\begin{array}{c} 0 \quad 1 \quad 2 \quad 3 \\ \downarrow \quad \downarrow \quad \swarrow \quad \downarrow \\ 0 \quad 1 \quad 2 \quad 3 \end{array} \\
g_1 \circ f_1 & g_2 \circ f_2 & g_3 \circ f_3
\end{array}$$

and $(g \circ f)^*: \mathcal{C}^{1 \cdot 2 \cdot 2} \rightarrow \mathcal{C}^{2 \cdot 2 \cdot 2}$ is defined by

$$\begin{aligned}
(A, B, C, D) &\mapsto (A, B, C, D, \perp, \perp, \perp, \perp) \mapsto (A, B, \perp, \perp, \perp, \perp, \perp, \perp) \mapsto \\
&\mapsto (A \wedge B, \top, \perp \wedge \perp, \top, \perp \wedge \perp, \top, \perp \wedge \perp, \top).
\end{aligned}$$

However, we have a natural transformation from $g^* \circ f^*$ to $(g \circ f)^*$ whose components are $c_{A,B,\perp,\perp}^k$, $\hat{w}_{\top}^{\rightarrow}$, $\hat{w}_{\perp}^{\leftarrow}$ (three times) and κ (three times). This natural transformation acts as $\omega_{g,f}$ from the definition of lax functor and since $\overline{WC}_{n,m}$ preserves identity arrows, there is no need for the natural transformation ω_A .

To show that $\overline{WC}_{n,m}$, for $n+m \geq 2$, is indeed a lax functor, we have to find for every composable pair of arrows f, g of $(\Delta^+)^{op} \times \dots \times (\Delta^+)^{op}$, a natural transformation $\omega_{g,f}: g^* \circ f^* \xrightarrow{\sim} (g \circ f)^*$, such that the following diagram commutes

$$\begin{array}{ccccc}
& & h^* \circ g^* \circ f^* & & \\
& \swarrow \omega_{h,g}(f^*) & & \searrow h^*(\omega_{g,f}) & \\
(lax) & (h \circ g)^* \circ f^* & & h^* \circ (g \circ f)^* & \\
& \searrow \omega_{h \circ g, f} & & \swarrow \omega_{h,g \circ f} & \\
& & (h \circ g \circ f)^* & &
\end{array}$$

For this, we rely on the category \mathcal{M}^{st} , which is strict monoidal *SMI* category freely generated by the same infinite set \mathcal{P} of generators we have used for the category \mathcal{M} in Section 3. The category \mathcal{M}^{st} is obtained from our category \mathcal{M} by factoring its objects through the smallest equivalence relation \equiv satisfying

$$\begin{aligned}
A \vee (B \vee C) &\equiv (A \vee B) \vee C, & A \wedge (B \wedge C) &\equiv (A \wedge B) \wedge C, \\
A &\equiv A \wedge \top \equiv \top \wedge A \equiv A \vee \perp \equiv \perp \vee A,
\end{aligned}$$

which is congruent with respect to \vee and \wedge , and by further factoring its arrow terms according to the new equations

$$\begin{aligned}
\check{b}_{A,B,C}^{\rightarrow} &= \check{b}_{A,B,C}^{\leftarrow} = \mathbf{1}_{A \vee B \vee C}, & \hat{b}_{A,B,C}^{\rightarrow} &= \hat{b}_{A,B,C}^{\leftarrow} = \mathbf{1}_{A \wedge B \wedge C}, \\
\check{\delta}_A^{\rightarrow} &= \check{\delta}_A^{\leftarrow} = \check{\sigma}_A^{\rightarrow} = \check{\sigma}_A^{\leftarrow} = \hat{\delta}_A^{\rightarrow} = \hat{\delta}_A^{\leftarrow} = \hat{\sigma}_A^{\rightarrow} = \hat{\sigma}_A^{\leftarrow} = \mathbf{1}_A.
\end{aligned}$$

(Hence, in writing objects of \mathcal{M}^{st} , we may omit parentheses tied to \vee , and the constant \perp in the immediate scope of another \vee and the same for \wedge and \top .) An object of \mathcal{M}^{st} is *pure* and *diversified* when it, as an equivalence class, consists of formulae that are pure and diversified. As a direct consequence of Theorem 3.1 we have

COROLLARY OF THEOREM 3.1. *If A and B are either pure and diversified or no letter occurs in them, then there is at most one arrow $f: A \rightarrow B$ in \mathcal{M}^{st} .*

The following lemma serves to reduce our problem to the category \mathcal{M}^{st} .

LEMMA 7.1. *If for $\overline{\mathcal{W}}\mathcal{M}_{n,m}^{st}$ the following holds:*

(1) *for every pair of arrows $f: (k_1, \dots, k_{n+m}) \rightarrow (l_1, \dots, l_{n+m})$ and $g: (l_1, \dots, l_{n+m}) \rightarrow (j_1, \dots, j_{n+m})$ of $(\Delta^+)^{op} \times \dots \times (\Delta^+)^{op}$, and every $k_1 \cdot \dots \cdot k_{n+m}$ -tuple of different letters $\vec{p} = p_{11\dots 1}, p_{11\dots 2}, \dots, p_{k_1 k_2 \dots k_{n+m}}$, there is an arrow $\omega_{f,g}(\vec{p}): g^* \circ f^*(\vec{p}) \rightarrow (g \circ f)^*(\vec{p})$ of $(\mathcal{M}^{st})^{j_1 \dots j_{n+m}}$, and*

(2) *for every sequence of composable arrows $f_1 \dots f_u$ of $(\Delta^+)^{op} \times \dots \times (\Delta^+)^{op}$, each coordinate of $f_u^* \circ \dots \circ f_1^*(\vec{p})$, is either pure and diversified or no letter occurs in it,*

then for every strict monoidal SMI category \mathcal{C} , we have that $\overline{\mathcal{W}}\mathcal{C}_{n,m}$ is a lax functor.

PROOF. Using the freedom of \mathcal{M}^{st} and (1) we define for every $k_1 \cdot \dots \cdot k_{n+m}$ -tuple $\vec{A} = (A_{11\dots 1}, A_{11\dots 2}, \dots, A_{k_1 \dots k_{n+m}})$ of objects of \mathcal{C} the arrow $\omega_{f,g}(\vec{A})$ as the image of $\omega_{f,g}(\vec{p})$ under the functor that extends the function sending generator $p_{i_1 \dots i_{n+m}}$ of \mathcal{M}^{st} to the object $A_{i_1 \dots i_{n+m}}$ of \mathcal{C} . From (2) ($u = 1$ and $u = 3$ are the only interesting cases), appealing again to the freedom of \mathcal{M}^{st} , and to Corollary of Theorem 3.1, we have that the diagram (lax) commutes. \dashv

To prove that (1) holds, we reason as in [1]. Since for every $i \in \{1, \dots, n+m\}$ we have that

$$(\mathbf{1}_{k_1}, \dots, \mathbf{1}_{k_{i-1}}, f_i, \mathbf{1}_{k_{i+1}}, \dots, \mathbf{1}_{k_{n+m}})^*$$

is a functor, it is sufficient to show that for every $1 \leq i < j \leq n+m$, and $f_i: k_i \rightarrow l_i$ and $f_j: k_j \rightarrow l_j$ arrows of $(\Delta^+)^{op}$ there is an arrow of $(\mathcal{M}^{st})^{k_1 \dots l_i \dots l_j \dots k_{n+m}}$ whose source is

$$(\mathbf{1}_{k_1}, \dots, f_i, \dots, \mathbf{1}_{l_j}, \dots, \mathbf{1}_{k_{n+m}})^* \circ (\mathbf{1}_{k_1}, \dots, \mathbf{1}_{k_i}, \dots, f_j, \dots, \mathbf{1}_{k_{n+m}})^*(\vec{p}),$$

and whose target is

$$(\mathbf{1}_{k_1}, \dots, \mathbf{1}_{l_i}, \dots, f_j, \dots, \mathbf{1}_{k_{n+m}})^* \circ (\mathbf{1}_{k_1}, \dots, f_i, \dots, \mathbf{1}_{k_j}, \dots, \mathbf{1}_{k_{n+m}})^*(\vec{p}).$$

Since it is sufficient to find each coordinate of this arrow, we may assume that all numbers except k_i and k_j are 1, and we write $k_i \cdot k_j$ tuple \vec{p} as $p_{11}, p_{12}, \dots, p_{k_i k_j}$.

Let $f_i: k_i \rightarrow 1$ and $f_j: k_j \rightarrow 1$ be the arrows of $(\Delta^+)^{op}$ graphically presented as



So, in the case when $i < j \leq n$ we need an arrow

$$\bigvee_{x=v+1}^{v+w} \bigvee_{y=t+1}^{t+u} p_{xy} \rightarrow \bigvee_{y=t+1}^{t+u} \bigvee_{x=v+1}^{v+w} p_{xy},$$

which is $\mathbf{1}_\perp$ when either u or w is 0, or it is built out of \check{c} , otherwise. In the case when $i \leq n < j$ we need an arrow

$$\bigvee_{x=v+1}^{v+w} \bigwedge_{y=t+1}^{t+u} p_{xy} \rightarrow \bigwedge_{y=t+1}^{t+u} \bigvee_{x=v+1}^{v+w} p_{xy},$$

which is built out of $\check{w}_\top^\rightarrow$, \hat{w}_\perp^\leftarrow and κ when u or v is 0, or it is built out of c^k , otherwise. In the case when $n < i < j$ we proceed as in the first case relying on $\mathbf{1}_\top$ and \hat{c} . So, (1) is proved.

To prove that (2) holds, note first that the equivalence relation used to factor the objects of \mathcal{M} in order to obtain the objects of \mathcal{M}^{st} is congruent with respect to the function ν defined in Section 3. So, ν may be considered as a function on the objects of \mathcal{M}^{st} . We say that an object $\vec{A} = (A_{i_1 \dots i_n}, A_{i_1 \dots i_{n+1}}, \dots, A_{i_1 \dots i_{n+m}})$ of $(\mathcal{M}^{st})^{k_1 \dots k_{n+m}}$ is (n, m) -coherent when the following holds for $1 \leq i_l, j_l \leq k_l$:

(*) Every $A_{i_1 i_2 \dots i_{n+m}}$ is either pure and diversified or no letter occurs in it, and $\text{let}(A_{i_1 i_2 \dots i_{n+m}}) \cap \text{let}(A_{j_1 j_2 \dots j_{n+m}}) = \emptyset$, when $i_1 i_2 \dots i_{n+m} \neq j_1 j_2 \dots j_{n+m}$;

(**) For every m -tuple $i_{n+1} \dots i_{n+m}$, if for some n -tuple i_1, \dots, i_n we have that $\nu(A_{i_1 \dots i_n i_{n+1} \dots i_{n+m}})$ is \top , then for every n -tuple j_1, \dots, j_n we have that $\nu(A_{j_1 \dots j_n i_{n+1} \dots i_{n+m}})$ is \top or \perp ;

(***) For every n -tuple i_1, \dots, i_n , if for some m -tuple i_{n+1}, \dots, i_{n+m} we have that $\nu(A_{i_1 \dots i_n i_{n+1} \dots i_{n+m}})$ is \perp , then for every m -tuple j_{n+1}, \dots, j_{n+m} we have that $\nu(A_{i_1 \dots i_n j_{n+1} \dots j_{n+m}})$ is \top or \perp .

The following lemma has (2) as an immediate corollary.

LEMMA 7.2. *For every $1 \leq i \leq n+m$ and every arrow $f_i: k_i \rightarrow l_i$ of $(\Delta^+)^{op}$, if \vec{A} is (n, m) -coherent object of $(\mathcal{M}^{st})^{k_1 \dots k_{n+m}}$, then*

$$\vec{B} = (\mathbf{1}_{k_1}, \dots, \mathbf{1}_{k_{i-1}}, f_i, \dots, \mathbf{1}_{k_{n+m}})^*(\vec{A})$$

is (n, m) -coherent object of $(\mathcal{M}^{st})^{k_1 \dots l_i \dots k_{n+m}}$.

PROOF. Since for every $i \in \{1, \dots, n+m\}$ we have that

$$(\mathbf{1}_{k_1}, \dots, \mathbf{1}_{k_{i-1}}, f_i, \mathbf{1}_{k_{i+1}}, \dots, \mathbf{1}_{k_{n+m}})^*$$

is a functor, it is sufficient to prove the lemma for f_i being $d_j^{k_i}: k_i \rightarrow k_i - 1$ or $s_j^{k_i+1}: k_i \rightarrow k_i + 1$. One can use the following table to verify that \vec{B} satisfies (*), (**) and (***). In this table the index α is an $n+m$ sequence of natural numbers, $\alpha_i \in [1, l_i]$ is its i -th component, e_i is the $n+m$ sequence with 1 as the i -th component and 0 everywhere else, and the addition-subtraction is componentwise.

f_i	B_α	
$d_0^{k_i}$	$A_{\alpha+e_i}$	
$d_{k_i}^{k_i}$	A_α	
$d_j^{k_i}$ $0 < j < k_i$	A_α	$\alpha_i < j$
	$A_\alpha \vee A_{\alpha+e_i}$	$\alpha_i = j \ \& \ 1 \leq i \leq n \quad (**)$
	$A_\alpha \wedge A_{\alpha+e_i}$	$\alpha_i = j \ \& \ n < i \leq n+m \quad (***)$
	$A_{\alpha+e_i}$	$\alpha_i > j$
$S_j^{k_i+1}$ $0 \leq j \leq k_i$	A_α	$\alpha_i < j+1$
	\perp	$\alpha_i = j+1 \ \& \ 1 \leq i \leq n$
	\top	$\alpha_i = j+1 \ \& \ n < i \leq n+m$
	$A_{\alpha-e_i}$	$\alpha_i > j+1$

In the case marked by $(**)$ we use essentially the property $(**)$ of \vec{A} to establish that $(*)$ holds for \vec{B} , and analogously for $(***)$. This is the reason why the properties $(**)$ and $(***)$ occur in the definition of (n, m) -coherent object. \dashv

Now (2) follows immediately since every $k_1 \cdot \dots \cdot k_{n+m}$ -tuple \vec{p} of different letters is obviously (n, m) -coherent and one has just to iterate Lemma 7.2 through the definition of $f_u^* \circ \dots \circ f_1^*(\vec{p})$, and eventually, to use the property $(*)$ of the obtained object. Hence, we conclude from Lemma 7.1 that every (n, m) -reduced bar construction based on a strict monoidal SMI category \mathcal{C} produces a lax functor.

8 Two questions

The following questions, to which I have no answer, may come to mind to a careful reader of this paper:

1) Do we have an unrestricted coherence for SMI categories, i.e. whether all diagrams (with diversified objects in the nodes) commute in \mathcal{M} ?

2) Since there is no need for $\tilde{w}_\top^\leftarrow$ and $\hat{w}_\perp^\rightarrow$ in the construction of $\omega_{f,g}$, is it possible to omit the assumptions that $\tilde{w}_\top^\rightarrow$ and \hat{w}_\perp^\leftarrow are isomorphisms from the definition of SMI categories, without loss of coherence necessary for (n, m) -reduced bar construction?

The first question is of lower interest, at least for the (n, m) -reduced bar construction, since we managed to work without unrestricted coherence for SMI categories. Some serious doubts about holding of such a coherence result may be found in [6], Section 7. However, the second question may be quite interesting for the matters of (n, m) -reduced bar construction. An affirmative answer says that this construction may be based on every category with finite coproducts and products without restriction to those categories having initial object as the product of initial objects, and terminal object as the coproduct of terminal objects.

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